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Qualitative Behavior of Solutions to Differential Equations in R^n and in Hilbert Space

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QUALITATIVE BEHAVIOR OF SOLUTIONS TO

DIFFERENTIAL EQUATIONS IN R^n AND IN HILBERT SPACE

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DIFFERENTIAL EQUATIONS IN R^n AND IN HILBERT SPACE

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ABSTRACT

The qualitative behavior of solutions of differential equations mainly addresses the various questions arising in the study of the long run behavior of solutions. The contents of this thesis are related to three of the major problems of the qualitative theory, namely the stability, the boundedness and the periodicity of the solution. Learning the qualitative behavior of such solutions is crucial part of the theory of differential equations. It is important to know if a solution is bounded or unbounded or if a solution is stable, i.e. $\lim_{t\to\infty} u(t) = 0$. Moreover, the periodicity of a solution is also of great significance for practical purposes.

PREFACE

In mathematics, different processes can be combined to help us prove a more comprehensive result. In fact, it is almost certain that when solving new problems, we use more knowledge that we have already acquired to reach a new conclusion. The qualitative behavior of solutions to differential equations mainly addresses various questions arising in the study of the long run behavior of solutions. The contents of this thesis are related to three of the major problems of the qualitative theory, namely stability, boundedness and periodicity of the solution.

It is our view that one of the most important problems in the study of homogeneous and non-homogeneous equations and their applications is that of describing the nature of the solutions for a large range of parameters involved. From a numerical point of view, the existence of a periodic solution of the population equation's approximation scheme must also be studied. The usual approach to fulfill such requirements is to have a set of differential equations which are as general as possible and for which explicit analytic conditions can be given.

Below, we are going to explain how to find the qualitative behavior of solutions to differential equations in R^n in three main chapters.

In Chapter 1, we analyze the non-homogeneous differential equation in 1-dimensional R with periodic solutions, then give applications of the asymptotic behavior of solutions of the ordinary differential equations in R with periodic solution in the real world and studying periodic solution of a population equation that represents real-world situations. We will also achieve some results for the population equation in 1-dimensional R as a good beginning of the multi-dimensional case. In this context, most of the attention has been given to one periodic solution in 1- dimensional R. A periodic solution with initial population y_0 ensures that the population cannot become extinct, provided $y(t+1) = y(t)$.

It is important to study not only in 1- dimension, but in multi-dimensional linear equations. Most obvious applications would be in the studies of Linear Algebra and Differential Equations where matrix functions are prevalent. To reach our final results, we are going to study space R^n, $n \times n$ matrices and their properties. In Chapter 2, we introduce a matrix-valued exponential function and properties of such exponential function. Using Riesz theory, we also introduce the matrix-valued function $f(A)$, where $f(z)$ is a given analytic function and A is a square matrix. If we look at $f(z) = e^z$, an exponential function, then we can define matrix e^A. Many properties of such functions are given. They are very important to the theory of matrix-valued differential equations and the behavior of their solutions. At the end of Chapter 3, we prove the Spectral Mapping Theorem, an exemplary theorem about the relationship between the eigenvalue set of a matrix A and the eigenvalue set of matrix $f(A)$.

Finally, we have the main results in Chapter 3 and Chapter 4. Learning the qualitative behavior of such solutions is an important part of the theory of differential equations. Namely given the system:

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases},$$

where *A* is a linear operator in a Hilbert space, it is important to know if a solution is bounded or unbounded, or if a solution is stable, i.e. $\lim_{t\to\infty} u(t) = 0$. Moreover, the periodicity of a solution is also of great significance for practical purposes. Among the results, we have the theorem about the stability of solutions of homogeneous equation. It gives four equivalent conditions to check if the system is stable. As a nice corollary of that result, if $\text{Re}(\lambda) < 0$ for each point in the spectrum of *A*, then we also have a result about the boundedness of the solution of non-homogeneous equation. Next, we study the periodicity of solutions of the non-homogeneous equation. If the function $f(t)$ (sometimes it is called the external force) is periodic, we want to find conditions on operator *A*, so that our solution is periodic. We find a nice condition on the spectrum of *A*, namely if $2\pi ni$ ($n \in Z$, the set of integers) are in the resolvent set of *A,* then it guarantees the existence and uniqueness of 1- periodic solutions. Finally, if the imaginary axis is a subset of the resolvent of *A,* then the existence and uniqueness of complete trajectory are stated.

CHAPTER 1:

Background: A Population Problem

In this chapter, we consider a population (such as human beings, bacteria, fish, etc.) model. In this population, we assume the birth rate = b, the death rate = d, then the growth rate: $r = b - d$. If we have external influence then each year $f(t)$ is added (or subtracted) to the population.

Let $y(t)$ be the population at time t. Then, we have the population equation:

$$\begin{cases} y'(t) = ry(t) + f(t) \\ y(0) = y_0 \end{cases} \tag{1}$$

We should use the method for solving linear differential equations. First, write the equation in the standard form:

$$y'(t) - ry(t) = f(t) \tag{2}$$

Using the integrating factor:

$$\mu(t) = e^{\int -r dt} = e^{-rt} \tag{3}$$

We have the solution:

$$y(t) = e^{rt} y_0 + \int_0^t e^{r(t-s)} f(s) ds \tag{4}$$

We consider the following question: Is $y(t)$ periodic if $f(t)$ is periodic? Recall, a function $f(t)$ is called p-periodic if $f(t + p) = f(t)$ for all t in the domain. For the sake of simplicity we choose $p = 1$. If $f(t)$ is 1-periodic then, in general, the solution $y(t)$ is not periodic. We now want to find certain initial value y_0, so that *y(t)* is periodic. We now are in the position to find the initial value, such that the solution $y(t)$ is periodic.

1.1 When the solution to population equation is periodic

Theorem 1.1 Suppose $f(t)$ is a periodic function with period 1. If $r \neq 0$, then there exists a unique initial value y_0, such that the solution of the population equation:

$$\begin{cases} y'(t) = ry(t) + f(t) \\ y(0) = y_0 \end{cases}$$

is 1-periodic .

Proof: Suppose the solution $y(t) = e^{rt} y_0 + \int_0^t e^{r(t-s)} f(s)ds$ is 1-periodic, then $y(1) = y_0$.

Hence,

$$e^r y_0 + \int_0^1 e^{r(1-s)} f(s)ds = y_0 \ .$$

Therefore,

$$(1 - e^r) y_0 = \int_0^1 e^{r(1-s)} f(s)ds \ .$$

Since $r \neq 0$, we have $(1 - e^r) \neq 0$. Hence,

$$y_0 = \frac{1}{1 - e^r} \int_0^1 e^{r(1-s)} f(s)ds \ .$$

So, if $y(t)$ is 1- periodic, then y_0 must be equal to $\frac{1}{1-e^r} \int_0^1 e^{r(1-s)} f(s)ds$, and hence, y_0 is unique.

Conversely, if $y_0 = \frac{1}{1-e^r}\int_0^1 e^{r(1-s)} f(s)ds$, we will show the

solution $y(t) = e^{rt} y_0 + \int_0^t e^{r(t-s)} f(s)ds$ is 1- periodic by showing $y(1) = y(0)$.

We have:

$$y(1) = e^r y_0 + \int_0^1 e^{r(1-s)} f(s)ds$$

$$= e^r \frac{1}{1-e^r}\int_0^1 e^{r(1-s)} f(s)ds + \int_0^1 e^{r(1-s)} f(s)ds$$

$$= \left(\frac{e^r}{1-e^r} + 1\right)\int_0^1 e^{r(1-s)} f(s)ds$$

$$= \frac{1}{1-e^r}\int_0^1 e^{r(1-s)} f(s)ds$$

$$= y_0,$$

so, we can easily to see that $y(t)$ is 1- periodic. QED

1.2 The initial value of the periodic solution

Remark: In the general case, if $f(t)$ is *p*- periodic, then the initial value of the unique *p*- periodic solution is:

$$y_0 = \frac{1}{1-e^{pr}}\int_0^p e^{r(p-s)} f(s)ds .$$

Next, we will use this result to solve problems in the real case.

1.3 Applications

1.3(a) Fish in a lake

The mass of fish in a lake, if left alone, increases 30% per year. However, commercial fishing removes fish with a constant rate of 15,000 tons per year. What is the amount of fish initially, so that there will still be fish in the lake?

Solution: We know, there is a unique initial value y_0 so that the fish in the lake is 1-periodic.

If the initial amount of fish > y_0, then the fish will grow.

If the initial amount of fish < y_0, then the fish will be gone.

What is y_0? We have the population equation:

$$\begin{cases} y'(t) = .3y(t) - 15{,}000 \\ y(0) = y_0 \end{cases}$$

The unique initial amount is:

$$y_0 = \frac{1}{1-e^r}\int_0^1 e^{r(1-s)} f(s)ds$$

$$= \frac{-1}{1-e^{.3}}\int_0^1 e^{.3(1-s)} 15{,}000 ds$$

$$= 50{,}000 \text{ (tons)}$$

1.3(b) Population in a village

Population of a village: $y(0) = y_0$. Let the birth rate = 2% , and the death rate = 1%, then the growth rate = 1%. However, each year the number of people leaving for cities is

$$- f(t) = 30 - \cos\frac{2\pi}{10}t \quad \text{(Period } p = 10\text{).}$$

What is the (initial) population of the village, so that the village won't become empty?

Solution: First, we need to find what the initial value y_0 is. We have the population equation:

$$\begin{cases} y'(t) = 0.01y(t) - 30 + \cos\frac{2\pi}{10}t \\ y(0) = y_0 \end{cases}.$$

The unique initial amount is:

$$y_0 = \frac{1}{1-e^{10\cdot(0.01)}}\int_0^{10} e^{0.01(10-s)} f(s)\,ds$$

$$= \frac{-1}{1-e^{0.1}}\int_0^{10} e^{0.01(10-s)}(30 - \cos\frac{2\pi}{10}s)\,ds$$

$$= \frac{-1}{1-e^{0.1}}[\int_0^{10} e^{0.01(10-s)}30ds - \int_0^{10} e^{0.01(10-s)}\cos(\frac{2\pi}{10}s)ds)]$$

$$= \frac{-1}{1-e^{0.1}}[30\frac{e^{0.01s}}{0.01}|_0^{10} - \int_0^{10} e^{0.01(10-s)}\cos(\frac{2\pi}{10}s)ds)]$$

$$= \frac{-1}{1-e^{0.1}}3000(e^{0.1}-1) - \frac{-1}{1-e^{0.1}}\int_0^{10} e^{0.01(10-s)}\cos(\frac{2\pi}{10}s)ds$$

$$= 3000 - \frac{-1}{1-e^{0.1}}e^{0.1}\int_0^{10} e^{-0.01s}\cos(\frac{2\pi}{10}s)ds$$

(Using $\int e^{au}\cos nu\,du = \frac{e^{au}(a\cos nu + n\sin nu)}{a^2+n^2}$)

$$=3000-\frac{-1}{1-e^{0.1}}\cdot\frac{-0.01(1-e^{0.1})}{(-0.01)^2+(\frac{2\pi}{10})^2}$$

$$=3000-\frac{0.01}{(0.01)^2+(\frac{2\pi}{10})^2}$$

$$=3000-0.0253\approx 3{,}000$$

When the village has 3,000 residents, then the population of the village is not decreasing.

