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Derivatives of the Dedekind Zeta Function Attached to a Complex Quadratic Field Extension

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DERIVATIVES OF THE DEDEKIND ZETA FUNCTION ATTACHED TO A
COMPLEX QUADRATIC FIELD EXTENSION

A Capstone/Experience Thesis Project

Presented in Partial Fulfillment of the Requirements for

the Degree Bachelor of Arts with

Honors College Graduate Distinction at Western Kentucky University

By

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2010

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ABSTRACT

The Riemann Zeta Function is a function of vital importance in the study of number theory and other branches of mathematics. This is primarily due to its intrinsic link with the prime numbers of the ring of integers. The value of the Riemann Zeta Function at 0 and the values of the first few derivatives at 0 have been determined by various mathematicians. Apostol obtained a closed expression for the n^{th} derivative of the Riemann Zeta Function at 0 that generalized previously known results. For higher derivatives, his result is useful for numerical computations. The Dedekind Zeta Function is a generalization of the Riemann Zeta Function which is used to study primes of more general rings of integers. In this paper we modify the methods of Apostol to study the derivatives of certain Dedekind Zeta Functions. In particular, we are interested in Dedekind Zeta Functions of complex quadratic extensions of the rational number field. We obtain a general, closed expression for the function and its first derivative evaluated at 0. We then provide a few explicit examples. The method allows us to obtain a closed expression for higher derivatives at 0 that is useful for explicit numerical computations.

Keywords: Dedekind Zeta Function, Riemann Zeta Function, Complex Quadratic Extension, Number Theory

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DERIVATIVES OF THE DEDEKIND ZETA FUNCTION ATTACHED TO A COMPLEX QUADRATIC FIELD EXTENSION

NATHAN SALAZAR

1. INTRODUCTION

The Riemann Zeta Function (RZF) was first introduced as a real function by Leonard Euler, and then generalized as a complex function by Bernhard Riemann in 1859. It is expressed as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s \in \mathbb{C}$ and $Re(s) > 1$. It was shown by Euler that this function could be linked to the prime numbers by the identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Using the ideas of Riemann, the RZF was later used to prove the prime number theorem. In 1859, Riemann made his famous conjecture, namely, that all non-trivial zeros of the function have real part equal to $1/2$. This remains unproven to this day.

Dedekind later generalized this function by means of his Dedekind Zeta Function. This is defined as a Dirichlet series attached to some number field. The RZF is just a Dedekind zeta function attached to the rationals \mathbb{Q} . For a general number field K , we have

$$\zeta_K(s) = \sum_{\mathfrak{n} \subset \mathcal{O}_K} \frac{1}{(N(\mathfrak{n}))^s} = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1},$$

where $N(\mathfrak{n})$ is the norm of \mathfrak{n} , \mathfrak{n} is an ideal in \mathcal{O}_K , and \mathcal{O}_K is the set of integers of K . Let K be a complex quadratic field extension of \mathbb{Q} . Then $K = \mathbb{Q}(\sqrt{-D})$, where $D > 0$ and a square-free integer. In this paper we study the Dedekind zeta function attached to K , or $\zeta_K(s)$.

Throughout the paper, $\Gamma(s)$ will denote the Gamma function, which is defined by $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$. Note that for a positive integer n , we have $\Gamma(n) = (n-1)!$, so $\Gamma(s)$ can be viewed as a generalization of the factorial. Also, “log” will denote the natural logarithm.

