

7-16-2001

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Kessler, Bruce. (2001). A Construction of Compactly-Supported Biorthogonal Scaling Vectors and Multiwavelets on \mathbb{R}^2 . *Journal of Approximation Theory*, 117 (2), 229-254.

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A Construction of Compactly-Supported Biorthogonal Scaling Vectors and Multiwavelets on \mathbf{R}^2

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Abstract

In [15], a construction was given for a class of orthogonal compactly-supported scaling vectors on \mathbf{R}^2 , called short scaling vectors, and their associated multiwavelets. The span of the translates of the scaling functions along a triangular lattice includes continuous piecewise linear functions on the lattice, although the scaling functions are fractal interpolation functions and possibly nondifferentiable. In this paper, a similar construction will be used to create biorthogonal scaling vectors and their associated multiwavelets. The additional freedom will allow for one of the dual spaces to consist entirely of the continuous piecewise linear functions on a uniform subdivision of the original triangular lattice.

1 Introduction

Much research has been done in the construction of orthogonal multiresolution analyses of $L^2(\mathbf{R})$ (see [5], [10], and [11]) and the associated multiwavelets (see [5], [8], and [18].) All of the multiwavelet constructions have involved the completion of a matrix satisfying certain conditions. A class of nonseparable, orthogonal dilation-3 scaling functions defined on uniform triangulations of \mathbf{R}^2 were constructed in [6] and [9]. Multiwavelets for this specific example were found in [7]. In [15], the author generalized the construction of the orthogonal scaling vectors and provided a construction of the associated multiwavelets. This paper generalizes those results to construct a class of biorthogonal scaling vectors and the associated multiwavelets. The construction of the multiwavelets is analogous to the construction of multiwavelets for short scaling vectors introduced in [15].

1.1 Orthogonal Scaling Vectors and Multiwavelets

Let ϵ_1 and ϵ_2 be linearly independent vectors in \mathbf{R}^2 and let $\epsilon_0 := (0, 0)$. For each $x \in \mathbf{R}^2$, there exist constants a and b such that $x = a\epsilon_1 + b\epsilon_2$. Then define the metric $|x|_\epsilon$ by

$$|x|_\epsilon = \begin{cases} |a + b| & \text{if both } a, b \text{ have the same sign} \\ \max\{|a|, |b|\} & \text{otherwise.} \end{cases}$$

Let \mathcal{T} be the 3-directional mesh with directions ϵ_1 , ϵ_2 , and $\epsilon_2 - \epsilon_1$. Define $\Delta_0 \in \mathcal{T}$ to be the triangular region with vertices ϵ_0 , ϵ_1 , and ϵ_2 , and $\nabla_0 \in \mathcal{T}$ to be the triangular region with vertices ϵ_1 , ϵ_2 , and $\epsilon_1 + \epsilon_2$. Define the translation function $t_{i,j}(x) := x - i\epsilon_1 - j\epsilon_2$ and the dilation function $d_{i,j}(x) := Nx - i\epsilon_1 - j\epsilon_2$ for some fixed integer dilation $N > 1$. Define the affine reflection function $r : \nabla_0 \rightarrow \Delta_0$ that maps the vertices ϵ_1 , ϵ_2 , and $\epsilon_1 + \epsilon_2$ to vertices ϵ_2 , ϵ_0 , and ϵ_1 , respectively. The notation $\tilde{f} := f \circ r$ is used for any f supported in Δ_0 .

Definition: A *multiresolution analysis* (MRA) of $L^2(\mathbf{R}^2)$ of multiplicity r is a set of closed linear subspaces such that

1. $\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$
2. $\bigcap_{n \in \mathbf{Z}} V_n = \{0\}$
3. $\overline{\bigcup_{n \in \mathbf{Z}} V_n} = L^2(\mathbf{R}^2)$
4. $f \in V_n \Leftrightarrow f(N^{-n}\cdot) \in V_0, n \in \mathbf{Z}$
5. There exists a set of functions $\{\phi^1, \phi^2, \dots, \phi^r\}$ such that $\{\phi^k \circ t_i : k = 1, \dots, r, i \in \mathbf{Z}^2\}$ forms a Riesz basis of V_0 .

The r -vector $\Phi := (\phi^1, \phi^2, \dots, \phi^r)^T$ is referred to as a *scaling vector* and the individual ϕ^k as *scaling functions*.

Conditions 1, 4, and 5 imply that a scaling vector Φ with compactly-supported ϕ^k satisfies the dilation equation

$$\Phi(x) = N \sum_{i \in \mathbf{Z}^2} g_i \Phi \circ d_i \tag{1.1}$$

for a finite number of $r \times r$ scalar matrices g_i .

Definition: A vector Φ of r linearly independent functions on \mathbf{R}^2 is *refinable* at dilation N if it satisfies (1.1) for some sequence of $r \times r$ scalar matrices c_i .

A simple example of a MRA of $L^2(\mathbf{R}^2)$ over the mesh \mathcal{T} is constructed by defining the “hat” function h by

$$h(x) := \begin{cases} 1 - |x|_\epsilon, & \text{for } |x|_\epsilon < 1 \\ 0, & \text{otherwise,} \end{cases} \quad (1.2)$$

and letting $\Phi = \{h\}$. Using the notation

$$S(H) := \text{clos}_{L^2} \text{span} \{f \circ t_i : i \in \mathbf{Z}^2, f \in H\} \text{ for } H \subset L^2(\mathbf{R}^2),$$

then define $V_0 := S(\Phi)$. It is easily verified that the scaling vector is refinable for any integer dilation $N > 1$, and that (V_p) is a MRA, where $V_p := S(\Phi(N^p \cdot))$.

For function vectors Γ and Λ with elements in $L^2(\mathbf{R}^2)$, define

$$\langle \Gamma, \Lambda \rangle = \int_{\mathbf{R}^2} \Gamma(x) \Lambda(x)^T dx.$$

Definition: If $\langle \Phi, \Phi \circ t_{i,j} \rangle = \delta_{0,i} \delta_{0,j} I$, then we say that Φ is an *orthogonal scaling vector*. If the ϕ^k are compactly-supported, then the multiresolution analysis generated by Φ is said to be *orthogonal*.

Define W_n to be the orthogonal complement of V_n in V_{n+1} , so that

$$V_{n+1} = V_n \oplus W_n \text{ for } n \in \mathbf{Z}.$$

The W_n , referred to as wavelet spaces, are necessarily pairwise orthogonal and are spanned by the orthogonal dilations and translations of a set of functions $\{\psi^1, \psi^2, \dots, \psi^t\}$, referred to as wavelets, that satisfy the equation

$$\Psi(x) = N \sum_{i \in \mathbf{Z}^2} h_i \Phi \circ d_i \quad (1.3)$$

for some h_i , where Ψ is the t-vector $(\psi^1, \psi^2, \dots, \psi^t)^T$.

Definition: A pair of n -dimensional function vectors Φ and $\tilde{\Phi}$ are said to be *biorthogonal* if

$$\langle \Phi, \tilde{\Phi} \circ t_{i,j} \rangle = \delta_{0,i} \delta_{0,j} I, \quad i, j \in \mathbf{Z}.$$

A necessary and sufficient condition for the construction of biorthogonal vectors was given in [12], and will be stated here without proof.

Lemma 1.1 *Suppose U and W are m -dimensional subspaces of \mathbf{R}^n . There exist dual (biorthogonal) bases for U and W if and only if $U \cap W^\perp = \{0\}$.*

If the criteria of Lemma 1.1 are met, then the Gram-Schmidt orthogonalization process can be modified to extract biorthogonal sets from bases for U and W .

1.2 Short Scaling Vectors

Throughout the paper, P_K denotes the orthogonal projection onto a subspace K of $L^2(\mathbf{R}^2)$. If C is a compact set of \mathbf{R}^2 and U is a space of functions on \mathbf{R}^2 , then define

$$U(C) := \{f \in U : \text{supp}(f) \subseteq C\}.$$

Definition: Suppose $\bar{\Phi} = (\phi^1, \dots, \phi^s)^T$ is refinable. If $w = (w^1, \dots, w^t)^T$ is such that $\Phi = (w^1, \dots, w^t, \phi^1, \dots, \phi^s)^T$ is refinable, then w is said to *extend* $\bar{\Phi}$.

The construction of short scaling vectors is given in [15], and is a generalization of a construction in [6]. Recall the nonorthogonal scaling vector generated by the ‘‘hat’’ function h defined in (1.2). Define $h_i := h(x - \epsilon_i)|_{\Delta_0}$, $i = 0, 1, 2$.

Definition: Suppose there is a subspace \mathcal{W} of $C_0(\mathbf{R}^2)$ with an orthogonal basis $\{w^1, \dots, w^t\}$ such that

1. $B := \{w^1, \dots, w^t, \check{w}^1, \dots, \check{w}^t\}$ extends $\{h\}$,
2. $\text{supp}(w^i) \subseteq \Delta_0$, $i = 1, \dots, t$,
3. $(I - P_{\mathcal{W}})h_i \perp (I - P_{\mathcal{W}})h_j$, $i \neq j$, $i, j \in \{0, 1, 2\}$.

Then

$$\Phi := \{w^1, \dots, w^t, \check{w}^1, \dots, \check{w}^t, (I - P_{S(B)})h\}^T$$

is called a *short* scaling vector, and generates a MRA (V_p) of multiplicity $q := 2t + 1$ such that V_0 still includes continuous piecewise linear functions on \mathcal{T} .

A dilation-3 example with $\mathcal{W} = \{w\}$ is given in [15] and is illustrated in Figure 1. The scaling

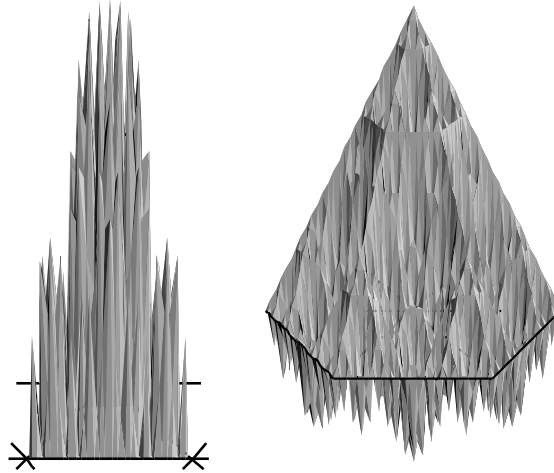


Figure 1: Approximations to scaling functions from a dilation-3 short scaling vector.

functions are fractal interpolation surfaces that are nondifferentiable and have a nonintegral box dimension greater than 2. (A full introduction to fractal interpolation surfaces can be found in [9] and [17].)

2 Main Results

Suppose that X and Y are spaces spanned by biorthogonal function vectors. Then define the projection operator P_X^Y such that $\ker P_X^Y = Y^\perp$ and $\text{range } P_X^Y = X$. If $\mathcal{X} := S(X)$ and $\mathcal{Y} := S(Y)$ are finite shift invariant spaces, then

$$P_X^Y f := \sum_{j \in \mathbf{Z}^2} \sum_{i=1}^n \frac{\langle f, y^i \circ t_j \rangle}{\langle x^i, y^i \rangle} x^i \circ t_j$$

where $x_i \in X$ and $y_i \in Y$.

Define h, h_0, h_1 , and h_2 as in Section 1.3. Then we have the following result.

Theorem 2.1 *Suppose there are function vectors $B := \{w^1, \dots, w^t\}$ and $\tilde{B} := \{\tilde{w}^1, \dots, \tilde{w}^t\}$ in $C_0(\mathbf{R}^2)$ such that*

1. B and \tilde{B} are biorthogonal,
2. $\{w^1, \dots, w^t, \check{w}^1, \dots, \check{w}^t\}$ and $\{\tilde{w}^1, \dots, \tilde{w}^t, \check{\tilde{w}}^1, \dots, \check{\tilde{w}}^t\}$ each extend $\{h\}$,
3. $\text{supp}(w^i), \text{supp}(\tilde{w}^i) \subseteq \Delta_0, i = 1, \dots, t$, and
4. $(I - P_{\tilde{W}}^W)h_i \perp (I - P_W^{\tilde{W}})h_j, i \neq j, i, j \in \{0, 1, 2\}$, where $W = S(B)$ and $\tilde{W} = S(\tilde{B})$.

