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Development of Nabla Fractional Calculus and a New Approach to Data Fitting in Time Dependent Cancer Therapeutic Study

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DEVELOPMENT OF NABLA FRACTIONAL CALCULUS AND A NEW APPROACH TO DATA FITTING IN TIME DEPENDENT CANCER THERAPEUTIC STUDY

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DEVELOPMENT OF NABLA FRACTIONAL CALCULUS AND A NEW APPROACH TO DATA FITTING IN TIME DEPENDENT CANCER THERAPEUTIC STUDY

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CONTENTS

DEVELOPMENT OF NABLA FRACTIONAL CALCULUS AND A NEW APPROACH TO DATA FITTING IN TIME DEPENDENT CANCER THERAPEUTIC STUDY

Nihan Acar May 2012 62 Pages

Directed by: Dr. Ferhan Atici, Dr. Ngoc Nguyen, Dr. Nancy Rice

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The aim of this thesis is to develop discrete fractional models of tumor growth for a given data and to estimate parameters of these models in order to have better data fitting. We use discrete nabla fractional calculus because we believe the discrete counterpart of this mathematical theory will give us a better and more accurate outcome.

This thesis consists of five chapters. In the first chapter, we give the history of the fractional calculus, and we present some basic definitions and properties that are used in this theory. We define nabla fractional exponential and then nabla fractional trigonometric functions. In the second chapter, we concentrate on completely monotonic functions on R, and we introduce completely monotonic functions on discrete domain. The third chapter presents discrete Laplace $\mathcal{N}$-transform table which is a great tool to find solutions of α-th order nabla fractional difference equations. Furthermore, we find the solution of nonhomogeneous up to first order nabla fractional difference equation using $\mathcal{N}$-transform. In the fourth chapter, first we give the definition of Casoration for the set of solutions up to n-th order nabla fractional equation. Then, we state and prove some basic theorems about linear independence of the set of solutions. We focus on the solutions of up to second order nabla fractional difference equation. We examine these solutions case by case namely, for the real and distinct characteristic roots, real and same, and complex ones. The fifth chapter emphasizes the aim of this thesis. First, we give a

brief introduction to parameter estimation with Gomperts and Logistic curves. In addition, we recall a statistical method called cross-validation for prediction. We state continuous, discrete, continuous fractional and discrete fractional forms of Gompertz and Logistic curves. We use the tumor growth data for twenty-eight mice for the comparison. These control mice were inoculated with tumors but did not receive any succeeding treatment. We claim that the discrete fractional type of sigmoidal curves have the best data fitting results when they are compared to the other types of models.

Chapter 1

INTRODUCTION AND PRELIMINARIES

In the last few decades, fractional calculus has become a research area where we see its applications in bioscience, engineering and applied mathematics [**30, 31, 32, 33, 34, 35**]. Fractional calculus is a branch of mathematical analysis that allows integrals and derivatives to have any positive real order. Discrete fractional calculus is the discrete version of fractional calculus which concerns any positive real order of sum and difference. For instance, one can calculate the $1/3 - th$ order difference or $\sqrt{5} - th$ sum of a function.

In Section 1.1, we present the historical background of fractional calculus. Then, in Section 1.2, we consider some important special functions such as Gamma function and Mittag-Leffler function. In addition, we introduce nabla fractional exponential and trigonometric functions.

1.1. Historical Background of Discrete Fractional Calculus

The idea of Fractional Calculus goes back to years when Marquis de L'Hospital $(1661 - 1704)$ and Gottfried Wilhelm Leibniz $(1646 - 1716)$ exchanged ideas through letters about the notations and basics of calculus. In L'Hospital's note, he was wondering of Leibniz's notation $d^n y/dx^n$ for the derivative of integer order $n > 0$ when $n = 1/2$. In Leibniz's reply, dated 30 September 1695, he wrote to L'Hospital as follows: "This is an apparent paradox from which, one day, useful consequences will be drawn." Thus, fractional calculus was born. In the following years, some famous mathematicians, such as Euler, Lagrange, Lacroix, Fourier, Liouville and Riemann, developed the theory of fractional calculus. In fact, in his 700− page textbook, S. F. Lacroix devoted two pages to fractional calculus, showing eventually that

$$\frac{d^{1\backslash 2}x}{dx^{1\backslash 2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

This important result is the same as Riemann-Liouville definition of fractional derivative. Furthermore, differences of fractional order were initially defined by Kuttner in 1957 [**36**]. The first work, devoted exclusively to the subject of fractional calculus, is the book published by Oldham and Spanier in 1974 [**29**]. Currently, the mathematicians, [**3, 6, 23, 24, 25**] have made many developments in the theory of fractional and discrete fractional calculus.

1.2. Special Functions of Fractional Calculus

In this section, we concentrate on some fundamental special functions which are quite important in the study of the theory of fractional calculus. First, we recall Gamma function and some basic properties of this function.

1.2.1. Gamma Function. Euler's Gamma function $\Gamma(x)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values, is one of the basic functions of the discrete fractional calculus. The Gamma function is defined by the integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \qquad x \in \mathbb{R}^+.$$

The Gamma function satisfies the following functional difference equation

$$\Gamma(x+1) = x\Gamma(x).$$

This can be proved by using integration by parts, so we have

$$\Gamma(x+1) = \int_0^\infty e^{-t}t^{x}dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-t} t^x dt$$

$$= \lim_{b\to\infty} [e^{-t} x]_0^b + \lim_{b\to\infty} \left[x \int_0^b e^{-t} t^{x-1} dt \right]$$

$$= x \int_0^\infty e^{-t} t^{x-1} dt$$

$$= x\Gamma(x).$$

In addition, for any natural number n, we have the following property

$$\Gamma(n) = (n-1)!.$$

Figure 1.2.1, shows the graph of $\Gamma(x)$, for the real values of x.

