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Balanced Biorthogonal Scaling Vectors Using Fractal Function Macroelements on [0,1]

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Running Head:
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Abstract

Geronimo, Hardin, et al have previously constructed orthogonal and biorthogonal scaling vectors by extending a spline scaling vector with functions supported on $[0, 1]$. Many of these constructions occurred before the concept of balanced scaling vectors was introduced. This paper will show that adding functions on $[0, 1]$ is insufficient for extending spline scaling vectors to scaling vectors that are both orthogonal and balanced. We are able, however, to use this technique to extend spline scaling vectors to balanced, biorthogonal scaling vectors, and we provide two large classes of this type of scaling vector, with approximation order two and three, respectively, with two specific constructions with desirable properties in each case. The constructions will use macroelements supported on $[0, 1]$, some of which will be fractal functions.

AMS Subject Classification Numbers: 42C40, 65D15

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1 Introduction

Geronimo, Hardin, and Massopust first extended a piecewise polynomial scaling function, the linear B-spline, to an orthogonal scaling vector (commonly referred to as the GHM scaling vector) in [7] by adding a function supported on $[0,1]$ (hence, automatically orthogonal to its integer translates) that, when its integer translates were projected out of the original function, made the resulting function orthogonal to its integer translates. Since then, that same basic idea has been used by Hardin and Marasovich in [10] to generalized the GHM scaling vector to a biorthogonal family of scaling vectors, by Donovan, Geronimo, and Hardin in [5] to create higher-approximation-order orthogonal scaling vectors, and by the author in [15] to extend the scaling vector of length 2 generating the spline space $\mathcal{S}_3^1(\mathbb{Z})$ to a differentiable, orthogonal scaling vector of length 4. Hardin and the author recast these types of constructions in a macroelement setting in [9]. Other researchers, notably Han and Jiang in [8], have worked on constructing multiwavelets on $[0,1]$.

Each of the above constructions exploits the general strengths of using multiwavelets, namely the ability to build symmetric scaling functions of relatively short support, but they also suffered from the general weakness of multiwavelets, namely that the filters associated with a general scaling vector of approximation order K do not necessarily preserve discrete-time polynomial data of degree $K-1$. One possible way of dealing with this shortcoming is to prefilter the raw data. (See [11] for a comprehensive introduction to the concept of prefiltering.) A more recent approach, initiated by Lebrun and Vetterli in [16, 17] and studied further by Chui and Jiang in [3, 4], Selesnick in [19, 20], Lian in [18], the author in [14], and others, is to design scaling vectors whose filters maintain polynomial order without prefiltering, called *balanced multiwavelets*.

The purpose of the research presented here is to determine whether the useful trick of adding functions on $[0,1]$ can be used to extend spline-based scaling vectors to scaling vectors that are both orthogonal and balanced up to their approximation order. The macroelement approach is natural since the functions considered are either supported completely on $[0,1]$ or piecewise polynomial on integer knots. Following a brief introduction of notation and terminology in Section 1, we shall show in Section 2 that the two conditions can not be met simultaneously using this type of construction. We can, however, use the technique to design dual **bi**orthogonal scaling vectors where the analysis basis is balanced. Two general constructions of scaling vectors with symmetry properties and approximation order two and three, respectively, will be shown in Section 3, with two concrete examples of each construction provided. The coefficient matrices satisfying the dilation equations for these scaling vectors will be provided in the appendix (Section 4.)

1.1 Scaling Vectors

A vector $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$ of functions defined on $\mathbb{R}^k$ is said to be *refinable* if

$$\Phi = N^{\frac{k}{2}} \sum g_i \Phi(N \cdot -i)$$

for some integer dilation $N > 1$, $i \in \mathbb{Z}^k$, and for some sequence of $r \times r$ matrices g_i. (The normalization factor $N^{\frac{k}{2}}$ can be dropped, but is convenient for applications.) A *scaling vector* is a refinable vector Φ of square-integrable functions where the set of the components of Φ and their integer translates are linearly independent. An *orthogonal* scaling vector Φ is a scaling vector where the functions $\phi_1, \ldots, \phi_r$ are compactly supported and satisfy

$$\langle \phi_i, \phi_j(\cdot - n) \rangle = \delta_{i,j}\delta_{0,n},\ i, j \in \{1, \ldots, r\},\ n \in \mathbb{Z}^k,$$

where the inner product is the standard $L^2(\mathbb{R}^k)$ integral inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^k} f(x)g(x)dx$$

and δ is Kronecker's delta (1 if indices are equal, 0 otherwise.) *Biorthogonal* scaling vectors Φ and $\tilde{\Phi}$ have compactly supported components that satisfy

$$\langle \phi_i, \tilde{\phi}_j(\cdot - n) \rangle = \delta_{i,j}\delta_{0,n},\ i, j \in \{1, \ldots, r\},\ n \in \mathbb{Z}^k.$$

We use the notation $P_A f$ to denote the orthogonal projection of f onto the subspace A with orthogonal basis $\{a_1, \ldots, a_n\}$, given by

$$P_A f = \sum_{k=1}^{n} \frac{\langle f, a_k \rangle}{\langle a_k, a_k \rangle} a_k.$$

A scaling vector Φ is said to *generate* a closed linear space denoted by

$$S(\Phi) = \mathrm{clos}_{L^2}\ \mathrm{span}\ \{\phi_i(\cdot - j) :\ i = 1, \ldots, r,\ j \in \mathbb{Z}\}.$$

Two scaling vectors Φ and Θ are *equivalent* if $S(\Phi) = S(\Theta)$. The scaling vector Θ is said to *extend* Φ, or be an *extension* of Φ, if $S(\Phi) \subset S(\Theta)$. A scaling vector Φ is said to have *approximation order* k if

$$x^j = \sum_n \alpha_j(n)\Phi(x - n)$$

for some sequence of $1 \times r$ row-vectors $\{\alpha_j(n)\}$ for $j = 0, \ldots, k-1$. For $r > 1$, a length-r scaling vector Φ of approximation order k is said to be K-*balanced* for $K \leq k$, or simply *balanced* if $K = k$, if it satisfies the conditions

$$M_j := \int_{\mathbb{R}} x^j \phi_1(x)\, dx = \int_{\mathbb{R}} \left(x - \frac{k-1}{r} \right)^j \phi_k(x)\, dx \quad k = 2, \ldots, r \tag{1}$$

for $j = 0, \ldots, K-1$, with $M_0 \neq 0$. Scaling vectors of length 1 are trivially balanced.

Scaling vectors are important because they provide a framework for analyzing functions in $L^2(\mathbb{R}^k)$. A *multiresolution analysis* (MRA) of $L^2(\mathbb{R}^k)$ of multiplicity r is a set of closed linear spaces (V_p) such that

1. $\cdots \supset V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset V_2 \cdots$,
2. $\overline{\bigcup_{p \in \mathbb{Z}} V_p} = L^2(\mathbb{R}^k)$,
3. $\bigcap_{p \in \mathbb{Z}} V_p = \{0\}$,

4. $f \in V_0$ iff $f(N^{-j}\cdot) \in V_j$, and

5. there exists a set of functions $\phi_1, \ldots, \phi_r$ whose integer translates form a Riesz basis of V_0.

From the above definitions, it is clear that scaling vectors can be used to generate MRA's, with $V_0 = S(\Phi)$. Jia and Shen proved in [13] that if the components of a scaling vector Φ are compactly-supported, then Φ will always generate an MRA. All the scaling vectors discussed in this paper will consist of compactly-supported functions, and therefore, will generate MRA's. A function vector $\Psi = (\psi_1, \ldots, \psi_{r(N^k-1)})^T$, such that $\psi_i \in V_{-1}$ for $i = 1, \ldots, r(N^k - 1)$ (see [12]) and such that $S(\Psi) = V_{-1} - V_0$, is called a *multiwavelet*, and the individual ψ_i are called *wavelets*.

1.2 Macroelements on $[0, 1]$

We will use the notation $f^{(j)}(x)$ to denote the j^{th} derivative of $f(x)$, with the convention $f^{(0)}(x) = f(x)$. As a convenience, we will use the notation $f^{(j)}(0)$ and $f^{(j)}(1)$ to denote $\lim_{x\to 0^+} f^{(j)}(x)$ and $\lim_{x\to 1^-} f^{(j)}(x)$, respectively, although the notation is not technically rigorous.

A C^k *macroelement* defined on $[0, 1]$ is a vector of the form $(l_1, \ldots, l_{k+1}, r_1, \ldots, r_{k+1}, m_1, \ldots, m_n)^T$, where the set of elements are linearly independent, square-integrable functions supported on $[0, 1]$ with k continuous derivatives such that

1. $l_i^{(j)}(0) = r_i^{(j)}(1) = 0$ for $i = 1, \ldots, k + 1$,

2. $m_i^{(j)}(0) = m_i^{(j)}(1) = 0$ for $i = 1, \ldots, n$, and

3. $l_i^{(j)}(1) = r_i^{(j)}(0) = \alpha_j \delta_{i-1,j}$ for $i = 1, \ldots, k + 1$, where $\alpha_j \neq 0$

for $j = 0, \ldots, k$. A C^k macroelement is *orthogonal* if $\langle l_i, r_j\rangle = 0$ for $i, j = 1, \ldots, k + 1$, and $\langle l_i, m_j\rangle = \langle r_i, m_j\rangle = 0$ and $\langle l_i, l_i\rangle = \langle r_i, r_i\rangle = \langle m_j, m_j\rangle = 1$ for $i = 1, \ldots, k + 1$ and $j = 1, \ldots, n$. Macroelements $\Lambda = (l_1, \ldots, l_{k+1}, r_1, \ldots, r_{k+1}, m_1, \ldots, m_n)^T$ and $\tilde{\Lambda} = (\tilde{l}_1, \ldots, \tilde{l}_{k+1}, \tilde{r}_1, \ldots, \tilde{r}_{k+1}, \tilde{m}_1, \ldots, \tilde{m}_n)^T$ are *biorthogonal* if

1. $\langle l_i, \tilde{r}_j\rangle = \langle \tilde{l}_i, r_j\rangle = 0$ for $i, j = 1, \ldots, k + 1$,

2. $\langle l_i, \tilde{m}_j\rangle = \langle \tilde{l}_i, m_j\rangle = 0$ for $i = 1, \ldots, k + 1$ and $j = 1, \ldots, n$,

3. $\langle r_i, \tilde{m}_j\rangle = \langle \tilde{r}_i, m_j\rangle = 0$ for $i = 1, \ldots, k + 1$ and $j = 1, \ldots, n$, and

4. $\langle l_i, \tilde{l}_i\rangle = \langle r_i, \tilde{r}_i\rangle = \langle m_j, \tilde{m}_j\rangle = 1$ for $i = 1, \ldots, k + 1$ and $j = 1, \ldots, n$.

A macroelement Λ is *refinable* if there are $(2k + 2 + n) \times (2k + 2 + n)$ matrices $p_0, \ldots, p_{N-1}$ such that

$$\Lambda(x) = \sqrt{N} p_i \Lambda(Nx - i) \text{ for } x \in \left[\frac{i}{N}, \frac{i+1}{N}\right], \; i = 0, \ldots, N - 1. \tag{2}$$

Because of the linear independence of the components of Λ, the matrix coefficients p_i will be unique if they exist.