Then there exist biorthogonal scaling vectors Φ and $\tilde{\Phi}$ of length $q := 2t + 1$ such that $V_0 := S(\Phi)$ and $\tilde{V}_0 := S(\tilde{\Phi})$ each contain continuous piecewise linear functions on the mesh \mathcal{T} .

Proof. The main issue is finding compactly-supported functions ϕ^i and $\tilde{\phi}^j$ that satisfy the biorthogonality conditions $\langle \phi^i, \tilde{\phi}^j \rangle = \delta_{i,j}$. Define the following:

$$\begin{aligned} \phi^i &:= w^i \text{ for } i = 1, \dots, t, & \tilde{\phi}^i &:= \tilde{w}^i \text{ for } i = 1, \dots, t, \\ \phi^{t+i} &:= \check{w}^i \text{ for } i = 1, \dots, t, \text{ and } \tilde{\phi}^{t+i} &:= \check{\tilde{w}}^i \text{ for } i = 1, \dots, t, \\ \phi^q &:= \frac{1}{\alpha}(I - P_W^{\tilde{W}})h, & \tilde{\phi}^q &:= \frac{1}{\beta}(I - P_{\tilde{W}}^W)h, \end{aligned}$$

where α, β are constants such that $\alpha\beta := \langle (I - P_W^{\tilde{W}})h, (I - P_{\tilde{W}}^W)h \rangle$. Let $\Phi := (\phi^1, \dots, \phi^q)^T$ and $\tilde{\Phi} := (\tilde{\phi}^1, \dots, \tilde{\phi}^q)^T$. Then set $V_p := S(\Phi(N^p \cdot))$ and $\tilde{V}_p := S(\tilde{\Phi}(N^p \cdot))$.

Condition (1) guarantees that

$$\begin{aligned} \langle \phi^i, \tilde{\phi}^j \rangle &= \delta_{i,j} \text{ for } i, j = 1, \dots, t, \text{ and} \\ \langle \phi^i, \tilde{\phi}^j \rangle &= \delta_{i,j} \text{ for } i, j = t+1, \dots, 2t. \end{aligned}$$

Condition (3) guarantees that

$$\begin{aligned} \langle \phi^i, \tilde{\phi}^j \rangle &= 0 \text{ for } i = 1, \dots, t, j = t+1, \dots, 2t, \text{ and} \\ \langle \phi^i, \tilde{\phi}^j \rangle &= 0 \text{ for } i = t+1, \dots, 2t, j = 1, \dots, t. \end{aligned}$$

Condition (4) establishes the remaining orthogonality conditions:

$$\begin{aligned} \langle \phi^q, \tilde{\phi}^i \rangle &= 0 \text{ for } i = 1, \dots, 2t, \text{ and} \\ \langle \phi^i, \tilde{\phi}^q \rangle &= 0 \text{ for } i = 1, \dots, 2t. \end{aligned}$$

Condition (2) guarantees that both Φ and $\tilde{\Phi}$ are refinable, and that $V_n \subset V_{n+1}$ and $\tilde{V}_n \subset \tilde{V}_{n+1}$. The requirements that $\cap_{j \in \mathbf{Z}} V_j = 0$, $\cap_{j \in \mathbf{Z}} \tilde{V}_j = 0$, $\overline{\cup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R})$, and $\overline{\cup_{j \in \mathbf{Z}} \tilde{V}_j} = L^2(\mathbf{R})$, and that the translates of Φ and $\tilde{\Phi}$ form Riesz bases, are trivially met by compactly-supported scaling vectors. Therefore, both (V_p) and (\tilde{V}_p) are MRA's. \square

While the restrictions on the spaces B and \tilde{B} are extensive, such spaces do exist. An example is provided in Section 3. It is important to note that, as with the short scaling vectors, the scaling functions and associated multiwavelets in these constructions may be nondifferentiable FIS. However, by relaxing the need for orthogonality, it is possible to construct a ‘‘smoother’’ basis that can be used in the reconstruction phase of applications.

Section 4 will give a detailed definition of the wavelet spaces $W_f, \tilde{W}_f, W_g, \tilde{W}_g, W_h,$ and \tilde{W}_h . W_f and \tilde{W}_f will have generators supported on triangles, W_g and \tilde{W}_g will have generators supported on parallelograms, and W_h and \tilde{W}_h will have generators supported on hexagons. The main theorem on the construction of the $q(N^2 - 1)$ wavelets will be stated and proven in that section.

3 A Construction of Biorthogonal Scaling Vectors

Set vectors $\epsilon_1 := (1, 0)$ and $\epsilon_2 := (1/2, \sqrt{3}/2)$, so that \mathcal{T} is a 3-directional mesh of equilateral triangles. Let h be the generalized hat function defined in (1.2) and fix $N = 3$. In order to construct a scaling vector Φ that satisfies Theorem 2.1, let w and u be continuous functions with (nonempty) support in Δ_0 and let $\check{w} := w \circ r$ and $\check{u} := u \circ r$. Let $G = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (0, 2)\}$ and let $\check{G} = \{(0, 0), (1, 0), (0, 1)\}$. With condition 2 of Theorem 2.1 in mind, we require that w and u satisfy the following dilation equations for some $\alpha, \beta, s_i, \check{s}_i, q_i, \check{q}_i$:

$$w = \alpha h \circ d_{1,1} + \sum_{i \in G} s_i w \circ d_i + \sum_{i \in \check{G}} \check{s}_i \check{w} \circ d_i. \quad (3.1)$$

$$u = \beta h \circ d_{1,1} + \sum_{i \in G} q_i u \circ d_i + \sum_{i \in \check{G}} \check{q}_i \check{u} \circ d_i. \quad (3.2)$$

The functions w and u are FIS, with interpolation points located uniformly over Δ_0 as illustrated in Figure 2, provided $|s_i| < 1$ and $|q_i| < 1$ for all $i \in \mathbf{Z}^2$ (this is necessary if w and u are to be continuous).

In order to construct $w, \check{w}, u,$ and \check{u} with rotational symmetry about the centroid of their support triangle, let

$$\begin{aligned} s_{0,0} = s_{2,0} = s_{0,2} &:= s_1, & q_{0,0} = q_{2,0} = q_{0,2} &:= q_1, \\ s_{1,0} = s_{0,1} = s_{1,1} &:= s_2, & \text{and } q_{1,0} = q_{0,1} = q_{1,1} &:= q_2, \\ \check{s}_{0,0} = \check{s}_{1,0} = \check{s}_{0,1} &:= s_3, & \check{q}_{0,0} = \check{q}_{1,0} = \check{q}_{0,1} &:= q_3, \end{aligned}$$

where $|s_i|, |q_i| < 1$ for $i = 1, 2, 3$. Then the only free parameters will be the scaling variables s_i, q_i , and α and β , the values of the functions w and u at the centroid of Δ_0 , respectively. Set $\alpha, \beta := 1$ for this construction.

Recall that $h_i = h(\cdot - \epsilon_i)|_{\Delta_0}$, where $i = 0, 1, 2$. Due to the rotational invariance of both w and the set of h_i 's, the six orthogonality conditions needed to satisfy condition 4 of Theorem 2.1 reduce to just

$$(I - P_W^{\check{W}})h_0 \perp (I - P_W^W)h_1, \quad (3.3)$$

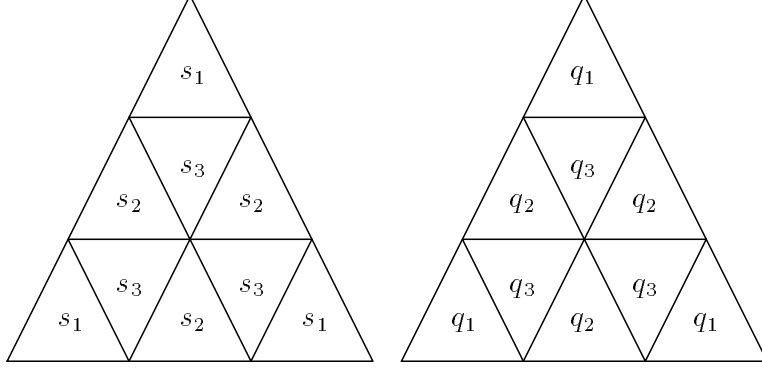


Figure 2: The domain points and scalings used in the example.

where $\mathcal{W} = S(w)$ and $\tilde{\mathcal{W}} = S(u)$. Since

$$P_{\tilde{\mathcal{W}}} h_0 = \frac{\langle h_0, u \rangle}{\langle w, u \rangle} w \text{ and } P_{\mathcal{W}} h_1 = \frac{\langle h_1, w \rangle}{\langle w, u \rangle} u,$$

then (3.3) reduces to

$$\langle h_0, h_1 \rangle = \frac{\langle h_0, u \rangle \langle h_1, w \rangle}{\langle w, u \rangle}. \quad (3.4)$$

Since $\langle h_0, w \rangle = \langle h_1, w \rangle = \langle h_2, w \rangle$ and $h_0 + h_1 + h_2 = 1$ on Δ_0 , we calculate $\langle w, 1 \rangle$ using (3.1):

$$\begin{aligned} \langle w, 1 \rangle &= \langle h \circ d_{1,1}, 1 \rangle + \sum_{i \in G} s_i \langle w \circ d_i, 1 \rangle + \sum_{i \in \tilde{G}} \check{s}_i \langle \check{w} \circ d_i, 1 \rangle \\ &= \frac{\langle h, 1 \rangle}{3 \left(3 - \sum_{i=1}^3 s_i \right)}. \end{aligned}$$

Likewise, from (3.2),

$$\langle u, 1 \rangle = \frac{\langle h, 1 \rangle}{3 \left(3 - \sum_{i=1}^3 q_i \right)}.$$

Since $\langle h, 1 \rangle = \frac{\sqrt{3}}{2}$, $\langle h_0, w \rangle = \frac{1}{3} \langle w, 1 \rangle$, and $\langle h_1, u \rangle = \frac{1}{3} \langle u, 1 \rangle$, then

$$\langle h_0, w \rangle = \frac{\sqrt{3}}{18 \left(3 - \sum_{i=1}^3 s_i \right)} \text{ and } \langle h_1, u \rangle = \frac{\sqrt{3}}{18 \left(3 - \sum_{i=1}^3 q_i \right)}. \quad (3.5)$$

Again, using both (3.1) and (3.2),

$$\langle w, u \rangle = \frac{\langle h, h \rangle + 3(s_2 + s_3) \langle h_0, w \rangle + 3(q_2 + q_3) \langle h_0, u \rangle}{3 \left(3 - \sum_{i=1}^3 s_i q_i \right)}.$$

Since $\langle h, h \rangle = \frac{\sqrt{3}}{4}$, then using (3.5),

$$\langle w, u \rangle = \frac{\sqrt{3} \left[3 \left(3 - \sum_{i=1}^3 s_i \right) \left(3 - \sum_{i=1}^3 q_i \right) + 2(s_2 + s_3) \left(3 - \sum_{i=1}^3 q_i \right) + 2(q_2 + q_3) \left(3 - \sum_{i=1}^3 s_i \right) \right]}{36 \left(3 - \sum_{i=1}^3 s_i \right) \left(3 - \sum_{i=1}^3 q_i \right) \left(3 - \sum_{i=1}^3 s_i q_i \right)}. \quad (3.6)$$

Substituting (3.5), (3.6), and $\langle h_0, h_1 \rangle = \frac{\sqrt{3}}{48}$ into (3.4) and requiring that (3.6) is nonzero provides the following necessary conditions on the s_i 's and q_i 's:

$$\begin{aligned} & 27s_1 + 9s_2 + 9s_3 + 27q_1 + 9q_2 + 9q_3 - 25s_1q_1 - 3s_1q_2 - 3s_1q_3 - 3s_2q_1 \\ & \quad - 13s_2q_2 + 3s_2q_3 - 3s_3q_1 + 3s_3q_2 - 13s_3q_3 - 33 = 0, \text{ and} \quad (3.7) \\ & 27 - 9q_1 - 3q_2 - 3q_3 - 9s_1 + 3q_1s_1 + q_2s_1 + q_3s_1 - 3s_2 + q_1s_2 \\ & \quad - q_2s_2 - q_3s_2 - 3s_3 + q_1s_3 - q_2s_3 - q_3s_3 \neq 0. \quad (3.8) \end{aligned}$$

By letting $s_i := 0$ for $i = 1, 2, 3$, w becomes piecewise linear and (3.7) and (3.8) reduce to

$$3(9q_1 + 3q_2 + 3q_3 - 11) = 0 \text{ and } 3(9 - 3q_1 - q_2 - q_3) \neq 0. \quad (3.9)$$

Furthermore, by letting $q =: q_i$ for $i = 1, 2, 3$, (3.9) reduces to $45q - 33 = 0$ and $27 - 15q \neq 0$, with the solution $q = 11/15$.