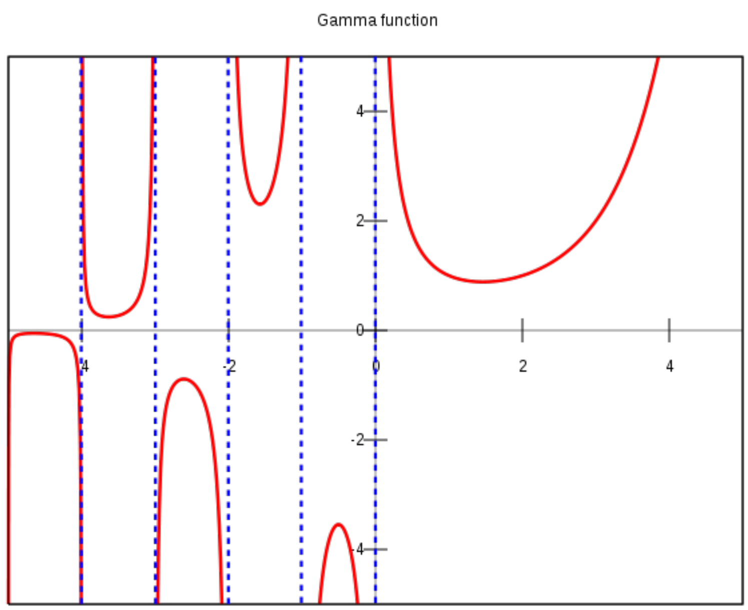

FIGURE 1.2.1. Gamma Function

For further reading about Gamma function, we refer to a book by Igor Podlubny [**26**].

1.2.2. Mittag-Leffler Function. The Mittag-Leffler function, which plays a significant role in the solutions of non-integer order differential equations, was first introduced by Gösta Mittag-Leffler in [**27**]. The Mittag-Leffler functions with one and two-parameters are defined by the series expansion as the following form

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)},$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)},$$

where α, β are positive real numbers. The two parameter function of Mittag-Leffler type was initially defined by Ravi P. Agarwal in 1953 [**28**].

Note that, for $\alpha = 1$ and $\beta = 1$, we obtain exponential function given as the following form

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

Therefore, it can be concluded that the Mittag-Leffler function is the generalization of the exponential function e^x.

In the literature, the discrete Mittag-Leffler functions with one and two parameters were defined as

$$F_{\alpha}(at) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k}}}{\Gamma(\alpha k + 1)},$$

$$F_{\alpha,\beta}(at) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k}}}{\Gamma(\alpha k + \beta)},$$

where α, β are positive real numbers and $|a| < 1$. In addition, for any real number ν, the discrete Mittag-Leffler is defined in [**3**] in the following way

$$F_{\alpha,\beta}(at^{\overline{\nu}}) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k\nu}}}{\Gamma(\alpha k + \beta)}.$$

1.3. Falling and Rising Factorials

The falling and rising factorial powers are the basic notions used in the theory of fractional calculus. Whereas, falling factorial is defined in delta (forward) fractional calculus, rising factorial is used in nabla (backward) fractional calculus. In our study, we are interested in using nabla fractional calculus, therefore we frequently see the notation of rising factorial power.

1.3.1. Falling Factorial. The falling factorial power $t^{\underline{r}}$ (read 'to the r falling') is defined as

$$t^{\underline{r}} = t(t-1)(t-2)\cdots(t-r+1) = \prod_{k=0}^{n-1}(t-k) = \frac{\Gamma(t+1)}{\Gamma(t+1-r)}, \qquad r \in \mathbb{N}.$$

The properties of the falling factorial (factorial polynomial) can be found in [**16**].

1.3.2. Rising Factorial. The rising factorial power $t^{\overline{r}}$ (read 'to the r rising') is defined in [**4**] as

$$t^{\overline{r}} = t(t+1)(t+2)\cdots(t+r-1) = \prod_{k=0}^{n-1}(t+k), \qquad r \in \mathbb{N}.$$

and $t^{\overline{0}} = 1$. This function is also known as the Pochhammer symbol in the theory of special functions.

Let α be any number. Then, $t^{\overline{\alpha}}$ is defined as

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},$$

where $t \in \mathbb{R}\setminus\{\cdots,-2,-1,0\}$, and $0^{\overline{\alpha}} = 0$.

Next, we recall some basic properties of the rising factorial power function. For further reading, we refer to the readers [**3**].

Lemma 1.3.1. $(i)\nabla t^{\overline{\alpha}} = \alpha t^{\overline{\alpha-1}}$.

$(ii) t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.

$(iii)\nabla_0^{-\alpha}(t+1)^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t+1)^{\overline{\alpha+\beta}}$. $(Power Rule)$

After seeing all the graphs and approximations, one can conclude that the closer to the continuous case is, the better job it does. In Figures (1.3.1), (1.3.2), (1.3.3), (1.3.4) we consider $t^{\alpha}, t^{\overline{\alpha}}, t^{\underline{\alpha}}$ for different values of α $(\alpha = 1, \alpha = .98, \alpha = .96, \alpha = .94)$.

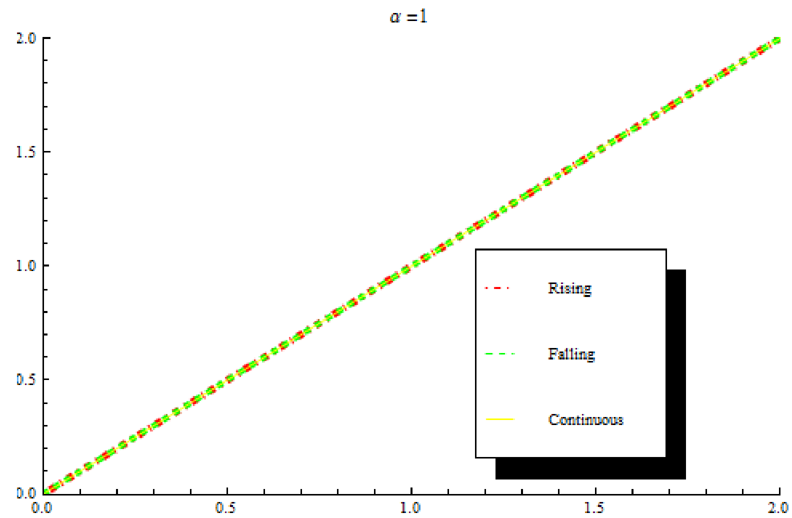

FIGURE 1.3.1. $t^{\alpha}, t^{\overline{\alpha}}, t^{\underline{\alpha}}$ for α=1