The following lemma unites the concepts of C^k macroelements and scaling vectors. The pivotal piece of the proof is that we may use the macroelements to construct scaling vectors by defining

$$\phi_i(x) = \frac{1}{\sqrt{2}} \begin{cases} l_i(x+1) \text{ for } x \in [-1,0] \\ r_i(x) \text{ for } x \in [0,1] \end{cases}, \ i = 1,\ldots,k, \text{ and} \tag{3}$$

$$\phi_{k+i}(x) = m_i(x) \text{ for } x \in [0,1],\ i = 1,\ldots,n. \tag{4}$$

The scaling vector $\Phi = (\phi_1,\ldots,\phi_{k+n})^T$ is called the scaling vector *associated* with Λ. The proof originally appeared in [15], but is shown here for completeness.

Lemma 1. *A refinable C^k macroelement $\Lambda = (l_1,\ldots,l_{k+1},r_1,\ldots,r_{k+1},m_1,\ldots,m_n)^T$ defined on $[0,1]$ has an associated scaling vector Φ of length $k+1+n$ and support $[-1,1]$. If the macroelement Λ is orthogonal, then the scaling vector Φ is equivalent to an orthogonal scaling vector.*

Proof. Let Λ satisfy equation (2) for some unique set of $(2k+2+n)\times(2k+2+n)$ matrices p_i, $i = 0,\ldots,N-1$ of the form

$$p_i = \begin{bmatrix} a_i & b_i & c_i \\ d_i & e_i & f_i \\ q_i & s_i & t_i \end{bmatrix}$$

for $(k+1)\times(k+1)$ matrices a_i, b_i, d_i, and e_i, $(k+1)\times n$ matrices c_i and f_i, $n\times(k+1)$ matrices q_i and s_i, and $n\times n$ matrices t_i, $i = 0,\ldots,N-1$. Note that due the continuity of the macroelement components, many of the matrices are redundant: $b_i = a_{i-1}$, $e_i = d_{i-1}$, and $s_i = q_{i-1}$ for $i = 1,\ldots,N-1$. Also, due to the endpoint condition on the C^k macroelement, $a_{N-1} = e_0$, and several of the matrices are zero: $b_0 = d_{N-1} = \mathbf{0}_{(k+1)\times(k+1)}$ and $s_0 = q_{N-1} = \mathbf{0}_{n\times(k+1)}$. Then the vector of functions as defined in (3) and (4) satisfy the dilation equation

$$\Phi(x) = \sqrt{N} \sum_{i=-N}^{N-1} g_i \Phi(Nx - i),$$

with

$$g_i = \begin{bmatrix} b_{N+i} & c_{N+i} \\ \mathbf{0}_{n\times(k+1)} & \mathbf{0}_{n\times n} \end{bmatrix},\ i = -N,\ldots,-1, \text{ and } g_i = \begin{bmatrix} e_i & f_i \\ s_i & t_i \end{bmatrix},\ i = 0,\ldots,N-1.$$

Hence, Φ is refinable, and supported completely in $[-1,1]$.

If Λ is orthonormal, then by definition, Φ meets the criteria of an orthogonal scaling vector, except that possibly $\langle\phi_i,\phi_j\rangle \neq 0$ for $i,j = 1,\ldots,k+1$, $i \neq j$, and for $i,j = k+2,\ldots,k+1+n$, $i \neq j$. However, we may replace $\{\phi_1,\ldots,\phi_{k+1}\}$ with an orthonormal set $\{\tilde{\phi}_1,\ldots,\tilde{\phi}_{k+1}\}$, and $\{\phi_{k+2},\ldots,\phi_{k+1+n}\}$ with an orthonormal set $\{\tilde{\phi}_{k+2},\ldots,\tilde{\phi}_{k+1+n}\}$, so that $\{\tilde{\phi}_1,\ldots,\tilde{\phi}_{k+1},\tilde{\phi}_{k+2},\ldots,\tilde{\phi}_{k+1+n}\}$ is an orthogonal scaling vector. □

The second part of this result is easily extended to the biorthogonal setting.

Lemma 2. *If the refinable C^k macroelements Λ and $\tilde{\Lambda}$ are biorthogonal, then the associated scaling vectors Φ and $\tilde{\Phi}$ are equivalent, respectively, to biorthogonal scaling vectors.*

Proof. If Λ is biorthogonal, then by definition, Φ and $\tilde{\Phi}$ meet the criteria for biorthogonal scaling vectors, except that possibly $\langle \phi_i, \tilde{\phi}_j \rangle \neq 0$ for$i \neq j$, $i, j = 1, \ldots, k+1$ and $i, j = k+2, \ldots, k+1+n\}$. However, we may replace $\{\phi_1, \ldots, \phi_{k+1}\}$ and $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_{k+1}\}$ with biorthogonal sets $\{\phi_1^*, \ldots, \phi_{k+1}^*\}$ and $\{\tilde{\phi}_1^*, \ldots, \tilde{\phi}_{k+1}^*\}$, respectively, using the biorthogonal version of the Gram-Schmidt process. Likewise, we can replace $\{\phi_{k+2}, \ldots, \phi_{k+1+n}\}$ and $\{\tilde{\phi}_{k+2}, \ldots, \tilde{\phi}_{k+1+n}\}$ with biorthogonal sets $\{\phi_{k+2}^*, \ldots, \phi_{k+1+n}^*\}$ and $\{\tilde{\phi}_{k+2}^*, \ldots, \tilde{\phi}_{k+1+n}^*\}$, respectively, so that $\{\phi_1^*, \ldots, \phi_{k+1}^*, \phi_{k+2}^*, \ldots, \phi_{k+1+n}^*\}$ and $\{\tilde{\phi}_1^*, \ldots, \tilde{\phi}_{k+1}^*, \tilde{\phi}_{k+2}^*, \ldots, \tilde{\phi}_{k+1+n}^*\}$ are biorthogonal scaling vectors. □

Let $\operatorname{span}\Lambda$ refer to the span of the elements of Λ. Two macroelements Λ and Γ are *equivalent* if $\operatorname{span}\Lambda = \operatorname{span}\Gamma$. The macroelement Γ is said to *extend* the C^k macroelement Λ, or be an *extension* of Λ, if $\Gamma = (\Lambda^T, M^T)^T$ is still linearly independent, where M is a set of square-integrable functions supported on $[0,1]$ with k continuous derivatives such that $m^{(j)}(0) = m^{(j)}(1) = 0$ for $m \in M$, $j = 0, \ldots, k$. In this paper, we will extend macroelements for the purpose of extending scaling vectors, using the following lemma. We use the notation $\chi_{[a,b]}$ to be the characteristic function defined by

$$\chi_{[a,b]} = \begin{cases} 1 \text{ for } x \in [a,b], \\ 0 \text{ otherwise.} \end{cases}$$

The following lemma with proof was given in [15], but is shown here for completeness.

Lemma 3. *Let Λ be a refinable C^p macroelement defined on $[0,1]$, and let Φ be the associated scaling vector as defined in (3) and (4). If Γ is a C^k macroelement extension of Λ, then the associated scaling vector Θ as defined in (3) and (4) is an extension of Φ.*

Proof. Let $\Lambda = \{l_1, \ldots, l_{k+1}, r_1, \ldots, r_{k+1}, m_1, \ldots, m_n\}$ and $\Gamma = \{l_1, \ldots, l_{k+1}, r_1, \ldots, r_{k+1}, m_1, \ldots, m_n, m_{n+1}, \ldots, m_{n+t}\}$, where Γ is an extension of Λ. Consider a basis element $\phi \in \{\phi_i(\cdot - j) : \phi_i \in \Phi, i \in \{1, \ldots, k+n\}, j \in \mathbb{Z}\}$ from $S(\Phi)$. If $\operatorname{supp}\phi \subset [j, j+2]$ for some $j \in \mathbb{Z}$, then from the definition of Θ in (3), $\phi(\cdot + j + 1) \in \Theta$ and $\phi \in S(\Theta)$. If $\operatorname{supp}\phi \subset [j, j+1]$ for some $j \in \mathbb{Z}$, then $\phi(x+j) \in \operatorname{span}\{m_1, \ldots, m_n\} \subset \operatorname{span}\{m_1, \ldots, m_{n+t}\}$. From the definition of Θ in (4), then $\phi \in S(\Theta)$. □

1.3 Fractal Interpolation Functions

Let $C_0([0,1])$ denote the space of continuous functions defined over $[0,1]$ that are 0 at $x = 0, 1$, and recall that the ∞-norm of a $n \times n$ matrix $A = (a_{ij})$ is given by $\|A\|_\infty = \max\limits_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$. Let Λ be a refinable macroelement of length n, and let Π be a function vector of length k defined by

$$\Pi(x) = \sqrt{N} p_i \Lambda(Nx - i) \text{ for } x \in \left[\frac{i}{N}, \frac{i+1}{N}\right], \; i = 0, \ldots, N-1,$$

for some $k \times n$ matrices p_i such that $\Pi(x) \in C_0([0,1])^k$. Then a vector Γ of the form

$$\Gamma(x) = \Pi(x) + \sum_{i=0}^{N-1} s_i \Gamma(Nx - i) \in C_0([0,1])^k,$$

where each s_i is a $k \times k$ matrix and $\max_i \|s_i\|_\infty < 1$, is a vector of *fractal interpolation functions* (FIF's). (See [1] and [2] for a more detailed introduction to FIF's.) By definition, the vector $\Lambda^* = (\Lambda^T, \Gamma^T)^T$ is a refinable C^0 macroelement that extends Λ.

Consider a C^k macroelement $\Lambda = (l_1, \ldots, l_{k+1}, r_1, \ldots, r_{k+1}, m_1, \ldots, m_n)^T$ defined on $[0,1]$ that is not orthogonal. We can not simply apply the Gram-Schmidt process to the components of Λ to obtain an orthonormal macroelement, since the resulting functions will not satisfy the endpoint criteria. In fact, we can not apply the process to any subset of elements that includes a l_i and r_j and still have the same type of macroelement. However, we can apply the Gram-Schmidt process to the set of functions $M = \{m_1, \ldots, m_n\}$ to get $\bar{M} = \{\bar{m}_1, \ldots, \bar{m}_n\}$, and then subtract P_M, the orthogonal projection onto the space spanned by M, from each of the other elements, giving the equivalent macroelement

$$\Gamma = ((I - P_M)l_1, \ldots, (I - P_M)l_k, (I - P_M)r_1, \ldots, (I - P_M)r_k, \bar{m}_1, \ldots, \bar{m}_n)^T.$$

If

$$\langle (I - P_M)l_i, (I - P_M)r_j \rangle = 0 \text{ for } i, j = 1, \ldots, k, \tag{5}$$

then Γ is an orthogonal macroelement. This is the fractal function approach for extending a macroelement to an orthogonal macroelement: add FIF's to the set M, hence the macroelement, so that (5) is satisfied. (See [6] for a broader discussion on constructing intertwined MRA's.) We may use the same basic approach to construct biorthogonal macroelements.

Example 1. The scaling vector shown in this example was originally constructed by Geronimo, Hardin, and Massopust in [7], although not in the macroelement context, and is reconstructed by Hardin and Kessler in detail using macroelements in [9]. It is widely known as the GHM scaling vector.