QED

From the above applications, we think it is important to study linear equations, not only in 1 dimension, but in the multi-dimensional case. Before doing that we are going to study the space R^n, the $n\times n$ matrices and their properties.

CHAPTER 2: Matrices

In this chapter, we will study *n*-dimensional space R^n, $n \times n$ matrices and their properties.

2.1 Space R^n , $n \times n$ Matrices and Their Properties

Definition 2.1: The space R^n is the set of all ordered n-tuples of the form:

$$u = \{u_1, u_2, \cdots, u_n\},$$

where $u_i \in R$ for $1 \le i \le n$ and $n \in N$ (the set of natural numbers). Elements in R^n are called vectors.

In R^n we define the dot product of two vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ as follows:

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

The norm of a vector *x* in R^n for each $x = (x_{1,} x_2, x_{3,} \cdots, x_n)$ is define by:

$$\| x \| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} .$$

The norm of a vector x in R^n has a lot of properties that make it useful in applications. In the following we collect some important properties of the norm.

Theorem2.2 (Properties of the norm) The following statements hold:

1) For each vector *x* in R^n, $x = 0$ if and only if $\| x \| = 0$.

2) If λ is a real number, then $\| \lambda x \| = | \lambda | \cdot \| x \|$.

3) **Triangle inequality:** For any two vectors x and y in R^n, we always have:

$$\| x+y \| \le \| x \| + \| y \| .$$

4) **Schwarz Inequality:** If x and y be vectors in R^n, $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$,

Then,

$$| x \cdot y | \le \| x \| \cdot \| y \| .$$

Definition2.3 The distance between x and y in R^n is defined by :

$$d(x,y) = \| x-y \| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

Definition2.4 (Convergence in R^n): We say that a sequence $\{x_n\}_{n\ge 1}$ of vectors converges to a vector x in R^n, written by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} \| x_n - x \| = 0$.

Next, I introduce the definition of norm of an $n\times n$ matrix A.

Definition2.5 Let $A=[a_{ij}]_{n\times n}$ be $n\times n$ matrix, $A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.We consider A as a vector in R^{n^2} by $A=\left(a_{11}, a_{12}, \cdots, a_{1n}, \cdots, a_{n1}, a_{n2}, \cdots, a_{nn}\right)$, ($n^2$ terms).Then, the norm of A, denoted by $\| A \|$ is $\| A \| = \sqrt{\sum_{\substack{i=1 \\ j=1}}^{n} a_{ij}^2}$.

Theorem 2.6 Let A and B be $n\times n$ matrices and x in R^n, then the following inequalities hold:

1) $\| Ax \| \le \| A \| \cdot \| x \|$,

2) $\| AB \| \leq \| A \| \cdot \| B \|$.

Proof:

1) The Schwarz Inequality says $| x \cdot y | \leq \| x \| \cdot \| y \|$, i.e.

$$\left(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n\right)^2 \leq \left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \cdot \left(y_1^2 + y_2^2 + \cdots + y_n^2\right).$$

Now we have

$$Ax = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{n1}x_1 + \cdots + a_{nn}x_n).$$

According to the Schwarz Inequality, we obtain:

$$(\| Ax \|)^2 = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)^2 + (a_{21}x_1 + \cdots + a_{2n}x_n)^2 + \cdots + (a_{n1}x_1 + \cdots + a_{nn}x_n)^2$$

$$\leq (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2)(x_1^2 + x_2^2 + \cdots + x_n^2) + (a_{21}^2 + \cdots + a_{2n}^2)(x_1^2 + x_2^2 + \cdots + x_n^2) + \cdots +$$

$$(a_{n1}^2 + \cdots + a_{nn}^2)(x_1^2 + x_2^2 + \cdots + x_n^n)$$

$$= (a_{11}^2 + a_{12}^2 + \cdots a_{1n}^2 + a_{21}^2 + \cdots + a_{2n}^2 + \cdots + a_{n1}^2 + \cdots + a_{nn}^2)(x_1^2 + x_2^2 + \cdots + x_n^n)$$

$$= \| A \|^2 \| x \|^2 = (\| A \| \cdot \| x \|)^2.$$

Taking the square roots, we have:

$$\| Ax \| \leq \| A \| \cdot \| x \|.$$

2) Let y_1, y_2, ..., y_n be the column vectors of matrix *B*. Then it is easy to see that Ay_1, Ay_2, …, Ay_n are the column vectors of matrix $A \cdot B$. Moreover, by definition we have:

$$\| B \|^2 = \sum_{i=1}^{n} \| y_i \|^2 \text{ and } \| A \cdot B \|^2 = \sum_{i=1}^{n} \| Ay_i \|^2.$$

On the other hand, using the above results we have:

$$\| A \cdot y_i \|^2 \leq \| A \|^2 \cdot \| y_i \|.^2$$

for $i = 1, 2, \ldots, n$. Hence, we obtain:

$$\| AB \|^2 = \sum_{i=1}^{n} \| Ay_i \|^2$$

$$\leq \sum_{i=1}^{n} \| A \|^2 \| y_i \|^2$$

$$= \| A \|^2 \cdot \sum_{i=1}^{n} \| y_i \|^2$$

$$= \| A \|^2 \cdot \| B \|^2 . \qquad \text{QED}$$

If $B = A$, then we have $\| A^2 \| = \| A \cdot A \| \leq \| A \| \cdot \| A \| = \| A \|^2$. With the same reasoning, we can conclude that $\| A^n \| \leq \| A \|^n$ for all natural number *n*.

Theorem 2.7 (Continuous Rule)

Let $F(t)$ be a matrix-valued continuous function and $x(t)$ be an n-dimensional continuous function (values in R^n), then the n-dimensional function $F(t)x(t)$ is continuous.

Proof: We show that $\lim_{t \to a} F(t)x(t) = F(a)x(a)$, which is equivalent to $\lim_{t \to a} \| F(t)x(t) - F(a)x(a) \| = 0$.

Then we have: $$\lim_{t \to a} \| F(t)x(t) - F(t)x(a) + F(t)x(a) - F(a)x(a) \|$$

$$= \lim_{t \to a} \| F(t)(x(t) - x(a)) + x(a)(F(t) - F(a))$$

$$\leq \lim_{t \to a} \| F(t)(x(t) - x(a)) \| + \lim_{t \to a} \| (F(t) - F(a))x(a) \|$$

$$\leq \lim_{t \to a} \| F(t) \| \ \| (x(t) - x(a)) \| + \lim_{t \to a} \| F(t) - F(a) \| \ \| x(a) \| = 0,$$

and the theorem is proved. QED

2.2 Derivative of n-dimensional function

Definition2.8 (Derivative of n-dimensional function)

We say $f : R \to R^n$ is differentiable at t if $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$ exists in R^n. The limit is called the derivative of $f(t)$, denoted by $f'(t)$.

It is easy to see that if $f(t)=\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$, then $f'(t)=\begin{pmatrix} f_1'(x) \\ f_2'(x) \\ \vdots \\ f_n'(x) \end{pmatrix}$. Here is an example: If $f(t)=\begin{pmatrix} t^2 \\ \cos t \end{pmatrix}$, then $f'(t)=\begin{pmatrix} 2t \\ -\sin t \end{pmatrix}$. Similarly, we say $F(t)$ is an anti-derivative of an n-dimensional function $f(t)$ if $F'(t)=f(t)$. Correspondingly, we define

$\int_a^b f(t)dt := \left(\int_a^b f_1(t)dt,\ \int_a^b f_2(t)dt,\ \dots,\ \int_a^b f_n(t)dt\right)$.

Theorem 2.9 (Product Rule)

If $F(t)$ is a matrix-valued function and $x(t)$ is an n-dimensional function, which are both continuously differentiable, then $y(t)=F(t)x(t)$ is continuously differentiable and:

$$\frac{d}{dt}F(t)x(t) = F'(t)x(t) + F(t)x'(t).$$

Proof: $\frac{d}{dt}F(t)x(t) = \lim_{h\to 0}\frac{F(t+h)x(t+h)-F(t)x(t)}{h}$

$$= \lim_{h\to 0}\frac{F(t+h)x(t+h)-F(t)x(t+h)+F(t)x(t+h)-F(t)x(t)}{h}$$

$$= \lim_{h\to 0}\frac{(F(t+h)-F(t))x(t+h)}{h} + \lim_{h\to 0}\frac{F(t)(x(t+h)-x(t))}{h}$$

$$= \lim_{h\to 0}\frac{F(t+h)-F(t)}{h}\cdot\lim_{h\to 0}x(t+h)+F(t)\cdot\lim_{h\to 0}\frac{x(t+h)-x(t)}{h}$$

$$= F'(t)x(t)+F(t)x'(t).$$ QED

To reach our end result, we need to know what the eigenvalues and eigenvectors of an $n\times n$ matrix A are.

2. 3 Eigenvalues and Eigenvectors of a Matrix

Definition 2.10 (Eigenvalues and Eigenvectors of a Matrix)

Let A be an $n\times n$ matrix. A scalar λ is called an eigenvalue of A if there exists a nonzero vector x such that

$$Ax = \lambda x.$$

The nonzero vector x is called an *eigenvector* corresponding to λ. We also note that any scalar multiple of the eigenvector is also an eigenvector. Finding eigenvalues can be simplified into a general process as follows: From $Ax = \lambda x$, we have $(\lambda - A)x = 0$ for $x \neq 0$. Therefore, $\lambda - A$ is a singular matrix, or equivalently:

$$\det(\lambda - A) = 0.$$

This equation is called the characteristic equation. Solutions to this equation will be eigenvalues.

For each eigenvalue, there is one or more corresponding eigenvectors (we disregard the multiplicity). Here is an example: Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Then,

$$|\lambda I - A| = \left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\right| = \begin{vmatrix}\lambda-1 & -4\\ -2 & \lambda-3\end{vmatrix} = (\lambda-1)(\lambda-3)-8 = \lambda^2-4\lambda-5$$

$$= (\lambda-5)(\lambda+1).$$

This gives two eigenvalues $\lambda_1 = 5$ *and* $\lambda_2 = -1$. Then, we need to find the corresponding eigenvectors. That means we need to solve the homogeneous linear system $(\lambda I - A)x = 0$ for each eigenvalue λ.

For $\lambda_1 = 5$ we have

$$(5I - A)x = (\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} - \begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}) \cdot \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}4 & -4\\ -2 & 2\end{bmatrix} \cdot \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Solving that system, we get $x = c\begin{bmatrix}1\\ 1\end{bmatrix}$.