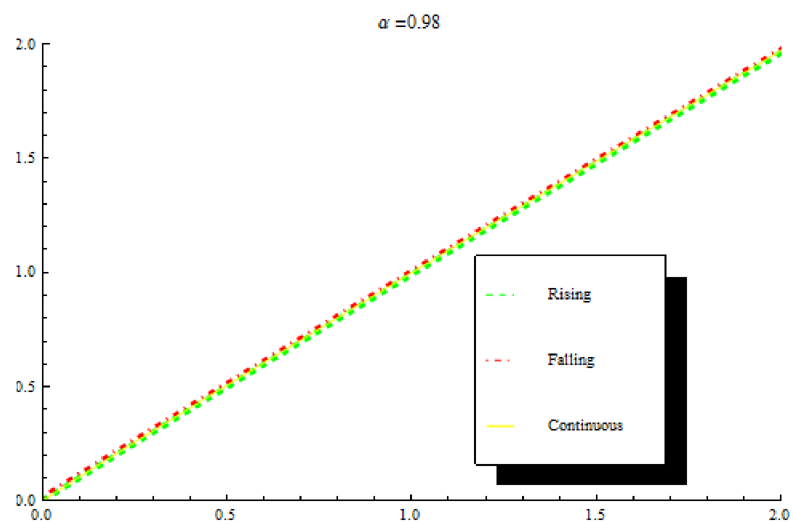

FIGURE 1.3.2. $t^{\alpha}, t^{\overline{\alpha}}, t^{\underline{\alpha}}$ for α=0.98

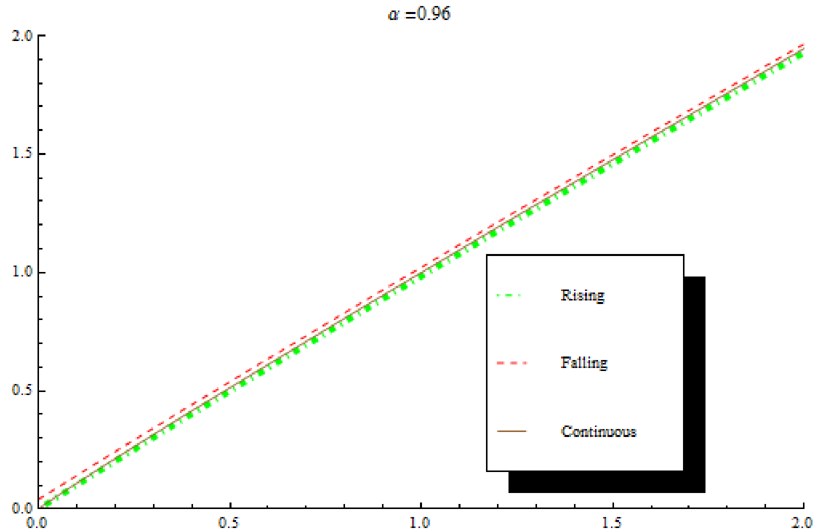

FIGURE 1.3.3. $t^{\alpha}, t^{\overline{\alpha}}, t^{\underline{\alpha}}$ for α=0.96

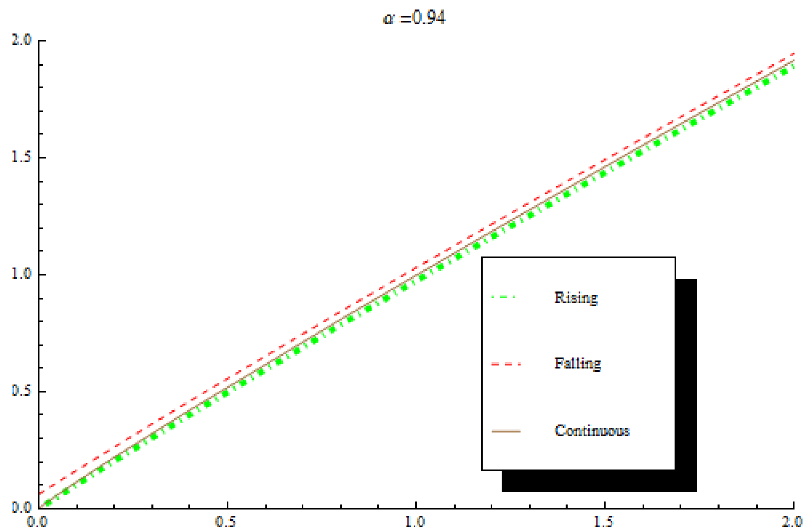

FIGURE 1.3.4. $t^{\alpha}, t^{\overline{\alpha}}, t^{\underline{\alpha}}$ for α=0.94

1.4. The Fractional Sum and Difference Operators

In this section, we recall definition of the fractional sum of a function f an arbitrary order $\alpha > 0$, denoted by $\nabla_a^{-\alpha} f$, starting from a. In addition, $\nabla_a^{\alpha} f$ will denote the fractional difference of a function f. First, we consider the $\alpha-$fractional sum of a function f.

Definition 1.4.1. *Let a be any real number and α be any positive real number. The $\alpha-th$ order fractional sum of f is defined* [**6**] *as*

$$\nabla_a^{-\alpha} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(s), \tag{1.1}$$

where $t = a+1, a+2, \ldots$ and $\rho(t) = t-1$ is backward jump operator on the time scale calculus [**15**].

Remark 1.4.2. *Note that for $\alpha = 1$, the equation (1.1) turns into discrete sum operator as given in this form*

$$\nabla_a^{-1} f(t) = \sum_{s=a}^{t} f(s).$$

Next, we proceed to the fractional difference of a function $f(t)$.

Definition 1.4.3. *Let a be any real number and α be any positive real number such that $0 < n-1 < \alpha < n$ where n is an integer. The $\alpha-th$ order fractional difference (a Riemann-Liouville fractional difference) of f is defined* [**6**] *by*

$$\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \nabla^n \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(s),$$

where f is defined on $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$.

Theorem 1.4.4. *(Commutative Property of the Fractional Sum and Difference) For any $\alpha > 0$, the following equality holds:*

$$\nabla_{a+1}^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a),$$

where f is defined on $\mathbb{N}_a$.

This property allows us to interchange the order of sum and difference operators and as you can see above, the result is slightly different as a constant. We refer to the readers [**37**] for the proof of this basic property.