2. DERIVATIVES OF THE RIEMANN AND DEDEKIND ZETA FUNCTIONS

Apostol, in his paper [1], was able to contain a closed formula for the derivatives $\frac{d^k}{ds^k} \zeta(0)$. To do this, he used the functional equation for $\zeta(s)$, namely,

$$\pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Apostol took the Taylor expansion of both sides and then equated coefficients. The values for $0 \leq k < 3$ were already known, these being $\zeta(0) = \frac{-1}{2}$, $\zeta'(0) = \frac{-1}{2} \log(2\pi)$, and $\zeta''(0) = \frac{-1}{2} \log^2(2\pi) - \frac{\pi^2}{24} + \frac{\gamma_0^2}{2} + \gamma_1$. This last value was first obtained by Ramanujan. Apostol obtained a closed expression for $\zeta^{(n)}(0)$ in terms of Stieltjes constants and other terms. In particular,

Theorem 2.1. [1] *Let $z_0 = -\log(2\pi) - \frac{\pi i}{2}$ and $n \geq 0$. Let*

$$a_n = \frac{\Gamma^{(n+1)}(1)}{(n+1)!} + \sum_{k=0}^n \frac{(-1)^k \gamma_k \Gamma^{(n-k)}(1)}{k! (n-k)!}.$$

Then

$$\zeta^{(n)}(0) = \frac{(-1)^n n!}{\pi} \left(\frac{\text{Im}(z_0^{n+1})}{(n+1)!} + \sum_{k=0}^{n-1} a_k \frac{\text{Im}(z_0^{n-k})}{(n-k)!} \right)$$

Here we have γ_k are the Stieltjes constants, which are defined by

$$(1) \quad \gamma_k = \lim_{N \rightarrow \infty} \left(\sum_{i=1}^N \frac{\log^k(i)}{i} - \frac{\log^{k+1}(N)}{k+1} \right)$$

and γ_0 is Euler’s constant.

For the Dedekind Zeta Function, $\zeta_K(s)$, we obtain the following result. This gives a closed expression for the 0^{th} and 1^{st} derivative, which is an analogue of Apostol’s result, Theorem 2.1, for those derivatives.

Theorem 2.2. Let $K = \mathbb{Q}(\sqrt{-D})$ be a complex quadratic extension of \mathbb{Q} and d_K be the discriminant of K . Let $\chi_d(n) = \left(\frac{d_K}{n}\right)$ be the Kronecker symbol, and d_0 be the conductor of χ_d . Then

$$\zeta_K(0) = \frac{-i\sqrt{|d_K|}\tau(\chi_d)}{2d_0^2} \sum_{j=1}^{d_0-1} \chi_d(j)j$$

and $\zeta'_K(0) =$

$$\frac{-\sqrt{|d_K|}}{2d_0^2} \left[\left\{ i\tau(\chi_d) \left(\gamma_0 - 2 \log \frac{\sqrt{|d_K|}}{2\pi} \right) + d_0^{1/2}(\gamma_0 + \log 2\pi) \right\} \sum_{j=1}^{d_0-1} \chi_d(j)j + d_0^{3/2} \log \prod_{j=1}^{d_0-1} \Gamma\left(\frac{j}{d_0}\right)^{\chi_d(j)} \right]$$

where $\tau(\chi_d) = \sum_{j=1}^{d_0-1} \chi_d(j)e^{2\pi ij/d_0}$ is the Gauss sum.

To derive this, we follow the same basic steps as Apostol. We utilize the Taylor expansion of the functional equation of $\zeta_K(s)$ to evaluate the derivatives of the Dedekind Zeta Function attached to K at zero. The functional equation for a general Dedekind zeta function is much more complicated than the one for $\zeta(s)$, and appears in Theorem 3.1.

3. FUNCTIONAL EQUATION AND TAYLOR EXPANSIONS

Theorem 3.1. [3] Let K be a number field of finite degree over \mathbb{Q} . Let r_1 be the number of real embeddings of K and r_2 be the number of complex embeddings. Then $\zeta_K(s)$ satisfies the functional equation

$$\zeta_K(1-s) = \zeta_K(s)|d_K|^{s-\frac{1}{2}} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} \left(\pi^{-s}2^{(1-s)r_2}\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\right)^n$$

where d_K is the discriminant of K .