For $\lambda_2 = -1$, we have

$$(-I - A)x = (\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix})\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}-2 & -4\\ -2 & -4\end{bmatrix} \cdot \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Solving that system, we get $x = c\begin{bmatrix}-2\\ 1\end{bmatrix}$.

Next, we will study the matrix exponential function e^{tA} .

2.4 Matrix Exponential Function e^{tA}

Let A be a $n \times n$ matrix. What are the matrix e^A and the function e^{tA} ? We have different approaches to define these matrices.

Definition 2.11 Suppose A is an $n \times n$ matrix. Then the matrix e^A is defined by:

$$e^A = \lim_{n\to\infty}(I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!}) = \sum_{n=0}^{\infty}\frac{A^n}{n!}$$

and

$$e^{tA} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n \quad .$$

The above definition is meaningful. Indeed, if we denote $S_n := I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!}$,

then

$e^A - S_n = \sum_{i=n+1}^{\infty} \frac{A^i}{i!}$ and hence,

$$\| e^A - S_n \| = \| \sum_{i=n+1}^{\infty} \frac{A^i}{i!} \| \le \sum_{i=n+1}^{\infty} \frac{\| A^i \|}{i!} \le \sum_{i=n+1}^{\infty} \frac{\| A \|^i}{i!} \to 0 \text{ as } n \to \infty .$$

Using the above definition we will find e^{tA} for some given matrix A.

Examples 2.12: Find e^{tA}, if a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

a) If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

From that pattern we have $A^5 = A$, $A^6 = A^2$,…., and so on. Hence we obtain

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{t^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!} & \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} \\ -\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!} \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Here we have used the facts that $\cos t = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!}$) and $\sin t = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!}$.

b) If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, $A^3 = A$, $A^4 = I, \cdots$. Hence,

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{t^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \ldots$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

(Using $\sinh t = \frac{e^t - e^{-t}}{2}$; $\cosh t = \frac{e^t + e^{-t}}{2}$.) .

c) If $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$A^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\cdots$.

Hence,

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{t^4}{4!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t \\ -t & -t \end{pmatrix} = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}.$$

a) Finally, if $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ \vdots & \cdots & 0 & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$, then

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Theorem 2.13 (Properties of the matrix exponential function)

Let A and B be $n \times n$ matrices and s and t be real numbers. Then

(a) $e^O = I$ (O is the zero $n \times n$ matrix);

(b) $e^{tI} = e^t I$;

(c) $e^{A(t+s)} = e^{At}e^{As}$;

(d) $(e^{At})^{-1} = e^{-At}$.

Proof: (a) Using the formula $e^{tA} = I + \frac{tA}{1!} + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \ldots$, we can calculate

$$e^O = I + \frac{O}{1!} + \frac{O^2}{2!} + \ldots + \frac{O^n}{n!} + \ldots..$$

$$= I.$$

(b) From $e^{tA} = I + \frac{t \cdot A}{1!} + \frac{(t \cdot A)^2}{2!} + ... + \frac{(t \cdot A)^n}{n!} + ...$ we have

$$e^{rI} = I + \frac{r \cdot I}{1!} + \frac{(r \cdot I)^2}{2!} + ... + \frac{(r \cdot I)^n}{n!} + ...$$

$$= I + \frac{r \cdot I}{1!} + \frac{r^2 I}{2!} + ... + \frac{r^n I}{n!} + ...$$

$$= I(1 + \frac{r}{1!} + \frac{r^2}{2!} + ... + \frac{r^n}{n!} + ...)$$

$$= e^r I .$$ QED

(c) 1) If A is diagonal, i.e. $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$. Then, $e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$

and $e^{As} = \begin{bmatrix} e^{\lambda_1 s} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 s} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n s} \end{bmatrix}$.

Hence, we can obtain

$$e^{At} e^{As} = \begin{bmatrix} e^{\lambda_1 (t+s)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 (t+s)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n (t+s)} \end{bmatrix} = e^{A(t+s)}.$$

If A is diagonalizable, then there exists S, S^{-1}, and $S \cdot S^{-1} = I$ and $SAS^{-1} = D$, a diagonal matrix. Then we have $A = S^{-1}DS$. We show now that $SA^2S^{-1} = D^2$, $SA^3S^{-1} = D^3$, …, $SA^nS^{-1} = D^n$. Indeed, we have

$$SA^2S^{-1} = SAAS^{-1} = SAS^{-1}SAS^{-1} = (SAS^{-1})^2 = D^2 .$$

Using the same reasoning, we can conclude that that $SA^nS^{-1} = (SAS^{-1})^n = D^n$, a diagonal matrix. Hence,

$$Se^{At}S^{-1} = S(\sum_{n=0}^{\infty} A^n \frac{t^n}{n!})S^{-1} = \sum_{n=0}^{\infty} \frac{SA^n t^n S^{-1}}{n!} = \sum_{n=0}^{\infty} \frac{t^n D^n}{n!} = e^{tD} .$$

Thus, we have $e^{At} = S^{-1}e^{tD}S$, $e^{As} = S^{-1}e^{sD}S$ and $e^{A(t+s)} = S^{-1}e^{D(t+s)}S$. Therefore,

$$\begin{aligned} e^{At}e^{As} &= S^{-1}e^{tD}S \cdot S^{-1}e^{sD}S \\ &= S^{-1}(e^{tD}e^{sD})S \\ &= S^{-1}e^{D(t+s)}S . \text{ (since } D \text{ is a diagonal matrix)} \\ &= e^{A(t+s)} . \end{aligned}$$

2) Let A be a Jordan block, then

$$A = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ \vdots & \cdots & 0 & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

$$= \lambda I + B \quad . \qquad .$$

. We observe that it is not hard to show λ is any constant, then $e^{\lambda I}e^{A} = e^{(\lambda I + A)}$. So we can obtain

$$e^{At} = e^{t(\lambda I+B)} = e^{t\lambda I}\cdot e^{tB} = e^{t\lambda I}\cdot\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$ and similarly,

$$e^{As} = e^{s(\lambda I+B)} = e^{s\lambda I}\cdot e^{sB} = e^{s\lambda I}\cdot\begin{bmatrix} 1 & s & \frac{s^2}{2!} & \cdots & \frac{s^{n-1}}{(n-1)!} \\ 0 & 1 & s & \cdots & \frac{s^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Hence, we obtain

$$e^{At}e^{As} = e^{t\lambda I}\cdot\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}\cdot e^{s\lambda I}\cdot\begin{bmatrix} 1 & s & \frac{s^2}{2!} & \cdots & \frac{s^{n-1}}{(n-1)!} \\ 0 & 1 & s & \cdots & \frac{s^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix},$$

$$= e^{t\lambda I}\cdot e^{s\lambda I}\cdot\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & s & \frac{s^2}{2!} & \cdots & \frac{s^{n-1}}{(n-1)!} \\ 0 & 1 & s & \cdots & \frac{s^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix},$$

$$=e^{(t+s)\lambda I}\cdot\begin{bmatrix}1 & (t+s) & \frac{(t+s)^2}{2!} & \cdots & \frac{(t+s)^{n-1}}{(n-1)!}\\ 0 & 1 & (t+s) & \cdots & \frac{(t+s)^{n-2}}{(n-2)!}\\ \vdots & \ddots & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & (t+s)\\ 0 & \cdots & \cdots & 0 & 1\end{bmatrix},$$

$$= e^{(t+s)\lambda I}\cdot e^{(t+s)B} = e^{(t+s)A}.$$

Finally, if A is similar to the Jordan block, i.e. $A = SJS^{-1}$ with J=Jordan block. Then we have,

$$e^{A(t+s)} = Se^{J(t+s)}S^{-1} = Se^{Jt+Js}S^{-1} = S(e^{Jt}\cdot e^{Js})S^{-1} = Se^{Jt}S^{-1}Se^{Js}S^{-1} = e^{At}\cdot e^{As}.$$

Therefore, the proof is completed.

c) We already proved $e^{A(t+s)} = e^{At}e^{As}$. Let $s = -t$, we have

$$e^{tA}.e^{-tA} = e^{A(t-t)} = e^{O} = I\ .$$

Hence, $(e^{At})^{-1} = e^{-At}$. QED

From property $e^{A(t+s)} = e^{At}e^{As}$ we have $e^{2tA} = e^{tA}e^{tA} = (e^{tA})^2$. Similarly, $e^{ntA} = (e^{tA})^n$ for any natural number n. This formula will be used often later.

Theorem2.14 If A is a $n\times n$ matrix, then $\frac{d}{dt}e^{tA} = Ae^{tA}$.

Proof: We know $e^{A} = 1+\frac{A}{1!}+\frac{A^2}{2!}+\cdots+\frac{A^n}{n!}+\cdots$. Then we have

$$e^{tA} = 1+\frac{tA}{1!}+\frac{(tA)^2}{2!}+\cdots+\frac{(tA)^n}{n!}+\cdots$$

Since the above Taylor series converges, and the series of derivatives of these terms converges too, we have:

$$\frac{d}{dt}e^{tA} = 0 + \frac{A}{1} + \frac{2tA^2}{2\times 1} + \frac{3t^2A^3}{3\times 2\times 1}\cdots + \frac{nt^{n-1}A^n}{n\times(n-1)\times\cdots\times 1} + \cdots$$

$$= A + \frac{tA^2}{1} + \frac{t^2A^3}{2\times 1} + \cdots + \frac{t^{n-1}A^n}{(n-1)!} + \cdots$$

$$= A(1 + \frac{tA}{1} + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^{n-1}}{(n-1)!}) + \cdots$$

$$= Ae^{tA}.$$

QED

2.5 The Cauchy Integral Formula of Exponential Function

The second approach to define e^{tA} is using the Riesz theory. Recall, the Cauchy Integral Formula is a useful tool for solving many problems in Complex Analysis. The Cauchy Integral Formula formally states that given a complex function $f(z)$ that is analytic everywhere inside and on a simple closed contour *C*, taken in the positive sense, with z_0 being interior to C, the following (the Cauchy Formula) is true:

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z_0}dz\ .$$

The Cauchy Integral Formula states that the value of $f(z_0)$ can be determined, if $f(z)$ on a closed contour around z_0 is known. An additional formula, the Cauchy Integral Formula for derivatives, is given below.

$$f'(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z - z_0)^2}dz.$$

In order to accommodate later use, the Cauchy Integral Formula can be rewritten as

$$f(z_0) = \frac{1}{2\pi i}\int_c f(z)\cdot(z - z_0)^{-1}dz.$$

Now, if A is a $n \times n$ matrix, and $f(z)$ is an analytic function in a domain containing eigenvalues of A., we define the matrix-valued function $f(A)$ by:

$$f(A) = \frac{1}{2\pi i} \int_c f(z) \cdot (z - A)^{-1} dz ,$$

where C is a closed contour containing the eigenvalues of A. By the above definition we have:

$$e^A = \frac{1}{2\pi i} \int_C e^z (z - A)^{-1} dz.$$

Note that the definition is independent of the choice of the contour C. We will find out that the two above definitions of e^A using the two approaches are the same. First, we study some properties of $f(A)$.