THEOREM 1.4.5. *(Leibniz Rule) For any $\alpha > 0$, $\alpha - th$ order fractional difference of the product fg is given in this form*

$$\overset{t}{\nabla}{}_a^{\alpha} f(t).g(t) = \sum_{n=0}^{t-a} \binom{\alpha}{n} \left[\overset{t-n}{\nabla}{}^{\alpha-n} f(t-n) \right] \left[\ \nabla^n g(t) \right]$$

where a be any real number and f, g are defined on $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. For the proof of Leibniz Rule, we refer to the readers [**5**].

1.5. Nabla Fractional Exponential and Trigonometric Functions

DEFINITION 1.5.1. *For any $\alpha > 0$, nabla exponential function is defined as the following form*

$$\hat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = \sum_{n=0}^{\infty} \frac{a^n (t+1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)},$$

where $|a| < 1$ and $t \geq 0$.

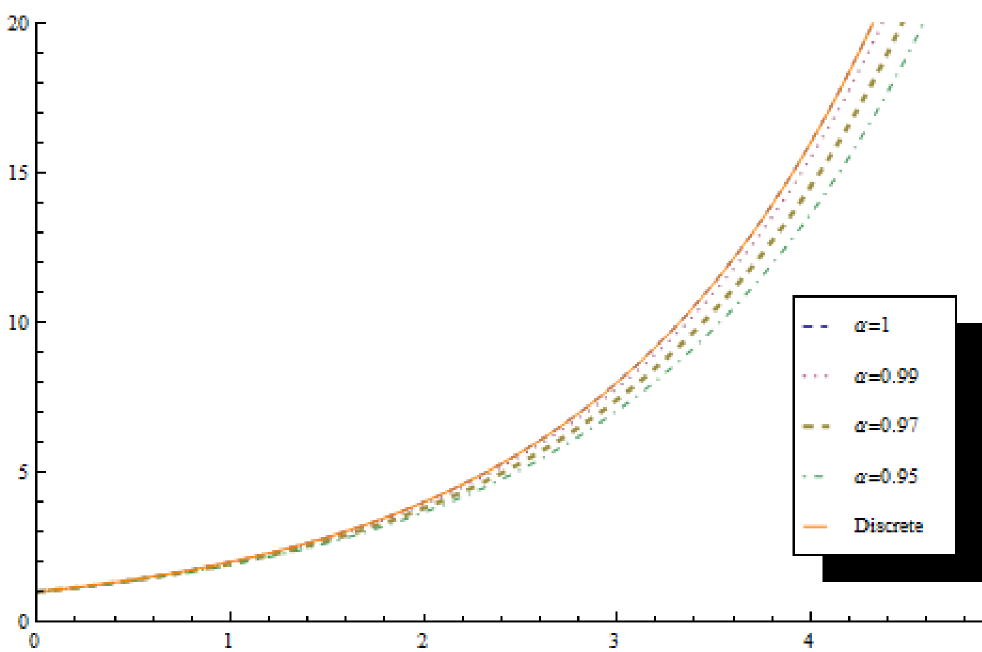

FIGURE 1.5.1. Nabla Exp. Growth Function

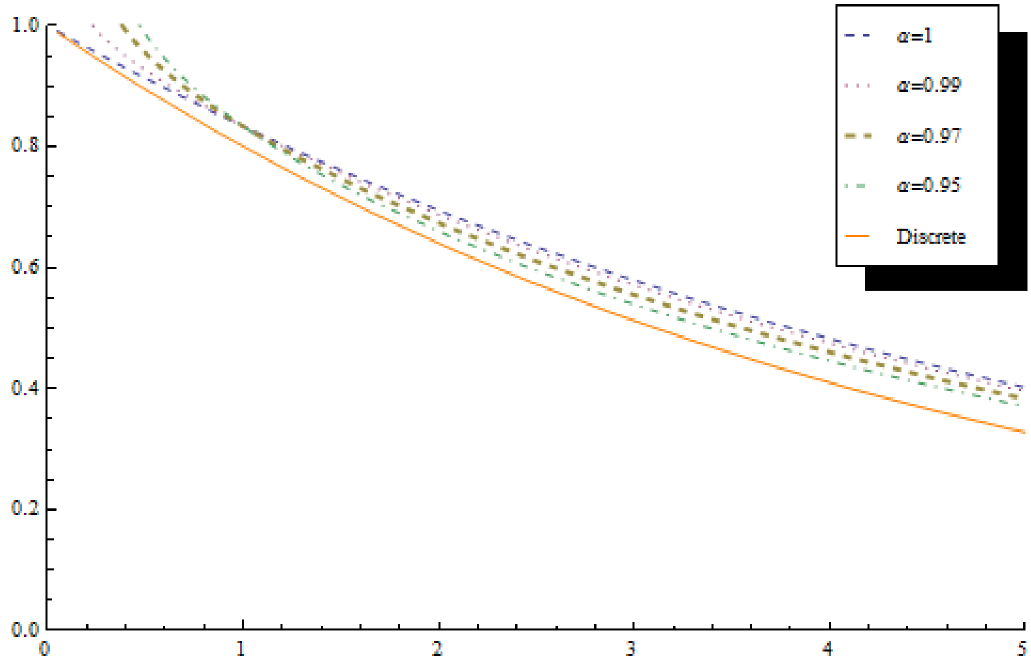

FIGURE 1.5.2. Nabla Exp. Decay Function

We know that trigonometric functions can be represented by the exponential function. In discrete fractional calculus, we employ the same idea to obtain nabla fractional trigonometric functions.

DEFINITION 1.5.2. *(Nabla Fractional Sine Function) For any* $\alpha > 0$, *nabla fractional sine function is given as*

$$\hat{\sin}_{\alpha,\alpha}(a,t) = \frac{\left[\hat{e}_{\alpha,\alpha}(ai, t^{\overline{\alpha}}) - \hat{e}_{\alpha,\alpha}(-ai, t^{\overline{\alpha}})\right]}{2i},$$

where $|a| < 1$ *and* t *is defined on* $\mathbb{N}_1 = \{1, 2, 3, \ldots\}$.

DEFINITION 1.5.3. *(Nabla Fractional Cosine Function) For any* $\alpha > 0$, *nabla fractional cosine function is given as*

$$\hat{\cos}_{\alpha,\alpha}(a,t) = \frac{\left[\hat{e}_{\alpha,\alpha}(ai, t^{\overline{\alpha}}) + \hat{e}_{\alpha,\alpha}(-ai, t^{\overline{\alpha}})\right]}{2},$$

where $|a| < 1$ *and* t *is defined on* $\mathbb{N}_1 = \{1, 2, 3, \ldots\}$.