If $K = \mathbb{Q}(\sqrt{-D})$, then

$$d_K = \begin{cases} -D, D \equiv 3 \pmod{4} \\ -4D, D \not\equiv 3 \pmod{4} \end{cases}$$

Also, $r_1 = 0$, $r_2 = 1$, and $n = r_1 + 2r_2 = 2$, which gives us

$$\left(\pi^{-s}2^{(1-s)r_2}\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\right)^n = \left(\pi^{-s}2^{(1-s)}\Gamma(s)\right)^2 = \pi^{-2s}2^{(2-2s)}\Gamma(s)^2.$$

And so, we derive our version of the functional equation for $\zeta_K(s)$:

$$\zeta_K(1-s) = |d_K|^{(s-\frac{1}{2})} 2^{(2-2s)} \pi^{-2s} \cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s}{2}\right) \Gamma(s)^2 \zeta_K(s)$$

$$(2) \quad \zeta_K(1-s) = \frac{4}{\sqrt{|d_K|}} \left(\frac{\sqrt{|d_K|}}{2\pi}\right)^{2s} \cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s}{2}\right) \Gamma(s)^2 \zeta_K(s),$$

which is analytic at $s = 1$.

This much simpler version allows the methods of Apostol to be applied. We split (2) into three parts: the first is $\frac{4}{\sqrt{|d_K|}} \left(\frac{\sqrt{|d_K|}}{2\pi}\right)^{2s} \cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s}{2}\right)$, the second is $\Gamma(s)^2$, and the third is $\zeta_K(s)$. We expand each part in a Taylor or Laurent series about $s = 1$.

We can rewrite the first part, leaving off the $\frac{4}{\sqrt{|d_K|}}$, as

$$\begin{aligned} \left(\frac{\sqrt{|d_K|}}{2\pi}\right)^{2s} \cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s}{2}\right) &= e^{2s \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right)} \left(\frac{e^{\frac{\pi i s}{2}} + e^{-\frac{\pi i s}{2}}}{2}\right) \left(\frac{e^{\frac{\pi i s}{2}} - e^{-\frac{\pi i s}{2}}}{2i}\right) \\ &= e^{2s \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right)} \left(\frac{e^{\pi i s} - e^{-\pi i s}}{4i}\right) = \frac{\exp\left[s\left(2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) + \pi i\right)\right] - \exp\left[s\left(2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) - \pi i\right)\right]}{4i} \\ &= \frac{e^{sz_0} - e^{s\bar{z}_0}}{4i}, \end{aligned}$$

where $z_0 = 2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) + \pi i$. Note that $e^{z_0} = e^{\bar{z}_0} = \frac{-|d_K|}{(2\pi)^2}$, so we can rewrite the above equation as

$$\begin{aligned} \frac{e^{(s-1)z_0} e^{z_0} - e^{(s-1)\bar{z}_0} e^{\bar{z}_0}}{4i} &= \frac{e^{z_0}}{4i} (e^{(s-1)z_0} - e^{(s-1)\bar{z}_0}) \\ &= \frac{-|d_K|}{4i(2\pi)^2} (e^{(s-1)z_0} - e^{(s-1)\bar{z}_0}). \end{aligned}$$

Taking the Taylor expansion we get

$$\frac{-|d_K|}{16\pi^2 i} \left(\sum_{n=0}^{\infty} \frac{z_0^n (s-1)^n}{n!} - \sum_{n=0}^{\infty} \frac{\bar{z}_0^n (s-1)^n}{n!} \right) = \frac{-|d_K|}{16\pi^2 i} \left(\sum_{n=0}^{\infty} \frac{(s-1)^n}{n!} (z_0^n - \bar{z}_0^n) \right).$$