Theorem2.15 The following statements hold:

1. If $f(z) = 1$, then $f(A) = I$.
2. If $f(z) = z$, then $f(A) = A$.
3. If $f(z) = z^n$, then $f(A) = A^n$.

Proof:

1. We know. if $\| A \| < 1$, then $(I - A)$ is invertible and

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n .$$

Let C be a contour with $| z | > \| A \|$ for all z on C. We have then $\left\| \frac{A}{z} \right\| < 1$ and hence,

$\left(I - \frac{A}{z} \right)$ is invertible. Thus, $z - A = z\left(I - \frac{A}{z} \right)$ is invertible and

$$(z-A)^{-1} = \frac{1}{z}\left(I - \frac{A}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{A}{z}\right)^{n}$$

$$= \sum_{n=0}^{\infty}\frac{A^n}{z^{n+1}}.$$

According to the definition of $f(A)$, if $f(z)=1$ then

$$f(A) = \frac{1}{2\pi i}\int_C f(z)(z-A)^{-1}dz$$

$$= \frac{1}{2\pi i}\int_C 1\cdot \sum_{n=0}^{\infty}\frac{A^n}{z^{n+1}}dz$$

$$= \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty} A^n \int_C \frac{1}{z^{n+1}}dz\right)$$

$$= \frac{1}{2\pi i}\cdot I\cdot 2\pi i$$

$$= I\ .$$

Here we have used the fact that $\int_C \frac{1}{z}dz = 2\pi \cdot i$ and $\int_C \frac{1}{z^n}dz = 0$ for all $n \geq 2$.

2. Similarly, if $f(z) = z^n$, then

$$f(A) = \frac{1}{2\pi i}\int_C z^n \sum_{m=0}^{\infty}\frac{A^m}{z^{m+1}}dz$$

$$= \frac{1}{2\pi i}\left(\sum_{m=0}^{\infty} A^m \int_C \frac{1}{z^{m-n+1}}dz\right)$$

$$= \frac{1}{2\pi i}\cdot A^n\cdot 2\pi i$$

$$= A^n.$$

Also, here we have used the fact that $\int_c \frac{1}{z} dz = 2\pi \cdot i$ and $\int_c \frac{1}{z^n} dz = 0$ for all $n \geq 2$. QED

From Theorem 2.5, it is expected that if $f(z) = z^2$, and then $f(A) = A^2$; if $f(z) = z^3$, then $f(A) = A^3$ and analogously, if $f(z) = z^n$, then $f(A) = A^n$. Additionally, these methods can be applied to more exotic functions which will be shown below. First, some addition and multiplication properties are stated.

Theorem 2.16

Suppose $f(t)$ and $g(z)$ are analytic functions on a domain containing eigenvalues of *A* and *c* is a complex constant. The following properties are true:

1. If $h(z) = f(z) \pm g(z)$, then $h(A) = f(A) \pm g(A)$;

2. If $h(z) = c \cdot f(z)$, then $h(A) = c \cdot f(A)$;

3. If $h(z) = f(z) \cdot g(z)$, then $h(A) = f(A) \cdot g(A)$;

4. If $f(z) \neq 0$ at all eigenvalues of *A*, then $f(A)$ is invertible and

$$f(A)^{-1} = \left(\frac{1}{f}\right)(A) \ .$$

Proof:

1. If $h(z) = f(z) \pm g(z)$, then

$$h(A) = \frac{1}{2\pi i} \int_c h(z)(z - A)^{-1} dz$$

$$= \frac{1}{2\pi i} \int_c \left(f(z) \pm g(z)\right)(z - A)^{-1} dz$$

$$=\frac{1}{2\pi i}\int_{c} f(z)(z-A)^{-1}dz \pm \frac{1}{2\pi i}\int_{c} g(z)(z-A)^{-1}dz$$

$$=f(A)\pm g(A).$$

2. If $h(z)=c\cdot f(z)$, then $h(A)=\frac{1}{2\pi i}\int_{c} h(z)(z-A)^{-1}dz$

$$=\frac{1}{2\pi i}\int_{c} c\cdot f(z)(z-A)^{-1}dz$$

$$=c\cdot\frac{1}{2\pi i}\int_{c} f(z)(z-A)^{-1}dz = c\cdot f(A).$$

3. If $h(z)=f(z)\cdot g(z)$ the expected result can again be obtained.

$$h(A)=f(A)\cdot g(A)=\frac{1}{2\pi i}\int_{c} f(z)g(z)(z-A)^{-1}dz.$$

Because the integral does not depend on the choice of the contour, the contours

$$f(A)=\frac{1}{2\pi i}\int_{C_1} f(z)\cdot(z-A)^{-1}dz$$

and

$$g(A)=\frac{1}{2\pi i}\int_{C_2} g(u)\cdot(u-A)^{-1}du$$

are chosen, where C_2 is a different contour which contains C_1.

$$f(A)\cdot g(A)=\frac{1}{(2\pi i)^2}\int_{C_1} f(z)(z-A)^{-1}dz\int_{C_2} g(u)(u-A)^{-1}du$$

$$=\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2} f(z)g(u)(z-A)^{-1}(u-A)^{-1}du\,dz$$

After multiplying the integrals together, the result can be rewritten after applying the Resolvent Identity (Resolvent Identity:

$$(z-A)^{-1}(u-A)^{-1}=\frac{1}{z-u}\left((u-A)^{-1}-(z-A)^{-1}\right))$$

as follows:

$$f(A)\cdot g(A)=\frac{1}{(2\pi i)^2}\int_{c_1}\int_{c_2} f(z)g(u)\frac{1}{z-u}\left[(u-A)^{-1}-(z-A)^{-1}\right]dudz$$

This can be split into two double integrals, which can be arranged as shown.

$$f(A)\cdot g(A)=\frac{1}{(2\pi i)^2}\int_{c_2} g(u)(u-A)^{-1}\int_{c_1}\frac{f(z)}{z-u}dzdu+\frac{1}{(2\pi i)^2}\int_{c_1} f(z)(z-A)^{-1}\int_{c_2}\frac{g(u)}{u-z}dudz.$$

However, by the Cauchy Integral Formula, $\int_{c_1}\frac{f(z)}{z-u}dz=0$, since u is not contained in C_1 for each u on C_2. Thus, the first integral is zero. Additionally,

$\int_{c_2}\frac{g(u)}{u-z}du=2\pi i g(z)$ by the Cauchy Integral Formula, since z is contained in C_2.

Hence,

$$\begin{aligned} f(A)\cdot g(A)&=\frac{1}{(2\pi i)^2}\int_{c_1} f(z)(z-A)^{-1}\cdot 2\pi i g(z)dz\\ &=\frac{1}{2\pi i}\int_{c_1} f(z)\cdot g(z)(z-A)^{-1}dz\\ &=\frac{1}{2\pi i}\int_{c_1} h(z)(z-A)^{-1}dz. \end{aligned}$$

This will again yield the desired result of $h(A)$.

4. Since $f(z)\neq 0$ at all eigenvalues of *A*, we can find a domain containing eigenvalues of *A* with $f(z)\neq 0$ for all z in this domain. Hence, $g(z):=\frac{1}{f(z)}$ exists in that domain and is also analytic. Hence we have $h(z)=1=f(z)g(z)$ and

$$h(A)=I=f(A)g(A).$$

That means $f(A)$ is invertible and $f(A)^{-1}=g(A)=\left(\frac{1}{f}\right)(A)$. QED

For example, if A is an invertible matrix, (i.e. 0 is not an eigenvalue of A) and $f(z)=\frac{1}{z}$ with the domain $D(f)=C\setminus\{0\}$, then $\frac{1}{f}(z)=z$ and $\frac{1}{f}(A)=A$. Hence

$$f(A)=\left(\frac{1}{f}(A)\right)^{-1}=A^{-1}.$$

If now $f(z)=e^{z}=\sum_{n=0}^{\infty}\frac{z^{n}}{n!}$, then, using the addition and multiplication properties, we have

$$e^{A}=\sum_{n=0}^{\infty}\frac{A^{n}}{n!}$$

$$=e^{A}.$$

2.6 Additional Functions

Example 2.17 (Sine and cosine function of a matrix)

Now, if $f(z)=\cos(z)=\frac{e^{iz}+e^{-iz}}{2}$, a new matrix function can be defined as

$$\cos(A)=\int_{C}\cos(z)(z-A)^{-1}dz$$

$$=\frac{1}{2\pi i}\int_{C}\frac{e^{iz}+e^{-iz}}{2}(z-A)^{-1}dz$$

$$=\frac{1}{2}\left(\frac{1}{2\pi i}\int_{C}e^{iz}(z-A)^{-1}dz+\frac{1}{2\pi i}\int_{C}e^{-iz}(z-A)^{-1}dz\right)$$

$$=\sum_{n=0}^{\infty}\frac{(iA)^{n}}{n!}+\sum_{n=0}^{\infty}\frac{(-iA)^{n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n}}{(2n)!} \quad = I - \frac{A^2}{2} + \frac{A^4}{4!} - \cdots$$

Likewise, if $f(z) = \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, then

$$\sin(A) = \frac{e^{iA} - e^{-iA}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n+1}}{(2n+1)!} = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots$$

Example 2.18 (A square root of a matrix)

An additional matrix that can be defined is $\sqrt{A}$. First let $f(z) = \sqrt{z}$, (take a certain branch) and $A = \begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix}$. The eigenvalues of this matrix A are 1 and 4. Now

$$f(A) = \sqrt{A} = \int_C \sqrt{z}(z - A)^{-1} dz$$

After performing the integration of each entry, we obtain a square root:

$$\sqrt{A} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

It is also easy to calculate that $\sqrt{A} \cdot \sqrt{A} = A$.

Before studying the properties of matrix-value function e^{tA}, we state a very useful theorem, the Spectral Mapping Theorem.

2.7 Spectral Mapping Theorem

Theorem 2.19 (Spectral Mapping Theorem)

Assume $f(z)$ is an analytic function in a domain containing all eigenvalues of A. If λ is an eigenvalue of matrix A, then $f(\lambda)$ is an eigenvalue of the matrix $f(A)$.Conversely, if

μ is an eigenvalue of $f(A)$ then there exists an eigenvalue λ of A, such that $f(\lambda)=\mu$. Indeed, if the set of eigenvalues of a matrix A is denoted by $EV(A)$, then we have

$$EV(f(A)) = f(EV(A)).$$

Proof: Let λ be an eigenvalue of A. Define

$$g(z) := \begin{cases} \dfrac{f(z)-f(\lambda)}{z-\lambda} & if \quad z \neq \lambda; \\ f'(\lambda) & if \quad z = \lambda. \end{cases}$$

Since $f(z)$ is analytic in a domain containing λ, we can write it as a Taylor series at λ as $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\lambda)}{n!}(z-\lambda)^n$. Hence, by definition, $g(z)=\sum_{n=0}^{\infty}\frac{f^{(n+1)}(\lambda)}{(n+1)!}(z-\lambda)^n$ is also analytic. Now, $f(z)-f(\lambda)=(z-\lambda)\cdot g(z)$ is a product of two analytic functions. Using Theorem 2.16(3), we have

$$f(A)-f(\lambda)I = (A-\lambda I)g(A)$$

and hence,

$$\begin{aligned} \det\big(f(A)-f(\lambda)I\big) &= \det\big((A-\lambda I)\cdot g(A)\big) \\ &= \det(A-\lambda I)\cdot\det(g(A)) \\ &= 0.\ \det(g(A)) \end{aligned}$$

by definition of an eigenvalue. Therefore, $\det(f(A)-f(\lambda)I)=0$ and $f(\lambda)$ is an eigenvalue of matrix $f(A)$.