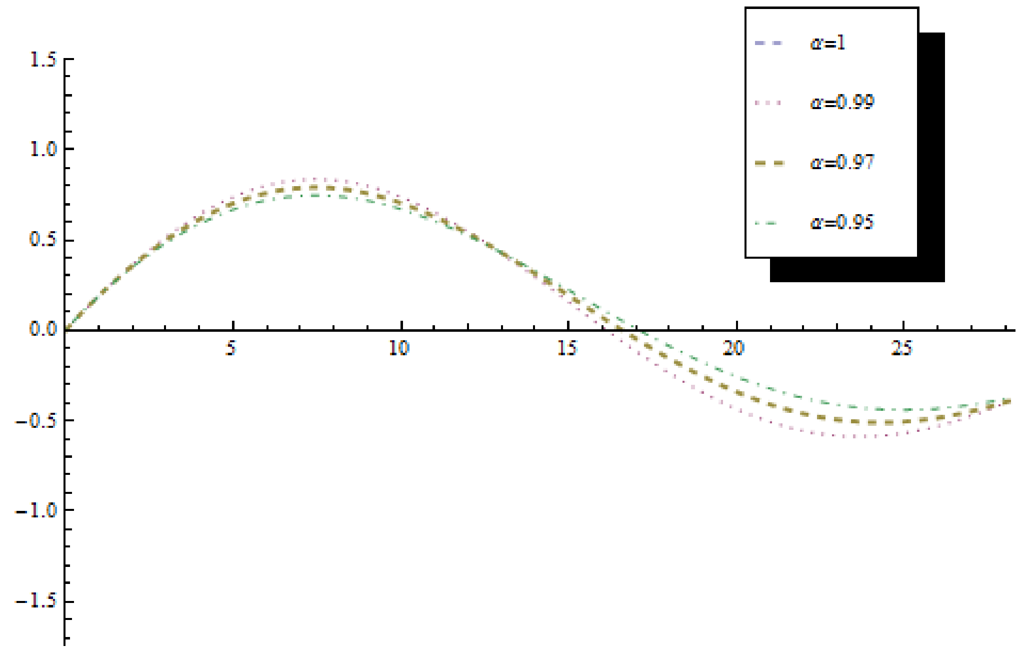

Figure 1.5.3. Fractional Sine Function

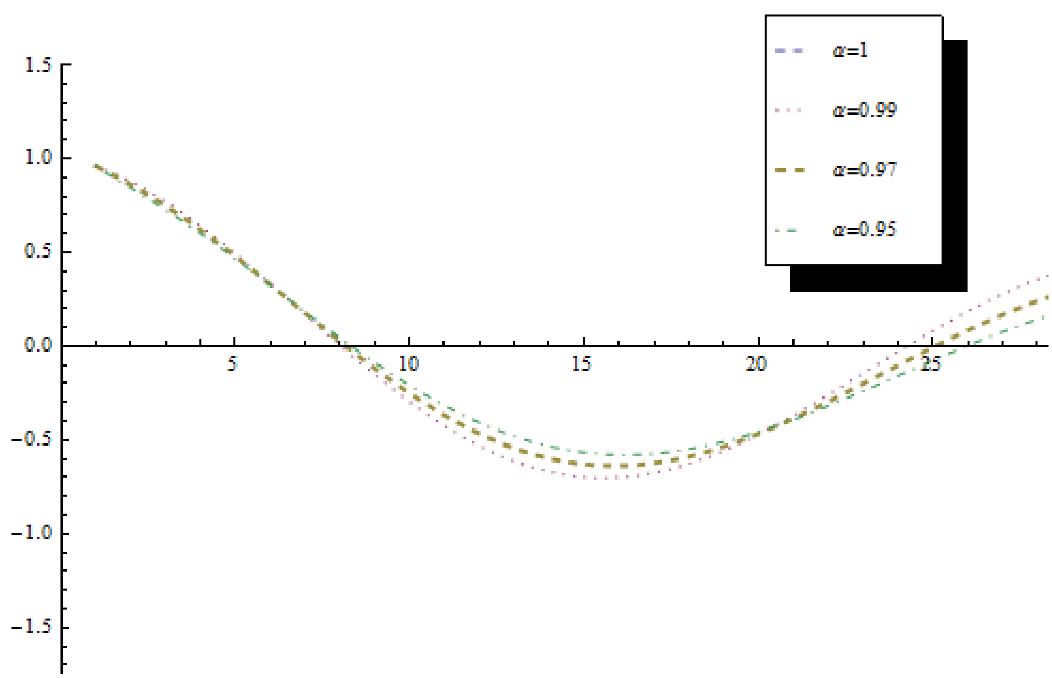

Figure 1.5.4. Fractional Cosine Function

Since we use the Putzer algorithm in Chapter 4, we recall this theorem and for further reading, we refer to the readers [**3**].

Theorem 1.5.4. *(Putzer Algorithm) Let A be a 2×2 matrix. If λ_1, λ_2 are the eigenvalues of A, then*

$$\Phi(A,t) = M_0 p_1(t) + M_1 p_2(t)$$

where $p_1(t)$ and $p_2(t)$ are chosen to satisfy the following system:

$$\begin{bmatrix} \nabla_a^\alpha p_1(t) \\ \nabla_a^\alpha p_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \quad \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and M_0, M_1 are defined by

$$M_0 = I,$$

$$M_1 = (A - \lambda_1 I)\, M_0.$$

Chapter 2
NABLA COMPLETELY MONOTONIC FUNCTIONS

Completely monotonic functions play an important role in a variety of branches of mathematics such as potential theory [**7**], probability theory [**8, 9**], physics [**10**], numerical and asymptotic analysis [**11**]. The theory on completely monotonic functions whose all order derivative exist was first given by Felix Hausdorff in 1921 [**1**]. Such a concept helps us to understand the qualitative behavior of a function in the given domain. In our study, we are interested in complete monotonicity of functions on a discrete domain in order to analyze the discrete Mittag-Lefler function.

In this chapter, we first consider nabla operator, also known as backwards difference operator. Then we give some properties about nabla operator, and we recall some theorems such as fundamental theorem of nabla calculus. We give the definition of completely monotonic functions on $\mathbb{R}$ and then we introduce completely monotonic functions on $\mathbb{Z}$ with nabla operator. In this study, we entitle such functions as "nabla completely monotonic functions". Then we state and prove some theorems about nabla completely monotonic functions.