Now, for any complex number a , we can write $a - \bar{a} = 2i\text{Im}(a)$. So, $z_0^n - \bar{z}_0^n = z_0^n - \overline{z_0^n} = 2i\text{Im}(z_0^n)$. Using this, the above equation becomes

$$(3) \quad \frac{-|d_K|}{8\pi^2} \left(\sum_{n=0}^{\infty} \frac{(s-1)^n}{n!} \text{Im}(z_0^n) \right).$$

Multiplying this by $\frac{4}{\sqrt{|d_K|}}$, we have the Taylor expansion for part 1, namely

$$(4) \quad \frac{4}{\sqrt{|d_K|}} \left(\frac{\sqrt{|d_K|}}{2\pi} \right)^{2s} \cos\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s}{2}\right) = \frac{-\sqrt{|d_K|}}{2\pi^2} \left(\sum_{n=0}^{\infty} \frac{(s-1)^n}{n!} \text{Im}(z_0^n) \right),$$

where $z_0 = 2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) + \pi i$. Since $\text{Im}(z_0^n) = 0$, equation (4) is actually equal to

$$\frac{-\sqrt{|d_K|}}{2\pi^2} \left(0 + \sum_{n=1}^{\infty} \frac{(s-1)^n}{n!} \text{Im}(z_0^n) \right),$$

which we note has a zero at $s = 1$.

In order to tackle part 2, we state a lemma that will help us exponentiate Taylor expansions.

Lemma 3.2. [4] *Let*

$$[a_1, \dots, a_n] = \det \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ (n-1) & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & (n-2) & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}$$

and let $f(x)$ be a positive function differentiable at $x = a$ so that

$$\log(f(x)) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{n} (x-a)^n.$$

Then

$$f(x) = e^{a_0} \left(1 + \sum_{n=1}^{\infty} [a_1, -a_2, \dots, (-1)^{n+1} a_n] \frac{(x-a)^n}{n!} \right).$$

Proof. Let

$$D_{j,n} = \det \begin{pmatrix} (-1)^{j-1}a_j & (-1)^j a_{j+1} & \cdots & (-1)^{n-1}a_n \\ n-j & a_1 & \cdots & (-1)^{n-j-1}a_{n-j} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \cdots 0 & 1 & a_1 \end{pmatrix}.$$

Then we have

$$D_{1,n} = [a_1, -a_2, \dots, (-1)^{n-1}a_n] = a_1[a_1, \dots, (-1)^{n-2}a_{n-1}] - (n-1)D_{2,n}$$

and for $j < n$,

$$D_{j,n} = (-1)^{j-1}a_j[a_1, \dots, (-1)^{n-j-1}a_{n-j}] - (n-j)D_{j+1,n}.$$

Applying these equations repeatedly, we get

$$\begin{aligned} D_{1,n} &= [a_1, -a_2, \dots, (-1)^{n-1}a_n] = a_1[a_1, \dots, (-1)^{n-2}a_{n-1}] - (n-1)D_{2,n} \\ &= a_1[a_1, \dots, (-1)^{n-2}a_{n-1}] + (n-1)a_2[a_1, \dots, (-1)^{n-3}a_{n-2}] + (n-1)(n-2)D_{3,n} \end{aligned}$$

Iterating, we obtain

$$(5) \quad D_{1,n} = \sum_{j=1}^n \frac{(n-1)!}{(n-j)!} a_j [a_1, \dots, (-1)^{n-j-1}a_{n-j}].$$

Setting $f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ then it is easy to see that $b_0 = e^{a_0}$ and $b_1 = e^{a_0}a_1$, so assume that

$$b_m = \frac{e^{a_0}}{m!} [a_1, -a_2, \dots, (-1)^{m-1}a_m]$$

for each $m < n$. Since $f'(x) = (\sum_{n=0}^{\infty} b_n(x-a)^n) (\sum_{n=1}^{\infty} a_n(x-a)^{n-1})$ then equating terms gives us

$$b_n n = \sum_{j=1}^n a_j b_{n-j}.$$

From the induction hypothesis we can rewrite this as

$$b_n n = e^{a_0} \left(\sum_{j=1}^n \frac{a_j}{(n-j)!} [a_1, \dots, (-1)^{n-j-1} a_{n-j}] \right).$$