Conversely, if μ is an eigenvalue of $f(A)$, i.e. $[\mu I - f(A)]$ is a singular matrix. By contradiction, we assume that $f(\lambda)\neq\mu$ for all eigenvalues λ of A. That means the analytic function $g(z)=\mu-f(z)$ is non-zero at all eigenvalues of A. By Theorem 2.6

(part 4) it implies that $g(A)$ is invertible. But $g(A) = \mu I - f(A)$ is singular. This is a contradiction, and the proof is complete. QED

If λ is an eigenvalue of A and x is the corresponding eigenvector, i.e., $Ax = \lambda x$, then we have $A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$. That means, x is the eigenvector of A^2 corresponding to λ^2. In the same manner, we have that the same vector x is the eigenvector of A^n corresponding to λ^n for all $n \geq 2$, that means $A^n x = \lambda^n x$. Hence,

$$e^{tA} x = \sum_{n=0}^{\infty} \frac{(tA)^n x}{n!} = \sum_{n=0}^{\infty} \frac{(t\lambda)^n x}{n!} = \left(\sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} \right) x = e^{t\lambda} x .$$

We now can conclude the above observation as follows:

Theorem 2.20 If x is an eigenvector of A corresponding to the eigenvalue λ, then x is also an eigenvector of e^{At} corresponding to the eigenvalue $e^{\lambda t}$.

CHAPTER 3:

Qualitative Behavior of Solutions of Differential Equations

In this Chapter we will study the qualitative behavior of solutions of the homogeneous differential equation

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases} \tag{3.1}$$

as well as of the non-homogeneous equation

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases}. \tag{3.2}$$

Studying the qualitative behavior of such solutions is an important part of the theory of differential equations. It is important to know if a solution is bounded or unbounded, or if a solution is stable, i.e. $\lim_{t\to\infty} y(t) = 0$. Moreover, the periodicity of a solution is also of great significance for practical purposes. Before studying the properties of such solutions, we give a statement for the existence and uniqueness of such solutions.

Theorem 3.1: (Existence and Uniqueness Theorem)

1. There is a unique solution of Equation (3.1) given by:

   $$y(t) = e^{tA} y_0 .$$

2. Suppose $f(t)$ is a continuous function on $[0, \ \infty)$. Then there is a unique solution of Equation (3.2) given by:

$$y(t)=e^{tA}y_0+\int_0^t e^{(t-s)A}f(s)ds\,.$$

Proof: We will prove part (2) , as part (1) is a special case of it, when $f(t)=0$.

Since (3.2) is a linear differential equation of the form:

$$y'(t)-Ay(t)=f(t)\,, \qquad (3.3)$$

the integrating factor $\mu(x)$ is given by

$$\mu(t)=e^{\int -Adt}=e^{-At}\,.$$

Multiplying both sides of (3.3) with $\mu(t)$ we have:

$$e^{-tA}y'(t)-Ae^{-tA}y(t)=e^{-tA}f(t)$$

or $$(e^{-tA}y(t))'=e^{-tA}f(t)$$

$$e^{-tA}y(t)=y(0)+\int_0^t e^{-sA}f(s)ds=y_0+\int_0^t e^{-sA}f(s)ds.$$

Multiplying both sides with e^{tA} we have:

$$y(t)=e^{tA}y_0+\int_0^t e^{(t-s)A}f(s)ds. \qquad \text{QED}$$

We now state the first result, in which the equivalence of part a. and part e. is the famous Lyapunov's Theorem.

Theorem 3.2 Consider the system of linear differential equations with initial condition:

$$\begin{cases} y'(t)=Ay(t) \\ y(0)=y_0 \end{cases}.$$

The solution of the system is $y(t)=e^{tA}y_0$.

Then the following statements are equivalent:

a. The system is stable: Solution $y(t)=e^{tA}y_0 \to 0$ as $t\to\infty$ for all vectors y_0 .

b. $\lim_{n\to\infty} \| e^{tA} \| = 0$.

c. There exist positive numbers M_1 and ω such that $\| e^{tA} \| \le M_1 e^{-\omega t}$.

d. There exists a number t_0 such that $\| e^{t_0 A} \| < 1$.

e. $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of A .

Proof: The implications (c. $\to$ b.), (b. $\to$ d.) and (b. $\to$ a.) are obvious. We now prove (a. $\to$ b.), (d. $\to$ c.) and (a. $\leftrightarrow$ e.) one at a time.

(a. $\to$ b.): Let $(e^{tA})_i$ be the i^{th} column of e^{tA}. Take now $y_0 = (1,\ 0,\dots,\ 0)$, then the solution is $y = e^{tA}y_0 = (e^{tA})_1$. Since $\lim_{t\to\infty} y(t) = 0$, we have $\lim_{t\to\infty} \| (e^{tA})_1 \| = 0$. With the same reasoning, we can prove that $\lim_{t\to\infty} \| (e^{tA})_i \| = 0$ for $i = 1,\ 2,\dots,\ n$. Hence,

$$\lim_{t\to\infty} \| e^{tA} \|^2 = \lim_{t\to\infty} \sum_{i=1}^{n} \| (e^{tA})_i \|^2 = 0 .$$

(d. $\to$ c.) Let t_0 be the number with $\| e^{t_0 A} \| = r_0 < 1$. For any number $t > t_0$, we write $t = mt_0 + s$, where m is a natural number and $0 \le s < t_0$. Since the function e^{tA} is continuous, the maximum $M = \max_{0\le t\le t_0} \| e^{tA} \|$ exists. We now have

$$\begin{aligned}
\| e^{tA} \| = \| e^{(mt_0+s)A} \| &= \| e^{mt_0 A} e^{sA} \| \\
&\le \| e^{mt_0 A} \| \cdot \| e^{sA} \| \\
&\le M \| e^{mt_0 A} \|
\end{aligned}$$

$$= M \| (e^{t_0 A})^m \|$$

$$\le M \| e^{t_0 A} \|^m$$

$$= M r_0{}^m = M e^{m \ln r_0} = M e^{(t-s)/t_0 \ln r_0} \text{ (since } m = (t-s)/t_0)$$

$$= M e^{-s/t_0 \ln r_0} e^{(\ln r_0 / t_0)t}. \tag{3.4}$$

Let $\omega = -(\ln r_0)/t_0$. Since, $r_0 < 1$, hence $\ln r_0 < 0$ and $\omega > 0$. From (3.3) we have

$$\| e^{tA} \| \le M e^{-s/t_0 \ln r_0} e^{-\omega t} \le M_1 e^{-\omega t}.$$

Here $M_1 = M \max\limits_{0 \le s \le t_0} e^{-(s/t_0)\ln r_0}$.

(a $\rightarrow$ e) We need to prove if $y(t) = e^{tA} y_0 \rightarrow 0$ as $t \rightarrow \infty$ for all the initial values y_0, then $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of A. On the contrary, suppose there exists an eigenvalue λ, where $\operatorname{Re}\lambda \ge 0$.

(1) If λ is a real eigenvalue with corresponding eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then the solution of the system is $y(t) = e^{\lambda t} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ (with the initial values $y(0) = x$), which does not approach to 0 as $t \rightarrow \infty$, since $x \ne 0$ and $\operatorname{Re}\lambda \ge 0$. This is a contradiction to the assumptions. Thus, $\operatorname{Re}\lambda < 0$.

(2) We need to use the following fact: Suppose $\alpha < 0$ and $P_m(t)$ is a real polynomial of degree m, then $\lim_{t\to\infty} e^{\alpha t} P_m(t) = 0$.

If $\lambda = \alpha + i\beta$ is a complex eigenvalue with corresponding eigenvector x, and $\alpha \ge 0$, then the function $y(t) = e^{\alpha t}(\cos \beta t \operatorname{Re} x - \sin \beta t \operatorname{Im} x)$ is a solution of the system (with the initial value $y(0) = \operatorname{Re} x$), which does not approach 0 as $t \to \infty$ since $\alpha \ge 0$. This is a contradiction to the assumptions. Thus, $\alpha < 0$.

(e $\to$ a) we need to show if $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of A, the $y(t) = e^{tA} y_0 \to 0$ as $t \to \infty$ for all the initial values y_0. Let $\operatorname{Re}\lambda < 0$ for each eigenvalue of the coefficient matrix. We denote the following:

(1) Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the real eigenvalues of A with multiplicity one, with corresponding eigenvector x_i;

(2) Let $\eta_1, \eta_2, \cdots, \eta_l$ be the real eigenvalues of A with multiplicity of n_i, with corresponding eigenvector y_i;

(3) Let $\mu_1, \mu_2, \cdots, \mu_m$ be the complex eigenvalues of A, where $\mu_i = \alpha_i + i\beta_i$, with corresponding eigenvector z_i.

Then the solution of the system is of the form:

$$
\begin{aligned}
y(t) = {} & \sum_{i=1}^{k} a_i e^{\lambda_i t} x_i + \sum_{i=1}^{l} e^{\eta_i t} P_{n_i}(t) y_i + \sum_{i=1}^{m} b_i e^{\alpha_i t} (\cos \beta_i \operatorname{Re} z_i - \sin \beta_i t \operatorname{Im} z_i) \\
& + \sum_{i=1}^{m} c_i e^{\alpha_i t} (\cos \beta_i \operatorname{Im} z_i + \sin \beta_i t \operatorname{Re} z_i) \\
= {} & I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where $P_{n_i}(t)$ are the corresponding polynomials of degree n_i . We need to consider each of I_1, I_2, I_3, I_4, so

$$I_1 = \sum_{i=1}^{k} a_i e^{\lambda_i t} x_i \to 0 \text{ as } t \to \infty \text{, since } \lambda_i < 0 \text{ for each } i = 1, 2, \cdots, k.$$

$$I_2 = \sum_{i=1}^{l} e^{\eta_i t} P_{n_i}(t) y_i \to 0 \text{ as } t \to \infty \text{, using the above mentioned fact.}$$

$$I_3 = \sum_{i=1}^{m} b_i e^{\alpha_i t} (\cos \beta_i \operatorname{Im} z_i - \sin \beta_i t \operatorname{Im} z_i \to 0 \text{ as } t \to \infty \text{, since } \alpha_i < 0 \text{ for each}$$

$i = 1, 2, \cdots, m$ and $|\sin \beta_i t| \le 1$ and $|\cos \beta_i t| \le 1$.

