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# TRANSITIONS OF TOPOLOGICAL RELATIONS BETWEEN PLANAR REGIONS

A Capstone Experience/Thesis Project

Presented in Partial Fulfillment of the Requirements for

the Degree Bachelor of Arts with

Honors College Graduate Distinction at Western Kentucky University

By

Kathleen Bell

*****

Western Kentucky University

2016

CE/T Committee:

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ABSTRACT

As two topological regions are morphed and translated, how does their intersection change? Previous research has been done on static configurations with planar spatial regions. I expand upon this research to include dynamically changing regions and intersections. I examine what forms of intersection are possible, and what transitions are directly possible, while considering such variables as the connectedness of the regions.

Keywords: Topology, GIS, Mathematics, Planar Regions

DEDICATION

Dedicated to Dr. Tom Richmond, whose patient explanations, hours and hours of hard work, and supportive guidance throughout my college career have been a great blessing.

ACKNOWLEDGEMENTS

My deepest thanks go to the many wonderful people who have helped me with this project in a myriad of ways. To my parents, who always placed great importance in my education and taught me a love of learning, and who have been supportive of every endeavor I have made. To my friends and classmates who encouraged me to keep going. To the Gatton Academy for providing me with opportunities I could not have imagined. And to the faculty at Western Kentucky University, whose helpfulness and dedication to their students has set the standard for the professor I want to be. You have all made an impact on this project and on me – thank you.

## VITA

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## PUBLICATIONS

Bell, Kathleen, Shania Polson, and Tom Richmond. "The Fastest Path Between Two Points, with a Symmetric Obstacle." *The College Mathematics Journal* 46, no. 2 (2015): 92-97.

## FIELDS OF STUDY

Major Field: Mathematics

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# CHAPTER 1

## INTRODUCTION

An understanding of this problem begins with a basic idea of Geographic Information Systems (GIS). GIS is a system of studying geographic data, including how different geographic regions intersect. Beyond just seeing whether two areas intersect, it is possible to see the manner of their intersection.

In order to categorize the different types of intersections, I examined four different factors of an intersection between two regions A and B, presented in "Introduction to Topology: Pure and Applied" and researched extensively by Max Egenhofer, Professor and Director of the School of Computing and Information Science and the National Center for Geographic Information and Analysis, along with his students and colleagues.[1] These four factors, known as the Four-Intersection Values, can be written as a binary 4-tuple, or stated in the form of four questions: Do the boundaries of A and B intersect? Do the interiors of A and B intersect? Does the boundary of A intersect the interior of B? Does the boundary of B intersect the interior of A? In this paper, $\partial A$ is the notation representing the boundary of a region A, and $A°$ is the notation to represent the interior of a region A.

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[1] Clementini, E., Sharma, J., & Egenhofer, M. Modelling topological spatial relations: Strategies for query processing. *Computers & Graphics*, Volume 18 (6), (1994): 815-822.

While these have been studied extensively[2] for static configurations, the dynamic transitions between these Four-Intersection forms is a relatively new idea.[3] Additionally, previously conducted studies primarily focus on planar spatial regions (regions that are nonempty, proper subsets of $\mathbb{R}^2$, the two-dimensional real plane, that are regularly closed and have connected interiors)[4].

A set is regularly closed if it equals the closure of its interior. So, for example, the set [-3, 3] is regularly closed, since it equals the closure of its interior (Closure of (-3, 3) is [-3, 3]). However, the set [-3, 3] ∪ {5} is not regularly closed, since its interior is (-3, 3) and the closure of its interior is [-3, 3], which does not equal the whole set. In this case, the point 5 is a boundary point that we would call a "whisker" – a boundary point that is not enclosing interior. A set can be regularly closed, however, without being a planar spatial region. One example of this is a bowtie-shaped region, as shown below. This is regularly closed, but its interior can be expressed as the union of two separate components.

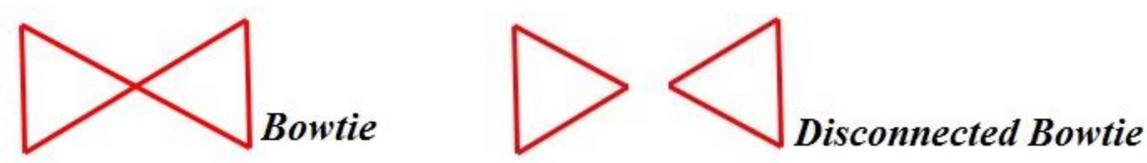


*Figure 1*

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[2] "Modelling topological spatial relations: Strategies for query processing." and Adams, C., & Franzosa, R. (2008). *Introduction to Topology: Pure and Applied*. Harlow: Prentice Hall.

[3] Egenhofer, M. and Al-Taha, K., 1992. Reasoning about gradual changes of topological relationships. *In:* A Frank, I. Campari, and U. Formentini (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space, Pisa, Italy, Lecture Notes in Computer Science*, 639. New York:Springer-Veralg, 196-219.

[4] Definition 2.20 from *Introduction to Topology: Pure and Applied*.

The overall approach was to use topological definitions to categorize the different types of planar spatial regions and to categorize what types of intersections are possible. It is known that there are only eight possible types of intersections using these regions[5]. I then considered how these intersections change as the regions become non-static; for example, is it possible for a certain boundary-interior intersection to directly transition to an interior-interior intersection?  It was necessary to define the regions, their intersections, and the transitions rigorously and mathematically in order to best apply those concepts.

This project was very topologically-based (many of the methods and techniques used in this project were ones borrowed from the homework issued in my topology class, Math 439). However, I believe this research not only has applications in the field of topology, but is also an innovative project in the study of GIS, as it incorporates the idea of elapsed time (a relatively new application of GIS).  This project, therefore, has relevance not only conceptually in the field of pure mathematics but also in real-world applications.  For example, the main question – what types of intersection-transitions are possible? – can be viewed in light of changing wildlife habitats; if a predator is slowly migrating in one direction, and a prey species is migrating in the same area, how do their habitats intersect over time?  Property lines, say of a park, may expand by acquiring privately owned property.  Yet another real-world application is the idea of Wifi (wireless internet) hotspots, which are constantly being created, expanded, and removed in different locations, and which often have overlap between two hotspots.

[5] *Introduction to Topology: Pure and Applied*

# CHAPTER 2

## RULES AND STRATEGIES

In order to maintain some consistency, we began with a set of rules and strategies for how regions could be created, changed, and deleted. We decided to consider any closed regions, whether they were connected or not, had nonempty interior or not, etc. In order to create (and, in reverse, delete) those regions, we used the following set of ideas, where $f(t)$ is the area function over time. Some of these types of regions could be created with a "first instant of occurrence", which we denoted "at an instant" or "instantly". Other transitions simply had a last instant at which they did *not* occur. We denoted these transitions "directly". A simple way to understand this system is to consider a continuous function such as $f(x) = -x \; for \; x \leq 0, f(x) = 0 \; for \; x > 0$. The inverse image of the closed set $\{0\}$ under the continuous function *f* is also closed[6], so there is a first instant (x=0) when the area f(x) is 0, but no first instant when the area f(x) is positive, so f becomes positive directly as x increases, but hits zero at an instant as x decreases.