From the equation (5) we have

$$\begin{aligned} b_n n! &= e^{a_0} \left(\sum_{j=1}^n a_j \frac{(n-1)!}{(n-j)!} [a_1, \dots, (-1)^{n-j-1} a_{n-j}] \right) \\ &= e^{a_0} [a_1, \dots, (-1)^{n-1} a_n], \end{aligned}$$

which is the result. □

We start with a well-known expression, from [2] for example,

$$\log \Gamma(s) = -\gamma_0(s-1) + \sum_{n=2}^{\infty} (-1)^n \zeta(n)(s-1)^n$$

where $\gamma_0 = \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \log(N) \right) \approx 0.577 \dots$ is Euler's constant. From here we get

$$\log(\Gamma(s)^2) = 2 \log(\Gamma(s)) = -2\gamma_0(s-1) + 2 \sum_{n=2}^{\infty} (-1)^n \zeta(n)(s-1)^n.$$

If we apply the lemma, taking $f(x) = \Gamma(s)^2$, $a_0 = 0$, $a_1 = -2\gamma_0$, and $a_n = (-1)^n 2n\zeta(n)$ for $n \geq 2$, then the first few terms of the Taylor expansion of $\Gamma(s)^2$ are

$$(6) \quad \Gamma(s)^2 = 1 - 2\gamma_0(s-1) + (4\gamma_0^2 + 4\zeta(2)) \frac{(s-1)^2}{2} - (8\gamma_0^3 + 24\gamma_0\zeta(2) + 12\zeta(3)) \frac{(s-1)^3}{3!} + \dots$$

which is sufficient for finding the first few derivatives of $\zeta_K(s)$.

Now, for part 3 we have the following factorization of the Dedekind zeta function from [3]:

$$(7) \quad \zeta_K(s) = \zeta(s) L(s, \chi_d)$$

where $d = d_K$, $L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$ which converges for $\text{Re}(s) > 0$, and $\chi_d(n) = \left(\frac{d}{n}\right)$ is the Kronecker symbol. As we are evaluating the derivatives at $s = 1$, we have from [3] the following theorem which allows us to compute $L(1, \chi_d)$:

Theorem 3.3. *If χ is odd, i.e. $\chi(-1) = -1$, then*

$$L(1, \chi) = \frac{\pi i \tau(\chi)}{d_0^2} \sum_{0 < a < d_0} \bar{\chi}(a) a,$$

where d_0 is the conductor of χ , and $\tau(\chi) = \sum_{x \pmod{d_0}} \chi(x) e^{2\pi i x / d_0}$.

The Kronecker symbol is defined so as $\left(\frac{a}{-1}\right) = -1$ if $a < 0$. Since $a = d_K$ is always negative for K a complex quadratic field extension, this theorem's condition is satisfied for our purposes. We can also make use of the following lemma to obtain values of the derivatives of $L(1, \chi_d)$.

Lemma 3.4. [2] *If $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is holomorphic and convergent for $\text{Re}(s) > \sigma$, then*

$$f^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{a_n (\log n)^k}{n^s}$$

with $\text{Re}(s) > \sigma$.

So, $L^{(k)}(1, \chi_d) = (-1)^k \sum_{n=2}^{\infty} \frac{\chi_d(\log n)^k}{n}$. Now, if we take the Laurent expansion of each term of the right-hand side of the decomposition (7) we get

$$\zeta(s) L(s, \chi_d) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} \right),$$

where the two series converge for $\text{Re}(s) > 1$. We take a Laurent expansion of the left series, and a Taylor expansion of the right series to get

$$(8) \quad \left(\frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \right) \left(\sum_{n=0}^{\infty} \frac{L^{(n)}(1, \chi_d)}{n!} (s-1)^n \right).$$

The following theorem gives a method for multiplying Dirichlet series:

Theorem 3.5. [2] Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ be holomorphic functions that converge for $\text{Re}(s) > \sigma$. Then, for $\text{Re}(s) > \sigma$, $f(s)g(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$ where $c_n = \sum_{d|n} a_n b_{n/d}$.