Define the scaling functions

$$\begin{aligned} \phi^1 & := w, \\ \phi^2 & := \check{w}, \\ \phi^3 & := h - \sum_{i \in H} \frac{\langle h, u \circ t_i \rangle}{\langle w, u \rangle} w \circ t_i - \sum_{i \in \check{H}} \frac{\langle h, \check{u} \circ t_i \rangle}{\langle w, u \rangle} \check{w} \circ t_i, \\ \tilde{\phi}^1 & := \frac{u}{\langle w, u \rangle}, \\ \tilde{\phi}^2 & := \frac{\check{u}}{\langle w, u \rangle}, \\ \tilde{\phi}^3 & := \frac{1}{\alpha} \left[h - \sum_{i \in H} \frac{\langle h, w \circ t_i \rangle}{\langle w, u \rangle} u \circ t_i - \sum_{i \in \check{H}} \frac{\langle h, \check{w} \circ t_i \rangle}{\langle w, u \rangle} \check{u} \circ t_i \right], \end{aligned}$$

where $H = \{(0, 0), (0, -1), (-1, 0)\}$ and $\check{H} = \{(0, -1), (-1, 0), (-1, -1)\}$ and

$$\alpha := 6 \left(\langle h_0, h_0 \rangle - \frac{\langle h_0, w \rangle \langle h_0, u \rangle}{\langle w, u \rangle} \right).$$

Then $\Phi := (\phi^1, \phi^2, \phi^3)^T$ and $\tilde{\Phi} := (\tilde{\phi}^1, \tilde{\phi}^2, \tilde{\phi}^3)^T$ are biorthogonal scaling vectors that generate the MRA's $V_p = S(\Phi(3^p(x, y)))$ and $\check{V}_p = S(\tilde{\Phi}(3^p(x, y)))$. Note that both V_0 and \check{V}_0 contain piecewise linears on the triangulation \mathcal{T} and with $s_i = 0$ for $i = 1, 2, 3$, V_0 is the set of piecewise linears on a uniform subdivision of \mathcal{T} . This set of scaling functions and their biorthogonal counterparts with $q_i = 11/15$ for $i = 1, 2, 3$ are illustrated in Figure 3 and Figure 4.

Notice that for any nonsingular linear map A , we may define the same set of scaling functions on the lattice generated by Ae_1 and Ae_2 , and the functions will maintain their biorthogonality.

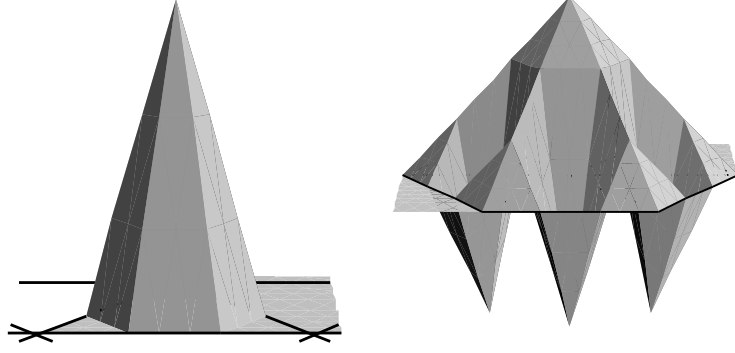


Figure 3: Scaling functions ϕ^1 and ϕ^3 with $s_i = 0$.

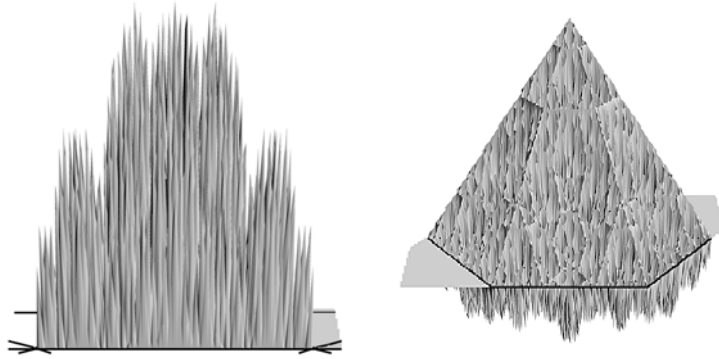


Figure 4: Approximations to scaling functions $\tilde{\phi}^1$ and $\tilde{\phi}^3$ with $q_i = 11/15$.

4 Construction of Associated Multiwavelets

Let Φ and $\tilde{\Phi}$ be the scaling vectors constructed in Theorem 2.1 and let (V_p) and (\tilde{V}_p) be the corresponding MRA's. Recall that $\text{supp } \phi^i, \text{supp } \tilde{\phi}^i \subseteq \Delta_0 = \Delta(\epsilon_0, \epsilon_1, \epsilon_2) \in \mathcal{T}$ for $i = 1, \dots, t$ and that $\text{supp } \phi^i, \text{supp } \tilde{\phi}^i \subseteq \nabla_0 = \Delta(\epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2) \in \mathcal{T}$ for $i = t + 1, \dots, 2t$. First consider wavelets supported in $\Delta \in \mathcal{T}$. Consider the $(tN^2 + \frac{(N-1)(N-2)}{2})$ -dimensional spaces $V_1(\Delta_0)$ and $\tilde{V}_1(\Delta_0)$, with the bases consisting of t dilated scaling functions on each of the N^2 subtriangles and $\frac{(N-1)(N-2)}{2}$ dilated ϕ^q and $\tilde{\phi}^q$.

Define the functions

$$g_i := P_{V_1(\Delta_0)}^{\tilde{V}_1(\Delta_0)}(\phi^q(\cdot - \epsilon_i)) \text{ and } \tilde{g}_i := P_{\tilde{V}_1(\Delta_0)}^{V_1(\Delta_0)}(\tilde{\phi}^q(\cdot - \epsilon_i)) \quad (4.1)$$

for $i = 0, 1, 2$. Then define the subspaces X of $V_1(\Delta_0)$ and \tilde{X} of $\tilde{V}_1(\Delta_0)$ by

$$\begin{aligned} X &:= \text{span} \left(\{g_i : i = 0, 1, 2\} \cup \{\phi^i : i = 1, \dots, t\} \right) \text{ and} \\ \tilde{X} &:= \text{span} \left(\{\tilde{g}_i : i = 0, 1, 2\} \cup \{\tilde{\phi}^i : i = 1, \dots, t\} \right). \end{aligned}$$

Let B be a basis for the space $(I - P_X^{\tilde{X}})V_1(\Delta_0)$ and let \tilde{B} be a basis for the space $(I - P_X^{\tilde{X}})\tilde{V}_1(\Delta_0)$. Note that the elements of B are orthogonal to \tilde{V}_0 and the elements of \tilde{B} are orthogonal to V_0 by definition. Also notice that due to their support, the elements of both B and \tilde{B} are orthogonal to their own translates.

A small lemma is needed before we proceed.

Lemma 4.1 $B \cap \tilde{B}^\perp = \{0\}$

Proof. Let $\mathcal{W} := \{\phi^i : i = 1, \dots, t\}$ and $\tilde{\mathcal{W}} := \{\tilde{\phi}^i : i = 1, \dots, t\}$ and notice that, from the construction of the scaling functions, \mathcal{W} and $\tilde{\mathcal{W}}$ are biorthogonal sets. Consider $\psi \in B \cap \tilde{B}^\perp$. Then $\text{supp}(\psi) \subset \Delta_0$ and $\psi \in P_X^{\tilde{X}}V_1(\Delta_0)$. Then ψ is a linear combination of elements in \tilde{X} orthogonal to X .

Consider $\langle \psi, \phi^i \rangle$, $i = 1, \dots, t$. Since $\{\tilde{g}_i : i = 0, 1, 2\} \perp \mathcal{W}$, ψ is a linear combination of elements in $\tilde{\mathcal{W}}$. But, by Lemma 1.1, $\tilde{\mathcal{W}} \cap \mathcal{W}^\perp = 0$, so $\psi = 0$. \square

Then from Lemma 4.1 and Lemma 1.1, there exist dual biorthogonal bases for B and \tilde{B} , denoted Ψ_f^Δ and $\tilde{\Psi}_f^\Delta$, respectively. Recall the notation $\check{f} = f \circ r$, where r is the affine transformation from ∇_0 to Δ_0 for $f \in L^2(\nabla_0)$. Define $\Psi_f^\nabla := \{\check{\psi} : \psi \in \Psi_f^\Delta\}$ and $\tilde{\Psi}_f^\nabla := \{\check{\tilde{\psi}} : \tilde{\psi} \in \tilde{\Psi}_f^\Delta\}$. Define $W_f := S(\Psi_f^\Delta \cup \Psi_f^\nabla) \subset W_0$ and $\tilde{W}_f := S(\tilde{\Psi}_f^\Delta \cup \tilde{\Psi}_f^\nabla) \subset \tilde{W}_0$. The spaces W_f and \tilde{W}_f each have

$$2 \left(tN^2 + \frac{(N-1)(N-2)}{2} - (t+3) \right) = q(N^2 - 1) - 3N - 3$$

generators.

Before proceeding, the following lemmas are needed.

Lemma 4.2 For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), $\langle g_i, \tilde{g}_j \rangle < -K$, $i \neq j$, where K is a positive constant.

Proof. Define $z_i := \phi^q(\cdot - \epsilon_i)|_{\Delta_0}$ and $\tilde{z}_i := \tilde{\phi}^q(\cdot - \epsilon_i)|_{\Delta_0}$ for $i = 0, 1, 2$. Recall that ϕ^q is the only scaling function with support larger than one $\Delta \in \mathcal{T}$. Notice that the z_i are still linear and nonnegative on all edges of Δ_0 .