To create a whisker at an instant, consider $[-r(t), \; r(t)] \; x \; [0,1]$, where r is the radius at time t. When r(t) < 0, [-r(t), r(t)] is the empty set, so we have Ø x [0,1], which

[6] By Theorem 4.8 from *Introduction to Topology: Pure and Applied*.

gives us no area or boundary. When r(t) = 0, this gives {0} $x$ [0,1] which is a whisker. Since there is a first instant when r(t) = 0, this creation of the whisker happens at an instant.

To directly create area by expanding boundary, consider $[0,\ f(t)]\ x\ [0,\ 1]$, where f(t) = t, starting with f(t) = 0. Here f(t) is an increasing function that serves as an area function. When f(t) = 0, we have {0} x [0,1] which is our boundary. When f(t) > 0, we have positive area. Since there is not a first instant of f(t) > 0, this creation of area by expanding boundary does not happen at a single instant, but rather straightaway, which we call "directly."

To delete area at an instant (by reducing it to a boundary point or whisker), consider $[0,\ f(t)]\ x\ [0,\ 1]$, where f(t) again serves as an area function. This time let f(t) be a decreasing function that starts positive. When f(t) > 0, this gives us a positive area,but when f(t) = 0, this gives us {0} x [0, 1], which is a whisker. These strategies, and the combination of them, were our original, "regular" system.

We discovered, however, that there was another approach that allowed more transitions to become possible. We called this the "closure system", since it was basically the approach of taking the closure of open sets. In order to delete a whisker at an instant, we considered $cl\left(\left(-f(t),\ f(t)\right)\right)\ x$ {0}; when $f(t) \leq 0$, this gives $\emptyset\ x$ [0,1] which is the empty set. When $f(t) > 0$, this gives a whisker from $-f(t)$ to $f(t)$.

To create Area-with-Boundary directly, we considered $Cl(-f(t),\ f(t))\ x$ [0,1]; when $f(t)\ \leq\ 0$, this gives the empty set. When $f(t)\ >\ 0$, it gives interior with boundary.

To delete Area-with-Boundary at an instant, we considered $Cl(-f(t),\ f(t))\ x$ [0,1]; when $f(t)\ \leq\ 0$, this gives the empty set, but when $f(t)\ >\ 0$, this gives interior with boundary.

Because any transitions could be reversed simply by performing the changes in "reverse-time", all transitions were bidirectional (for example, if form five couldtransition to form seven, then form seven could transition to form five simply by reversing the operations).

## CHAPTER 3

In this section, we will assume all regions are spatial regions with positive finite area. This will be satisfied if the spatial regions are compact (closed and bounded). The adjacency graph is shown in Figure 1. It is easy to see that the adjacencies shown are possible, with the surprising exception of Disjoint being adjacent to Overlaps. Our next example shows this adjacency.

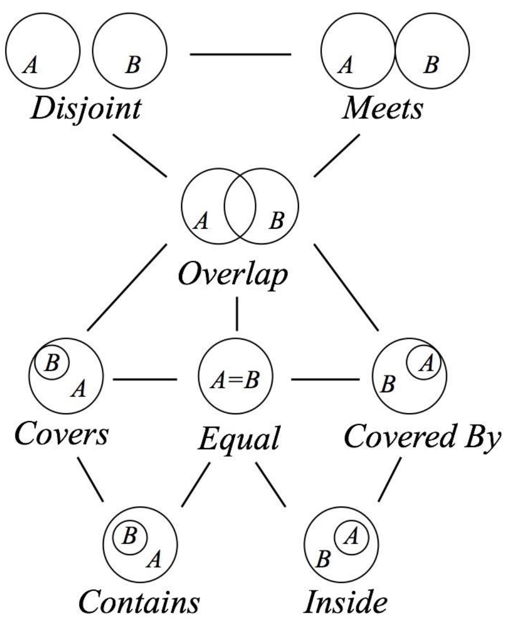



*Figure 2 - Adjacency graph for spatial regions with finite positive areas.*

**Example 1**: For spatial regions A and B, Disjoint is adjacent to Overlaps. One may initially be tempted to believe that as disjoint regions A and B morph continuously from Form 1 (0, 0, 0, 0) = Disjoint to Form 16 (1, 1, 1, 1) = Overlaps, they should pass through

Form 9 (1, 0, 0, 0) = Meets. Included are two examples which show that we need not pass through Form 9.

Consider a static set $A = \{(x, y) \in R^2 : x > 0, y \geq 1/x\}$ and a dynamic set $B(t) = \{(x, y) \in R^2 : y \leq t\}$. Now A and B are Disjoint for t ≤ 0 and Overlap for t > 0, showing that Disjoint transforms directly to Overlap, and Overlap transforms instantly to Disjoint. This example depends on the choice of spatial regions which are not compact. To see that it is possible with compact spatial regions, let $A = [-1, 1]$

$\times [1/2, 2]$ and $B = ([-1, 1] \times [-1, 0]) \cup (cl(-r, r) \times [0, 1])$. Note that B employs the closure system. When r ≤ 0, A and B are Disjoint rectangles. For r > 0, A and B Overlap.

The following proposition shows that the adjacencies shown in the graph are the only ones possible, assuming some natural continuity conditions on the area. We want our regions A and B to morph in such a way that not only is the area of each region changing continuously with time, but also the area of that part of B inside A and the area of that part of A inside B changes continuously. For example, if A = $[-3, 3]^2$ is static, and $B(t) = \bar{B}((0, 0), 1)$ for t < 0 and B(t) = $\bar{B}$((6, 0), 1) for t ≥ 0 (where $\bar{B}(x, e)$ is the closure of the ball centered at x with radius e) then the areas of A and B are constant, and thus continuous, as time changes, but the area of A ∩ B is not continuous. Our assumptions on continuity of these areas is a minimal assumption quantifying that the regions should morph “continuously”.

**Proposition 1**: Suppose A and B are spatial regions with positive finite areas, and the areas of A, B, and $A \cap B$ change continuously with time. Then Disjoint and Meet are not adjacent to any of Covers, Equal, Covered By, Contains, or Inside. Neither of Covers or Contains is adjacent to either Covered By or Inside.