Similarly, $I_4 = \sum_{i=1}^{m} c_i e^{\alpha_i t} (\cos \beta_i \operatorname{Im} z_i + \sin \beta_i t \operatorname{Re} z_i) \to 0$ as $t \to \infty$.

Therefore, $y(t) = e^{tA} y_0 \to 0$ as $t \to \infty$ for all the initial values y_0 is proved, and the proof is complete. QED

As corollary of the above theorem, if $\operatorname{Re}(\lambda) < 0$ for each eigenvalue of A and the non-homogeneity term $f(t)$ is bounded, then each solution of the non-homogeneous equation is bounded. The following theorem shows the equivalence of the two properties.

Corollary 3.3(Boundedness of solutions of Non-homogeneous DE)

Consider the system of linear differential equation

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases}.$$

The solution of the system is $y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} f(s) ds.$.

Then the following statements are equivalent:

a. For each bounded function $f(t)$ (i.e. $\| f(t) \| < C$ for all t), solution $y(t)$ of the non-homogeneous equation $y'(t) = Ay(t) + f(t)$ is bounded.

b. $\operatorname{Re}\lambda < 0$ for all eigenvalue λ of A.

Proof (b. $\rightarrow$ a.) Suppose $\operatorname{Re}\lambda < 0$ for all eigenvalue λ of A. Then, by Theorem 3.2, there is a positive number M, such that $\| e^{tA} \| < Me^{-\omega t}$ of all $t \geq 0$. Hence,

$$\| y(t) \| = \| e^{tA} y_0 + \int_0^t e^{(t-s)A} f(s) ds \|$$

$$\leq \| e^{tA} y_0 \| + \| \int_0^t e^{(t-s)A} f(s) ds \|$$

$$\leq \| e^{tA} \| \cdot \| y_0 \| + \int_0^t \| e^{(t-s)A} \| \cdot \| f(s) \| ds$$

$$\leq Me^{-\omega t} \cdot \| y_0 \| + \int_0^t Me^{-\omega(t-s)} \cdot C ds$$

$$= Me^{-\omega t} \| y_0 \| + MC \frac{1 - e^{-\omega t}}{\omega}$$

$$\leq M \| y_0 \| + \frac{MC}{\omega}.$$

Hence, $y(t)$ is bounded.

(a,. $\rightarrow$ b.) Suppose for each bounded function $f(t)$, solution $y(t)$ of the non-homogeneous equation is bounded. On the contrary, assume that there exists an eigenvalue λ of A such that $\operatorname{Re}\lambda \geq 0$. By Theorem2.20, $e^{tA} x = e^{\lambda t} x$ for all real numbers t, where x is the eigenvector corresponding to λ.

We now choose $f(t) \equiv x$, then $f(t)$ is bounded , and choose the initial value $y_0 = 0$.

Then we have

$$y(t) = \int_0^t e^{(t-s)A} x ds = \int_0^t e^{(t-s)\lambda} x ds = (\int_0^t e^{(t-s)\lambda} ds)x = \frac{e^{t\lambda} - 1}{\lambda} x.$$

Hence,

$$\| y(t) \| = \| \frac{e^{t\lambda} - 1}{\lambda} x \| = \frac{| e^{t\lambda} - 1 |}{| \lambda |} \| x \| \geq \frac{| e^{t\lambda} | - 1}{| \lambda |} \| x \| = \frac{e^{(\operatorname{Re}\lambda)t} - 1}{| \lambda |} \| x \| \to \infty$$

as $t \to \infty$ (since $e^{(\operatorname{Re}\lambda)t} \to \infty$ for $\operatorname{Re}\lambda > 0$). This is a contradiction, and the proof is complete. QED

Next we study the periodicity of solutions of the non-homogeneous equation. First we state that if $y(1) = y(0)$ then the solution $y(t)$ is 1- periodic.

Theorem 3.4 (Periodicity Function)

Suppose $y(t)$ is a solution of Equation (3.2):

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} f(s) ds$$

where $f(t)$ is periodic with period 1. If $y(1) = y(0)$, then $y(t)$ is a periodic function with period 1.

Proof: We need to prove: $y(t+1) = y(t)$ for all *t*. Then we have

$$y(t+1) = e^{A(t+1)} y_0 + \int_0^{t+1} e^{A(t+1-s)} f(s) ds$$

$$= e^{At} e^{A} y_0 + \int_0^1 e^{A(t+1-s)} f(s) ds + \int_1^{1+t} e^{A(t+1-s)} f(s) ds$$

$$= e^{At}e^{A}y_0 + e^{At}\int_0^1 e^{A(1-s)}f(s)ds + \int_0^t e^{A(t-s')}f(s'+1)ds' \quad \text{(using } s = s'+1\text{)}$$

$$= e^{At}\left(e^{A}y_0 + \int_0^1 e^{A(1-s)}f(s)ds\right) + \int_0^t e^{A(t-s')}f(s')ds' \quad \text{(using } f(s'+1) = f(s')\text{)}$$

$$= e^{At}y(1) + \int_0^t e^{A(t-s')}f(s')ds = e^{At}y(0) + \int_0^t e^{A(t-s')}f(s')ds = y(t).$$

Thus, $y(t+1) = y(t)$, and hence $y(t)$ is a 1- periodic function if $y(1) = y(0)$.

QED

We now can state the Theorem about the periodicity of solutions of non-homogeneous equation.

Theorem 3.5 (Existence of Periodic Solutions)

The following statements are equivalent

a) (**Existence and uniqueness of periodic solution**) For each periodic function $f(t)$ with period 1, there exists a unique initial value y_0, such that the solution of equation(3.2)

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases}$$

is 1-periodic ;

b) Number 1 is not an eigenvalue of e^{A};

c) The numbers $2k\pi \cdot i$, ($k = 0, \ \pm 1, \ \pm 2, ...$) are not eigenvalues of A.

Proof: The equivalence between b) and c) is actually the content of the Spectral Mapping Theorem, when $f(z) = e^z$. We now prove the equivalence of a) and b).

"b) $\Rightarrow$ a)" Suppose 1 is not an eigenvalue of e^A, we know $(I - e^A)$ has inverse $(I - e^A)^{-1}$. Take

$$y_0 = (I - e^A)^{-1} \int_0^1 e^{A(1-s)} f(s)ds .$$

We will show that the solution

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} f(s)ds$$

is periodic by showing $y(1) = y(0)$.

We have

$$y(1) = e^A y_0 + \int_0^1 e^{A(1-s)} f(s)ds$$

$$= e^A (I - e^A)^{-1} \int_0^1 e^{A(1-s)} f(s)ds + \int_0^1 e^{A(1-s)} f(s)ds$$

$$= \left(e^A (I - e^A)^{-1} + I\right) \int_0^1 e^{A(1-s)} f(s)ds$$

$$= (I - e^A)^{-1} \int_0^1 e^{A(1-s)} f(s)ds = y_0$$

Hence, by Theorem 3.4, $y(t)$ is 1-periodic. If there were now another initial value $y_2 \neq y_0$ such that the solution

$$y_2(t) = e^{tA} y_2 + \int_0^t e^{(t-s)A} f(s)ds$$

is also 1-periodic, then $y(t) - y_2(t) = e^{tA}(y_0 - y_2)$ is also a 1-periodic function, i.e.

$$y(1) - y_2(1) = y(0) - y_2(0)$$

or $$e^{A}(y_0 - y_2) = y_0 - y_2 \neq 0\,.$$

This means 1 is an eigenvalue of e^{A}, which is a contradiction to the assumption.

"a) $\Rightarrow$ b)" Let

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} f(s)ds$$

be the unique 1-periodic solution of differential equation (3.2). We use the contradiction method: Suppose 1 is an eigenvalue of e^{A}, i.e. there is a non-zero vector x_0 such that $e^{A} x_0 = x_0$. We will show that the solution

$$y_2(t) = e^{At}(y_0 + x_0) + \int_0^t e^{A(t-s)} f(s)ds$$

with another initial value $y_2(0) = (y_0 + x_0)$ is also 1-periodic. Indeed, we have

$$\begin{aligned} y_2(1) &= e^{A}(y_0 + x_0) + \int_0^1 e^{(1-s)A} f(s)ds \\ &= e^{A} x_0 + (e^{A} y_0 + \int_0^1 e^{(1-s)A} f(s)ds) \\ &= x_0 + (y_0) \\ &= y_2(0)\,. \end{aligned}$$

So, $y_2(t)$ is also 1-periodic, and that is the contradiction to the uniqueness of the 1-periodic solution. Hence, 1 is not an eigenvalue of e^{A}. QED

In physics and biology sometimes we consider the solutions on the whole line *R*. Such solution is called a complete trajectory.

Definition 3.6 (Complete trajectory) A complete trajectory of the equation $y'(t) = Ay(t) + f(t)$, where $t \in R$, is the solution

$$y(t) = e^{tA} y(0) + \int_0^t e^{(t-s)A} f(s) ds$$

where $-\infty < t < \infty$. Note that, if $t < 0$ then we define $\int_0^t f(s)ds := -\int_t^0 f(s)ds$. It is not hard to show that a function $y(t)$ is a complete trajectory of the equation $y'(t) = Ay(t) + f(t)$ if and only if it satisfies the following formula:

$$y(t) = e^{(t-s)A} u(s) + \int_s^t e^{(t-\tau)A} f(\tau) d\tau$$

for all $t, s \in R$ with $t > s$.

Theorem 3.7 (Boundedness of the complete trajectory)

The following statements are equivalent:

a) For each bounded function $f(t)$ in R, there exists a unique bounded complete trajectory.

b) There is no eigenvalue of A on the imaginary line iR;

c) There is no eigenvalue of $e^{t_0 A}$ in the unit circle for a number t_0.

d) There is no eigenvalue of e^{tA} in the unit circle for each positive number *t*.

Proof: We will prove this Theorem in Chapter 4 (see Theorem 4.22).

CHAPTER 4: Extension of Results to

Hilbert Spaces

In this Chapter we try to extend some results in Chapter 3 to Hilbert space. First we introduce Hilbert space and its properties.

1. Hilbert Space and its Properties

Definition 4.1 Let X be a vector space over a field F. An inner product on X is a function $u: X \times X \to F$ such that for all α, β in F, and x, y, z in X, the following are satisfied:

(a) $u(\alpha x + \beta y, z) = \alpha u(x, z) + \beta u(y, z)$,

(b) $u(x, \alpha y + \beta z) = \overline{\alpha} u(x, y) + \overline{\beta} u(x, z)$,

(c) $u(x, x) \geq 0$,

(d) $u(x, y) = \overline{u(y, x)}$.