Consider $\langle g_0, \tilde{g}_1 \rangle$. Let $\alpha := \phi^q(0, 0) = \tilde{\phi}^q(0, 0)$ and express both z_0 and \tilde{z}_1 in terms of basis functions for $V_1|_{\Delta_0}$ and the \tilde{g}_i :

$$\begin{aligned} z_0 &= \phi^q \circ d_{0,0}|_{\Delta_0} + \frac{1}{N} \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i}|_{\Delta_0} \\ &\quad + \frac{1}{N} \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{i,0}|_{\Delta_0} + g_0 \\ \tilde{z}_1 &= \tilde{\phi}^q \circ d_{N,0}|_{\Delta_0} + \frac{1}{N} \sum_{i=1}^{N-1} i\tilde{\phi}^q \circ d_{i,0}|_{\Delta_0} + \frac{1}{N} \sum_{i=1}^{N-1} i\tilde{\phi}^q \circ d_{i,N-i}|_{\Delta_0} + \tilde{g}_1. \end{aligned}$$

Recall that ϕ^q is orthogonal to the translates of $\tilde{\phi}^q$, even when restricted to bounded domains, so the same will be true for $\phi^q \circ d_{0,0}$ and $\tilde{\phi}^q \circ d_{0,0}$. Therefore, many of the terms of $\langle g_0, \tilde{g}_1 \rangle$ will vanish:

$$\langle z_0, \tilde{z}_1 \rangle = \frac{1}{N^2} \sum_{i=1}^{N-1} i(N-i) \langle \phi^q \circ d_{i,0}|_{\Delta_0}, \tilde{\phi}^q \circ d_{i,0}|_{\Delta_0} \rangle + \langle g_0, \tilde{g}_1 \rangle.$$

Since

$$\begin{aligned}\langle \phi^q \circ d_{i,0}|_{\Delta_0}, \tilde{\phi}^q \circ d_{i,0}|_{\Delta_0} \rangle &= \frac{1}{2} \langle \phi^q \circ d_{i,0}, \tilde{\phi}^q \circ d_{i,0} \rangle \\ &= \frac{1}{2N^2} \langle \phi^q, \tilde{\phi}^q \rangle = \frac{1}{2N^2},\end{aligned}$$

then

$$\langle z_0, \tilde{z}_1 \rangle = \frac{1}{2N^4} \sum_{i=1}^{N-1} i(N-i) + \langle g_0, \tilde{g}_1 \rangle.$$

Using the identities

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

then

$$\langle z_0, \tilde{z}_1 \rangle = \frac{N^2 - 1}{12N^3} + \langle g_0, \tilde{g}_1 \rangle. \quad (4.2)$$

Let $K := K(N)$ be the scalar dependent on N on the right side of (4.2). Note that $K(N) > 0$ for $N > 1$, so that

$$\langle g_0, \tilde{g}_1 \rangle = -K(N) = -K < 0$$

since $z_0 \perp \tilde{z}_1$. It is easily verified that the same result holds for the remaining $\langle g_i, \tilde{g}_j \rangle$, $i \neq j$. \square

Lemma 4.3 *For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), the sets $\{g_0, g_1, g_2\}$ and $\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$ are each linearly independent.*

Proof. This proof hinges on the linear algebra result that for an n -dimensional space A and a space B where $A \cap B = \{0\}$, then $(I - P_B)A$ is an n -dimensional space. Recall the linear polynomials h_i , $i = 0, 1, 2$, supported on Δ_0 and define the 3-dimensional space $H := \text{span}\{h_0, h_1, h_2\}$. Let $H^* := P_{V_1(\Delta_0)}H$. Since $H \cap (H - H^*) = \{0\}$, then $H^* = (I - (I - P_{V_1(\Delta_0)}))H$ is a 3-dimensional space.

Recall the space \mathcal{W} used in the construction of ϕ^q . Since $H^* \cap \mathcal{W} = \{0\}$, then

$$G := \text{span}\{g_0, g_1, g_2\} = (I - P_{\mathcal{W}})H^*$$

is a 3-dimensional space. An analogous proof holds for $\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$. \square

Lemma 4.4 *For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), there exist σ_i and $\tilde{\sigma}_i$, $i = 0, 1, 2$, such that*

1. $\text{span}\{\sigma_0, \sigma_1, \sigma_2\} = \text{span}\{g_0, g_1, g_2\}$
2. $\text{span}\{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\} = \text{span}\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$
3. $\{\sigma_0, \sigma_1, \sigma_2\}$ and $\{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$ are biorthogonal sets, and
4. $\sigma_i \perp \tilde{g}_i$ and $\tilde{\sigma}_i \perp g_i$ for $i = 0, 1, 2$.

Proof. Note that, from Lemma 4.2 and Lemma 4.3, the sets $G := \text{span}\{g_0, g_1, g_2\}$ and $\tilde{G} := \text{span}\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$ are each 3-dimensional, but not biorthogonal. Define the following biorthogonal bases for G and \tilde{G} :

$$\begin{aligned} v_2 &:= g_0, & \tilde{v}_2 &:= \tilde{g}_0, \\ v_0 &:= g_1 - \frac{\langle g_1, \tilde{v}_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} v_2, & \tilde{v}_0 &:= \tilde{g}_1 - \frac{\langle \tilde{g}_1, v_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} \tilde{v}_2, \\ v_1 &:= g_2 - \frac{\langle g_2, \tilde{v}_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} v_2 - \frac{\langle g_2, \tilde{v}_0 \rangle}{\langle v_0, \tilde{v}_0 \rangle} v_0, & \text{and } \tilde{v}_1 &:= \tilde{g}_2 - \frac{\langle \tilde{g}_2, v_2 \rangle}{\langle v_2, \tilde{v}_2 \rangle} \tilde{v}_2 - \frac{\langle \tilde{g}_2, v_0 \rangle}{\langle v_0, \tilde{v}_0 \rangle} \tilde{v}_0. \end{aligned}$$

Let $u_i := v_i$ and $\tilde{u}_i := \tilde{v}_i / \langle v_i, \tilde{v}_i \rangle$ for $i = 0, 1, 2$. Then the transformations $T : G \rightarrow \mathbf{R}^3$ and $\tilde{T} : \tilde{G} \rightarrow \mathbf{R}^3$ defined by

$$T(f) := (\langle f, \tilde{u}_0 \rangle, \langle f, \tilde{u}_1 \rangle, \langle f, \tilde{u}_2 \rangle)^T \text{ and } \tilde{T}(f) := (\langle f, u_0 \rangle, \langle f, u_1 \rangle, \langle f, u_2 \rangle)^T$$

are isometries and

$$\begin{aligned} T(g_0) &= (0, 0, \langle g_0, \tilde{u}_2 \rangle)^T, & \tilde{T}(\tilde{g}_0) &= (0, 0, \langle \tilde{g}_0, u_2 \rangle)^T, \\ T(g_1) &= (\langle g_1, \tilde{u}_0 \rangle, 0, \langle g_1, \tilde{u}_2 \rangle)^T, & \tilde{T}(\tilde{g}_1) &= (\langle \tilde{g}_1, u_0 \rangle, 0, \langle \tilde{g}_1, u_2 \rangle)^T, \\ T(g_2) &= (\langle g_2, \tilde{u}_0 \rangle, \langle g_2, \tilde{u}_1 \rangle, \langle g_2, \tilde{u}_2 \rangle)^T, & \text{and } \tilde{T}(\tilde{g}_2) &= (\langle \tilde{g}_2, u_0 \rangle, \langle \tilde{g}_2, u_1 \rangle, \langle \tilde{g}_2, u_2 \rangle)^T. \end{aligned}$$

Define $\omega_0 \in G$ and $\tilde{\omega}_0 \in \tilde{G}$ by $T(\omega_0) := (\cos \theta, \sin \theta, 0)^T$ and $\tilde{T}(\tilde{\omega}_0) := (\cos \tilde{\theta}, \sin \tilde{\theta}, 0)^T$, respectively, so that $\omega_0 \perp \tilde{g}_0$ for all θ and $\tilde{\omega}_0 \perp g_0$ for all $\tilde{\theta}$. Then define $\omega_1 \in G$ and $\tilde{\omega}_1 \in \tilde{G}$ by

$$\begin{aligned} T(\omega_1) &:= T(\tilde{\omega}_0) \times T(\tilde{g}_1) \\ &= (\langle \tilde{g}_1, u_2 \rangle \sin \tilde{\theta}, -\langle \tilde{g}_1, u_2 \rangle \cos \tilde{\theta}, -\langle \tilde{g}_1, u_0 \rangle \sin \tilde{\theta})^T \text{ and} \\ T(\tilde{\omega}_1) &:= T(\omega_0) \times T(g_1) \\ &= (\langle g_1, \tilde{u}_2 \rangle \sin \theta, -\langle g_1, \tilde{u}_2 \rangle \cos \theta, -\langle g_1, \tilde{u}_0 \rangle \sin \theta)^T \end{aligned}$$

so that $\omega_1 \perp \tilde{\omega}_0$, $\omega_1 \perp \tilde{g}_1$, $\tilde{\omega}_1 \perp \omega_0$, and $\tilde{\omega}_1 \perp g_1$. Also, define $\omega_2 \in G$ and $\tilde{\omega}_2 \in \tilde{G}$ by

$$\begin{aligned} T(\omega_2) &:= T(\tilde{\omega}_0) \times T(\tilde{g}_2) \\ &= (\langle \tilde{g}_2, u_2 \rangle \sin \tilde{\theta}, -\langle \tilde{g}_2, u_2 \rangle \cos \tilde{\theta}, \langle \tilde{g}_2, u_1 \rangle \cos \tilde{\theta} - \langle \tilde{g}_2, u_0 \rangle \sin \tilde{\theta})^T \text{ and} \\ T(\tilde{\omega}_2) &:= T(\omega_0) \times T(g_2) \\ &= (\langle g_2, \tilde{u}_2 \rangle \sin \theta, -\langle g_2, \tilde{u}_2 \rangle \cos \theta, \langle g_2, \tilde{u}_1 \rangle \cos \theta - \langle g_2, \tilde{u}_0 \rangle \sin \theta)^T \end{aligned}$$

so that $\omega_2 \perp \tilde{\omega}_0$, $\omega_2 \perp \tilde{g}_2$, $\tilde{\omega}_2 \perp \omega_0$, and $\tilde{\omega}_2 \perp g_2$. Then $\omega_1 \perp \tilde{\omega}_2$ and $\tilde{\omega}_1 \perp \omega_2$ provided that there exist θ and $\tilde{\theta}$ such that $\langle T(\omega_1), T(\tilde{\omega}_2) \rangle = 0$ and $\langle T(\tilde{\omega}_1), T(\omega_2) \rangle = 0$; that is,

$$\begin{aligned} (\langle \tilde{g}_1, u_0 \rangle \langle g_2, \tilde{u}_0 \rangle + \langle \tilde{g}_1, u_2 \rangle \langle g_2, \tilde{u}_2 \rangle) \sin \theta \sin \tilde{\theta} + \langle \tilde{g}_1, u_2 \rangle \langle g_2, \tilde{u}_2 \rangle \cos \theta \cos \tilde{\theta} \\ - \langle \tilde{g}_1, u_0 \rangle \langle g_2, \tilde{u}_1 \rangle \sin \tilde{\theta} \cos \theta = 0. \end{aligned} \tag{4.3}$$

$$\begin{aligned} (\langle g_1, \tilde{u}_0 \rangle \langle \tilde{g}_2, u_0 \rangle + \langle g_1, \tilde{u}_2 \rangle \langle \tilde{g}_2, u_2 \rangle) \sin \theta \sin \tilde{\theta} + \langle g_1, \tilde{u}_2 \rangle \langle \tilde{g}_2, u_2 \rangle \cos \theta \cos \tilde{\theta} \\ - \langle g_1, \tilde{u}_0 \rangle \langle \tilde{g}_2, u_1 \rangle \sin \theta \cos \tilde{\theta} = 0. \end{aligned} \tag{4.4}$$

respectively.

Since $\langle g_i, \tilde{g}_j \rangle = -K$ for $i \neq j$ from Lemma 4.2, many of these inner products can be simplified. Let $M_i := \langle g_i, \tilde{g}_i \rangle$ for $i = 0, 1, 2$. Note that

$$\begin{aligned} \langle g_1, \tilde{u}_0 \rangle &= 1, & \langle \tilde{g}_1, u_0 \rangle &= \frac{M_0 M_1 - K^2}{M_0}, \\ \langle g_1, \tilde{u}_2 \rangle &= -\frac{K}{M_0}, & \langle \tilde{g}_1, u_2 \rangle &= -K, \\ \langle g_2, \tilde{u}_0 \rangle &= 1, & \langle \tilde{g}_2, u_0 \rangle &= \frac{K(K + M_0)}{K^2 - M_0 M_1}, \\ \langle g_2, \tilde{u}_1 \rangle &= 1, & \langle \tilde{g}_2, u_1 \rangle &= \frac{2K^3 + K^2(M_0 + M_1 + M_2) - M_0 M_1 M_2}{K^2 - M_0 M_1}, \\ \langle g_2, \tilde{u}_2 \rangle &= -\frac{K}{M_0}, & \text{and } \langle \tilde{g}_2, u_2 \rangle &= -K. \end{aligned}$$

Then equations (4.3) and (4.4) reduce to

$$\frac{1}{M_0} \left(-K M_0 \sin \theta \sin \tilde{\theta} + K^2 \cos \theta \cos \tilde{\theta} + (K^2 - M_0 M_1) \sin \tilde{\theta} \cos \theta \right) = 0, \quad \text{and} \quad (4.5)$$

$$-\frac{K \sin \theta \sin \tilde{\theta}}{K^2 - M_0 M_1} + \frac{K^2 \cos \theta \cos \tilde{\theta}}{M_0} - \frac{(2K^3 + K^2(M_0 + M_1 + M_2) - M_0 M_1 M_2) \sin \theta \cos \tilde{\theta}}{K^2 - M_0 M_1} = 0, \quad (4.6)$$

respectively.