Proof : Suppose that Disjoint is adjacent to Covers. Then there exists a time interval [-a, a] during which A and B always satisfy Disjoint or Covers, and no other states, and without loss of generality, A and B satisfy Disjoint for $t < 0$ and Covers for $t > 0$. Let $f(t)$ be the area of B outside of A at time t and let $g(t)$ be the area of B inside A at time t. That is, $f(t)$ is the area of B minus the area of $A \cap B$, and $g(t)$ is the area of $A \cap B$. Now f and g are continuous, $g(t) = 0$ for $t < 0$, and $g(t) > 0$ for $t > 0$. Since g is continuous, $g^{-1}(0)$ is a closed set containing (-1, 0), so $g(0) = 0$, and thus A and B are still Disjoint at $t = 0$. Now $f(t) - g(t)$ is positive for $t < 0$ and negative for $t > 0$, so $f(0) - g(0) = f(0) = 0$. But $f(0)$ is the area of B at time 0, contradicting the assumption that B have positive area. The same argument shows that Disjoint and Meet are not adjacent to any of Covers, Equal, Covered By, Contains, or Inside.

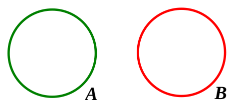
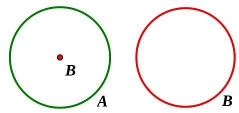
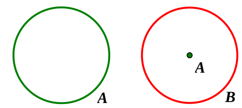
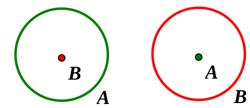
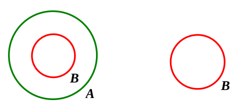
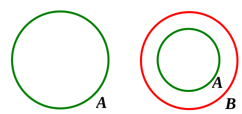
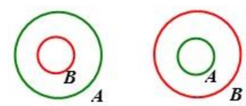
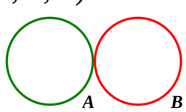
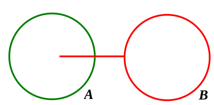
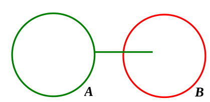
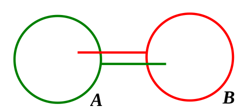
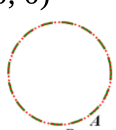
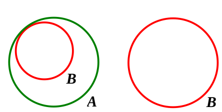
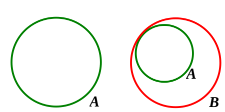
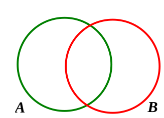
To see that neither of Covers or Contains is adjacent to either Covered By or Inside, we will show that Covers is not adjacent to Covered By. The same argument works for the other proofs. Suppose A and B satisfy Covers for $t < 0$ and Covered By for $t > 0$, and do not assume a third form at $t = 0$. Let $a(t)$ and $b(t)$ be the areas of A and B, respectively. Now $b(t) < a(t)$ for $t < 0$ and $b(t) > a(t)$ for $t > 0$. The continuity conditions imply $a(0) = b(0)$, which is not possible if A and B satisfy Covers or Covered By. Thus, there is not a direct transition between these two forms.

The general question that I investigated is how the form of the intersection between two planar regions changes as they are translated (continuously changed in position) and morphed (continuously changed in shape and/or size). I began by only considering planar spatial regions, but decided to also explore the transitions possible if our regions were not simply connected. What if we allowed whiskers and disconnected sets? This removes some of the restrictions we previously had (for example, with connected regions, if the boundary of A moves to intersect the interior of B, the interiors of A and B must intersect.) Allowing the new types of regions opened up sixteen possibilities for our four-intersection values ($\partial A \cap \partial B, A^\circ \cap B^\circ, \partial A \cap B^\circ, A^\circ \cap \partial B$). Within the sixteen forms, we began to explore which intersection forms could transition to which other intersection forms.

With the exception of Four-Intersection Value (0, 1, 0, 0), which we labelled form number five, all of our sets can be chosen to be closed and bounded and thus compact.

We gave each of the four-intersection values an assigned number, and the eight that were possible with spatial regions were labelled using the same names that Dr. Egenhoffer used[7]. These names are explained in the next table, which shows the sixteen intersection forms with examples to illustrate some ways these forms can be realized. Note that most of the forms have multiple ways to be realized, so I am just including one illustration per form.

[7] "Modelling topological spatial relations: Strategies for query processing."

| | | | |
|---|---|---|---|
| 1. Disjoint (no intersecting values)<br>(0, 0, 0, 0)<br> | 2.<br>(0, 0, 0, 1)<br> | 3.<br>(0, 0, 1, 0)<br> | 4.<br>(0, 0, 1, 1)<br> |
| 5. Real Plane (this Four-Intersection value has only one way to be realized, which is the entire Real Plane)<br>(0, 1, 0, 0)<br> = $\mathbb{R}^2$ | 6. Contains (Some component of A contains a component of B)<br>(0, 1, 0, 1)<br> | 7. Inside (some component of A is inside a component of B)<br>(0, 1, 1, 0)<br> | 8.<br>(0, 1, 1, 1)<br> |
| 9. Meets (some component of A has a boundary-boundary intersection with some component of B)<br>(1, 0, 0, 0)<br> | 10.<br>(1, 0, 0, 1)<br> | 11.<br>(1, 0, 1, 0)<br> | 12.<br>(1, 0, 1, 1)<br> |
| 13. Equals (some component of A is equal in size, location, and shape to some component of B)<br>(1, 1, 0, 0)<br> | 14. Covers (some component of A covers some component of B)<br>(1, 1, 0, 1)<br> | 15. CoveredBy (some component of A is covered by some component of B)<br>(1, 1, 1, 0)<br> | 16. Overlaps (all four types of intersection are occurring)<br>(1, 1, 1, 1)<br> |

The chart below shows the sixteen different Four-Intersection Value possibilities and what they can transition to using the regular transitions and using the closure-system. We were surprised to find that, once the closure-system was allowed and area changes were permitted to be discontinuous, all transitions were possible. This was largely due to the fact that, with the closure-system being allowed, whiskers and boundary points can be not only added and deleted instantaneously, but by taking the reverse-time of those operations, can be added and deleted directly. With such a variety of techniques at our disposal for the manipulation of boundary points and whiskers, we could operate them however was necessary to work with the type of transition the area-intersections called for.