So, the product of two convergent Dirichlet series gives a convergent Dirichlet series. Thus we can give the Laurent expansion of both sides. Applying Theorem 3.5, the above product (8) can be written as

$$\frac{L(1, \chi_d)}{s-1} + \sum_{n=0}^{\infty} \left(\frac{L^{(n+1)}(1, \chi_d)}{(n+1)!} + \sum_{j+k=n} \frac{(-1)^j \gamma_j}{j!k!} L^{(k)}(1, \chi_d) \right) (s-1)^n.$$

Again, we need only find the first few terms of this series in order to calculate the first few derivatives of $\zeta_K(s)$. These are

$$(9) \quad \frac{L(1, \chi_d)}{s-1} + \left(L^{(1)}(1, \chi_d) + \gamma_0 L(1, \chi_d) \right) + \left(\frac{L^{(2)}(1, \chi_d)}{2} - \gamma_1 L(1, \chi_d) + \gamma_0 L^{(1)}(1, \chi_d) \right) (s-1) + \dots$$

Now we multiply (4), (6), and (9), equate coefficients, and get the first few derivatives of $\zeta_K(0)$. The first result we get is the following:

Proposition 3.6.

$$\zeta_K(0) = \frac{-\sqrt{|d_K|}}{2\pi} L(1, \chi_d),$$

$$\zeta'_K(0) = \frac{\sqrt{|d_K|}}{2\pi} L^{(1)}(1, \chi_d) - \frac{\sqrt{|d_K|}}{2\pi} \left(\gamma_0 - 2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) \right) L(1, \chi_d),$$

and

$$\zeta''_K(0) = \frac{-\sqrt{|d_K|}}{2\pi} L^{(2)}(1, \chi_d) + \frac{\sqrt{|d_K|}}{\pi} \left(\gamma_0 - 2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) \right) L^{(1)}(1, \chi_d) +$$

$$\frac{\sqrt{|d_K|}}{\pi} \left(\gamma_1 - \zeta(2) + 2\gamma_0 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) - 2 \log^2\left(\frac{\sqrt{|d_K|}}{2\pi}\right) \right) L(1, \chi_d).$$

Proof. If we expand the functional equation (1) and equate coefficients with the Laurent expansions of each part of the right hand side, given by multiplying (4), (6), and (9) together, we get

$$\begin{aligned} & \zeta_K(0) - \zeta'_K(0)(s-1) + \frac{\zeta''_K(0)}{2!}(s-1)^2 - \dots = \\ & \left[\frac{-\sqrt{|d_K|}}{2\pi^2} \left(0 + \pi(s-1) + 4\pi \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) \frac{(s-1)^2}{2!} + (12\pi \log^2\left(\frac{\sqrt{|d_K|}}{2\pi}\right) - \pi^3) \frac{(s-1)^3}{3!} + \dots \right) \right] \\ & \times \left[1 - 2\gamma_0(s-1) + (4\gamma_0^2 + 4\zeta(2)) \frac{(s-1)^2}{2} - (8\gamma_0^3 + 24\gamma_0\zeta(2) + 12\zeta(3)) \frac{(s-1)^3}{3!} + \dots \right] \times \\ & \left[\frac{L(1, \chi_d)}{s-1} + \left(L^{(1)}(1, \chi_d) + \gamma_0 L(1, \chi_d) \right) + \left(\frac{L^{(2)}(1, \chi_d)}{2} - \gamma_1 L(1, \chi_d) + \gamma_0 L^{(1)}(1, \chi_d) \right) (s-1) + \dots \right] \end{aligned}$$