If $M_0 > 0$, then let $\tilde{\theta} = \theta$. Equations (4.5) and (4.6) reduce to

$$f_1(\theta) := \frac{1}{2M_0} \left(K(K - M_0) + K(K + M_0) \cos(2\theta) + (K^2 - M_0 M_1) \sin(2\theta) \right) = 0, \quad \text{and}$$

$$f_2(\theta) := \frac{K(K - M_0)}{2M_0} + \frac{K(K + M_0)}{2M_0} \cos(2\theta) - \frac{2K^3 + K^2(M_0 + M_1 + M_2) - M_0 M_1 M_2}{2(K^2 - M_0 M_1)} \sin(2\theta) = 0.$$

Since f_1 and f_2 are continuous, $f_1(0) = f_2(0) = \frac{K^2}{M_0} > 0$, and $f_1(\pi/2) = f_2(\pi/2) = -K < 0$, then f_1 and f_2 each have a zero in the interval $(0, \pi/2)$ by the Intermediate Value Theorem. If $M_0 < 0$, then let $\tilde{\theta} = -\theta$. Equations (4.5) and (4.6) reduce to

$$f_1(\theta) := \frac{1}{2M_0} \left(K(K + M_0) + K(K - M_0) \cos(2\theta) - (K^2 - M_0 M_1) \sin(2\theta) \right) = 0, \quad \text{and}$$

$$f_2(\theta) := \frac{K(K + M_0)}{2M_0} + \frac{K(K - M_0)}{2M_0} \cos(2\theta) - \frac{2K^3 + K^2(M_0 + M_1 + M_2) - M_0 M_1 M_2}{2(K^2 - M_0 M_1)} \sin(2\theta) = 0.$$

Since f_1 and f_2 are continuous, $f_1(0) = f_2(0) = \frac{K^2}{M_0} < 0$, and $f_1(\pi/2) = f_2(\pi/2) = K > 0$, then f_1 and f_2 each have a zero in the interval $(0, \pi/2)$ by the Intermediate Value Theorem.

Once θ and $\tilde{\theta}$ are found that satisfy (4.5) and (4.6), find the ω_i and $\tilde{\omega}_i$, $i = 0, 1, 2$, by the inverse transformations $T^{-1} : \mathbf{R}^3 \rightarrow G$ and $\tilde{T}^{-1} : \mathbf{R}^3 \rightarrow \tilde{G}$ defined by

$$T^{-1}(c_0, c_1, c_2) = c_0 u_0 + c_1 u_1 + c_2 u_2 \quad \text{and} \quad \tilde{T}^{-1}(c_0, c_1, c_2) = c_0 \tilde{u}_0 + c_1 \tilde{u}_1 + c_2 \tilde{u}_2.$$

As a final step, define $\sigma_i := \omega_i$ and $\tilde{\sigma}_i := \tilde{\omega}_i / \langle \omega_i, \tilde{\omega}_i \rangle$ for $i = 0, 1, 2$. □

This lemma shows that each g_i is a linear combination of the two σ_j where $j \neq i$, and likewise for the \tilde{g}_i .

Consider the spaces

$$\begin{aligned} Y_0 &:= \text{span} \{ \sigma_0, \check{\sigma}_1, \phi^q \circ d_{N-i,i} : i = 1, \dots, N-1 \} \text{ and} \\ \tilde{Y}_0 &:= \text{span} \{ \tilde{\sigma}_0, \check{\tilde{\sigma}}_1, \tilde{\phi}^q \circ d_{N-i,i} : i = 1, \dots, N-1 \}. \end{aligned}$$

Functions in Y_0 will be orthogonal to \tilde{W}_f and all translates of $\tilde{\phi}^i$, $i = 1, \dots, q-1$ and $\tilde{\phi}^q$ except $\tilde{\phi}^q \circ t_{1,0}$ and $\tilde{\phi}^q \circ t_{0,1}$. Likewise, functions in \tilde{Y}_0 will be orthogonal to W_f and all translates of ϕ^i , $i = 1, \dots, q-1$ and ϕ^q except $\phi^q \circ t_{1,0}$ and $\phi^q \circ t_{0,1}$. Define $X_0 := P_{Y_0}^{\tilde{Y}_0} V_0$ and $\tilde{X}_0 := P_{\tilde{Y}_0}^{Y_0} \tilde{V}_0$ to be two-dimensional subspaces of Y_0 and \tilde{Y}_0 , respectively. Let Ψ_0 and $\tilde{\Psi}_0$ be biorthogonal bases for the $(N-1)$ -dimensional complements $(I - P_{X_0}^{\tilde{X}_0})Y_0$ and $(I - P_{\tilde{X}_0}^{X_0})\tilde{Y}_0$. The elements of Ψ_0 satisfy all existing orthogonality conditions necessary to belong to the wavelet space W_0 , and likewise for $\tilde{\Psi}_0$.

The same construction can be used across the other two edges of Δ_0 using the spaces

$$\begin{aligned} Y_1 &:= \text{span} \{ \sigma_2, \check{\sigma}_2 \circ t_{0,-1}, \phi^q \circ d_{i,0} : i = 1, \dots, N-1 \}, \\ \tilde{Y}_1 &:= \text{span} \{ \tilde{\sigma}_2, \check{\tilde{\sigma}}_2 \circ t_{0,-1}, \tilde{\phi}^q \circ d_{i,0} : i = 1, \dots, N-1 \}, \\ Y_2 &:= \text{span} \{ \sigma_1, \check{\sigma}_0 \circ t_{-1,0}, \phi^q \circ d_{0,i} : i = 1, \dots, N-1 \}, \text{ and} \\ \tilde{Y}_2 &:= \text{span} \{ \tilde{\sigma}_1, \check{\tilde{\sigma}}_0 \circ t_{-1,0}, \tilde{\phi}^q \circ d_{0,i} : i = 1, \dots, N-1 \} \end{aligned}$$

and analogous subspaces $X_1, \tilde{X}_1, X_2,$ and \tilde{X}_2 to build biorthogonal pairs Ψ_1 and $\tilde{\Psi}_1$ and also Ψ_2 and $\tilde{\Psi}_2$. Define $\Psi_g := \Psi_0 \cup \Psi_1 \cup \Psi_2$ and $\tilde{\Psi}_g := \tilde{\Psi}_0 \cup \tilde{\Psi}_1 \cup \tilde{\Psi}_2$. The wavelets in Ψ_g and their translates will be orthogonal to the wavelets in $\tilde{\Psi}_g$ and their translates due to the biorthogonality of the σ_i and $\tilde{\sigma}_i$. Define $W_g := S(\Psi_g) \subset W_0$ and $\tilde{W}_g := S(\tilde{\Psi}_g) \subset \tilde{W}_0$. The spaces W_g and \tilde{W}_g each have $3(N-1)$ generators.

Let D_0, \dots, D_5 be the parallelogram-shaped regions of \mathbf{R}^2 defined in Figure 5. Define

$$\nu_i := P_{V_1(D_i)}^{\tilde{V}_1(D_i)} \phi^q \text{ and } \tilde{\nu}_i := P_{\tilde{V}_1(D_i)}^{V_1(D_i)} \tilde{\phi}^q$$

for $i = 0, \dots, 5$ and consider for the moment ν_0 . Notice that ν_0 meets several orthogonality conditions required of wavelets in W_0 : $\nu_0 \perp \tilde{\phi}^j$ for $j = 1, \dots, \frac{q-1}{2}$, $\nu_0 \perp (\tilde{\phi}^j \circ t_{-1,0})$ for $j = \frac{q+1}{2}, \dots, q-1$, $\nu_0 \perp (\tilde{\phi}^q \circ t_{0,1})$, and ν_0 is perpendicular to \tilde{W}_f . Also, ν_0 is perpendicular to generators of \tilde{W}_g that are built across the edges (ϵ_0, ϵ_2) , (ϵ_1, ϵ_2) , and $(\epsilon_2, \epsilon_2 - \epsilon_1)$. Similar results can be found for the other ν_i and $\tilde{\nu}_i$. The goal is to alter the ν_i and $\tilde{\nu}_i$ in such a way that these orthogonalities are maintained, while achieving the other necessary orthogonalities.

Define

$$\mu_i := (I - P_{W_g(D_i)}^{\tilde{W}_g(D_i)}) \nu_i + c_i \phi^q \circ d_{0,0} \text{ and } \tilde{\mu}_i := (I - P_{\tilde{W}_g(D_i)}^{W_g(D_i)}) \tilde{\nu}_i + \tilde{c}_i \tilde{\phi}^q \circ d_{0,0},$$

for $i = 0, \dots, 5$, where nonzero c_i and \tilde{c}_i satisfy $\langle \mu_i, \tilde{\phi}^q \rangle = 0$ and $\langle \tilde{\mu}_i, \phi^q \rangle = 0$, respectively. From Lemma 4.4, there exist biorthogonal sets $\Sigma = \{\sigma_0, \sigma_1, \sigma_2\}$ and $\tilde{\Sigma} = \{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$ such that $\text{span}(\Sigma) = \text{span}\{g_0, g_1, g_2\}$, $\text{span}(\tilde{\Sigma}) = \text{span}\{\tilde{g}_0, \tilde{g}_1, \tilde{g}_2\}$, and $\sigma_j \perp \tilde{g}_j$ and $\tilde{\sigma}_j \perp g_j$ for $j = 0, 1, 2$. Then

$$\mu_0 := \nu_0 - \langle \nu_0, \tilde{\sigma}_2 \rangle \sigma_2 - \langle \nu_0, \tilde{\sigma}_1 \circ t_{-1,0} \rangle \tilde{\sigma}_1 \circ t_{-1,0} + c_0 \phi^q \circ d_{0,0}$$

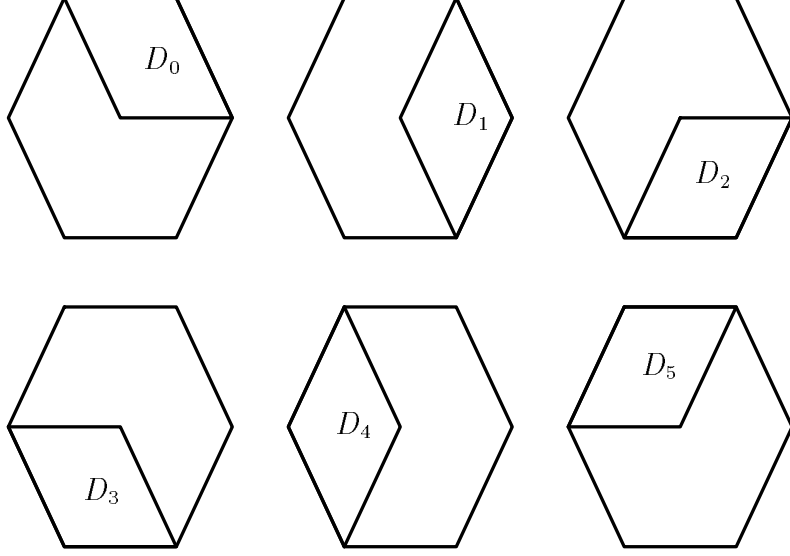


Figure 5: Domains used in the construction of Ψ_h and $\tilde{\Psi}_h$.

$$= c_0 \phi^q \circ d_{0,0} + \langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 + \langle g_2, \tilde{\sigma}_0 \rangle \check{\sigma}_0 + \frac{1}{N} \sum_{j=1}^{N-1} (N-j) \phi^q \circ d_{0,j}, \text{ and} \quad (4.7)$$

$$\begin{aligned} \tilde{\mu}_1 &:= \tilde{\nu}_1 - \langle \tilde{\nu}_1, \sigma_1 \rangle \tilde{\sigma}_1 - \langle \tilde{\nu}_1, \sigma_1 \circ t_{0,-1} \rangle \check{\tilde{\sigma}}_1 \circ t_{0,-1} + \tilde{c}_1 \tilde{\phi}^q \circ d_{0,0} \\ &= \tilde{c}_1 \tilde{\phi}^q \circ d_{0,0} + \langle \tilde{g}_0, \sigma_2 \rangle \tilde{\sigma}_2 + \langle \tilde{g}_0, \sigma_2 \rangle \check{\tilde{\sigma}}_2 \circ t_{0,-1} + \frac{1}{N} \sum_{j=1}^{N-1} (N-j) \tilde{\phi}^r \circ d_{j,0}. \end{aligned} \quad (4.8)$$