| Name | Four-Intersection Value | Possible Regular Transitions: | Possible with Closure-System |
|---|---|---|---|
| 1. Disjoint | (0, 0, 0, 0) | 2, 3, 4, 9, 10, 11, 12 | 5, 6, 7, 8, 13, 14, 15, 16 |
| 2. | (0, 0, 0, 1) | 1, 4, 6, 9, 10, 12 | 5, 3, 7, 11, 13, 14, 15, 16 |
| 3. | (0, 0, 1, 0) | 1, 4, 7, 9, 11, 12 | 2, 5, 6, 10, 13, 14, 15, 16 |
| 4. | (0, 0, 1, 1) | 1, 2, 3, 6, 7, 8, 9, 10, 11, 12 | 5, 13, 14, 15, 16 |
| 5. Real Plane | (0, 1, 0, 0) | 6, 7, 8, 13, 14, 15, 16 | 1, 2, 3, 4, 9, 10, 11, 12 |
| 6. Contains | (0, 1, 0, 1) | 2, 4, 5, 8, 9, 10, 13, 14, 16 | 1, 3, 7, 11, 12, 15 |
| 7. Inside | (0, 1, 1, 0) | 3, 4, 5, 8, 9, 11, 13, 15, 16 | 1, 3, 6, 10, 12, 14 |
| 8. | (0, 1, 1, 1) | 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16 | 1, 2, 3 |
| 9. Meet | (1, 0, 0, 0) | 1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16 | 5 |
| 10. | (1, 0, 0, 1) | 1, 2, 4, 6, 8, 9, 12, 14, 16 | 3, 5, 7, 11, 13, 15 |
| 11. | (1, 0, 1, 0) | 1, 3, 4, 7, 8, 9, 12, 15, 16 | 2, 5, 6, 10, 13, 14 |
| 12. | (1, 0, 1, 1) | 1, 2, 3, 4, 8, 9, 10, 11, 16 | 5, 6, 7, 13, 14, 15 |
| 13. Equal | (1, 1, 0, 0) | 5, 6, 7, 8, 9, 14, 15, 16 | 1, 2, 3, 4, 10, 11, 12 |
| 14. Covers | (1, 1, 0, 1) | 5, 6, 8, 9, 10, 13, 15, 16 | 1, 2, 3, 4, 7, 11, 12 |
| 15. CoveredBy | (1, 1, 1, 0) | 5, 8, 9, 11, 13, 14, 16 | 1, 2, 3, 4, 6, 10, 12 |
| 16. Overlaps | (1, 1, 1, 1) | 5, 6, 7, 9, 10, 11, 12, 13, 14, 15 | 1, 2, 3, 4 |

This table summarizes the main results from my project. Starting with Four-Intersection Value form 1, I had to check if it could transition to 2, 3 … 16. I then checked if Four-Intersection Value form 2 could transition to 3, 4 … 16. This gave 15 + 14 + 13 + 12 … + 1 = 120 possibilities to check and justify. Developing these 120 examples provided insights about what kinds of morphing could take place, and summarizing the results in the above table organized the information I found.

Our main findings can be summarized by saying that the closure system allows all transitions. It also allows discontinuous area functions; by adding infinitely many components simultaneously, a direct, discontinuous jump from no area to area > 1 is possible. Even a single closure-system operation, though it changes area continuously, will not necessarily change arc-length of the boundary continuously. While the closure system initially seemed reasonable, this consequence seems unnatural, as the continuous morphing of area would intuitively have continuous area functions.

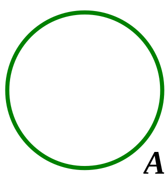
This suggests refinements of the models and methods we wish to allow — should we allow the closure system, but only in cases that do not cause the area function to be discontinuous? Or should we entirely disallow the closure system? How do we quantify continuous morphing of subsets of the plane? — and provides the launching point for further studies and projects.

## CHAPTER 4

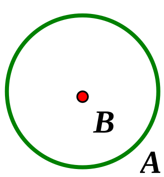
## ILLUSTRATING EXAMPLES

In this section, I show explanations and illustrations of some of the transitions. Due to time and page constraints, I have limited the number of transitions being displayed to fifty of the one-hundred twenty total transitions, in order to provide a variety of examples. This is enough to demonstrate well the techniques required.

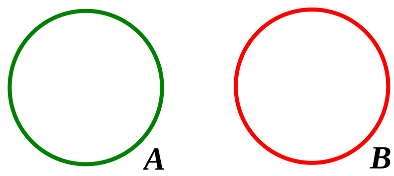
1 (Disjoint) to 2: Create a boundary point of B inside A.




to




1 (Disjoint) to 4: Create a boundary point of A inside B while simultaneously creating a boundary point of B inside A.

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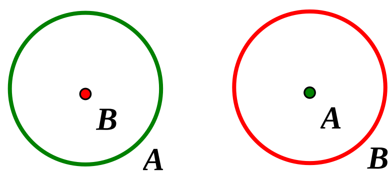
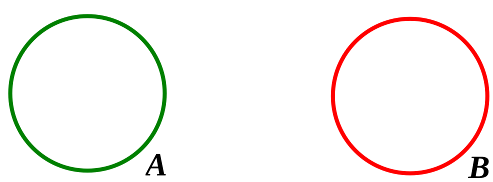
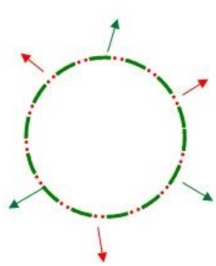


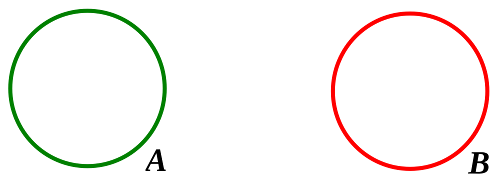

1 (Disjoint) to 5 (Real Plane): Create infinitely many equal components of A and B simultaneously (we can create interior-with-boundary directly using the closure system). Note that this is a transition with discontinuous area change.



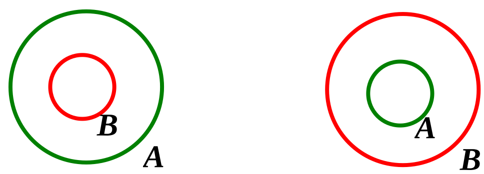

to



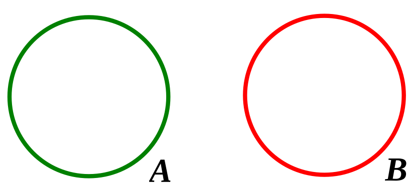
1 (Disjoint) to 8: Create a connected component of A in B and a connected component of B in A (we can create interior-with-boundary directly using the closure system).



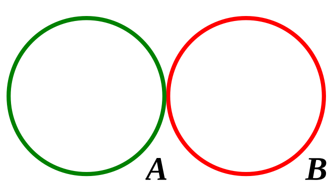

to



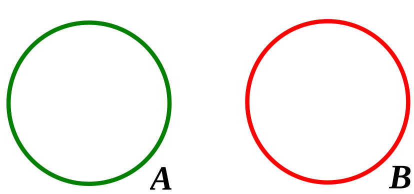

1 (Disjoint) to 9 (Meets): Translate regions A and B towards each other until the boundaries intersect.




to




1 (Disjoint) to 12: Create a whisker of B that intersects the boundary of A and the interior of A while simultaneously creating a whisker of A that intersects the boundary of B and the interior of B.