Let

$$l_1 = \sqrt{3}x\chi_{[0,1]}, \quad r_1 = \sqrt{3}(1-x)\chi_{[0,1]}, \text{ and } \phi_1^{\mathcal{S}}(x) = \frac{1}{\sqrt{2}} \begin{cases} l_1(x+1), & x \in [-1, 0] \\ r_1(x), & x \in [0, 1]. \end{cases} \tag{6}$$

Consider the C^0 macroelement $\Lambda^{\mathcal{S}} = (l_1, r_1)$ and the scaling vector $\Phi^{\mathcal{S}} = (\phi_1^{\mathcal{S}})$ shown on the left in Figure 1, and note that $S(\Phi^{\mathcal{S}}) = \mathcal{S}_1^0(\mathbb{Z}) \cap L^2(\mathbb{R})$. In order to extend $\Lambda^{\mathcal{S}}$ to an orthogonal C^0 macroelement, we construct an FIF satisfying

$$u(x) = \phi_1^{\mathcal{S}}(2x - 1) + s_0 u(2x) + s_1 u(2x - 1), \ \max_{i=0,1} |s_i| < 1,$$

that satisfies (5), which reduces to the one condition $\langle (I - P_u)l_1, (I - P_u)r_1 \rangle = 0$. It was shown in [7] and [9] that the orthogonality condition is satisfied by $s_0 = s_1 = -\frac{1}{5}$. By letting

$$l_1^* = (I - P_u)l_1, \quad r_1^* = (I - P_u)r_1, \text{ and } m_1^* = u,$$

we have the orthogonal C^0 macroelement $\Lambda = (l_1^*, r_1^*, m_1^*)^T$, that is equivalent to $(l_1, r_1, u)^T$ and is an extension of $\Lambda^{\mathcal{S}}$. The associated scaling vector $\Phi = (\phi_1, \phi_2)^T$, defined in (3) and

(4) and normalized, is the orthogonal GHM scaling vector, and is illustrated on the right in Figure 1. Hardin and Marasovich generalized this construction to biorthogonal duals of multiplicity-2 in [10], by extending the original scaling vector $\Phi^{\mathcal{S}}$ with two fractal functions defined on $[0, 1]$.

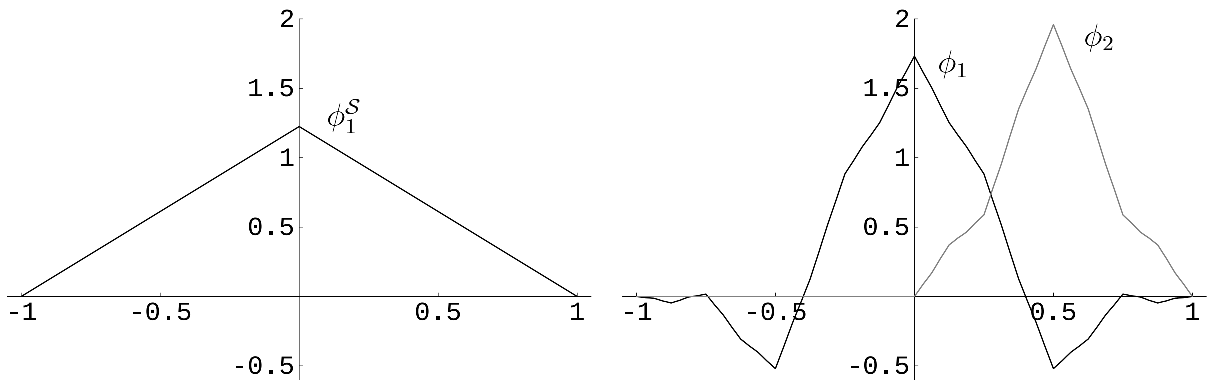

Figure 1: The original $\mathcal{S}_1^0(\mathbb{Z})$ scaling vector $\Phi^{\mathcal{S}}$ at left, and its extension, the orthogonal GHM scaling vector Φ at right.

Example 2. The scaling vector shown in this example was originally constructed by Donovan, Geronimo, and Hardin in [5], although not in the macroelement context, and again by Hardin and Kessler in detail in [9] using a macroelement approach.

Let l_1, r_1, and $\phi_1^{\mathcal{S}}$ be defined as in (6), and let

$$\phi_2^{\mathcal{S}} = m_1 = \sqrt{30}x(1-x)\chi_{[0,1]}.$$

Consider the C^0 macroelement $\Lambda^{\mathcal{S}} = (l_1, r_1, m_1)^T$ and the associated scaling vector $\Phi^{\mathcal{S}} = (\phi_1^{\mathcal{S}}, \phi_2^{\mathcal{S}})^T$ shown on the left in Figure 2, and note that $S(\Phi^{\mathcal{S}}) = \mathcal{S}_2^0(\mathbb{Z}) \cap L^2(\mathbb{R})$. In order to extend $\Lambda^{\mathcal{S}}$ to an orthogonal C^0 macroelement, we construct a FIF satisfying

$$u(x) = \phi_2^{\mathcal{S}}(2x) - \phi_2^{\mathcal{S}}(2x-1) + su(2x) + su(2x-1) \text{ for } |s| < 1,$$

that satisfies (5), which reduces to the one condition $\langle (I - P_M)l_1, (I - P_M)r_1 \rangle = 0$, where $M = \{m_1, u\}$. (Note that $\langle m_1, u \rangle = 0$, since m_1 and u are symmetric and antisymmetric, respectively, about $x = \frac{1}{2}$.) It was shown in [5] and [9] that the orthogonality condition is satisfied by $s = \frac{2-\sqrt{10}}{6} \approx -0.1937$. By letting

$$l_1^* = (I - P_M)l_1, \quad r_1^* = (I - P_M)r_1, \quad m_1^* = m_1, \text{ and } m_2^* = u,$$

we have the orthogonal C^0 macroelement $\Lambda = (l_1^*, r_1^*, m_1^*, m_2^*)$, equivalent to $(l_1, r_1, m_1, u)^T$ and an extension of $\Lambda^{\mathcal{S}}$. The associated scaling vector $\Phi = (\phi_1, \phi_2, \phi_3)^T$, defined in (3) and (4) and normalized, is an orthogonal scaling vector, and is illustrated in Figure 2.

For an example of a multiplicity-4 C^1 orthogonal scaling vector with approximation order 3 built using the same basic technique, see [15].

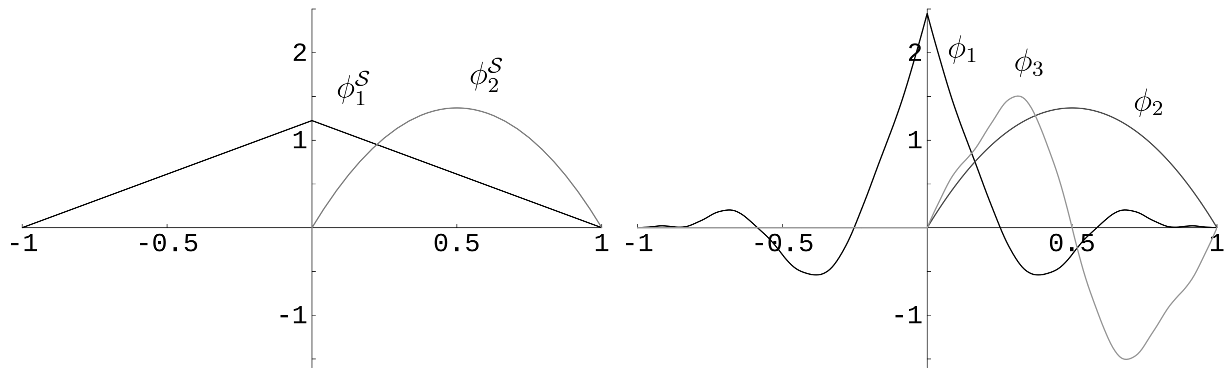

Figure 2: The original $\mathcal{S}_2^0(\mathbb{Z})$ scaling vector $\Phi^{\mathcal{S}}$ at left, and its extension, the orthogonal scaling vector Φ at right.

2 Main Results

Examples 1 and 2 demonstrate how extending spline scaling vectors with fractal functions defined on $[0,1]$ is a powerful tool for creating orthogonal scaling vectors without greatly increasing the multiplicity of the scaling vector. Our intuition would lead us to think that with the addition of more fractal functions, hence more free parameters, we should be able to create an orthogonal scaling vector that is at least 2-balanced. This turns out to be impossible, but we can use the technique to generate biorthogonal dual scaling vectors with the analysis basis balanced up to the approximation order of the original scaling vector.

Let $l_1 = \sqrt{3}x\chi_{[0,1]}$ and $r_1 = \sqrt{3}(1-x)\chi_{[0,1]}$, so that $\langle l_1, l_1\rangle = \langle r_1, r_1\rangle = 1$ and $\langle l_1, r_1\rangle = \frac{1}{2}$. Then the normalized piecewise-linear spline $\phi^{\mathcal{S}}$ can be defined as

$$\phi^{\mathcal{S}}(x) = \frac{1}{\sqrt{2}}\begin{cases} l_1(x+1) & \text{if } x \in [-1,0] \\ r_1(x) & \text{if } x \in [0,1], \end{cases} \tag{7}$$

and $S(\{\phi^{\mathcal{S}}\}) = \mathcal{S}_1^0(\mathbb{Z}) \cap L^2(\mathbb{R})$. Note that $1\chi_{[0,1]} = \frac{1}{\sqrt{3}}(l_1 + r_1)$ and $x\chi_{[0,1]} = \frac{1}{\sqrt{3}}l_1$, and so $(x-1)\chi_{[0,1]} = -\frac{1}{\sqrt{3}}r_1$.

We will use the following lemma in our proof of the main theorem.

Lemma 4. *Let $M = \{m_1, \ldots, m_n\}$ be an orthonormal set of continuous functions supported on $[0,1]$ satisfying $m_k(0) = m_k(1) = 0$ for $k = 1, \ldots, n$. If*

$$l_1^* = (I - P_M)l_1, \quad r_1^* = (I - P_M)r_1, \text{ and } \phi^*(x) = \begin{cases} l_1^*(x+1) & \text{if } x \in [-1,0] \\ r_1^*(x) & \text{if } x \in [0,1], \end{cases}$$

then $\langle x, \phi^*\rangle = 0$.

Proof. Note that

$$\begin{aligned}
\langle x, \phi^* \rangle &= \frac{1}{\sqrt{3}}\langle -r_1, l_1^* \rangle + \frac{1}{\sqrt{3}}\langle l_1, r_1^* \rangle \\
&= \frac{1}{\sqrt{3}}\left(-\langle l_1, r_1 \rangle + \langle r_1, P_M l_1 \rangle + \langle l_1, r_1 \rangle - \langle l_1, P_M r_1 \rangle\right) \\
&= \frac{1}{\sqrt{3}}\left(\sum_{k=1}^{n}\langle l_1, m_k \rangle\langle r_1, m_k \rangle - \sum_{k=1}^{n}\langle l_1, m_k \rangle\langle r_1, m_k \rangle\right) \\
&= 0.
\end{aligned}$$

□

Theorem 1. *Let ϕ be the linear B-spline $\phi^{\mathcal{S}}$ defined in (7), and let $\Phi^{\mathcal{S}} = \{\phi^{\mathcal{S}}\}$. Then $\Phi^{\mathcal{S}}$ can not be extended to a 2-balanced orthogonal scaling vector with continuous functions supported on* $[0, 1]$.