2.1. Definition of Nabla Operator and Some Properties

In this section, we summarize basic definitions and notations from the nabla difference calculus.

Definition 2.1.1. *The backward difference operator, or nabla operator* (∇), *for a function* $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ *is defined by*

$$(\nabla f)(t) = f(t) - f(\rho(t)) = f(t) - f(t-1),$$

where $\mathbb{N}_a = \{a,\ a+1,\ a+2, \ldots\}$ *and* $\rho(t) = t - 1$*, known as backward jump operator on time scale calculus* [**15**].

Definition 2.1.2. *The definite nabla sum of* $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ *is given by*

$$\int_c^d f(t)\nabla t = \begin{cases} \sum\limits_{t=c+1}^{d} f(t), & \text{if } c < d \\ 0, & \text{if } c = d \\ -\sum\limits_{t=c+1}^{d} f(t), & \text{if } d < c \end{cases} \quad \text{where } c,\ d \in \mathbb{N}_a.$$

Next, we state the fundamental theorem of nabla calculus.

Theorem 2.1.3. *Let* $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ *and* F *be an anti-nabla difference of* f *on* $\mathbb{N}_a$ *, that is* $\nabla F(t) = f(t)$ *for* $t \in \mathbb{N}_{a+1}$*, then for any* $c,\ d \in \mathbb{N}_a$*, we have*

$$\int_c^d f(t)\nabla t = F(d) - F(c).$$

Definition 2.1.4. *The nabla product of two functions* $u, v : \mathbb{N}_a \longrightarrow \mathbb{R}$ *and* $t \in \mathbb{N}_{a+1}$*, is given by*

$$\nabla(u(t)v(t)) = u(t)\nabla v(t) + v(\rho(t))\nabla u(t).$$

Lemma 2.1.5. *If* $\nabla f(t) \leq 0$*, then* $f(t)$ *is decreasing for all* $t \in \mathbb{N}_{a+1}$.

2.2. Introduction to Completely Monotonic Functions

In this section, we give a brief introduction to completely monotonic functions on $\mathbb{R}$ and then we proceed to complete monotonicity of a real valued function on a discrete domain.

DEFINITION 2.2.1. *A function $f : (0, \infty) \longrightarrow \mathbb{R}$ is said to be completely monotonic, if f has derivatives of all orders and if it satisfies the following condition*

$$(-1)^n f^{(n)}(x) \geq 0 \text{ and } n = 0, 1, 2, 3, \ldots$$

for all $x > 0$.

Many examples and theorems about completely monotonic functions can be found in a paper by Miller and Samko. [**2**].

Now, we define completely monotonic functions on a discrete domain.

DEFINITION 2.2.2. *A function $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ is said to be nabla completely monotonic, for any function f on $\mathbb{N}_a$ (with nabla derivatives of all orders) and if it satisfies for each n=0,1,2,. . .*

$$(-1)^n \nabla^n f(x) \geq 0$$

where $x \in \{n + 1, n + 2, \cdots\}$, and a is a real number.

REMARK 2.2.3. *If a real valued function f on a discrete domain is nabla completely monotonic, then it can be easily seen for $m = 0, 1, 2, \cdots$ and for $x \in \{m + 1, m + 2, \cdots\}$*

$$\nabla^{2m} f(x) \quad \text{and} \quad -\nabla^{2m+1} f(x)$$

are also nabla completely monotonic.

2.3. Theorems on Nabla Completely Monotonic Functions

In this section, we will state and prove some basic theorems for nabla completely monotonic functions to have some ideas about how the concept is related to the stability of fractional difference equations.

THEOREM 2.3.1. *If $f(x)$ and $g(x)$ are nabla completely monotonic real valued functions, then*

i) $af(x) + bg(x)$ is also nabla completely monotonic where a and b are nonnegative constants.

ii) $f(x)g(x)$ is nabla completely monotonic.

PROOF. i) If $f(x)$ and $g(x)$ are nabla completely monotonic functions, then by Definition 2.2.2 we have

$$(-1)^n \nabla^n f(x) \geq 0 \text{ and } (-1)^n \nabla^n g(x) \geq 0$$

for $n = 0, 1, 2, 3, \cdots$ and $x \in \{n+1, n+2, \cdots\}$.

Since a and b are nonnegative constants, the following holds,

$$a(-1)^n \nabla^n f(x) \geq 0 \text{ and } b(-1)^n \nabla^n g(x) \geq 0$$

Therefore we have,

$$a(-1)^n \nabla^n f(x) + b(-1)^n \nabla^n g(x) \geq 0$$

for $n = 0, 1, 2, 3, \cdots$ and $x \in \{n+1, n+2, \cdots\}$.

Thus, $af(x) + bg(x)$ is nabla completely monotonic.

ii) To prove this part, we use Leibniz formula for fractional discrete calculus.

Leibniz rule [**5**] states, If m is a nonnegative integer,

$$\nabla^m f(x)g(x) = \sum_{n=0}^{m} \binom{m}{n} [\ \nabla^{m-n} f(x-n)] [\ \nabla^n g(x)].$$

We need to show that

$$(-1)^m \nabla^m f(x)g(x) \geq 0$$

for $m = 0, 1, 2, 3, \cdots$ and $x \in \{m+1, m+2, \cdots\}$.