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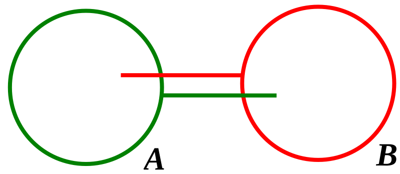
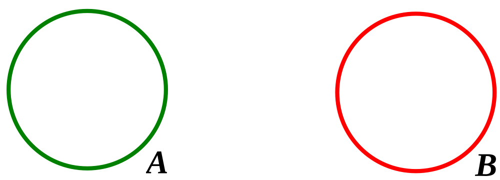



1 (Disjoint) to 13 (Equals): Create a connected component of A and a connected component of B that are equal (we can create interior with boundary directly using the closure system).

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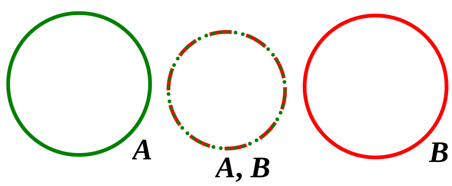
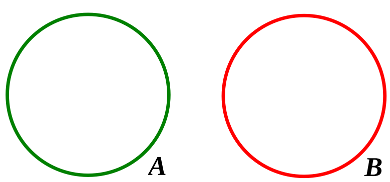



1 (Disjoint) to 14 (Covers): Using the closure system, directly create an area-with-boundary component of B whose boundary intersects both the interior of A and the boundary of A, and whose interior intersects the interior of A.

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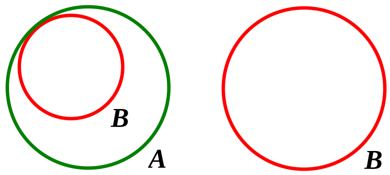
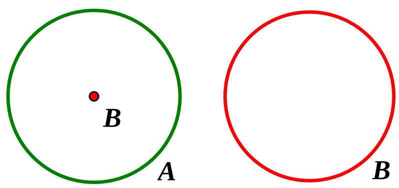



2 to 3: Delete the boundary point of B in A interior at the same instant as we create a boundary point of A in B interior.

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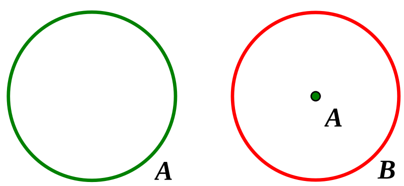
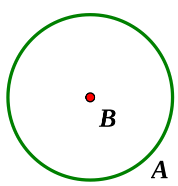



2 to 6 (Contains): Create area of B directly from the boundary point of B.

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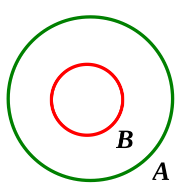
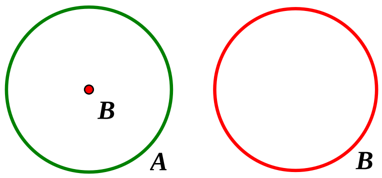



2 to 7 (Inside): Using the closure system, directly delete the boundary point of B in A at the same time as creating an area-with-boundary component of A inside the region of B,

such that both the boundary and the interior of the component of A intersect the interior of B.

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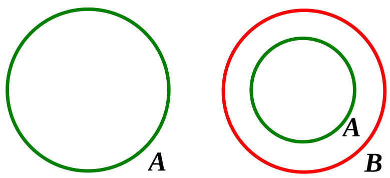
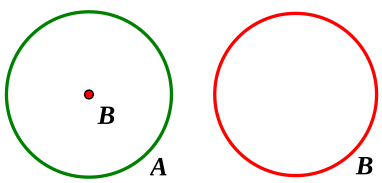



2 to 8: Create area-with-boundary of A directly in the interior of B.

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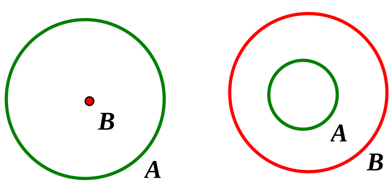
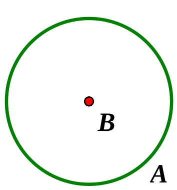



2 to 9 (Meets): Translate the boundary point of B until it intersects the boundary of A.

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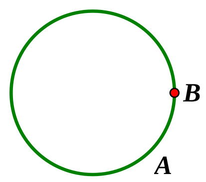
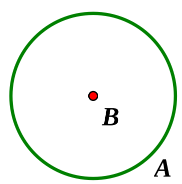



2 to 10: Add a boundary point of B on the boundary of A.

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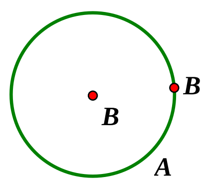
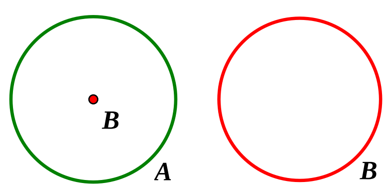



2 to 11: At the same instant that we delete the boundary point of B in the interior of A, add a whisker of A that intersects B boundary and B interior.

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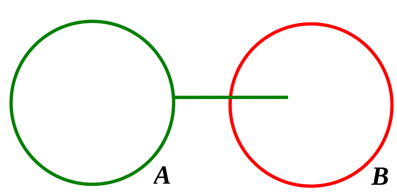
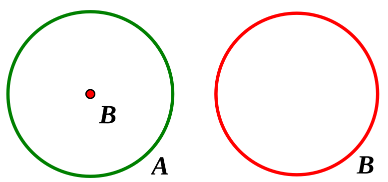


2 to 12: Create a whisker of B in A (such that it intersects both A interior and the boundary of A) at the same instant that we create a boundary point of A in the interior of B.

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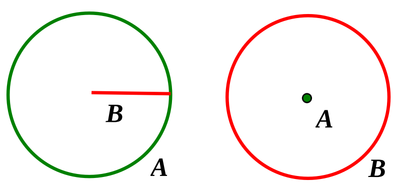
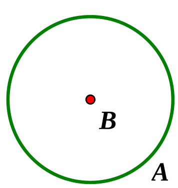



2 to 13 (Equals): Directly delete the boundary point (using reverse-time of instantly adding a boundary point) and simultaneously directly create an area-with-boundary component of B that is equal to the component of region of A.