Proof. Let M, l_1^*, r_1^*, and ϕ^* be defined as in the statement of Lemma 4, and let $\phi_p = \phi^*$ for fixed integer p, $1 \le p \le n+1$. Then the orthogonality condition (5) is equivalent to

$$\sum_{k=1}^{n}\langle l_1, m_k \rangle\langle r_1, m_k \rangle = \frac{1}{2}, \tag{8}$$

and we have

$$\|\phi_p\| = \sqrt{\langle \phi_p, \phi_p \rangle} = \sqrt{2 - \sum_{k=1}^{n}\langle l_1, m_k \rangle^2 - \sum_{k=1}^{n}\langle r_1, m_k \rangle^2}. \tag{9}$$

The 1-balancing conditions from (1) are independent of the order of the functions in the scaling vector, so that, given (8) and (9),

$$\begin{aligned}
M_0 &= \int_{\mathbb{R}} \frac{\phi_p(x)}{\|\phi_p\|}dx = \left\langle \frac{\phi_p}{\|\phi_p\|}, 1 \right\rangle = \frac{1}{\sqrt{3}\|\phi_p\|}\left(\langle l_1^*, l_1 + r_1 \rangle + \langle r_1^*, l_1 + r_1 \rangle\right) \\
&= \sqrt{\frac{1}{3}\left(2 - \sum_{k=1}^{n}\langle l_1, m_k \rangle^2 - \sum_{k=1}^{n}\langle r_1, m_k \rangle^2\right)}.
\end{aligned} \tag{10}$$

The remaining 1-balancing conditions are

$$\langle 1, \phi_k \rangle = \frac{1}{\sqrt{3}}\left(\langle l_1, m_k \rangle + \langle r_1, m_k \rangle\right) = M_0 \text{ for } k = 1, \ldots, n,$$

so that

$$\langle l_1, m_k \rangle + \langle r_1, m_k \rangle = \sqrt{3}M_0 \text{ for } k = 1, \ldots, n. \tag{11}$$

Squaring both sides of (11) and adding each case, we have

$$\sum_{k=1}^{n}\langle l_1, m_k \rangle^2 + \sum_{k=1}^{n}\langle r_1, m_k \rangle^2 + 2\sum_{k=1}^{n}\langle l_1, m_k \rangle\langle r_1, m_k \rangle = 3nM_0. \tag{12}$$

Also, squaring both sides of (10), we have

$$M_0^2 = \frac{1}{3}\left(2 - \sum_{k=1}^{n}\langle l_1, m_k\rangle^2 - \sum_{k=1}^{n}\langle r_1, m_k\rangle^2\right),$$

so that

$$\sum_{k=1}^{n}\langle l_1, m_k\rangle^2 + \sum_{k=1}^{n}\langle r_1, m_k\rangle^2 = 2 - 3M_0. \tag{13}$$

Substituting (13) and (8) into (12), we have

$$3nM_0^2 = 2 - 3M_0^2 + 1,$$

or

$$M_0^2 = \frac{1}{n+1}. \tag{14}$$

The order of the functions in the scaling vector can affect the 2-balancing conditions, so we consider two cases.

Case 1: Let $p = 1$. The 2-balancing constant M_1 defined in equation (1) is

$$M_1 = \int_{\mathbb{R}} x \frac{\phi_1(x)}{\|\phi_1\|} dx = \frac{\langle x, \phi_1(x)\rangle}{\sqrt{2 - \sum_{k=1}^{n}\langle l_1, m_k\rangle^2 - \sum_{k=1}^{n}\langle r_1, m_k\rangle^2}} = 0$$

by Lemma 4. Then the remaining 2-balancing conditions are

$$\int_{\mathbb{R}} \left(x - \frac{k}{n+1}\right)\phi_{k+1}(x)dx = \frac{1}{\sqrt{3}}\langle l_1, m_k\rangle - \frac{k}{n+1}M_0 = 0 \text{ for } k = 1, \ldots, n.$$

Note that

$$\langle l_1, m_k\rangle = \frac{\sqrt{3}k}{n+1}M_0 \text{ for } k = 1, \ldots, n, \tag{15}$$

so by substituting this into (11), we can solve for $\langle r_1, m_k\rangle$:

$$\langle r_1, m_k\rangle = \frac{\sqrt{3}(n-k+1)}{n+1}M_0 \text{ for } k = 1, \ldots, n. \tag{16}$$

Substituting (15) and (16) into the orthogonality condition (5) and summing over $k = 1, \ldots, n$, we get

$$\frac{3M_0^2}{(n+1)^2}\sum_{k=1}^{n}((n+1)k - k^2) = \frac{1}{2},$$

$$\frac{M_0^2}{2(n+1)}(n^2 + 2n) = \frac{1}{2}, \text{ and}$$

$$M_0^2 = \frac{n+1}{n(n+2)}. \tag{17}$$

Combining (14) and (17) to solve for n, we have

$$n^2 + 2n + 1 = n^2 + 2n,$$

which has no solution.
Case 2: Let $1 < p \leq n+1$. Then our scaling vector would have $\phi_k = m_k$ for $k = 1, \ldots, p-1$, and $\phi_{k+1} = m_k$ for $k = p, \ldots, n$. The 2-balancing constant M_1 defined in equation (1) is

$$M_1 = \int_{\mathbb{R}} x\phi_1(x)dx = \langle x, m_1 \rangle = \frac{1}{\sqrt{3}}\langle l_1, m_1 \rangle.$$

Then,

$$\int_{\mathbb{R}} \left(x - \frac{p-1}{n+1}\right) \frac{\phi_p(x)}{\|\phi_p\|}dx = \left\langle x - \frac{p-1}{n+1}, \frac{\phi_p}{\|\phi_p\|} \right\rangle = -\frac{p-1}{n+1}M_0 = \frac{1}{\sqrt{3}}\langle l_1, m_1 \rangle$$

by Lemma 4, so that

$$\langle l_1, m_1 \rangle = \frac{\sqrt{3}(1-p)}{n+1}M_0 \text{ and } M_1 = \frac{1-p}{n+1}M_0.$$

For $k = 1, \ldots, p-1$, the 2-balancing conditions are

$$\int_{\mathbb{R}} \left(x - \frac{k-1}{n+1}\right) \phi_k(x)dx = \left\langle x - \frac{k-1}{n+1}, m_k \right\rangle = \frac{1}{\sqrt{3}}\langle l_1, m_k \rangle - \frac{k-1}{n+1}M_0 = \frac{1-p}{n+1}M_0.$$

Then, in light of (11), we have

$$\langle l_1, m_k \rangle = \frac{\sqrt{3}(k-p)}{n+1}M_0 \text{ and } \langle r_1, m_k \rangle = \frac{\sqrt{3}(n-k+p+1)}{n+1}M_0 \text{ for } k = 1, \ldots, p-1.$$

Summing their products over $k = 1, \ldots, p-1$, we have

$$\begin{aligned} \sum_{k=1}^{p-1} \langle l_1, m_k \rangle \langle r_1, m_k \rangle &= \frac{3M_0^2}{(n+1)^2}(-k^2 + k(n+2p+1) - p(n+p+1)) \\ &= \frac{M_0^2 p(1-p)(3n+2p+2)}{2(n+1)^2}. \end{aligned} \tag{18}$$

For $k = p, \ldots, n$, the 2-balancing conditions are

$$\int_{\mathbb{R}} \left(x - \frac{k}{n+1}\right) \phi_{k+1}(x)dx = \left\langle x - \frac{k}{n+1}, m_k \right\rangle = \frac{1}{\sqrt{3}}\langle l_1, m_k \rangle - \frac{k}{n+1}M_0 = \frac{1-p}{n+1}M_0.$$

Then, in light of (11), we have

$$\langle l_1, m_k \rangle = \frac{\sqrt{3}(k-p+1)}{n+1}M_0 \text{ and } \langle r_1, m_k \rangle = \frac{\sqrt{3}(n-k+p)}{n+1}M_0 \text{ for } k = p, \ldots, n.$$

Summing their products over $k = p, \ldots, n$, we have

$$\begin{aligned} \sum_{k=p}^{n} \langle l_1, m_k \rangle \langle r_1, m_k \rangle &= \frac{3M_0^2}{(n+1)^2}(-k^2 + k(n+2p-1) + (n+p)(1-p)) \\ &= \frac{M_0^2 (n-p+1)(n-p+2)(n+2p)}{2(n+1)^2}. \end{aligned} \tag{19}$$

Given (18) and (19), the orthogonality condition (8) is equivalent to

$$\frac{M_0^2(n^2+2n+6p(1-p))}{2(n+1)} = \frac{1}{2},$$

so that

$$M_0^2 = \frac{n+1}{n^2+2n+6p(1-p)}. \tag{20}$$

Combining (14) and (20) to solve for n, we have

$$n^2+2n+1 = n^2+2n+6p(1-p),$$

which is equivalent to $6p^2-6p+1=0$, which has no positive integer roots. □

3 Balanced Biorthogonal Scaling Vectors

While we can not use the insertion of fractal functions on $[0,1]$ to create 2-balanced orthogonal scaling vectors, we can use them to create balanced biorthogonal scaling vectors. The following section contains a construction for biorthogonal duals that contain the square-integrable elements of the classic spline space $\mathcal{S}_1^0$, where the analysis scaling vector $\tilde{\Phi}$ is 2-balanced. The last section contains a construction for biorthogonal duals that contain the square-integrable elements of the spline space $\mathcal{S}_2^0$, where the analysis scaling vector $\tilde{\Phi}$ is 3-balanced. In each construction, we have some freedom in actually constructing the scaling vector elements, so different concrete examples will be illustrated. The matrix coefficients of the dilation equation (hence the analysis and reconstruction filters) are given in the appendix.

3.1 2-Balanced, Approximation Order 2

Hardin and Marasovich generalized the GHM scaling vector to a class of biorthogonal duals in [10]. This construction is undoubtedly a subclass of their construction. It is presented here to illustrate the use of macroelements as a tool for extending scaling vectors to biorthogonal scaling vectors. Also, the concept of balanced scaling vectors had not been introduced at the time of their paper, so none of the bases they used as illustrations were actually balanced. We will further restrict our construction to bases that have symmetry properties.

Let $l_1 = \sqrt{3}x\chi_{[0,1]}$ and $r_1 = \sqrt{3}(1-x)\chi_{[0,1]}$, so that $\langle l_1, l_1\rangle = \langle r_1, r_1\rangle = 1$ and $\langle l_1, r_1\rangle = \frac{1}{2}$, and let $\phi^{\mathcal{S}}$ be defined by

$$\phi^{\mathcal{S}}(x) = \begin{cases} l_1(x+1) & \text{if } x \in [-1,0] \\ r_1(x) & \text{if } x \in [0,1], \end{cases}$$

Let m_1 and $\tilde{m}_1$ satisfy the inhomogenous dilation equations

$$\begin{aligned} m_1(x) &= \phi^{\mathcal{S}}(2x-1) + sm_1(2x) + sm_1(2x-1) \text{ and} & (21)\\ \tilde{m}_1(x) &= \tilde{a}\phi^{\mathcal{S}}(2x-1) + \tilde{s}\tilde{m}_1(2x) + \tilde{s}\tilde{m}_1(2x-1) & (22) \end{aligned}$$

for some $|s|, |\tilde{s}| < 1$, with $\tilde{a}$ chosen so that $\langle m_1, \tilde{m}_1\rangle = 1$. Note that both m_1 and $\tilde{m}_1$ are symmetrical with respect to $x = \frac{1}{2}$, so $\langle r_1, m_1\rangle = \langle l_1, m_1\rangle$ and $\langle r_1, \tilde{m}_1\rangle = \langle l_1, \tilde{m}_1\rangle$. Using the equations (21) and (22) and the fact that

$$l_1(x) = \begin{cases} \frac{1}{2}l_1(2x) & \text{if } x \in [0, \frac{1}{2}] \\ l_1(2x-1) + \frac{1}{2}r_1(2x-1) & \text{if } x \in [\frac{1}{2}, 1], \end{cases} \tag{23}$$

we have conditions that must be satisfied by $\langle l_1, m_1\rangle$, $\langle l_1, \tilde{m}_1\rangle$, and the remaining parameters $\tilde{a}$, s, and $\tilde{s}$:

$$\begin{cases} \frac{3}{2} + 2s\langle l_1, m_1\rangle = 2\langle l_1, m_1\rangle \\ \frac{3}{2}\tilde{a} + 2\tilde{s}\langle l_1, \tilde{m}_1\rangle = 2\langle l_1, \tilde{m}_1\rangle \\ \tilde{a}(1 + s\langle l_1, m_1\rangle) + \tilde{s}(s + \langle l_1, \tilde{m}_1\rangle) = 1. \end{cases} \tag{24}$$