By using Leibniz rule we have

$$(-1)^m \nabla^m f(x) g(x) = (-1)^m \sum_{n=0}^{m} \binom{m}{n} [\ \nabla^{m-n} f(x-n)] [\ \nabla^n g(x)].$$

We expand our series and thus we get

$$(-1)^m \left\{ \binom{m}{0} [\ \nabla^m f(x)]\, g(x) + \binom{m}{1} \left[\ \nabla^{m-1} f(x-1)\right] \nabla g(x) + \cdots \right.$$
$$\left. + \binom{m}{m} [\ f(x-m)]\, \nabla^m g(x) \right\}.$$

Let $x \in \{m+1, m+2, \cdots\}$, so the nonnegativity holds and we obtain the following

$$\binom{m}{0} \underbrace{[\ (-1)^m \nabla^m f(x)]}_{\geq 0} g(x) + \binom{m}{1} \underbrace{\left[\ (-1)^{m-1} \nabla^{m-1} f(x-1)\right]}_{\leq 0} \underbrace{(-1) \nabla g(x)}_{\leq 0} + \cdots$$

$$+ \binom{m}{m}\ f(x-m) \underbrace{[(-1)^m\ \nabla^m g(x)]}_{\geq 0}.$$

Also, note that

$$f(x) \geq 0 \text{ and } g(x) \geq 0 \text{ for all } \ x \in \mathbb{N}_a \text{ where } \mathbb{N}_a = \{a,\ a+1, \cdots\}.$$

Thus, we get

$$(-1)^m \nabla^m f(x) g(x) \geq 0 \text{ for } x \in \{m+1, m+2, \cdots\}.$$

This implies that $f(x)g(x)$ is nabla completely monotonic. □

THEOREM 2.3.2. *Let $y = f(x)$ be a nabla completely monotonic function and let the power series*

$$\varphi(y) = \sum_{k=0}^{\infty} a_k y^{\overline{k}}$$

converge for all y in the range of the function $y = f(x)$. *If* $a_k \geq 0$ *for all* $k = 0, 1, 2, \ldots$, *then* $\varphi[f(x)]$ *is nabla completely monotonic.*

PROOF. We need to show that $(-1)^n \nabla^n (\varphi \circ f)(x) \geq 0$ for all $x \in \{n+1, n+2, \ldots\}$. By Definition 2.2.2 we have

$$(-1)^n \nabla^n (\varphi \circ f)(x) = (-1)^n \nabla^n \sum_{k=0}^{\infty} a_k \left[f^{\overline{k}}(x) \right].$$

Since the power series is convergent, we can take the n^{th} order nabla difference operator inside the sum notation. So, we obtain

$$\sum_{k=0}^{\infty} a_k (-1)^n \nabla^n \left[f^{\overline{k}}(x) \right].$$

It is sufficient to prove that $f^{\overline{k}}$ is nabla completely monotonic. We will prove this by Mathematical induction.

For $k = 1$, $f^{\overline{1}}$ is nabla completely monotonic.

For $k = 2$, $f^{\overline{2}} = f(f+1) = f^2 + f$ is nabla completely monotonic using Theorem 2.3.1.

Let's assume that for $k = n$ is true. By Induction assumption, $f^{\overline{n}}$ is nabla completely monotonic.

For $k = n+1$, we will use the Lemma 1.3.1 and by the Induction assumption we have

$$f^{\overline{n+1}} = f^{\overline{n}} (f+n)^{\overline{1}} \text{ is nabla completely monotonic.}$$

So, $f^{\overline{k}}(x)$ is nabla completely monotonic.

Thus, we obtain

$$(-1)^n \nabla^n \left[f^{\overline{k}}(x) \right] \geq 0 \text{ and } \sum_{k=0}^{\infty} a_k (-1)^n \nabla^n \left[f^{\overline{k}}(x) \right] \geq 0.$$

This implies that $\varphi[f(x)]$ is nabla completely monotonic. □

Next, we prove nabla complete monotonicity of generalized Mittag-Leffler functions. In our study, we are interested in discrete Mittag-Leffler function using complete monotonicity so that one can examine the stability of nonlinear fractional difference equation, in addition, such a study will help us to decide the range of α in our sigmoidal models for tumor growth data in Chapter 5.

THEOREM 2.3.3. *The generalized discrete Mittag-Leffler function* $F_{\alpha,\beta}\left(\lambda, \frac{1}{x}\right)$ *is nabla completely monotonic for all* $\alpha > 0$, $\beta > 0$ *and* $\lambda \geq 0$.

PROOF. Recall that the generalized discrete Mittag-Leffler function is given in the following form

$$F_{\alpha,\beta}\left(\lambda, \frac{1}{x}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k + \beta)} \left(\frac{1}{x}\right)^{\overline{k}}.$$

We use the same idea that we had in the previous theorem. Therefore, it is sufficient to show that $\frac{1}{x}$ is nabla completely monotonic.

For $m = 1$, the following holds,

$$(-1)\nabla\left(\frac{1}{x}\right) = -\left(\frac{1}{x} - \frac{1}{x-1}\right) = \frac{1}{x(x-1)} = \frac{1}{x^{\overline{2}}} \geq 0.$$

For $m = 2$, the nonnegativity also holds,

$$(-1)^2\nabla^2\left(\frac{1}{x}\right) = \nabla\left(-\frac{1}{x^{\overline{2}}}\right) = \frac{2}{x(x-1)(x-2)} = \frac{2}{x^{\overline{3}}} \geq 0.$$

For $m = n$, we have

$$(-1)^n\nabla^n\left(\frac{1}{x}\right) = \frac{(-1)^{2n} n!}{x^{\overline{n+1}}} \geq 0.$$

Thus, $\frac{1}{x}$ is nabla completely monotonic and since we have the following

$$\left(\frac{1}{x}\right)^{\overline{k}} = \left(\frac{1}{x}\right)\left(\frac{1}{x}+1\right)\left(\frac{1}{x}+2\right)\cdots\left(\frac{1}{x}+k-1\right).$$

So, $\left(\frac{1}{x}\right)^{\overline{k}}$ is also nabla completely monotonic using Theorem 2.3.1.

Also, we know that Gamma function is nonnegative on the interval $(0, \infty)$. Finally, we obtain

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k + \beta)} \left[(-1)^n \nabla^n \left(\frac{1}{x}\right)^{\overline{k}} \right] \geq 0.$$

This implies that, the generalized discrete Mittag-Leffler function $F_{\alpha,\beta}\left(\frac{1}{x}\right)$ is nabla completely monotonic. □

From the literature, we see that the papers give a connection between the concept of complete monotonicity of a function and the generalized Mittag-Leffler stability [**13, 14**]. Thus, this chapter will lead us or even others to find the stability of discrete Mittag-Leffler function.

Chapter 3
$\mathcal{N}$-TRANSFORM TABLE

3.1. A Brief Introduction to Laplace Transform

The Laplace Transform is a well-known mathematical method for solving differential equations which arise in physics, biology, economics and engineering problems. By applying the Laplace transform, one can convert an ordinary differential equation into an algebraic equation and obviously, an algebraic equation is generally easier to deal with. In addition, thanks to the availability of large computers, its applications have become increasingly significant tools in the numerical solution of mathematical problems [**12**]. In order to make this mathematical method clear, we present Laplace transform:

Definition 3.1.1. *The Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined by*

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt = F(s), \qquad (s \in \mathbb{C}).$$

In the next section, we recall the discrete Laplace transform also known as $\mathcal{N}$ transform in order to find the solutions of nabla fractional difference equation. For further reading about $\mathcal{N}$ transform, we refer to the readers [**3**].