(Note: Since (4) has a zero of at least order 1 at $s = 1$, and (9) has a pole at $s = 1$, the product is analytic at $s = 1$, resulting in a Taylor series for $\zeta_K(0)$). We first take the sum of the products of all terms that will multiply out to give us coefficients of $(s-1)^0 = 1$. There is only one such term, and it is given by $\zeta_K(0) = \left(\frac{-\sqrt{|d_K|}}{2\pi^2} \pi(s-1) \right) (1) \left(\frac{L(1, \chi_d)}{s-1} \right) = \frac{-\sqrt{|d_K|}}{2\pi} L(1, \chi_d)$. To get the first derivative, we find all the terms that are coefficients of $(s-1)$. These are given by

$$\begin{aligned} -\zeta'_K(0)(s-1) &= \left(\frac{-\sqrt{|d_K|}}{2\pi^2} \pi(s-1) \right) (1) \left(L^{(1)}(1, \chi_d) + \gamma_0 L(1, \chi_d) \right) + \\ & \left(\frac{-\sqrt{|d_K|}}{2\pi^2} \pi(s-1) \right) (-2\gamma_0(s-1)) \frac{L(1, \chi_d)}{s-1} + \left(\frac{-\sqrt{|d_K|}}{2\pi^2} 4\pi \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) \frac{(s-1)^2}{2!} \right) (1) \frac{L(1, \chi_d)}{s-1}. \end{aligned}$$

After simplifying we obtain the first derivative, given by

$$\zeta'_K(0) = \frac{\sqrt{|d_K|}}{2\pi} L^{(1)}(1, \chi_d) - \frac{\sqrt{|d_K|}}{2\pi} \left(\gamma_0 - 2 \log\left(\frac{\sqrt{|d_K|}}{2\pi}\right) \right) L(1, \chi_d).$$

The second derivative can be obtained in a similar manner. □

We know from Lemma 3.4 that $L^{(1)}(1, \chi_d) = -\sum_{n=2}^{\infty} \frac{\chi_f(n) \log(n)}{n}$, and so the following lemma allows us to obtain an exact expression for this value.

Lemma 3.7. [6] *Let $\chi_f(n)$ be a nonprincipal character with $\chi_f(f-1) = -1$. Then*

$$\sum_{n=2}^{\infty} \frac{\chi_f(n) \log(n)}{n} = \frac{\pi}{f^{1/2}} \left\{ \log \prod_{p=1}^f \Gamma(p/f)^{\chi_f(p)} + \frac{1}{f} (\gamma_0 + \log(2\pi)) \sum_{p=1}^f p \chi_f(p) \right\}$$

If we take $f = d_0$, then substituting the right-hand side of the equation of Lemma 3.7 into the expression for Proposition 3.6 gives the value for $\zeta'_K(0)$ given by Theorem 2.2.

4. EXAMPLES

For a first example, we take $K = \mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(i)$. Then we have $d_{\mathbb{Q}(i)} = -4$, and $L(1, \chi_{-4}) = \frac{\pi}{4}$. Thus, by our theorem we have

$$\zeta_{\mathbb{Q}(i)}(0) = \frac{-\sqrt{|-4|}}{2\pi} \left(\frac{\pi}{4} \right) = \frac{-1}{4}.$$