Let $l_1^* = l_1 - \langle l_1, \tilde{m}_1\rangle m_1$, $r_1^* = r_1 - \langle l_1, \tilde{m}_1\rangle m_1$, $\tilde{l}_1^* = l_1 - \langle l_1, m_1\rangle \tilde{m}_1$, and $\tilde{r}_1^* = r_1 - \langle l_1, m_1\rangle \tilde{m}_1$. Then the macroelements $\Lambda = (l_1^*, r_1^*, m_1)^T$ and $\tilde{\Lambda} = (\tilde{l}_1^*, \tilde{r}_1^*, \tilde{m}_1)^T$ are biorthogonal provided m_1, $\tilde{m}_1$ satisfy $\langle l_1^*, \tilde{r}_1^*\rangle = \langle r_1^*, \tilde{l}_1^*\rangle = 0$, or equivalently,

$$\langle l_1, m_1\rangle\langle l_1, \tilde{m}_1\rangle = \frac{1}{2},$$

which is satisfied by setting

$$\langle l_1, \tilde{m}_1\rangle = \frac{1}{2\langle l_1, m_1\rangle}.$$

Then from Lemma 2, we may define the biorthogonal scaling vectors $\Phi = (\phi_1, \phi_2)^T$ and $\tilde{\Phi} = (\tilde{\phi}_1, \tilde{\phi}_2)^T$, with components

$$\phi_1(x) = \alpha \begin{cases} l_1^*(x+1) & \text{if } x \in [-1, 0] \\ r_1^*(x) & \text{if } x \in [0, 1], \end{cases} \qquad \tilde{\phi}_1(x) = \tilde{\alpha} \begin{cases} \tilde{l}_1^*(x+1) & \text{if } x \in [-1, 0] \\ \tilde{r}_1^*(x) & \text{if } x \in [0, 1], \end{cases}$$

$$\phi_2(x) = \beta m_1, \quad \text{and} \quad \tilde{\phi}_2(x) = \frac{1}{\beta}\tilde{m}_1.$$

The condition $\langle \phi_2, \tilde{\phi}_2\rangle = 1$ is automatically satisfied, and the condition $\langle \phi_1, \tilde{\phi}_1\rangle = 1$ is satisfied when $\tilde{\alpha} = \frac{1}{\alpha}$.

The 1-balancing constant is

$$M_0 = \int_{\mathbb{R}} \tilde{\phi}_1(x)dx = \langle 1, \alpha\tilde{l}_1^*\rangle + \langle 1, \alpha\tilde{r}_1^*\rangle = \frac{1}{\alpha\sqrt{3}},$$

so the 1-balancing condition is

$$\int_{\mathbb{R}} \tilde{\phi}_2(x)dx = \langle 1, \frac{1}{\beta}\tilde{m}_1\rangle = \frac{1}{\beta\sqrt{3}\langle l_1, m_1\rangle} = M_0,$$

which is satisfied by setting $\langle l_1, m_1\rangle = \frac{\alpha}{\beta}$. The 2-balancing constant is $M_1 = 0$ from Lemma 4, and the 2-balancing condition

$$\int_{\mathbb{R}} \left(x - \frac{1}{2}\right) \tilde{\phi}_2(x)dx = 0$$

is automatically satisfied. The system in (24) now reduces to a system in $\tilde{a}$, s, $\tilde{s}$, and β:

$$\begin{cases} 3\tilde{a}\alpha + 2\beta(\tilde{s}-1) = 0 \\ 4\alpha(s-1) + 3\beta = 0 \\ 2\tilde{a}s\alpha^2 + 2\alpha\beta(\tilde{a} + s\tilde{s} - 1) + \tilde{s}\beta^2 = 0, \end{cases}$$

which has the solution

$$\beta = \frac{4\alpha}{3}(1-s), \quad \tilde{a} = \frac{8(1-s)^2}{6-15s}, \quad \text{and} \quad \tilde{s} = \frac{2s+1}{5s-2},$$

with $\langle l_1, m_1\rangle = \frac{3}{4(1-s)}$ and $\langle l_1, \tilde{m}_1\rangle = \frac{2(1-s)}{3}$, and $|s|, |\tilde{s}| < 1$ for $-1 < s < \frac{1}{7}$.

We choose $\alpha = 1$ so that $\phi_1(0) = \tilde{\phi}_1(0)$ and look at two interesting examples from this class of functions. If we choose $s = 0$, then $\tilde{s} = -\frac{1}{2}$ and $S(\Phi) = \mathcal{S}_1^0(\mathbb{Z}/2) \cap L^2(\mathbb{R})$. The balanced biorthogonal duals for this choice of s are illustrated in Figure 3, and their matrix coefficients appear in the appendix. If we choose $s = \tilde{s} = -\frac{1}{5}$, then one may verify that

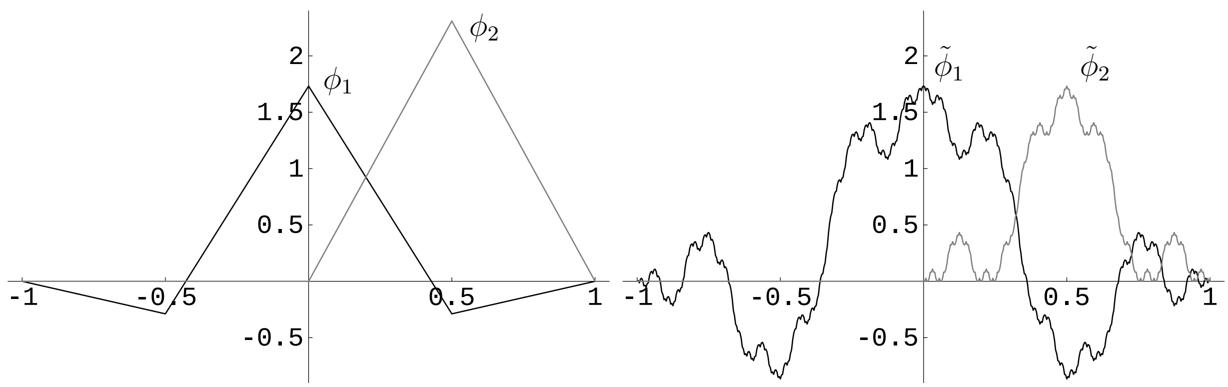

Figure 3: The 2-balanced biorthogonal scaling vectors with $s = 0$, Φ at left, $\tilde{\Phi}$ at right.

$\langle l_1^*, r_1^*\rangle = \langle \tilde{l}_1^*, \tilde{r}_1^*\rangle = 0$, $\langle l_1^*, m_1\rangle = \langle r_1^*, m_1\rangle = 0$, and $\langle \tilde{l}_1^*, \tilde{m}_1\rangle = \langle \tilde{r}_1^*, \tilde{m}_1\rangle = 0$, using (21), (22), and (23). The balanced biorthogonal duals for this choice of s are illustrated in Figure 4, and their matrix coefficients appear in the appendix. Note that the elements of this particular dual are merely scaled elements of the orthogonal GHM scaling vector.

3.2 3-Balanced, Approximation Order 3

We believe the following construction to be a new class of scaling vectors. We have neglected some generality here by restricting our construction to bases that have symmetry properties, since the ability to build bases with symmetry properties is one of the major advantages of using scaling vectors over a single scaling function.

Let $l_1 = \sqrt{3}x\chi_{[0,1]}$ and $r_1 = \sqrt{3}(1-x)\chi_{[0,1]}$, so that $\langle l_1, l_1\rangle = \langle r_1, r_1\rangle = 1$ and $\langle l_1, r_1\rangle = \frac{1}{2}$, and let $\phi^{\mathcal{S}}$ be defined by

$$\phi^{\mathcal{S}}(x) = \begin{cases} l_1(x+1) & \text{if } x \in [-1, 0] \\ r_1(x) & \text{if } x \in [0, 1], \end{cases}$$

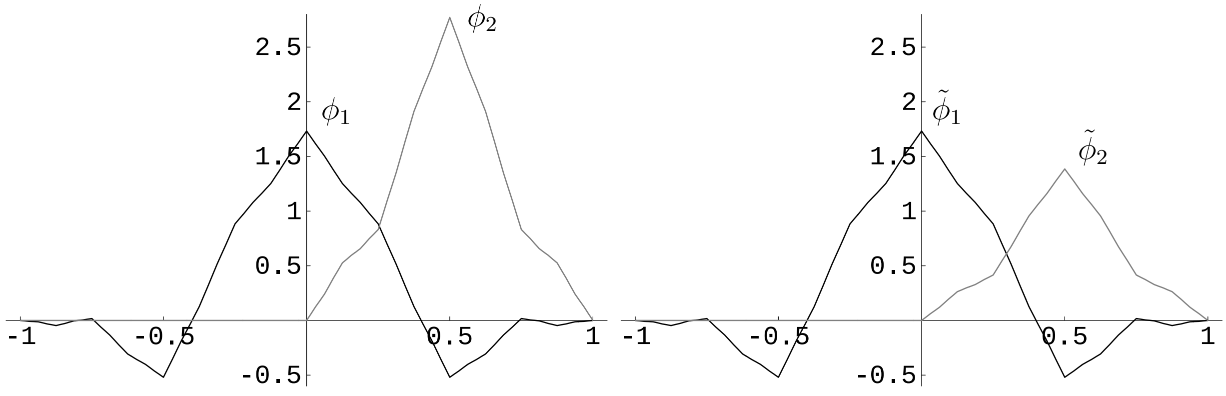

Figure 4: The 2-balanced biorthogonal scaling vectors with $s = -\frac{1}{5}$, Φ at left, $\tilde{\Phi}$ at right.

Let $m_1 = \sqrt{30}x(1-x)\chi_{[0,1]}$, so that $S((\phi^{\mathcal{S}}, m_1)^T) = \mathcal{S}_2^0(\mathbb{Z}) \cap L^2(\mathbb{R})$. Note that $\langle m_1, m_1 \rangle = 1$ and $\langle r_1, m_1 \rangle = \langle l_1, m_1 \rangle = \frac{\sqrt{5}}{2\sqrt{2}}$ due to the symmetry of m_1 about $x = \frac{1}{2}$.