Notice that the μ_i and $\tilde{\mu}_i$ maintain the orthogonalities of the ν_i and $\tilde{\nu}_i$. Also, by definition, $\mu_i \perp \tilde{\phi}^q$, $\tilde{\mu}_i \perp \phi^q$, $\mu_i \perp \tilde{W}_g$, and $\tilde{\mu}_i \perp W_g$. Finally, note from (4.7) and (4.8),

$$\langle \mu_0 \circ t_{1,-1}, \tilde{\mu}_1 \rangle = \langle g_2, \tilde{\sigma}_0 \rangle \langle \tilde{g}_0, \sigma_2 \rangle \langle \check{\sigma}_0, \check{\tilde{\sigma}}_2 \rangle = 0 \text{ and}$$

$$\langle \mu_0, \tilde{\mu}_1 \rangle = c_0 \tilde{c}_1 + \langle g_0, \tilde{\sigma}_1 \rangle \langle \tilde{g}_0, \sigma_2 \rangle \langle \sigma_1, \tilde{\sigma}_2 \rangle = c_0 \tilde{c}_1 \neq 0.$$

Also, it is trivially established that $(\mu_0 \circ t_{1,0}) \perp \tilde{\mu}_1$ and $(\mu_0 \circ t_{0,1}) \perp \tilde{\mu}_1$. Similarly, it is established that the μ_i and $\tilde{\mu}_i$ satisfy the condition $\mu_i \perp (\tilde{\mu}_j \circ t_{m,n})$ for $m, n \neq 0$, $i, j = 0, \dots, 5$, $i \neq j$, and that the sets $\{\mu_i\}$ and $\{\tilde{\mu}_i\}$ satisfy Lemma 1.1. Let Ψ_h and $\tilde{\Psi}_h$ be biorthogonal bases for $\text{span}\{\mu_i : i = 0, \dots, 5\}$ and $\text{span}\{\tilde{\mu}_i : i = 0, \dots, 5\}$, respectively. Define $W_h := S(\Psi_h) \subset W_0$ and $\tilde{W}_h := S(\tilde{\Psi}_h) \subset \tilde{W}_0$.

Before establishing that all the wavelets necessary to “build” V_1 and \tilde{V}_1 have been found, a lemma is needed.

Lemma 4.5 For g_i and \tilde{g}_i , $i = 0, 1, 2$, as defined in (4.1), $g_i \in V_0 \oplus W_g \oplus W_h$ and $\tilde{g}_i \in \tilde{V}_0 \oplus \tilde{W}_g \oplus \tilde{W}_h$ for $i = 0, 1, 2$.

Proof. By Lemma 4.4, it suffices to show that $\sigma_i \in V_0 \oplus W_g \oplus W_h$ and $\tilde{\sigma}_i \in \tilde{V}_0 \oplus \tilde{W}_g \oplus \tilde{W}_h$ for $i = 0, 1, 2$. Recall the space Y_0 is spanned by σ_1 , $\check{\sigma}_0 \circ t_{-1,0}$, and the functions $\phi^q \circ d_{0,i}$, $i = 1, \dots, N-1$,

and, likewise, \tilde{Y}_0 is spanned by $\tilde{\sigma}_1, \tilde{\sigma}_0 \circ t_{-1,0}$, and the functions $\tilde{\phi}^q \circ d_{0,i}, i = 1, \dots, N-1$. Recall that X_0 is the span of $P_{Y_0}^{\tilde{Y}_0} \phi^q$ and $P_{Y_0}^{\tilde{Y}_0}(\phi^q \circ t_{0,1})$ and likewise, \tilde{X}_0 is the span of $P_{\tilde{Y}_0}^{Y_0} \tilde{\phi}^q$ and $P_{\tilde{Y}_0}^{Y_0}(\tilde{\phi}^q \circ t_{0,1})$. By definition, $\sigma_1 \in Y_0$ and $(I - P_{X_0}^{\tilde{X}_0})Y_0 \subset W_g$. It suffices to show that $X_0 \subset V_0 \oplus W_h$.

It is tediously verified that

$$\begin{aligned}
& c_0(\phi^q - \mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5) - (1 - c_1 - c_2 - c_3 - c_4 - c_5)\mu_0 = \\
& c_0\langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 + c_0\langle g_2, \tilde{\sigma}_0 \rangle \tilde{\sigma}_0 \circ t_{-1,0} + \frac{c_0}{N} \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i} \\
& + (c_1 + c_2 + c_3 + c_4 + c_5 - 1)\langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 \\
& + (c_1 + c_2 + c_3 + c_4 + c_5 - 1)\langle g_2, \tilde{\sigma}_0 \rangle \tilde{\sigma}_0 \circ t_{-1,0} \\
& + \left(\frac{c_1 + c_2 + c_3 + c_4 + c_5 - 1}{N} \right) \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i} \\
& = (c_0 + c_1 + c_2 + c_3 + c_4 + c_5 - 1)[\langle g_0, \tilde{\sigma}_1 \rangle \sigma_1 \\
& + \langle g_2, \tilde{\sigma}_0 \rangle \tilde{\sigma}_0 \circ t_{-1,0} + \frac{1}{N} \sum_{i=1}^{N-1} (N-i)\phi^q \circ d_{0,i}] \\
& = (c_0 + c_1 + c_2 + c_3 + c_4 + c_5 - 1)P_{Y_0}^{\tilde{Y}_0} \phi^q.
\end{aligned}$$

Likewise,

$$\begin{aligned}
& c_3(\phi^q - \mu_0 - \mu_1 - \mu_2 - \mu_4 - \mu_5) - (1 - c_0 - c_1 - c_2 - c_4 - c_5)\mu_3 \circ t_{0,1} = \\
& (c_0 + c_1 + c_2 + c_3 + c_4 + c_5 - 1)P_{Y_0}^{\tilde{Y}_0}(\phi^q \circ t_{0,1}).
\end{aligned}$$

Thus, $X_0 \subset V_0 \oplus W_h$ and $\sigma_1 \in V_0 \oplus W_g \oplus W_h$. Analogous arguments establish $\sigma_0, \sigma_2 \in V_0 \oplus W_g \oplus W_h$ and $\tilde{\sigma}_i \in \tilde{V}_0 \oplus \tilde{W}_g \oplus \tilde{W}_h$. \square

Theorem 4.6 *Let (V_p) and (\tilde{V}_p) be biorthogonal MRA of multiplicity r in \mathbf{R}^2 constructed from Theorem 2.1. Define $W_f, \tilde{W}_f, W_g, \tilde{W}_g, W_h,$ and \tilde{W}_h as above. Then $V_1 = V_0 \oplus W_0$ and $\tilde{V}_1 = \tilde{V}_0 \oplus \tilde{W}_0$ where $W_0 = W_f \oplus W_g \oplus W_h, \tilde{W}_0 = \tilde{W}_f \oplus \tilde{W}_g \oplus \tilde{W}_h,$ and W_0 and \tilde{W}_0 each have $q(N^2 - 1)$ generators.*

Proof. Define $W := W_f \oplus W_g \oplus W_h, \tilde{W} := \tilde{W}_f \oplus \tilde{W}_g \oplus \tilde{W}_h, V := V_1(\Delta_0),$ and $\tilde{V} := \tilde{V}_1(\Delta_0)$. Certainly $V_1 \supseteq V_0 \oplus W$ by nature of the wavelet constructions. At issue is whether $V_1 \subseteq V_0 \oplus W$.

For $N > 2$, generators $\phi^i \circ d_{0,0}, i = 1, \dots, q-1$ of V_1 can be found in the space V . Notice that

$$\dim V = tN^2 + \frac{(N-1)(N-2)}{2},$$

where $t = \frac{q-1}{2}$. The scaling functions and the definitions of W_f, \tilde{W}_f, W_g and \tilde{W}_g , along with Lemma 4.5, provide biorthogonal bases

$$\{\phi^1, \dots, \phi^t\} \cup \Psi_f \cup \{\sigma_0, \sigma_1, \sigma_2\} \text{ and } \{\tilde{\phi}^1, \dots, \tilde{\phi}^t\} \cup \tilde{\Psi}_f \cup \{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$$

of V and \tilde{V} , each with cardinality

$$t + (tN^2 + \frac{(N-1)(N-2)}{2} - t - 3) + 3 = tN^2 + \frac{(N-1)(N-2)}{2}.$$

Since the linear systems have full rank, each $f \in V$ is a linear combination of elements of $V_0 \oplus W$ and each $\tilde{f} \in \tilde{V}$ is a linear combination of elements of $\tilde{V}_0 \oplus \tilde{W}$. Thus, $\phi^i \circ d_{0,0} \in V_0 \oplus W$ and $\tilde{\phi}^i \circ d_{0,0} \in \tilde{V}_0 \oplus \tilde{W}$ for $i = 1, \dots, q-1$.

Also notice that

$$\phi^q - \sum_{i=0}^5 \mu_i = (1 - \sum_{i=0}^5 c_i) \phi^q \circ d_{0,0}.$$

Thus, $V_1 \subseteq V_0 \oplus W_f \oplus W_g \oplus W_h$ and $W = W_0$. The analogous results hold for \tilde{W} . The number of generators is the sum of the generators for W_f , W_g , and W_h :

$$(q(N^2 - 1) - 3N - 3) + (3(N - 1)) + 6 = q(N^2 - 1).$$

□

Corollary 4.7 *Let (V_p) and (\tilde{V}_p) be biorthogonal MRA of multiplicity q in \mathbf{R}^2 constructed from Theorem 2.1. Define $W_f, \tilde{W}_f, W_g,$ and \tilde{W}_g as above. Let D be the hexagonal support of ϕ^q , and let $X := (V_0 \oplus W_f \oplus W_g)(D)$ and $\tilde{X} := (\tilde{V}_0 \oplus \tilde{W}_f \oplus \tilde{W}_g)(D)$. Then W_h and \tilde{W}_h are generated by biorthogonal bases for $P_X^{\tilde{X}} V_1(D)$ and $P_{\tilde{X}}^X \tilde{V}_1(D)$.*

While the definitions of W_h and \tilde{W}_h provide an explicit construction, Corollary 4.7 says that after finding the generators of W_f, \tilde{W}_f, W_g and \tilde{W}_g , the generators of W_h are whatever is left in V_1 with the support of ϕ^q , and likewise for \tilde{W}_h .

5 Wavelets for Scaling Functions in Section 3

By Theorem 4.6, W_0 and \tilde{W}_0 will each have $3(3^2 - 1) = 24$ generators. Let D_0 be the overlap domain with vertices at $(0, 0)$, ϵ_1 , ϵ_2 , and $\epsilon_2 - \epsilon_1$ and let D_i be clockwise rotation of D_0 about the origin by $\frac{\pi}{3}i$ for $i = 1, \dots, 5$, as in Figure 5.

5.1 Wavelets in W_f and \tilde{W}_f

By definition, W_f and \tilde{W}_f each have 12 generators, 6 supported on Δ_0 and 6 supported on ∇_0 . Define the 10-dimensional spaces $V := V_1(\Delta_0)$ and $\tilde{V} := \tilde{V}_1(\Delta_0)$, and the 4-dimensional spaces

$$X := P_V^{\tilde{V}}(\text{span}\{\phi^1, \phi^3, \phi^3 \circ t_{1,0}, \phi^3 \circ t_{0,1}\}) \text{ and } \tilde{X} := P_{\tilde{V}}^V(\text{span}\{\tilde{\phi}^1, \tilde{\phi}^3, \tilde{\phi}^3 \circ t_{1,0}, \tilde{\phi}^3 \circ t_{0,1}\}).$$

The ψ^i , $i = 1, \dots, 6$, illustrated in Figure 6, were chosen as a spanning set for $(I - P_X^{\tilde{X}})V$ that met some symmetry conditions. The associated $\tilde{\psi}^i$, illustrated in Figure 7, were chosen so that $\tilde{\psi}^i \in \ker(\text{span}\{\psi^j : j \neq i\} \cup X)$ and $\langle \psi^i, \tilde{\psi}^i \rangle > 0$ for $i = 1, \dots, 6$. Each of the above is “normalized” by the factor $\sqrt{\langle \psi^i, \tilde{\psi}^i \rangle}$.