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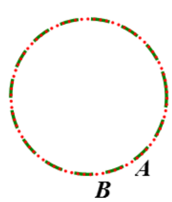
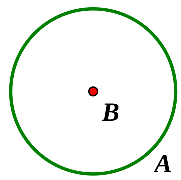



2 to 14 (Covers): Using the closure system, directly create an area-with-boundary component of B such that its interior and boundary intersect the interior of A, and its

boundary also intersects the boundary of A. Simultaneously delete the boundary point of B (using reverse-time of instantly adding a boundary point).

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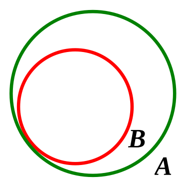
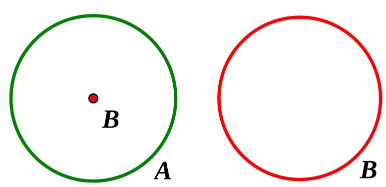



2 going to 15 (CoveredBy): Directly delete the boundary point of B in A (using reverse-time of instantly adding a boundary point). At the same time, directly create an area-with-boundary component of A in B, such that the boundary of the component of A intersected both the boundary and interior of B, and the interior of the component of A intersected the interior of B.

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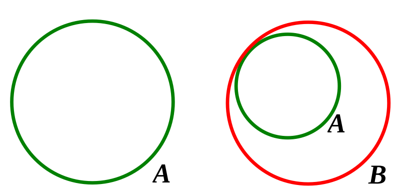
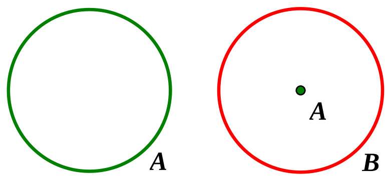



3 to 4: Create a boundary point of B in the interior of A.

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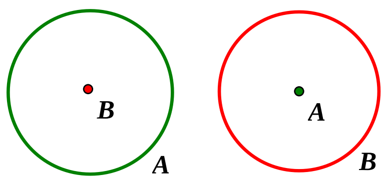
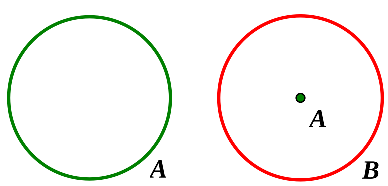



3 going to 6 (Contains): Directly delete the boundary point of A in B (using reverse-time of instantly adding a boundary point) and simultaneously directly create an area-with-boundary component of B such that both the interior and the boundary of the component of B intersect the interior of A.

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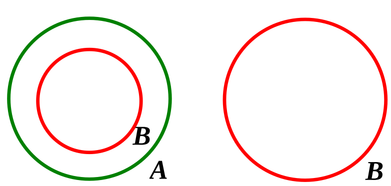
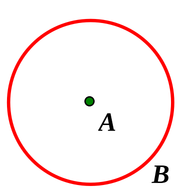



3 to 7 (Inside): Directly create area from the boundary point of A.



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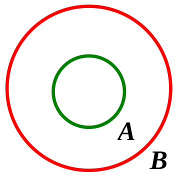
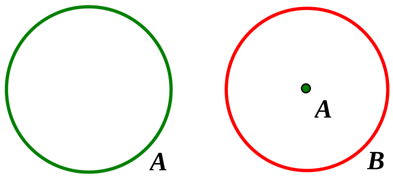



3 to 8: Directly create area-with-boundary using the closure system.

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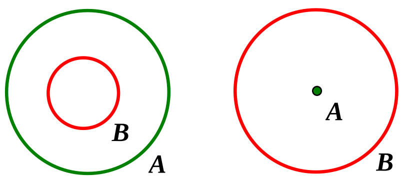
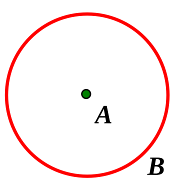



3 to 9 (Meet): Using the closure system, instantaneously delete the boundary point of A in the interior of B. At the same instant, add a boundary point of A on the boundary of B. (Alternatively, translate the point over until it intersects the boundary of B.)

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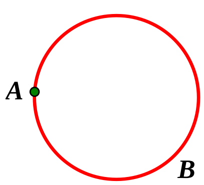
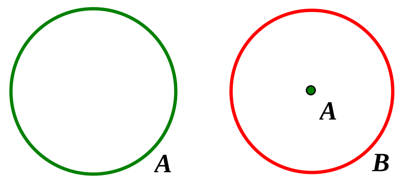


3 to 10: Using the closure system, delete the boundary point of A in the interior B at an instant. At the same instant, add a whisker of B that intersects the boundary of A and the interior of A.

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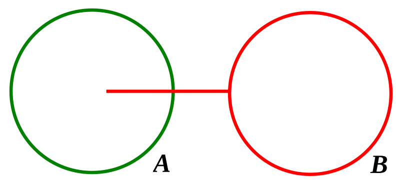
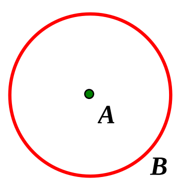



3 to 11: Instantaneously add a boundary point of A on the boundary of B.

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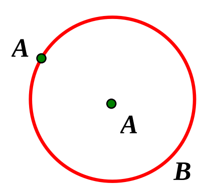
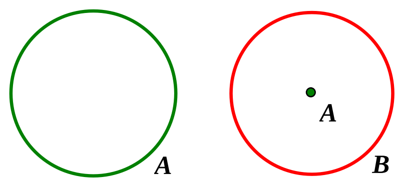



3 to 12: Instantaneously add a whisker of B to a component of A such that the whisker of B intersects both the interior of A and the boundary of A.

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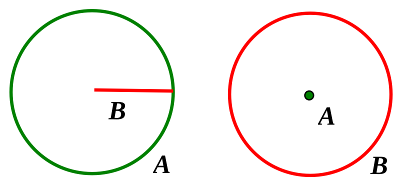
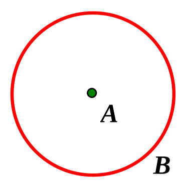



3 to 13 (Equal): Directly delete the boundary point of A (using reverse-time of instantly adding a boundary point) and simultaneously directly create an area-with-boundary component of A that is equal to the region of B.

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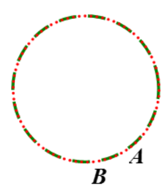
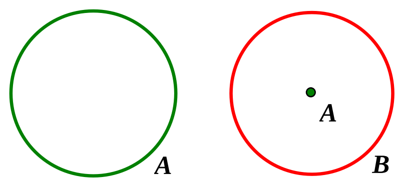



3 going to 14 (Covers): Directly delete the boundary point of A in B (using reverse-time of instantly adding a boundary point) and simultaneously directly create an area-with-boundary component of B whose boundary intersects both the interior and the boundary of A, and whose interior intersects the interior of A.