Let m_2, m_3, $\tilde{m}_2$, and $\tilde{m}_3$ satisfy the inhomogenous matrix dilation equations

$$\begin{bmatrix} m_2 \\ m_3 \end{bmatrix}(x) = \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}\begin{bmatrix} \phi^{\mathcal{S}} \\ m_1 \end{bmatrix}(2x) + \begin{bmatrix} a & c \\ a & b \end{bmatrix}\begin{bmatrix} \phi^{\mathcal{S}} \\ m_1 \end{bmatrix}(2x-1) + \begin{bmatrix} q & r \\ t & s \end{bmatrix}\begin{bmatrix} m_2 \\ m_3 \end{bmatrix}(2x) + \begin{bmatrix} s & t \\ r & q \end{bmatrix}\begin{bmatrix} m_2 \\ m_3 \end{bmatrix}(2x-1) \tag{25}$$

and

$$\begin{bmatrix} \tilde{m}_2 \\ \tilde{m}_3 \end{bmatrix}(x) = \begin{bmatrix} 0 & \tilde{b} \\ 0 & \tilde{c} \end{bmatrix}\begin{bmatrix} \phi^{\mathcal{S}} \\ m_1 \end{bmatrix}(2x) + \begin{bmatrix} \tilde{a} & \tilde{c} \\ \tilde{a} & \tilde{b} \end{bmatrix}\begin{bmatrix} \phi^{\mathcal{S}} \\ m_1 \end{bmatrix}(2x-1) + \begin{bmatrix} \tilde{q} & \tilde{r} \\ \tilde{t} & \tilde{s} \end{bmatrix}\begin{bmatrix} \tilde{m}_2 \\ \tilde{m}_3 \end{bmatrix}(2x) + \begin{bmatrix} \tilde{s} & \tilde{t} \\ \tilde{r} & \tilde{q} \end{bmatrix}\begin{bmatrix} \tilde{m}_2 \\ \tilde{m}_3 \end{bmatrix}(2x-1) \tag{26}$$

where

$$\left\| \begin{bmatrix} q & r \\ t & s \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} s & t \\ r & q \end{bmatrix} \right\|_\infty < 1 \quad \text{and} \quad \left\| \begin{bmatrix} \tilde{q} & \tilde{r} \\ \tilde{t} & \tilde{s} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{s} & \tilde{t} \\ \tilde{r} & \tilde{q} \end{bmatrix} \right\|_\infty < 1,$$

and the parameters are chosen so that $\langle m_2, \tilde{m}_2 \rangle = \langle m_3, \tilde{m}_3 \rangle = 1$ and $\langle m_2, \tilde{m}_3 \rangle = \langle m_3, \tilde{m}_2 \rangle = 0$. Note that while m_2, m_3, $\tilde{m}_2$, and $\tilde{m}_3$ are not themselves symmetric, $m_3(x) = m_2(1-x)$ and $\tilde{m}_3(x) = \tilde{m}_2(1-x)$, so

$$\begin{array}{lll} \langle r_1, m_3 \rangle = \langle l_1, m_2 \rangle & \langle r_1, m_2 \rangle = \langle l_1, m_3 \rangle & \langle m_1, m_3 \rangle = \langle m_1, m_2 \rangle \\ \langle r_1, \tilde{m}_3 \rangle = \langle l_1, \tilde{m}_2 \rangle & \langle r_1, \tilde{m}_2 \rangle = \langle l_1, \tilde{m}_3 \rangle & \langle m_1, \tilde{m}_3 \rangle = \langle m_1, \tilde{m}_2 \rangle \end{array}$$

Using the equations (25), (26), and (23), we have the following conditions that must be satisfied by $\langle l_1, m_2 \rangle$, $\langle l_1, \tilde{m}_2 \rangle$, $\langle l_1, m_3 \rangle$, $\langle l_1, \tilde{m}_3 \rangle$, $\langle m_1, m_2 \rangle$, $\langle m_1, \tilde{m}_2 \rangle$, and the remaining parameters:

$$\begin{cases} \frac{1}{8}(12a+\sqrt{10}b+3\sqrt{10}c+4(\langle l_1,m_2\rangle(q+2s+t)+\langle l_1,m_3\rangle(r+s+2t))) = 2\langle l_1,m_2\rangle \\ \frac{1}{8}(12a+3\sqrt{10}b+\sqrt{10}c+4(\langle l_1,m_2\rangle(q+2r+t)+\langle l_1,m_3\rangle(2q+r+s))) = 2\langle l_1,m_3\rangle \\ \frac{1}{8}(12\tilde{a}+\sqrt{10}\tilde{b}+3\sqrt{10}\tilde{c}+4(\langle l_1,\tilde{m}_2\rangle(\tilde{q}+2\tilde{s}+\tilde{t})+\langle l_1,\tilde{m}_3\rangle(\tilde{r}+\tilde{s}+2\tilde{t}))) = 2\langle l_1,\tilde{m}_2\rangle \\ \frac{1}{8}(12\tilde{a}+3\sqrt{10}\tilde{b}+\sqrt{10}\tilde{c}+4(\langle l_1,\tilde{m}_2\rangle(\tilde{q}+2\tilde{r}+\tilde{t})+\langle l_1,\tilde{m}_3\rangle(2\tilde{q}+\tilde{r}+\tilde{s}))) = 2\langle l_1,\tilde{m}_3\rangle \\ \frac{1}{8}(5\sqrt{10}a+7b+7c+2(\sqrt{10}\langle l_1,m_2\rangle(q+t)+\sqrt{10}\langle l_1,m_3\rangle(r+s)+ \\ \qquad \langle m_1,m_2\rangle(q+r+s+t))) = 2\langle m_1,m_2\rangle \\ \frac{1}{8}(5\sqrt{10}\tilde{a}+7\tilde{b}+7\tilde{c}+2(\sqrt{10}\langle l_1,\tilde{m}_2\rangle(\tilde{q}+\tilde{t})+\sqrt{10}\langle l_1,\tilde{m}_3\rangle(\tilde{r}+\tilde{s})+ \\ \qquad \langle m_1,\tilde{m}_2\rangle(\tilde{q}+\tilde{r}+\tilde{s}+\tilde{t}))) = 2\langle m_1,\tilde{m}_2\rangle \\ 2a\tilde{a}+b\tilde{b}+c\tilde{c}+\frac{\sqrt{5}}{2\sqrt{2}}(\tilde{a}(b+c)+a(\tilde{b}+\tilde{c}))+q\tilde{q}+r\tilde{r}+s\tilde{s}+t\tilde{t}+\tilde{a}\langle l_1,m_2\rangle(q+t) \\ \qquad +\tilde{a}\langle l_1,m_3\rangle(r+s)+a\langle l_1,\tilde{m}_2\rangle(\tilde{q}+\tilde{t})+a\langle l_1,\tilde{m}_3\rangle(\tilde{r}+\tilde{s}) \\ \qquad +\langle m_1,m_2\rangle(\tilde{b}(q+r)+\tilde{c}(s+t))+\langle m_1,\tilde{m}_2\rangle(b(\tilde{q}+\tilde{r})+c(\tilde{s}+\tilde{t})) = 2 \\ 2a\tilde{a}+\tilde{b}c+b\tilde{c}+\frac{\sqrt{5}}{2\sqrt{2}}(\tilde{a}(b+c)+a(\tilde{b}+\tilde{c}))+\tilde{r}s+r\tilde{s}+\tilde{q}t+q\tilde{t}+\tilde{a}\langle l_1,m_2\rangle(q+t) \\ \qquad +\tilde{a}\langle l_1,m_3\rangle(r+s)+a\langle l_1,\tilde{m}_2\rangle(\tilde{q}+\tilde{t})+a\langle l_1,\tilde{m}_3\rangle(\tilde{r}+\tilde{s}) \\ \qquad +\langle m_1,m_2\rangle(\tilde{c}(q+r)+\tilde{b}(s+t))+\langle m_1,\tilde{m}_2\rangle(c(\tilde{q}+\tilde{r})+b(\tilde{s}+\tilde{t})) = 0 \end{cases} \tag{27}$$

Let

$$m_1^* = m_1 - \langle m_1,\tilde{m}_2\rangle m_2 - \langle m_1,\tilde{m}_2\rangle m_3 \text{ and } \tilde{m}_1^* = k(m_1 - \langle m_1,m_2\rangle\tilde{m}_2 - \langle m_1,m_2\rangle\tilde{m}_3).$$

Setting $\langle m_1^*,\tilde{m}_1^*\rangle = 1$ and solving for k, we have

$$k = \frac{1}{1-2\langle m_1,m_2\rangle\langle m_1,\tilde{m}_2\rangle}.$$

Then $\{m_1^*, m_2, m_3\}$ and $\{\tilde{m}_1^*, \tilde{m}_2, \tilde{m}_3\}$ are normalized biorthogonal duals, and

$$\langle l_1,m_1^*\rangle = \langle r_1,m_1^*\rangle = \frac{1}{4}(\sqrt{10}-4(\langle l_1,m_2\rangle+\langle l_1,m_3\rangle)\langle m_1,\tilde{m}_2\rangle) \text{ and}$$

$$\langle l_1,\tilde{m}_1^*\rangle = \langle r_1,\tilde{m}_1^*\rangle = \frac{\sqrt{10}-4(\langle l_1,\tilde{m}_2\rangle+\langle l_1,\tilde{m}_3\rangle)\langle m_1,m_2\rangle}{4(1-2\langle m_1,m_2\rangle\langle m_1,\tilde{m}_2\rangle)}$$

Let

$$\begin{aligned} l_1^*(x) &= l_1 - \langle l_1,\tilde{m}_1^*\rangle m_1^* - \langle l_1,\tilde{m}_2\rangle m_2 - \langle l_1,\tilde{m}_3\rangle m_3, \\ r_1^*(x) &= r_1 - \langle l_1,\tilde{m}_1^*\rangle m_1^* - \langle l_1,\tilde{m}_3\rangle m_2 - \langle l_1,\tilde{m}_2\rangle m_3, \\ \tilde{l}_1^*(x) &= l_1 - \langle l_1,m_1^*\rangle\tilde{m}_1^* - \langle l_1,m_2\rangle\tilde{m}_2 - \langle l_1,m_3\rangle\tilde{m}_3, \text{ and} \\ \tilde{r}_1^*(x) &= r_1 - \langle l_1,m_1^*\rangle\tilde{m}_1^* - \langle l_1,m_3\rangle\tilde{m}_2 - \langle l_1,m_2\rangle\tilde{m}_3. \end{aligned}$$

Then the macroelements $\Lambda = (l_1^*, r_1^*, m_1^*, m_2, m_3)^T$ and $\tilde{\Lambda} = (\tilde{l}_1^*, \tilde{r}_1^*, \tilde{m}_1^*, \tilde{m}_2, \tilde{m}_3)^T$ are biorthogonal provided $\langle l_1^*, \tilde{r}_1^*\rangle = \langle r_1^*, \tilde{l}_1^*\rangle = 0$, which is satisfied by setting

$$\langle m_1,\tilde{m}_2\rangle = \frac{1+8(\langle l_1,m_2\rangle\langle l_1,\tilde{m}_3\rangle+\langle l_1,m_3\rangle\langle l_1,\tilde{m}_2\rangle)-2\sqrt{10}\langle m_1,m_2\rangle(\langle l_1,\tilde{m}_2\rangle+\langle l_1,\tilde{m}_3\rangle)}{2(\sqrt{10}(\langle l_1,m_2\rangle+\langle l_1,m_3\rangle)-4\langle m_1,m_2\rangle(1+(\langle l_1,m_2\rangle-\langle l_1,m_3\rangle)(\langle l_1,\tilde{m}_2\rangle-\langle l_1,\tilde{m}_3\rangle)))}.$$

We define the biorthogonal scaling vectors $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $\tilde{\Phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4)^T$, with components

$$\phi_1(x) = \alpha \begin{cases} l_1^*(x+1) & \text{if } x \in [-1,0] \\ r_1^*(x) & \text{if } x \in [0,1], \end{cases} \qquad \tilde{\phi}_1(x) = \tilde{\alpha} \begin{cases} \tilde{l}_1^*(x+1) & \text{if } x \in [-1,0] \\ \tilde{r}_1^*(x) & \text{if } x \in [0,1], \end{cases}$$

$$\phi_2(x) = \beta m_2, \quad \tilde{\phi}_2(x) = \frac{1}{\beta}\tilde{m}_2, \quad \phi_3(x) = \gamma m_1^*, \quad \tilde{\phi}_3(x) = \frac{1}{\gamma}\tilde{m}_1^*,$$

$$\phi_4(x) = \beta m_3, \quad \text{and} \quad \tilde{\phi}_4(x) = \frac{1}{\beta}\tilde{m}_3,$$

where α, $\tilde{\alpha}$ are chosen so that $\langle \phi_1, \tilde{\phi}_1 \rangle = 1$.