Here, for α in F, $\overline{\alpha} = \alpha$ if $F = R$ and $\overline{\alpha}$ is the complex conjugate of α if $F = C$. If $\alpha \in C$, the statement that $\alpha \geq 0$ means that $\alpha \in R$ and α is non-negative. An inner product will be denoted by $\langle x, y \rangle = u(x, y)$.

Definition4.2 A Hilbert space is a vector space H over F together with an inner product $u(\cdot,\cdot)$ such that relative to the metric $d(x, y) = \| x - y \|$ induced by the norm, H is a complete metric space.

Example 4.3 Let C^n be complex n-dimensional space of vectors $x = (x_1, x_2, ..., x_n)$ with the inner product

$$x \cdot y = x_1 \overline{y_1} + x_2 \overline{y_2} + ... + x_n \overline{y_n} .$$

Then, C^n is a Hilbert space.

Example 4.4 Let $L^2([a,b])$ be the space of all complex-valued integrable functions from $[a,b]$. We define the function:

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt ,$$

Then this defines an inner product on $L^2([a,b])$. $L^2([a,b])$ is a Hilbert space with the norm

$$\| f \|^2 = \int_a^b | f(t) |^2 \ dt$$

Next we list some important properties of Hilbert spaces.

Theorem 4.5 (The Cauchy Bunyakowsky -Schwarz Inequality) If X is a Hilbert space, then

$$|< x, y >|^2 \leq <\| x \|^2 \| y \|^2$$

for all x and y in X. Moreover, equality occurs if and only if there are scalars α and β, both not 0, such that $< \beta x + \alpha y, \beta x + ay >= 0$.

Corollary4.6 If X is a Hilbert space, then

(a) $\| x \| = 0$ implies $x = 0$.

(b) $\| \alpha x \| = | \alpha | \| x \|$ for α in F and x in X.

(c) **(Triangle Inequality)** $\| x + y \| \leq \| x \| + \| y \|$ for x, y in X,

Next we define linear operators on Hilbert spaces.

Definition4.7 An operator A from a Hilbert space H to another Hilbert space K is called linear if it satisfies the following conditions:

a) $A(x + y) = Ax + Ay$;

b) $A(\alpha x) = \alpha Ax$

For all *x*, *y* in *H* and α in *F*.

Definition4.8 A linear operator *A* is said to be continuous, if $x_n \to x$ in *H* implies $Ax_n \to Ax$.

By using the standard argument we can prove this theorem.

Proposition 4.9 ([2], Proposition II.1.1) Let H and K be Hilbert spaces and *A*: $H \to K$ a linear operator. The following statements are equivalent.

(a) *A* is continuous.

(b) *A* is continuous at 0.

(c) *A* is continuous at some point.

(d) There is a constant $c > 0$ such that $\| Ah \| \le c \| h \|$ for all *h* in H. We say in this case that *A* is a bounded operator.

Definition4.10 (The norm of a bounded operator) Let *A* be a bounded operator. The norm of *A*, denoted by $\| A \|$ is defined by:

$$\| A \| = \sup\{\| Ah \| : h \in H, \| h \| \le 1\}.$$

Remark: It is not hard to see that

$$\| A \| = \sup\{\| A \| : \| h \| = 1\}$$

$$= \sup\{\| Ah \| / \| h \| : h \neq 0\}$$

$$= \inf\{c > 0 : \| Ah \| \le c \| h \|, h \text{ } in \text{ } H\}.$$

The following theorem is about properties of the norm of bounded operators. First, the set of all bounded operators from *H* to *K* is denoted by $B(H, K)$. If $K = H$, then we denote $B(H)$ the set of all bounded operators from *H* to itself.

Proposition 4.11 ([2], Proposition II.1.2)

(a) If A and $B \in B(H,K)$, then $A+B \in B(H,K)$, and $\| A+B \| \le \| A \| + \| B \|$.

(b) If $\alpha \in F$ and $A \in B(H,K)$, then $\alpha A \in B(H,K)$ and $\| \alpha A \| = | \alpha | \| A \|$.

(c) If $A \in B(H,K)$ and $B \in B(K,L)$, then $BA \in B(H,L)$ and $\| BA \| \le \| B \| \| A \|$.

We also can define a vector-valued function $f(t): R \to H$. The continuity and the derivative of such functions are defined as in R^n. Also, the continuity and the product rule in Theorem 2.7 and in Theorem 2.9 also hold in Hilbert space.

2. The Spectrum of an Operator

Definition 4.12 Let H be a Hilbert space and $A \in B(H)$. The resolvent of *A*, denoted by $\rho(A)$, is the set of all complex numbers λ such that $(\lambda I - A)$ has an inverse and $(\lambda I - A)^{-1}$ is also a bounded operator.

The complement set of the resolvent in C is called the spectrum of A, denoted by $\sigma(A)$;

$$\sigma(A) = C \setminus \rho(A).$$

An operator A is called injective if $Ax \neq Ay$ for $x \neq y$, and called surjective if the range of *A* is the whole space *H*. It is not hard to see that λ is in resolvent set if and only if $(\lambda I - A)$ is both injective and surjective.

Let now denote the kernel space of *A*.

$$\ker(A) = \{x \in H : \quad Ax = 0\}.$$

It is easy to see that $0 \in \ker(A)$ for every operator *A*. If $\ker(A) = \{0\}$, then *A* is injective.

Definition 4.13 The point spectrum of A, $\sigma_p(A)$, is defined by

$$\sigma_p(A) = \{\lambda \in C : \ker(A - \lambda) \neq (0)\}$$

As in the case of operators on a Hilbert space, elements of $\sigma_p(A)$ are called eigenvalues.

If $\lambda \in \sigma_p(A)$, non-zero vectors in $\ker(A-\lambda)$ are called eigenvectors; $\ker(A-\lambda)$ is called the eigenspace of A at λ.

It is well known that. if $\|A\|<1$, then $(I-A)$ is invertible and

$$(I-A)^{-1}=\sum_{n=0}^{\infty}A^n .$$

If λ now is a complex number with $|\lambda|>\|A\|$, then we have $\left\|\frac{A}{\lambda}\right\|<1$ and hence,

$\left(I-\frac{A}{\lambda}\right)$ is invertible. Thus, $\lambda-A=\lambda\left(I-\frac{A}{\lambda}\right)$ is invertible and

$$(\lambda-A)^{-1}=\frac{1}{\lambda}\left(I-\frac{A}{\lambda}\right)^{-1},$$

$$=\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^{n},$$

$$=\sum_{n=0}^{\infty}\frac{A^n}{\lambda^{n+1}} .$$

Hence, we have

Theorem 4.14 The spectrum of a bounded operator A lies inside the disk with radius $\|A\|$, i.e., if λ is in the spectrum of A, then $|\lambda|\le\|A\|$.

Since the spectrum of a bounded operator is bounded, we can define operator-valued functions using the Cauchy formula as follows.

Definition 4.15 Let C be the closed contour which contains all the spectrum of A and $f(\lambda)$ be an analytic function everywhere inside and on C. We define an operator by the following:

$$f(A) = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda - A)^{-1} d\lambda.$$

As with matrices in R^n, we can show that these functions have all properties in Theorem 2.15. and 2.16. Moreover, we have

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!}.$$

3. Spectral Mapping Theorem in Hilbert Space

Theorem 4.16 (Spectral Mapping Theorem) If A is a bounded operator on *H*, and $f(\lambda)$ is an analytic functions in a domain containing the spectrum of *A*, then

$$\sigma(f(A)) = f(\sigma(A)).$$

Proof: If $\lambda \in \sigma(A)$, define

$$g(z) := \begin{cases} \dfrac{f(z) - f(\lambda)}{z - \lambda} & if \quad z \neq \lambda; \\ f'(\lambda) & if \quad z = \lambda. \end{cases}$$

Then $g(z)$ is analytic in the same domain as $f(z)$ and $f(z) - f(\lambda) = (z - \lambda)g(z)$. If it were the case that $f(\lambda) \notin \sigma(f(A))$, then $(A - \lambda)$ would be invertible with inverse $g(A)[f(A) - f(\lambda)]^{-1}$, since

$$(A - \lambda)[g(A)(f(A) - f(\lambda))^{-1}] = [f(A) - f(\lambda)](f(A) - f(\lambda))^{-1} = I$$

and

$$\begin{aligned}[g(A)(f(A) - f(\lambda)]^{-1}(A - \lambda) &= (f(A) - f(\lambda))^{-1} g(A)(A - \lambda) \\ &= (f(A) - f(\lambda))^{-1}[f(A) - f(\lambda)] \\ &= I.\end{aligned}$$

Hence, $f(\lambda) \in \sigma(f(A))$; that is, $f(\sigma(A)) \subseteq \sigma(f(A))$.

The converse part is proved the same as in Theorem 2.19 and the theorem is proved.

QED

Next we introduce the **Spectral Projectors** in Hilbert space. Let $\sigma(A)$ be the spectrum set of A, σ_1 and σ_2 be closed, disjoint subsets of $\sigma(A)$ such that $\sigma_1+\sigma_2=\sigma(A)$. Let C_1 be a closed contour containing σ_1 and C_2 be a closed contour containing σ_2, which have no intersect. We now define the following operators:

$$P_1 := \frac{1}{2\pi i}\int_{C_1}(\lambda - A)^{-1}\,d\lambda$$

and

$$P_2 := \frac{1}{2\pi i}\int_{C_2}(\lambda - A)^{-1}\,d\lambda\,.$$

Then P_1 and P_2 have the following properties:

Theorem 4.17 ([6], Theorem 48.2) P_1 and P_2 are the orthogonal projectors in H, that means,

a) $P_1^2 = P_1$ and $P_2^2 = P_2$;

b) $P_1P_2 = P_2P_1 = 0$;

c) $P_1 + P_2 = I$.

Moreover, if $A_1 = P_1A = AP_1$ and $A_2 = P_2A = AP_2$, then

a) $A_1 + A_2 = A$ and

b) $f(A_1) + f(A_2) = f(A)$;

c) $\sigma(A_1) = \sigma_1$ and $\sigma(A_2) = \sigma_2$ and

d) $\sigma(f(A_1)) = f(\sigma_1)$ and $f(\sigma(A_2)) = f(\sigma_2)$.

We now consider the equation

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases} \quad (4.1)$$

as well as the non-homogeneous equation

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases}. \quad (4.2)$$

where A is a bounded operator on a Hilbert space H. As in Chapter 3, we can obtain the existence and uniqueness of the solutions of the above two equations as the following.

4. Extension the Main Results to Hilbert Space

Theorem 4.18

1. There is a unique solution of Equation (4.1) given by:

$$y(t) = e^{tA} y_0.$$

2. Suppose $f(t)$ is a continuous function on $[0, \infty)$. Then there is a unique solution of Equation (4.2) given by:

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} f(s) ds.$$

We now want to extend some results in Chapter 3 to Hilbert space. The problem we need to overcome is, in Hilbert space the spectrum and the set of eigenvalues are not the same, and we do not have the determinant concept. If the proof of any result does not use eigenvalues and determinant, then we can extend that result to Hilbert space, as the following Lyapunov's Theorem.