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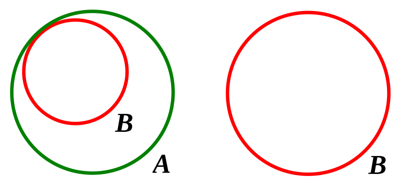
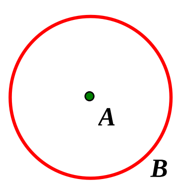



3 to 15 (CoveredBy): Directly delete the boundary point of A in B (using reverse-time of instantly adding a boundary point) and simultaneously directly create an area-with-boundary component of A whose boundary intersects both the interior and the boundary of B, and whose interior intersects the interior of B.

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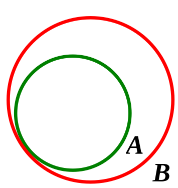
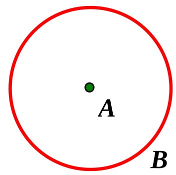



3 to 16 (Overlaps): Directly create an area-with-boundary component of A whose boundary intersects both B interior and the boundary of B, and whose interior intersects both the interior of B and the boundary of B.

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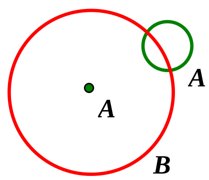
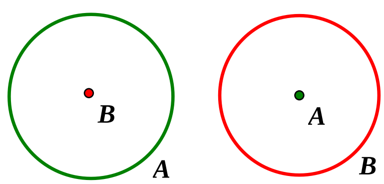



4 going to 6: Directly create area of B by expanding the boundary point of B in the interior of A to be a regularly closed, connected component of B that is still contained in the interior of A.  At the same time, directly delete the boundary point of A in B.

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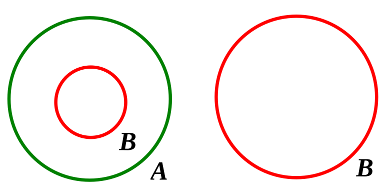
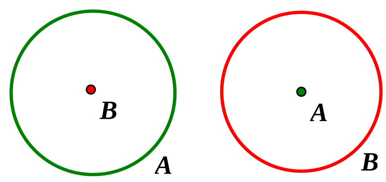
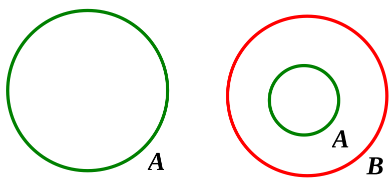


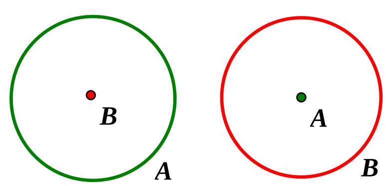

4 going to 7: Directly create area of A by expanding the boundary point of A in the interior of B to be a regularly closed, connected component of A that is still contained in the interior of B.  At the same time, directly delete the boundary point of B in A.




to




4 going to 8: Directly create area of B by expanding the boundary point of B in the interior of A to be a regularly closed, connected component of B that is still contained in the interior of A.

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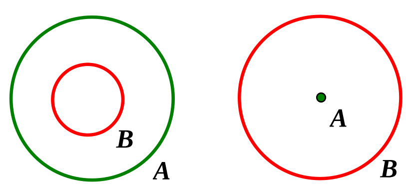
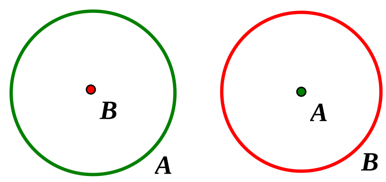



4 going to 9: Using the closure system, instantaneously delete the boundary points of A in B and of B in A. At the same instant, create a boundary point of A that is on the boundary of B.

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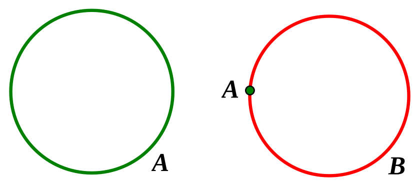
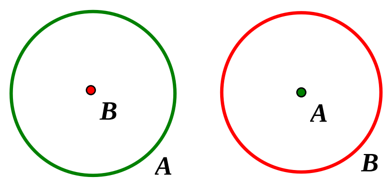



4 going to 10: Using the closure system, instantaneously delete the boundary points of A in B and of B in A. At the same instant, create a whisker of B that intersects both the interior of A and the boundary of A.

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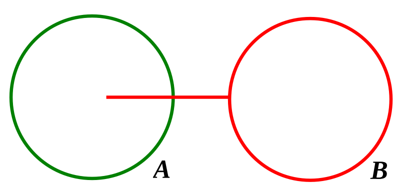
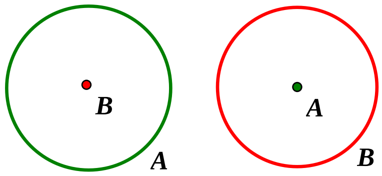



4 going to 11: Using the closure system, instantaneously delete the boundary point of B from the interior of A. At the same instant, create a boundary point of A on the boundary of B.

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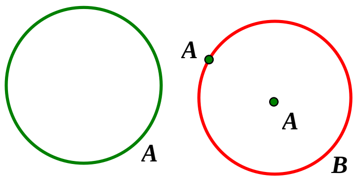
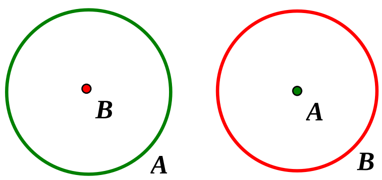



4 going to 12: Instantaneously add a whisker of B that intersects the boundary of A.

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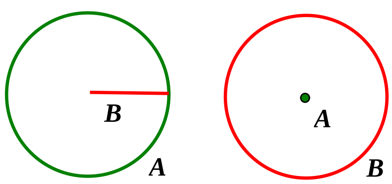
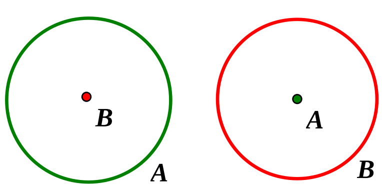



4 going to 13: Directly delete the boundary point of A in B and the boundary point of B in A (using reverse-time of instantly adding a boundary point). At the same time, directly create area-with-boundary components of A and B that are equal.