These wavelets reflected onto ∇_0 will span $W_f|_{\chi_{\nabla_0}}$ and $\tilde{W}_f|_{\chi_{\nabla_0}}$. Define

$$\psi^{i+6} := \psi^i \circ r \circ t_{-1,0} \text{ and } \tilde{\psi}^{i+6} := \tilde{\psi}^i \circ r \circ t_{-1,0}$$

for $i = 1, \dots, 6$.

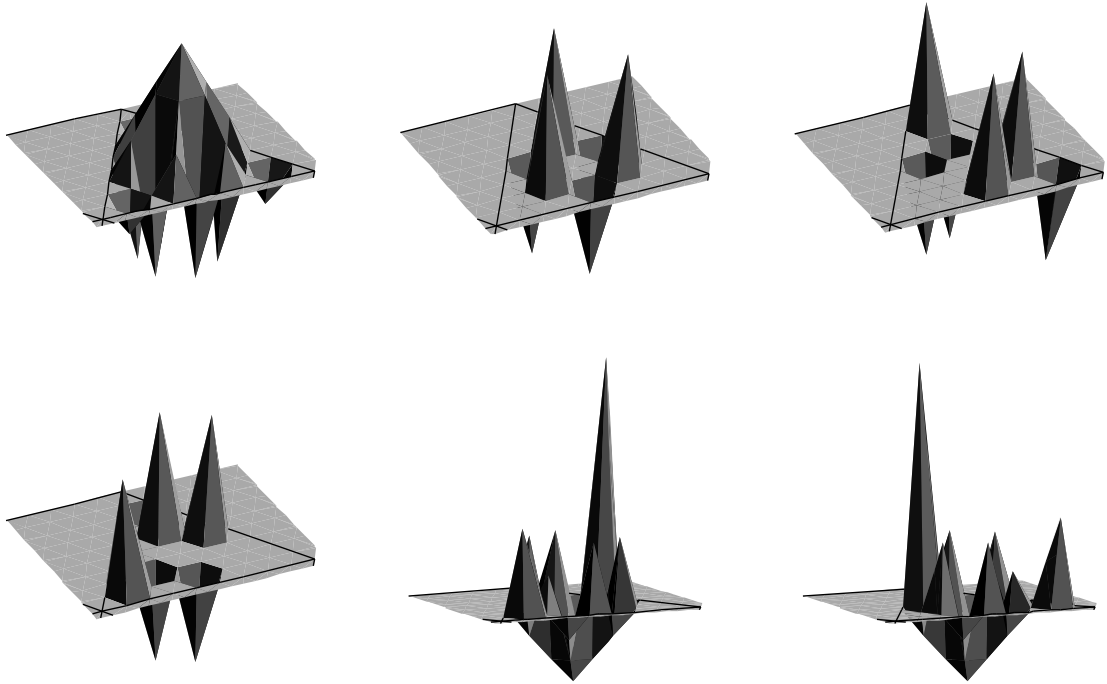


Figure 6: Wavelets ψ^1 through ψ^6 .

5.2 Wavelets in W_g and \tilde{W}_g

By definition, W_g and \tilde{W}_g each have 6 generators. Following the construction in Lemma 4.4, biorthogonal sets $\{\sigma_0, \sigma_1, \sigma_2\}$ and $\{\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2\}$ can be found such that $\sigma_i \perp \tilde{g}_i$ and $\tilde{\sigma}_i \perp g_i$, $i = 0, 1, 2$. Then, following the construction in Section 3, functions in W_g with support on the parallelogram $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_2 - \epsilon_1)$ will be linear combinations of σ_1 , $\check{\sigma}_0 \circ t_{-1,0}$, $\phi^3 \circ d_{0,1}$, and $\phi^3 \circ d_{0,2}$. These functions are orthogonal to all translates of ϕ^1 , ϕ^2 , and ψ^i for $i = 1, \dots, 12$. Also, $\sigma_2 \perp \phi^3 \circ t_{1,0}$ and $\check{\sigma}_0 \circ t_{-1,0} \perp \phi^3 \circ t_{-1,1}$. Only two other orthogonality conditions must be met.

It is possible to construct symmetric-antisymmetric pairs of wavelets. Define

$$\nu_1 := \sigma_1 + \check{\sigma}_0 \circ t_{-1,0} + c_1 \phi^3 \circ d_{0,1} + c_2 \phi^3 \circ d_{0,2}$$

and solve the system of equations

$$\begin{cases} \langle \nu_1, \tilde{\phi}^3 \rangle & = 0 \\ \langle \nu_1, \tilde{\phi}^3 \circ t_{0,1} \rangle & = 0 \end{cases}$$

for c_1 and c_2 . Likewise, define

$$\tilde{\nu}_1 := \tilde{\sigma}_1 + \check{\check{\sigma}}_0 \circ t_{-1,0} + \tilde{c}_1 \tilde{\phi}^3 \circ d_{0,1} + \tilde{c}_2 \tilde{\phi}^3 \circ d_{0,2}$$

and solve the system of equations

$$\begin{cases} \langle \tilde{\nu}_1, \phi^3 \rangle & = 0 \\ \langle \tilde{\nu}_1, \phi^3 \circ t_{0,1} \rangle & = 0 \end{cases}$$

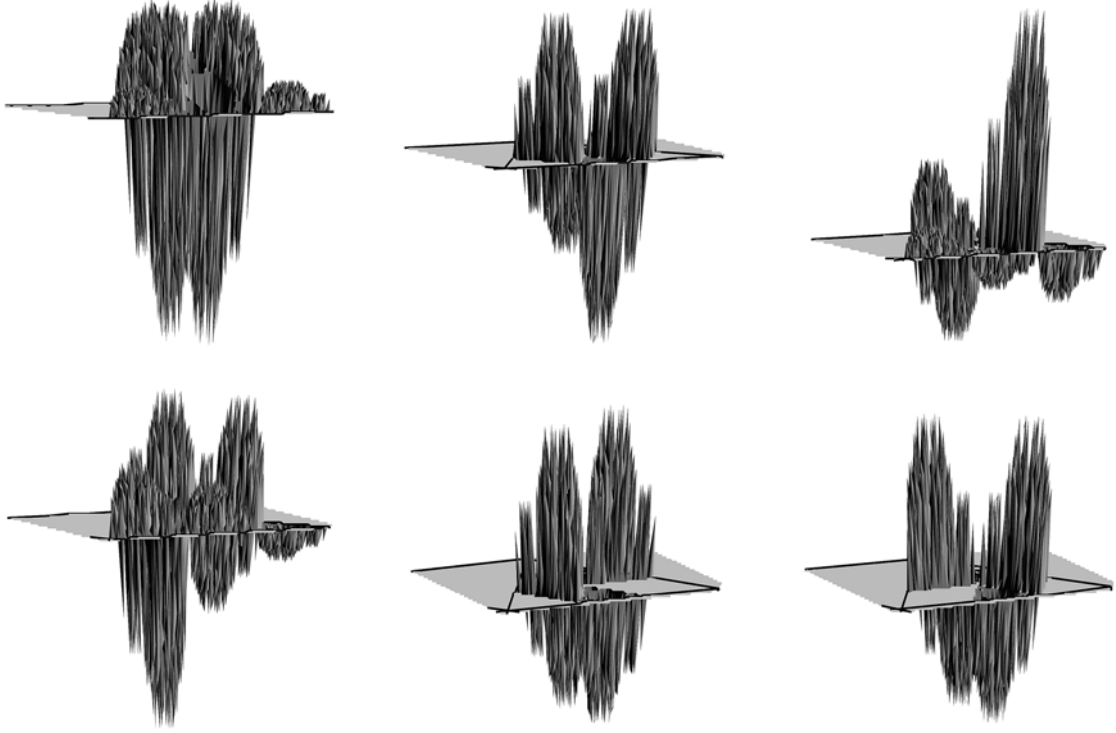


Figure 7: Wavelets $\tilde{\psi}^1$ through $\tilde{\psi}^6$.

for \tilde{c}_1 and \tilde{c}_2 . If $\langle \nu_1, \tilde{\nu}_1 \rangle < 0$, then change $\tilde{\nu}_1$ to its additive inverse. Also, define ν_2 and $\tilde{\nu}_2$ by

$$\nu_2 := \sigma_1 - \check{\sigma}_0 \circ t_{-1,0} \text{ and } \tilde{\nu}_2 := \tilde{\sigma}_1 - \check{\tilde{\sigma}}_0 \circ t_{-1,0}.$$

Then $\nu_1 \perp \tilde{\nu}_2$ and $\tilde{\nu}_1 \perp \nu_2$ by nature of their symmetry properties.

The remaining wavelets generating W_g are merely $2\pi/3$ rotations of ν_1 and ν_2 about ϵ_0 , denoted $\bar{\tau}$. Define

$$\begin{aligned} \omega^{13} &:= \nu_1, & \tilde{\omega}^{13} &:= \tilde{\nu}_1 \\ \omega^{14} &:= \nu_2, & \tilde{\omega}^{14} &:= \tilde{\nu}_2, \\ \omega^{15} &:= \nu_1 \circ \bar{\tau}, & \tilde{\omega}^{15} &:= \tilde{\nu}_1 \circ \bar{\tau}, \\ \omega^{16} &:= \nu_2 \circ \bar{\tau}, & \omega^{16} &:= \tilde{\nu}_2 \circ \bar{\tau}, \\ \omega^{17} &:= \nu_1 \circ \bar{\tau} \circ \bar{\tau}, & \tilde{\omega}^{17} &:= \tilde{\nu}_1 \circ \bar{\tau} \circ \bar{\tau} \\ \omega^{18} &:= \nu_2 \circ \bar{\tau} \circ \bar{\tau} \text{ and } \tilde{\omega}^{18} &:= \tilde{\nu}_2 \circ \bar{\tau} \circ \bar{\tau}. \end{aligned}$$

Normalize by defining $\psi^i := \omega^i / \sqrt{\langle \omega^i, \tilde{\omega}^i \rangle}$ and $\tilde{\psi}^i := \tilde{\omega}^i / \sqrt{\langle \omega^i, \tilde{\omega}^i \rangle}$ for $i = 13, \dots, 18$. Wavelets ψ^{13} , ψ^{14} , $\tilde{\psi}^{13}$, and $\tilde{\psi}^{14}$ are illustrated in Figure 8.