Theorem 4.19 Consider the system of linear differential equation with the initial condition:

$$\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases}.$$

The solution of the system is $y(t) = e^{tA} y_0$.

Then the following statements are equivalent:

a. The system is stable: Solution $y(t) = e^{tA} y_0 \to 0$ as $t \to \infty$ for all vectors y_0 .

b. $\lim_{n\to\infty} \| e^{tA} \| = 0$.

c. There exist positive numbers M_1 and ω such that $\| e^{tA} \| \le M_1 e^{-\omega t}$.

d. There exists a number t_0 such that $\| e^{t_0 A} \| < 1$.

e. $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of A .

Next we extend Theorem 3.5 with a new proof.

Theorem 4.20 The following statements are equivalent

a)(**Existence and uniqueness of periodic solution**) For each periodic function $f(t)$ with period 1, there exists a unique initial value y_0, such that the solution of equation(3.2)

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases}$$

is 1-periodic .

b) 1 is in the resolvent set of e^A

c) The numbers $2k\pi \cdot i$, ($k = 0, \ \pm 1, \ \pm 2,.....$) are in the resolvent set of *A*.

Proof: The equivalence between b) and c) is actually the content of the Spectral Mapping Theorem, when $f(z) = e^z$, and the proof of "b) $\Rightarrow$ a)" is the same as in Theorem 3.5. We now prove "a) $\Rightarrow$ b)". From the same reasoning as in the proof of Theorem 3.5 we can prove that 1 is not an eigenvalue of e^A, i.e. $(I - e^A)$ is injective. We now need only to show that $(I - e^A)$ is surjective, i.e., if *u* is any vector in *H*, there is a vector *v*, such that $(I - e^A)v = u$.

To do so, let $f(t) = e^{At}u$ for $0 \le t \le 1$ and let $y(t)$ be the periodic solution corresponding to $f(t)$. We have

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} e^{As} u ds$$

$$= e^{At} y_0 + te^{At} u \,.$$

Hence, $y_0 = y(1) = e^A y_0 + e^A u$. This implies $(I - e^A) y_0 = e^A u$ and

$$u = (I - e^A)e^{-A} y_0 \,.$$

It means $(I - e^A)$ is surjective, and the proof is complete. QED

If $f(t)$ is a 1-periodic function on $[0, \ \infty)$, then we can extend it to R by defining $f(-t) = f(t)$. This extension is called the periodic extension. It is easy to see that the periodic extension of $u(t)$ from above theorem is the complete trajectory corresponding to the periodic extension of $f(t)$. Hence, we have

Theorem 4.21 (**Existence and uniqueness of periodic complete trajectory**) The following statements are equivalent

a) For each 1- periodic function $f(t)$ on R there exists a unique 1-periodic complete trajectory.

b) 1 is in resolvent set of e^A

c) The numbers $2k\pi \cdot i$, ($k = 0, \pm 1, \pm 2, ...$) are in the resolvent set of A.

Finally, we have the result about the existence and uniqueness of bounded complete trajectories.

Theorem 4.22 (Existence and uniqueness of bounded complete trajectory): The following statements are equivalent:

a) For each continuous, bounded function $f(t)$ from R to H, there exists a unique bounded complete trajectory.

b) The imaginary line iR lies in the resolvent set of A.

c) The unit circle lies in the resolvent of $e^{t_0 A}$ for a number t_0.

d) The unit circle is a subset of the resolvent of e^{tA} for each positive number t.

Proof: The equivalence among b), c), and d) are actually the content of the Spectral Mapping Theorem. We now prove 'b) $\Rightarrow$ a)" and " a)$\Rightarrow$c)".

"$b) \rightarrow a)$": Let σ_1 be the subset of $\sigma(A)$ which lies to the left of the imaginary axis, and σ_2 be the subset of $\sigma(A)$ which lies to the right of the imaginary axis, then $\operatorname{Re}\lambda < 0$ for $\lambda \in \sigma_1$ and $\operatorname{Re}\lambda > 0$ for $\lambda \in \sigma_2$. Let P_1 and P_2 be the orthogonal projectors in H corresponding to σ_1 and σ_2. Let $A_1 = AP_1$ and $A_2 = AP_2$, then, by Theorem 4.17, $\sigma(A_1) = \sigma_1$ and $\sigma(A_2) = \sigma_2$. It is not hard to see that $\sigma(-A_2) = -\sigma_2$, and hence, $\operatorname{Re}\lambda < 0$ for $\lambda \in \sigma(-A_2)$.

Let $f(t)$ be now a bounded function from R to Hilbert space H. We define the function

$$y(t) = \int_{-\infty}^{t} e^{A_1(t-s)} f(s)ds + \int_{t}^{\infty} e^{A_2(t-s)} f(s)ds .$$

First we show $y(t)$ is complete trajectory. We have for $s < t$:

$$y(t) - e^{A(t-s)} y(s) = \left(\int_{-\infty}^{t} e^{A_1(t-\tau)} f(\tau)d\tau + \int_{t}^{\infty} e^{A_2(t-\tau)} f(\tau)d\tau \right)$$

$$- \left(e^{A(t-s)} \int_{-\infty}^{s} e^{A_1(s-\tau)} f(\tau)d\tau + e^{A(t-s)} \int_{s}^{\infty} e^{A_2(s-\tau)} f(\tau)d\tau \right)$$

$$= \left(\int_{-\infty}^{t} e^{A_1(t-\tau)} f(\tau)d\tau - e^{A(t-s)} \int_{-\infty}^{s} e^{A_1(s-\tau)} f(\tau)d\tau \right)$$

$$+ \left(\int_{t}^{\infty} e^{A_2(t-\tau)} f(\tau)d\tau - e^{A(t-s)} \int_{s}^{\infty} e^{A_2(s-\tau)} f(\tau)d\tau \right)$$

$$= \int_{s}^{t} e^{A_1(t-\tau)} f(\tau)d\tau + \int_{s}^{t} e^{A_2(t-\tau)} P_2 f(\tau)d\tau$$

$$= \int_{s}^{t} e^{A(t-\tau)} f(\tau)d\tau .$$

Hence, $y(t) = e^{A(t-s)} y(s) + \int_{s}^{t} e^{A(t-\tau)} f(\tau)d\tau$ and therefore, is a complete trajectory.

To prove $u(t)$ is bounded, using Theorem 4.19, we have $\| e^{A_1 t} \| < Me^{-\omega t}$ and

$\| e^{-A_2 t} \| < Me^{-\omega t}$ for $t > 0$. Hence,

$$\| y(t) \| = \| \int_{-\infty}^{t} e^{A_1(t-s)} f(s)ds + \int_{t}^{\infty} e^{A_2(t-s)} f(s)ds \|$$

$$\leq \| \int_{-\infty}^{t} e^{A_1(t-s)} f(s)ds \| + \| \int_{t}^{\infty} e^{A_2(t-s)} f(s)ds \|$$

$$\leq \int_{-\infty}^{t} \| e^{A_1(t-s)} \| \cdot \| f(s) \| ds + \int_{t}^{\infty} \| e^{A_2(t-s)} \| \cdot \| f(s) \| ds$$

$$\leq \int_{-\infty}^{t} Me^{-\omega(t-s)} \cdot Cds + \int_{t}^{\infty} Me^{-\omega(s-t)} Cds$$

$$= MCe^{-\omega t} \int_{-\infty}^{t} e^{\omega s} ds + MCe^{\omega t} \int_{t}^{\infty} e^{-\omega s} ds$$

$$= MCe^{-\omega t} \cdot \frac{e^{\omega t}}{\omega} + MCe^{\omega t} \cdot \frac{e^{-\omega t}}{\omega}$$

$$= \frac{2MC}{\omega} \ .$$

Since M is a positive number and C is a constant, $y(t)$ is bounded.

"$a) \to c)$" Assume that for each bounded function $f(x)$, there exists a unique bounded complete trajectory. We prove that the unit circle lies in $\rho(A)$, or equivalently, $e^{i\alpha} \in \rho(A)$ for all $\alpha \in R$.

First, we prove that $1 \in \rho(A)$. Let $f(t)$ be 1 periodic and

$$y(t) = e^{A(t-s)} y(s) + \int_{s}^{t} e^{A(t-\tau)} f(\tau) d\tau$$

the bounded trajectory corresponding to $f(t)$. We put $v(t) = y(t+1)$, then it is not hard to see that $v(t)$ is the bounded complete trajectory corresponding to $f(t+1)$. But $f(t+1) = f(t)$, hence, $v(t)$ is another bounded trajectory corresponding to $f(t)$. Because $y(t)$ is unique, we have $y(t) = v(t)$. So, $y(t) = y(t+1)$ for all t, that means $y(t)$ is 1-periodic. By Theorem 4.21(b), $1 \in \rho(A)$.

Next, let $\mu = e^{i\alpha}$, $\alpha \in R$. We will show that $e^{i\alpha} \in \rho(e^{A})$. We have

$$e^{i\alpha} - e^{A} = e^{i\alpha}(I - e^{A-i\alpha}),$$

that means $i\alpha \in \rho(e^A)$ if and only if $1 \in \rho(e^{A-i\alpha})$. Thus, it is enough now to prove that $1 \in \rho(e^{A-i\alpha})$ and we prove it by showing: For each bounded function $g(t)$, there exists a unique complete trajectory $v(t)$ of the differential equation

$$v'(t) = (A - i\alpha)v(t) + g(t). \qquad (4.3)$$

In order to do that, let $f(t) = e^{i\alpha t} g(t)$, then $f(t)$ is bounded. Let $y(t)$ be the bounded complete trajectory of the equation $y'(t) = Ay(t) + f(t)$. We show $v(t) = e^{-i\alpha t} y(t)$ is the bounded complete trajectory of equation (4.3). Because $v(t) = e^{-i\alpha t} y(t)$, we have

$$\begin{aligned} v'(t) &= e^{-i\alpha t} y'(t) - i\alpha e^{-i\alpha t} y(t), \\ &= e^{-i\alpha t}(Ay(t) + f(t)) - i\alpha e^{-i\alpha t}(e^{i\alpha t} v(t)), \\ &= e^{-i\alpha t}(Ae^{i\alpha t} v(t) + f(t)) - i\alpha e^{-i\alpha t}(e^{i\alpha t} v(t)), \\ &= (A - i\alpha)v(t) + f(t)e^{-i\alpha t}. \\ &= (A - i\alpha)v(t) + g(t). \end{aligned}$$

That means $v(t)$ is the bounded complete trajectory of equation (4.3). By the above argument, we obtain $1 \in \rho(A - i\alpha)$ and hence, $e^{i\alpha t} \in \rho(A)$. QED

In conclusion, for the system

$$\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases},$$

it is very interesting to find conditions on the operator A, so that we can check the qualitative behavior of solutions. These results can be applied in other areas such as in physics, biology, medicine, and so on.

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