The 1-balancing constant is

$$M_0 = \frac{\tilde{\alpha}}{\sqrt{3}}(1 - 2(\langle l_1, m_2\rangle - \langle l_1, m_3\rangle)(\langle l_1, \tilde{m}_2\rangle - \langle l_1, \tilde{m}_3\rangle)),$$

so the 1-balancing conditions are

$$\int_{\mathbb{R}} \tilde{\phi}_2(x)dx = \int_{\mathbb{R}} \tilde{\phi}_4(x)dx = \frac{\langle l_1, \tilde{m}_2\rangle + \langle l_1, \tilde{m}_3\rangle}{\sqrt{3}\beta} = M_0, \text{ and}$$

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\phi}_3(x)dx \;=\; & (5(\langle l_1, m_2\rangle + \langle l_1, m_3\rangle) - 2\sqrt{10}\langle m_1, m_2\rangle(1 - 2(\langle l_1, m_2\rangle\langle l_1, \tilde{m}_2\rangle \\ & +\langle l_1, m_3\rangle\langle l_1, \tilde{m}_3\rangle)) + 8(1 + (\langle l_1, m_2\rangle - \langle l_1, m_3\rangle)(\langle l_1, \tilde{m}_2\rangle - \langle l_1, \tilde{m}_3\rangle)) \\ & (\langle l_1, \tilde{m}_2\rangle + \langle l_1, \tilde{m}_3\rangle)\langle m_1, m_2\rangle^2)/(\sqrt{3}\gamma(\sqrt{10}(\langle l_1, m_2\rangle + \langle l_1, m_3\rangle) - 5\langle m_1, m_2\rangle \\ & -4(\langle l_1, m_2\rangle + \langle l_1, m_3\rangle)(\langle l_1, \tilde{m}_2\rangle + \langle l_1, \tilde{m}_3\rangle)\langle m_1, m_2\rangle + 2\sqrt{10}(\langle l_1, \tilde{m}_2\rangle \\ & +\langle l_1, \tilde{m}_3\rangle)\langle m_1, m_2\rangle^2)) = M_0\end{aligned}$$

From Lemma 4, the 2-balancing constant is $M_1 = 0$, so the 2-balancing condition is

$$\int_{\mathbb{R}} \left(x - \frac{1}{4}\right) dx = -\int_{\mathbb{R}} \left(x - \frac{3}{4}\right) dx = \frac{3\langle l_1, \tilde{m}_2\rangle - \langle l_1, \tilde{m}_3\rangle}{4\sqrt{3}\beta} = 0.$$

The remaining 2-balancing condition

$$\int_{\mathbb{R}} \left(x - \frac{1}{2}\right) \tilde{\phi}_2(x)dx = 0$$

is automatically satisfied. The 3-balancing constant is $M_2 = 0$, so the 3-balancing conditions are

$$\int_{\mathbb{R}} \left(x - \frac{1}{4}\right)^2 dx = \int_{\mathbb{R}} \left(x - \frac{3}{4}\right)^2 dx = 0, \text{ and } \int_{\mathbb{R}} \left(x - \frac{1}{2}\right)^2 dx = 0.$$

(The actual equations are omitted due to their messiness.) One may show that we satisfy all of the biorthogonality and balancing conditions by setting

$$\tilde{\alpha} = \frac{3}{3\alpha - 2\beta\langle l_1, m_3\rangle}, \quad \gamma = \frac{2\sqrt{2}\alpha}{\sqrt{5}}, \quad \langle l_1, m_2\rangle = \frac{1}{3}\langle l_1, m_3\rangle,$$

$$\langle l_1, \tilde{m}_2\rangle = \frac{\beta}{4\alpha}, \quad \langle l_1, \tilde{m}_3\rangle = \frac{3\beta}{4\alpha}, \quad \langle m_1, m_2\rangle = \frac{\sqrt{2}\alpha}{\sqrt{5}\beta}, \text{ and } \langle m_1, \tilde{m}_2\rangle = \frac{3\sqrt{5}\beta}{8\sqrt{2}\alpha}.$$

The system in (27) is still extremely underdetermined, which allows for some flexibility in the construction of the bases. If we choose $a = c = q = r = s = t = 0$ and $b = 1$, then $S(\Phi) = \mathcal{S}_2^0(\mathbb{Z}/2) \cap L^2(\mathbb{R})$, and the system in (27) is satisfied by setting

$$\langle l_1, m_3\rangle = \frac{3\sqrt{5}}{8\sqrt{2}}, \quad \beta = \frac{16\sqrt{2}\alpha}{7\sqrt{5}}, \quad \tilde{a} = -\frac{2\sqrt{10}(85 + 56\tilde{r} + 72\tilde{t})}{1085}, \quad \tilde{b} = \frac{2(315 - 128\tilde{r} + 48\tilde{t})}{217},$$

$$\tilde{c} = \frac{2(11 + 16\tilde{r} - 192\tilde{t})}{217}, \quad \tilde{q} = \frac{42\tilde{r} - 8\tilde{t} - 37}{62}, \text{ and } \tilde{s} = \frac{21 + 8\tilde{r} + 90\tilde{t}}{62}.$$

We choose $\alpha = \sqrt{\frac{7}{3}}$ so that $\phi_1(0) = \tilde{\phi}_1(0) = \sqrt{7}$. By setting $\tilde{r} = 0$ and $\tilde{t} = -\frac{29}{80}$, we minimize the infinity norm of the coefficient matrices in (26) so that

$$\left\| \begin{bmatrix} \tilde{q} & \tilde{r} \\ \tilde{t} & \tilde{s} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{s} & \tilde{t} \\ \tilde{r} & \tilde{q} \end{bmatrix} \right\|_\infty = \frac{11}{20} < 1.$$

The balanced biorthogonal duals for this choice of parameters are illustrated in Figure 5, and their matrix coefficients appear in the appendix.

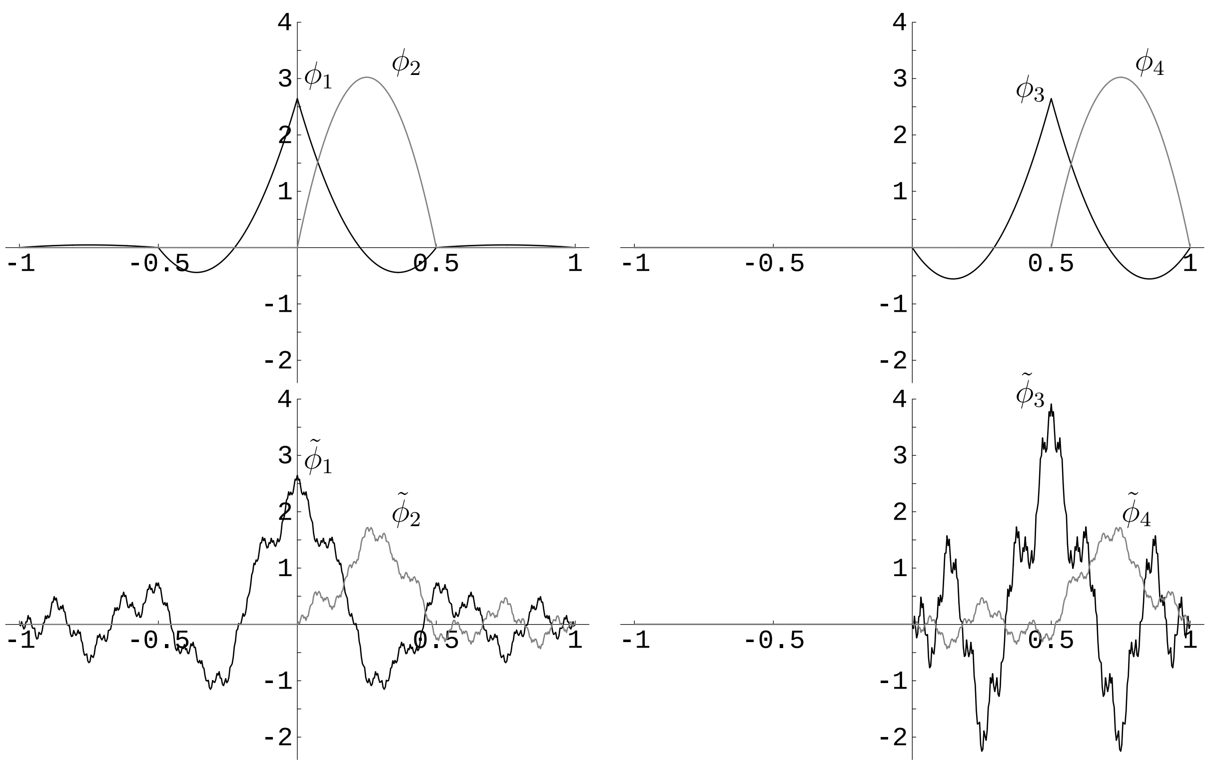

Figure 5: The 3-balanced biorthogonal scaling vectors where $S(\Phi) = \mathcal{S}_2^0(\mathbb{Z}/2) \cap L^2(\mathbb{R})$, Φ at top, $\tilde{\Phi}$ at bottom.

We may also satisfy (27) and construct bases that are pairwise orthogonal (although not normalized) by setting

$$\langle l_1, m_3 \rangle = -\frac{9}{2}, \quad \beta = -\frac{8\alpha}{45}, \quad a = 1, \quad b = -\frac{39\sqrt{5}}{7\sqrt{2}}, \quad c = -\frac{3\sqrt{5}}{7\sqrt{2}}, \quad \tilde{a} = \frac{4}{135},$$

$$\tilde{b} = -\frac{26\sqrt{2}}{63\sqrt{5}}, \quad \tilde{c} = -\frac{2\sqrt{2}}{63\sqrt{5}}, \quad q = -r = \tilde{q} = -\tilde{r} = -\frac{5}{21}, \text{ and } s = -t = \tilde{s} = -\tilde{t} = \frac{2}{21},$$

so that

$$\left\| \begin{bmatrix} q & r \\ t & s \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} s & t \\ r & q \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{q} & \tilde{r} \\ \tilde{t} & \tilde{s} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{s} & \tilde{t} \\ \tilde{r} & \tilde{q} \end{bmatrix} \right\|_\infty = \frac{10}{21} < 1.$$

We choose $\alpha = \sqrt{\frac{15}{7}}$ so that $\phi_1(0) = \tilde{\phi}_1(0) = \frac{3\sqrt{5}}{\sqrt{7}}$. The balanced biorthogonal duals for this choice of parameters are illustrated in Figure 6, and their matrix coefficients appear in the appendix.