5.3 Wavelets in W_h and \tilde{W}_h

By definition, W_h has 6 generators. Following the construction in Section 3, construct μ_i and $\tilde{\mu}_i$, $i = 0, \dots, 5$ that span W_h and \tilde{W}_h , respectively. It can be verified that $c_0 = c_2 = c_4$, $c_1 = c_3 = c_5$,

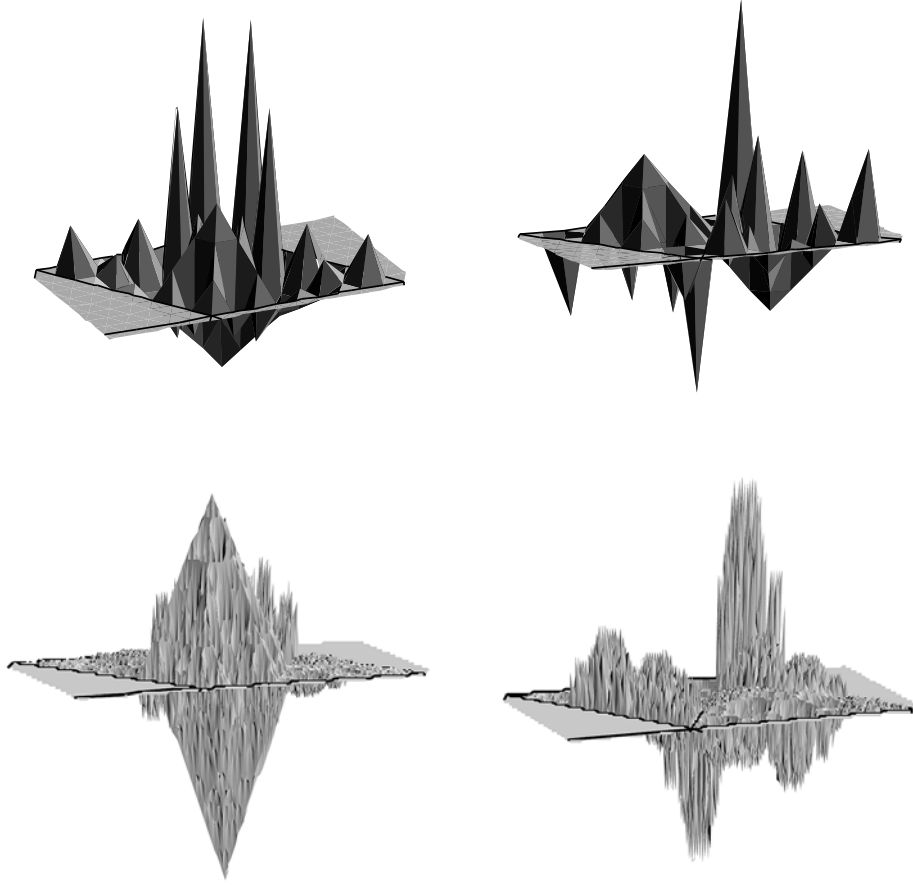


Figure 8: Wavelets ψ^{13} , ψ^{14} , $\tilde{\psi}^{13}$, and $\tilde{\psi}^{14}$.

$\tilde{c}_0 = \tilde{c}_2 = \tilde{c}_4$, and $\tilde{c}_1 = \tilde{c}_3 = \tilde{c}_5$ due to the rotational invariance of both the g_i and \tilde{g}_i . To construct biorthogonal sets with some symmetric properties, first define the following

$$\begin{aligned}
 \gamma_1 &:= \sum_{i=0}^5 \mu_i & \tilde{\gamma}_1 &:= \sum_{i=0}^5 \tilde{\mu}_i \\
 \gamma_2 &:= \sum_{i=0}^5 (-1)^i \mu_i & \tilde{\gamma}_2 &:= \sum_{i=0}^5 (-1)^i \tilde{\mu}_i \\
 \gamma_3 &:= \mu_0 - \mu_2 & \tilde{\gamma}_3 &:= \tilde{\mu}_0 - \tilde{\mu}_2 \\
 \gamma_4 &:= \mu_1 - \mu_3 & \tilde{\gamma}_4 &:= \tilde{\mu}_1 - \tilde{\mu}_3 \\
 \gamma_5 &:= \mu_0 + \mu_2 & \tilde{\gamma}_5 &:= \tilde{\mu}_0 + \tilde{\mu}_2 \\
 \gamma_6 &:= \mu_1 + \mu_3 & \tilde{\gamma}_6 &:= \tilde{\mu}_1 + \tilde{\mu}_3
 \end{aligned}$$

Then construct the biorthogonal sets $\{\omega_1, \dots, \omega_6\}$ and $\{\tilde{\omega}_1, \dots, \tilde{\omega}_6\}$ using the biorthogonal Gram-Schmidt process so that $\langle \omega_i, \omega_i \rangle > 0$, $i = 1, \dots, 6$. Define $\psi^{i+18} := \frac{\omega_i}{\sqrt{\langle \omega_i, \omega_i \rangle}}$ and $\tilde{\psi}^{i+18} := \frac{\tilde{\omega}_i}{\sqrt{\langle \tilde{\omega}_i, \tilde{\omega}_i \rangle}}$ for $i = 1, \dots, 6$. These wavelets are illustrated in Figure 9 and Figure 10, respectively. The sets $S(\{\psi^i : i = 1, \dots, 24\})$ and $S(\{\tilde{\psi}^i : i = 1, \dots, 24\})$ form biorthogonal bases for W_0 and \tilde{W}_0 .

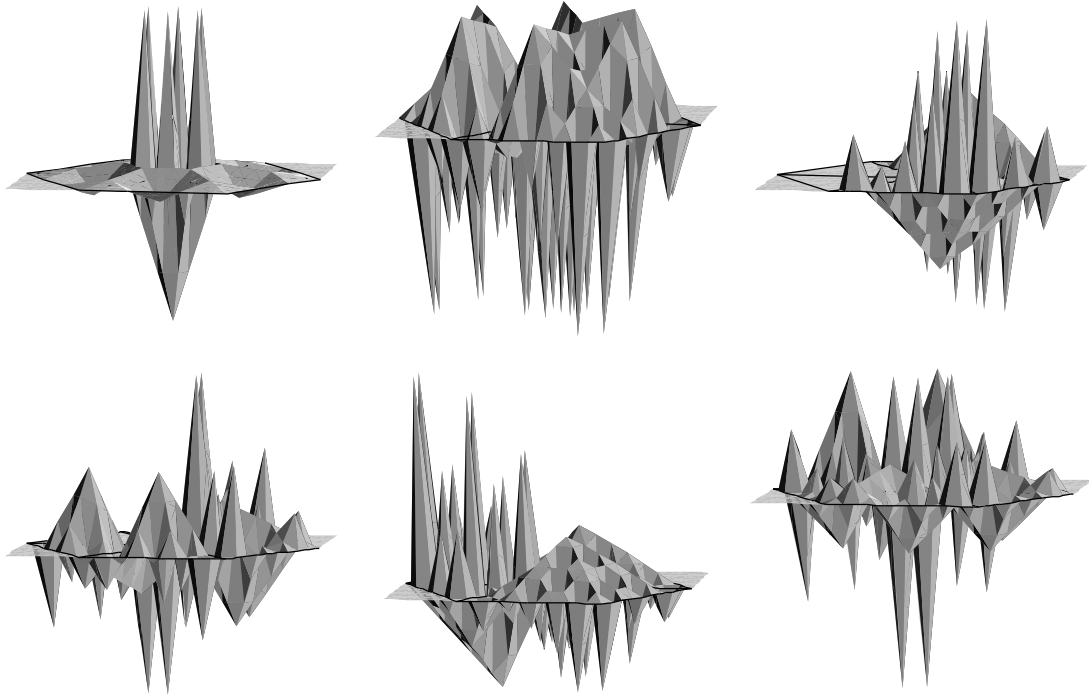


Figure 9: Wavelets ψ^{19} through ψ^{24} .

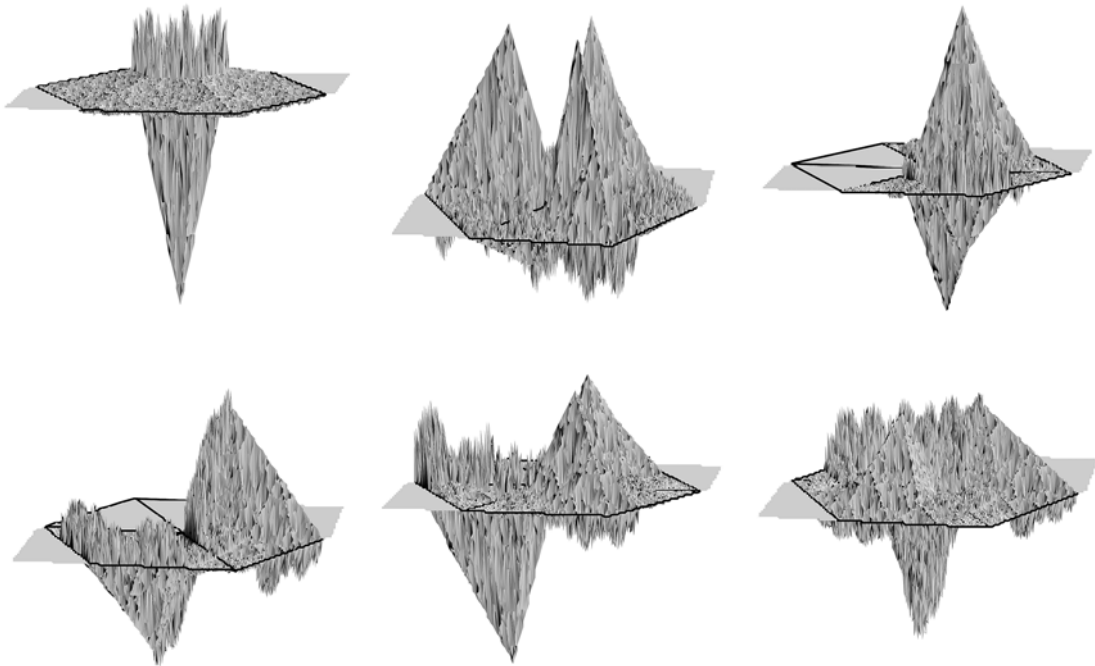


Figure 10: Wavelets $\tilde{\psi}^{19}$ through $\tilde{\psi}^{24}$.

References

- [1] M. Barnsley, “Fractals Everywhere”, Academic Press, San Diego, CA, 1988.
- [2] E. Belogay and Y. Wang, Arbitrarily smooth orthogonal nonseparable wavelets in \mathbf{R}^2 , *SIAM J. Math. Anal.* **30:3**, (1999) 678-697.
- [3] I. Daubechies, “Ten Lectures on Wavelets”, SIAM, Philadelphia, Pennsylvania, 1992.
- [4] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, Compactly supported, piecewise affine scaling functions on triangulations, *Constr. Approx.* **16**, (2000) 201-219.
- [5] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, Intertwining multiresolution analyses and the construction of piecewise polynomial wavelets, *SIAM J. Math. Anal.* **27:6**, (1996) 1791-1815.
- [6] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, A Class of Orthogonal Multiresolution Analyses in 2D, *Mathematical Methods for Curves and Surfaces*, Daehlen, Lyche, & Schumaker, eds., 1995.
- [7] G. C. Donovan, J. S. Geronimo, D. P. Hardin, and B. Kessler, Construction of two-dimensional multiwavelets on a triangulation, *Wavelet Application in Signal and Image Processing*, Laine & Unser, editors, SPIE Conf. Proc., Denver, 1996.
- [8] G. C. Donovan, J. S. Geronimo, D. P. Hardin, and P. Massopust, Construction of Orthogonal Wavelets Using Fractal Interpolation Functions, *SIAM J. Math. Anal.*, **27** (1996), 1158-1192.
- [9] J. S. Geronimo and D. Hardin, Fractal Interpolation Surfaces and a Related 2-D Multiresolution Analysis, *J. Math. Anal. Appl.* **176** (1993), 561-586.
- [10] J. Geronimo, D. Hardin and P. Massopust, “Fractal functions and wavelet expansions based on several scaling functions,” *J. Approx. Theory* **78:3** (1994), 373-401.
- [11] D. P. Hardin, B. Kessler, and P. R. Massopust, Multiresolution analyses and fractal functions, *J. Approx. Theory* **71** (1992), 104-120.
- [12] D. P. Hardin and J. A. Marasovich, Biorthogonal multiwavelets on $[-1,1]$, *Appl. Comput. Harmonic Anal.* **7** (1999), 34-53.
- [13] D. P. Hardin and P. R. Massopust, Fractal interpolation functions from \mathbf{R}^n into \mathbf{R}^m and their projections, *Zeitshchrift fur Analysis u. i. Anwendungen* **12** (1993), 535-548.
- [14] R. Jia, “Refinable shift-invariant spaces: from splines to wavelets,” *Approximation Theory VIII*, **2** (1995), 179-208.
- [15] B. Kessler, “A construction of orthogonal compactly-supported multiwavelets on \mathbf{R}^2 ,” *Appl. Comput. Harmonic Anal.* **9** (2000), 146-165.
- [16] M. J. Lai and L. Schumaker, Course notes from “Splines on Triangulations”, 1996.

- [17] P. R. Massopust, Fractal Surfaces, *J. Math. Anal. Appl.* **151** (1990), 275-290.
- [18] G. Strang and V. Strela, Short wavelets and matrix dilation equations, *IEEE Trans. on SP* **33** (1994), 2104-2107.