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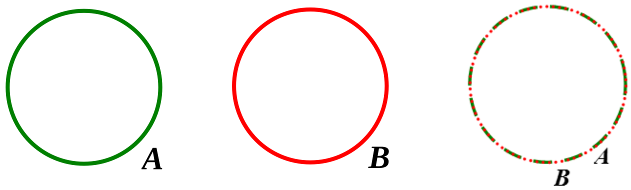
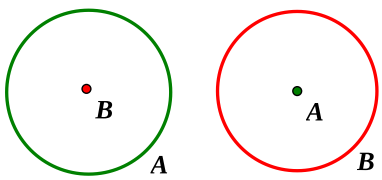
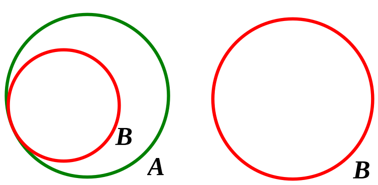


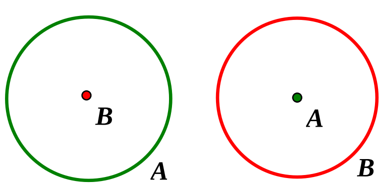

4 going to 14: Directly delete the boundary point of A in B and the boundary point of B in A (using reverse-time of instantly adding a boundary point). At the same time, directly create an area-with-boundary component of B such that its boundary intersects both the boundary of A and the interior of A, and its interior intersects only the interior of A.



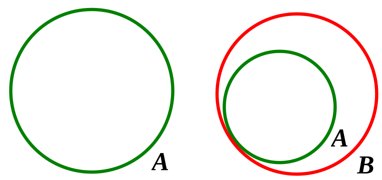

to



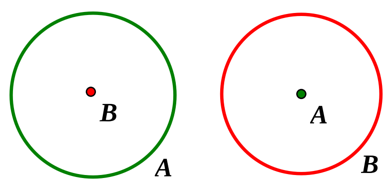

4 going to 15: Directly delete the boundary point of A in B and the boundary point of B in A (using reverse-time of instantly adding a boundary point). At the same time, directly create an area-with-boundary component of A such that its boundary intersects both the boundary of B and the interior of B, and its interior intersects only the interior of B.



to




4 going to 16: Using the closure system, directly add an area-with-boundary component of B such that both its boundary and its interior intersect both the boundary and the interior of A.

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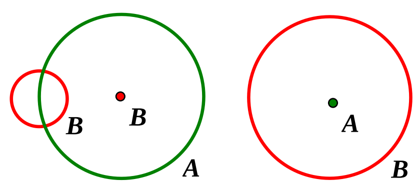
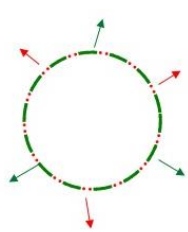
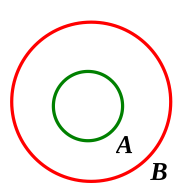


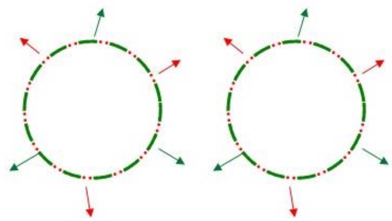

5 going to 7: scale both regions down in size, but scale region A at a higher rate.



to



5 going to 8: In both pairs of regions, scale both regions down in size, but in the first pair, scale the component of B down at a higher rate, and in the second pair, scale the component of A at a higher rate.



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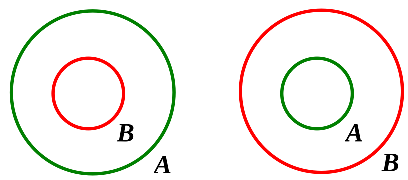
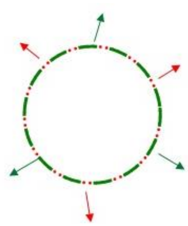



5 going to 13: scale both regions down at the same rate.



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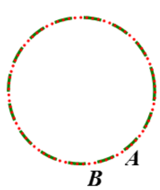
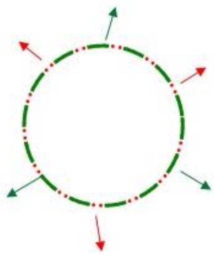



5 going to 16: simultaneously scale both regions down while translating in opposite directions.



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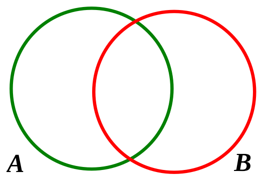
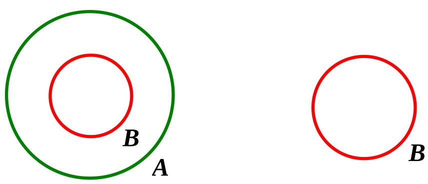
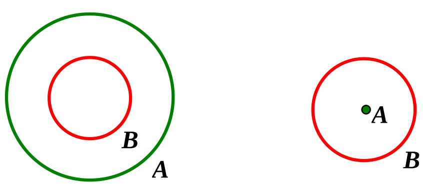


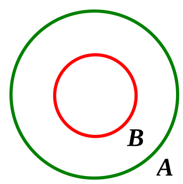

6 (Contains) going to 8: Instantly create a boundary point of A intersecting the interior of B.



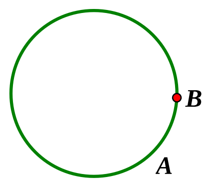

to



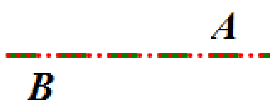

6 (Contains) going to 9: At the same instant, delete the area-with-boundary component of B in A (using the closure system) and create a boundary point of B on the boundary of A.




to




9 (Meets) going to 13 (Equals): Using the usual system, create area directly from the intersecting boundary of A and B, using the same formula $([0, 1]\ x\ [0, t])$ for the area of A and for the area of B. This gives an equal area and boundary.

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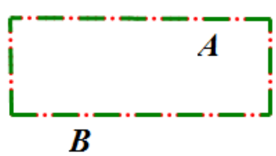
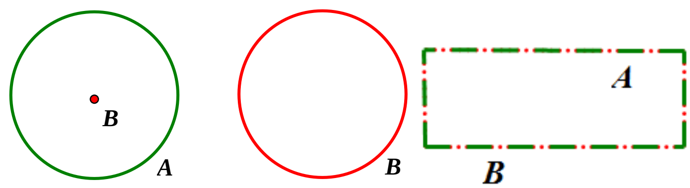



14 (Covers) going to 15 (CoveredBy): Begin with two connected, area-with-boundary components of A and two connected, area-with-boundary components of B, such that one of the components of A is equal in size, shape, and location to one of the components of B, and with a boundary point of B contained in the interior of the other component of A. Delete the boundary point of B (using the closure system) at the same instant as creating a boundary point of A that intersects the interior of B.

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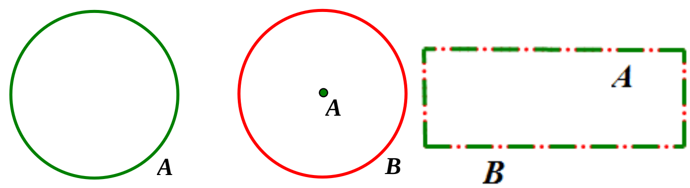

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