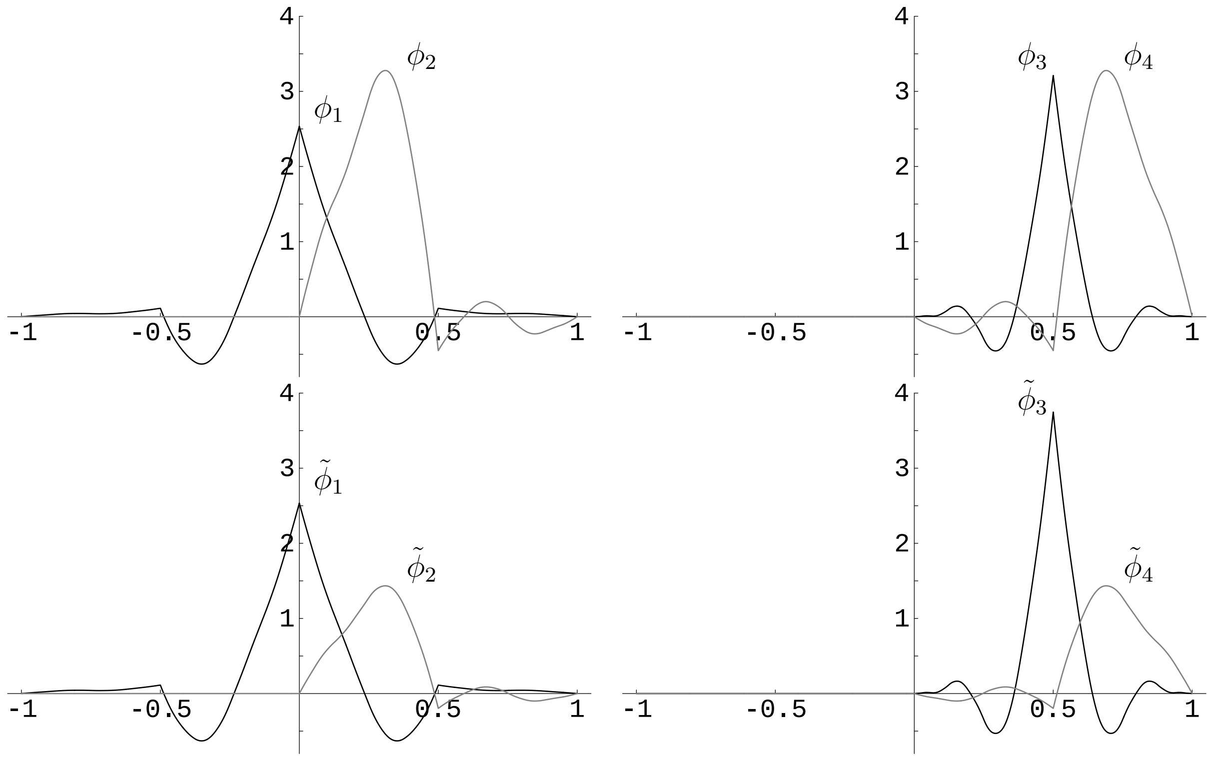

Figure 6: The 3-balanced pairwise orthogonal biorthogonal scaling vectors, Φ at top, $\tilde{\Phi}$ at bottom.

4 Appendix

The matrix coefficients of the first scaling vector constructed in Section 3.1 (illustrated in Figure 3) satisfying

$$\Phi(x) = \sqrt{2}\sum_{i=-2}^{1} g_i \Phi(2x-i) \text{ and } \tilde{\Phi}(x) = \sqrt{2}\sum_{i=-2}^{1} \tilde{g}_i \tilde{\Phi}(2x-i) \tag{28}$$

are given below.

$$g_{-2} = \begin{bmatrix} 0 & -\frac{1}{12\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad g_{-1} = \begin{bmatrix} -\frac{1}{6\sqrt{2}} & \frac{5}{12\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad g_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{5}{12\sqrt{2}} \\ 0 & \frac{\sqrt{2}}{3} \end{bmatrix} \quad g_1 = \begin{bmatrix} -\frac{1}{6\sqrt{2}} & -\frac{1}{12\sqrt{2}} \\ \frac{2\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \end{bmatrix}$$

$$\tilde{g}_{-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \tilde{g}_{-1} = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad \tilde{g}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} \quad \tilde{g}_1 = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

The matrix coefficients of the second scaling vector constructed in Section 3.1 (illustrated in Figure 4) satisfying (28) are given below.

$$g_{-2} = \begin{bmatrix} 0 & -\frac{1}{20\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad g_{-1} = \begin{bmatrix} -\frac{3}{10\sqrt{2}} & \frac{9}{20\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad g_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{9}{20\sqrt{2}} \\ 0 & \frac{3}{5\sqrt{2}} \end{bmatrix} \quad g_1 = \begin{bmatrix} -\frac{3}{10\sqrt{2}} & -\frac{1}{20\sqrt{2}} \\ \frac{4\sqrt{2}}{5} & \frac{3}{5\sqrt{2}} \end{bmatrix}$$

$$\tilde{g}_{-2} = \begin{bmatrix} 0 & -\frac{1}{10\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad \tilde{g}_{-1} = \begin{bmatrix} -\frac{3}{10\sqrt{2}} & \frac{9}{10\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad \tilde{g}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{9}{10\sqrt{2}} \\ 0 & \frac{3}{5\sqrt{2}} \end{bmatrix} \quad \tilde{g}_1 = \begin{bmatrix} -\frac{3}{10\sqrt{2}} & -\frac{1}{10\sqrt{2}} \\ \frac{2\sqrt{2}}{5} & \frac{3}{5\sqrt{2}} \end{bmatrix}$$

The matrix coefficients of the first scaling vector constructed in Section 3.2 (illustrated in Figure 5) satisfying (28) are given below.

$$g_{-2} = \begin{bmatrix} 0 & \frac{3}{224\sqrt{2}} & \frac{1}{56\sqrt{2}} & \frac{3}{224\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad g_{-1} = \begin{bmatrix} 0 & -\frac{37}{224\sqrt{2}} & -\frac{3}{56\sqrt{2}} & \frac{75}{224\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$g_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{75}{224\sqrt{2}} & -\frac{3}{56\sqrt{2}} & -\frac{37}{224\sqrt{2}} \\ 0 & \frac{3\sqrt{2}}{7} & \frac{4\sqrt{2}}{7} & \frac{3\sqrt{2}}{7} \\ 0 & -\frac{23}{112\sqrt{2}} & -\frac{3}{28\sqrt{2}} & \frac{33}{112\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad g_1 = \begin{bmatrix} 0 & \frac{3}{224\sqrt{2}} & \frac{1}{56\sqrt{2}} & \frac{3}{224\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{33}{112\sqrt{2}} & -\frac{3}{28\sqrt{2}} & -\frac{23}{112\sqrt{2}} \\ 0 & \frac{3\sqrt{2}}{7} & \frac{4\sqrt{2}}{7} & \frac{3\sqrt{2}}{7} \end{bmatrix}$$

$$\tilde{g}_{-2} = \begin{bmatrix} 0 & \frac{29}{420\sqrt{2}} & -\frac{29}{140\sqrt{2}} & \frac{29}{140\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{g}_{-1} = \begin{bmatrix} \frac{39}{140\sqrt{2}} & -\frac{19}{35\sqrt{2}} & -\frac{8\sqrt{2}}{35} & \frac{121}{105\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{g}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{121}{105\sqrt{2}} & -\frac{8\sqrt{2}}{35} & \frac{121}{105\sqrt{2}} \\ 0 & \frac{163}{280\sqrt{2}} & \frac{317}{560\sqrt{2}} & \frac{279}{280\sqrt{2}} \\ 0 & \frac{73}{280\sqrt{2}} & -\frac{219}{280\sqrt{2}} & \frac{219}{280\sqrt{2}} \\ 0 & -\frac{59}{560\sqrt{2}} & \frac{9}{70\sqrt{2}} & -\frac{37}{560\sqrt{2}} \end{bmatrix} \quad \tilde{g}_1 = \begin{bmatrix} \frac{39}{140\sqrt{2}} & \frac{29}{140\sqrt{2}} & -\frac{29}{140\sqrt{2}} & \frac{29}{420\sqrt{2}} \\ -\frac{57}{560\sqrt{2}} & -\frac{37}{560\sqrt{2}} & \frac{9}{70\sqrt{2}} & -\frac{59}{560\sqrt{2}} \\ \frac{207}{140\sqrt{2}} & \frac{219}{280\sqrt{2}} & -\frac{219}{280\sqrt{2}} & \frac{73}{280\sqrt{2}} \\ -\frac{57}{560\sqrt{2}} & \frac{279}{280\sqrt{2}} & \frac{317}{560\sqrt{2}} & \frac{163}{280\sqrt{2}} \end{bmatrix}$$

The matrix coefficients of the second scaling vector constructed in Section 3.2 (illustrated in Figure 6) satisfying (28) are given below.

$$g_{-2} = \begin{bmatrix} 0 & \frac{127}{10080\sqrt{2}} & \frac{41}{2520\sqrt{2}} & \frac{11}{480\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad g_{-1} = \begin{bmatrix} \frac{\sqrt{2}}{45} & -\frac{109}{480\sqrt{2}} & -\frac{139}{2520\sqrt{2}} & \frac{4087}{10080\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$g_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{4087}{10080\sqrt{2}} & -\frac{139}{2520\sqrt{2}} & -\frac{109}{480\sqrt{2}} \\ 0 & \frac{407}{630\sqrt{2}} & \frac{181\sqrt{2}}{315} & \frac{31}{30\sqrt{2}} \\ 0 & \frac{1}{240\sqrt{2}} & -\frac{7}{60\sqrt{2}} & \frac{11}{80\sqrt{2}} \\ 0 & -\frac{43}{630\sqrt{2}} & \frac{\sqrt{2}}{315} & \frac{1}{30\sqrt{2}} \end{bmatrix} \quad g_1 = \begin{bmatrix} \frac{\sqrt{2}}{45} & \frac{11}{480\sqrt{2}} & \frac{41}{2520\sqrt{2}} & \frac{127}{10080\sqrt{2}} \\ -\frac{4\sqrt{2}}{45} & \frac{1}{30\sqrt{2}} & \frac{\sqrt{2}}{315} & -\frac{43}{630\sqrt{2}} \\ \frac{19}{15\sqrt{2}} & \frac{11}{80\sqrt{2}} & -\frac{7}{60\sqrt{2}} & \frac{1}{240\sqrt{2}} \\ -\frac{4\sqrt{2}}{45} & \frac{31}{30\sqrt{2}} & \frac{181\sqrt{2}}{315} & \frac{407\sqrt{2}}{630\sqrt{2}} \end{bmatrix}$$

$$\tilde{g}_{-2} = \begin{bmatrix} 0 & \frac{127}{4410\sqrt{2}} & \frac{41}{2940\sqrt{2}} & \frac{11}{210\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{g}_{-1} = \begin{bmatrix} \frac{\sqrt{2}}{45} & -\frac{109}{210\sqrt{2}} & -\frac{139}{2940\sqrt{2}} & \frac{4087}{4410\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{g}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{4087}{4410\sqrt{2}} & -\frac{139}{2940\sqrt{2}} & -\frac{109}{210\sqrt{2}} \\ 0 & \frac{407}{630\sqrt{2}} & \frac{181}{420\sqrt{2}} & \frac{31}{30\sqrt{2}} \\ 0 & \frac{1}{90\sqrt{2}} & -\frac{7}{60\sqrt{2}} & \frac{11}{30\sqrt{2}} \\ 0 & -\frac{43}{630\sqrt{2}} & \frac{1}{420\sqrt{2}} & \frac{1}{30\sqrt{2}} \end{bmatrix} \quad \tilde{g}_1 = \begin{bmatrix} \frac{\sqrt{2}}{45} & \frac{11}{210\sqrt{2}} & \frac{41}{2940\sqrt{2}} & \frac{127}{4410\sqrt{2}} \\ -\frac{7}{90\sqrt{2}} & \frac{1}{30\sqrt{2}} & \frac{1}{420\sqrt{2}} & -\frac{43}{630\sqrt{2}} \\ \frac{133}{90\sqrt{2}} & \frac{11}{30\sqrt{2}} & -\frac{7}{60\sqrt{2}} & \frac{1}{90\sqrt{2}} \\ -\frac{7}{90\sqrt{2}} & \frac{31}{30\sqrt{2}} & \frac{181}{420\sqrt{2}} & \frac{407}{630\sqrt{2}} \end{bmatrix}$$

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