Now, in order to obtain an exact expression for $\zeta'_K(0)$ we need to find $L^{(1)}(1, \chi_d)$. We do this using Lemma 3.7, and get $L^{(1)}(1, \chi_{-4}) = \frac{-\pi}{2} \left\{ \log \left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \right) - \frac{1}{2} (\gamma_0 + \log 2\pi) \right\}$. So

$$\begin{aligned} \zeta_{\mathbb{Q}(i)}^{(1)}(0) &= \frac{-\sqrt{|-4|}}{2\pi} \left(\gamma_0 - 2 \log \left(\frac{\sqrt{|-4|}}{2\pi} \right) \right) \left(\frac{\pi}{4} \right) + \frac{\sqrt{|-4|}}{2\pi} \frac{-\pi}{2} \left\{ \log \left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \right) - \frac{1}{2} (\gamma_0 + \log 2\pi) \right\} \\ &= \frac{-1}{2} \log \pi - \frac{1}{2} \log \left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \right) + \frac{1}{2} \log \sqrt{2\pi} \\ &= \frac{1}{2} \log \left(\frac{\sqrt{2}\Gamma(\frac{3}{4})}{\sqrt{\pi}\Gamma(\frac{1}{4})} \right) \approx -0.655266463\dots \end{aligned}$$

An exact expression for $\zeta_{\mathbb{Q}(i)}^{(2)}(0)$ is given by

$$\frac{-1}{\pi} L^{(2)}(1, \chi_4) + \frac{2}{\pi} (\gamma_0 + 2 \log \pi) L^{(1)}(1, \chi_4) + \frac{1}{2} (\gamma_1 - \zeta(2) - 2\gamma_0 \log \pi - 2 \log^2 \pi)$$

which has a numerical approximation of $-2.4286\dots$

For $K = \mathbb{Q}(\sqrt{-3})$, we have $d_{\mathbb{Q}(\sqrt{-3})} = -3$, $d_0 = 3$, and $L(1, \chi_3) = \frac{\pi\sqrt{3}}{9}$. Applying Proposition 3.6 and Lemma 3.7 we get

$$\zeta_{\mathbb{Q}(\sqrt{-3})}(0) = \frac{-1}{6}$$

$$\zeta_{\mathbb{Q}(\sqrt{-3})}^{(1)}(0) = \frac{1}{2} \log \left(\frac{\sqrt[3]{3}\Gamma(2/3)}{\sqrt[3]{2\pi}\Gamma(1/3)} \right) \approx -0.464345982 \dots$$

$$\begin{aligned} \zeta_{\mathbb{Q}(\sqrt{-3})}^{(2)}(0) &= \frac{-\sqrt{3}}{2\pi} L^{(2)}(1, \chi_{-3}) + \frac{\sqrt{3}}{\pi} \left(\gamma_0 - 2 \log \frac{\sqrt{3}}{2\pi} \right) L^{(1)}(1, \chi_{-3}) + \\ &\frac{1}{3} \left(\gamma_1 - \zeta(2) + 2\gamma_0 \log \frac{\sqrt{3}}{2\pi} - 2 \log^2 \frac{\sqrt{3}}{2\pi} \right) \approx -1.7614 \dots \end{aligned}$$

For $K = \mathbb{Q}(\sqrt{-11})$ we have $d_{\mathbb{Q}(\sqrt{-11})} = -11$, $d_0 = 11$, and $L(1, \chi_{11}) = \frac{\pi\sqrt{11}}{11}$. For our first value, we have

$$\zeta_{\mathbb{Q}(\sqrt{-11})}(0) = \frac{-1}{2}.$$

The multiplication theorem for the gamma function states that

$$\Gamma(z)\Gamma\left(z + \frac{1}{m}\right)\Gamma\left(z + \frac{2}{m}\right) \dots \Gamma\left(z + \frac{m-1}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{1/2-mz} \Gamma(mz).$$

Using this, the first derivative may be simplified down to the following form:

$$\zeta_{\mathbb{Q}(\sqrt{-11})}^{(1)}(0) = \log \left(\frac{\sqrt[4]{11}4\pi^2}{\Gamma\left(\frac{1}{11}\right)\Gamma\left(\frac{3}{11}\right)\Gamma\left(\frac{4}{11}\right)\Gamma\left(\frac{5}{11}\right)\Gamma\left(\frac{9}{11}\right)} \right) \approx -0.9696464 \dots$$

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