3.2. Definition of $\mathcal{N}$-Transform and Some Properties

In this section, we give a table called $\mathcal{N}-$transform table for functions which are defined on $\mathbb{Z}$. This table is a great tool for us to find solutions of nabla fractional difference equations. First, we recall the definition of $\mathcal{N}$ transform:

Definition 3.2.1. *Discrete Laplace transform ($\mathcal{N}$-transform) for a function $f : \mathcal{N}_a \to \mathbb{R}$ is defined by*

$$\mathcal{N}_a(f(t))(s) = \sum_{t=a}^{\infty} (1-s)^{t-1} f(t).$$

We use the notation $\mathcal{N}$ for $\mathcal{N}_1$, If domain of the function f is $\mathbb{N}_1$. Now we proceed the following properties which can be found in [**3**].

Lemma 3.2.2. *For any $\alpha \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$,*

(i) $\mathcal{N}_1(t^{\overline{\alpha-1}})(s) = \dfrac{\Gamma(\alpha)}{s^{\alpha}}$, $|1-s| < 1$, *and*

(ii) $\mathcal{N}_1(t^{\overline{\alpha-1}}\nu^{-t})(s) = \dfrac{\nu^{\alpha-1}\Gamma(\alpha)}{(s+\nu-1)^{\alpha}}$, $|1-s| < \nu$.

(iii) $\mathcal{N}_1(t^{\overline{\alpha}})(s) = \dfrac{\alpha}{s}\mathcal{N}_1(t^{\overline{\alpha-1}})(s)$.

(iv) $\mathcal{N}_a(f(\sigma(t)) = (1-s)^{-1}\mathcal{N}_{a+1}f(t)$.

(v) $\mathcal{N}_a(\nabla_a^{\alpha} f(t))(s) = s^{-\alpha}\mathcal{N}_a(f(t))(s)$.

(vi) $\mathcal{N}_{a+1}(\nabla_a^{\alpha} f(t))(s) = s^{-\alpha}\mathcal{N}_a(f(t))(s) - (1-s)^{a-1}f(a)$, *where* $0 < \alpha < 1$.

3.3. $\mathcal{N}$-Transform of Discrete Fractional Exponential Function

First, recall the nabla exponential function as the following

$$\widehat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}}) = \sum_{n=0}^{\infty} \frac{\lambda^n (t-a+1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}. \tag{3.1}$$

We apply the $\mathcal{N}$-transform to each side of (3.1) to obtain

$$\mathcal{N}_a\left(\widehat{e}_{\alpha,\alpha}\left(\lambda, (t-a)^{\overline{\alpha}}\right)\right)(s) = \mathcal{N}_a\left(\sum_{n=0}^{\infty} \frac{\lambda^n (t-a+1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}\right)(s). \tag{3.2}$$

Expand our series on the right hand side of (3.2), thus we get

$$\mathcal{N}_a\left(\left\{\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \frac{\lambda(t-a+1)^{\overline{2\alpha-1}}}{\Gamma(2\alpha)} + \frac{\lambda^2(t-a+1)^{\overline{3\alpha-1}}}{\Gamma(3\alpha)} + \cdots\right\}\right)(s). \tag{3.3}$$

By the linearity property of $\mathcal{N}$-transform, (3.3) can be written as

$$\mathcal{N}_a\left(\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right)(s)+\mathcal{N}_a\left(\frac{\lambda(t-a+1)^{\overline{2\alpha-1}}}{\Gamma(2\alpha)}\right)(s)+\mathcal{N}_a\left(\frac{\lambda^2(t-a+1)^{\overline{3\alpha-1}}}{\Gamma(3\alpha)}\right)(s)+\cdots.$$

Using the Lemma 3.2.2, we have

$$\left\{\frac{(1-s)^{a-1}}{s^{\alpha}}+\frac{(1-s)^{a-1}\lambda}{s^{2\alpha}}+\frac{(1-s)^{a-1}\lambda^2}{s^{3\alpha}}+\cdots\right\}.$$

Factor $\dfrac{(1-s)^{a-1}}{s^{\alpha}}$ out, thus we get

$$\frac{(1-s)^{a-1}}{s^{\alpha}}\left\{1+\frac{\lambda}{s^{\alpha}}+\frac{\lambda^2}{s^{2\alpha}}\cdots\right\}, \qquad |\lambda|<1.$$

Finally, the infinite sum above can be written in the following form by using the geometric series expansion

$$\frac{(1-s)^{a-1}}{s^{\alpha}\left(1-\frac{\lambda}{s^{\alpha}}\right)}=\frac{(1-\lambda)(1-s)^{a-1}}{s^{\alpha}-\lambda},$$

Thus, we have the following result

$$\mathcal{N}_a\left(\widehat{e}_{\alpha,\alpha}\left(\lambda,(t-a)^{\overline{\alpha}}\right)\right)(s)=\frac{(1-s)^{a-1}}{s^{\alpha}-\lambda}.$$

3.4. $\mathcal{N}$-Transform of Discrete Fractional Trigonometric Functions

From the literature, we see that trigonometric functions play a significant role in a variety of branches of mathematical analysis. Of particular importance is their place in Fourier analysis which has a broad range in mathematics and especially applications in engineering. Fourier transform is an important tool to separate an image into its sine and cosine components. Recently, we encounter the discrete Fourier transform which is a discrete version of Fourier transform. The discrete Fourier transform requires an input function that is discrete. Therefore, discrete trigonometric functions are quite essential in Fourier analysis as well. In this section, we define nabla fractional trigonometric functions. Furthermore, we consider $\mathcal{N}$-Transform of discrete fractional trigonometric functions. We start proving $\mathcal{N}_a$-transform of nabla fractional sine function. Apply the $\mathcal{N}_a$-transform to each side of the nabla fractional sine function to obtain

$$\mathcal{N}_a\left(\left\{\widehat{\sin}_{\alpha,\alpha}(b,t-a)\right\}\right)(s)=\mathcal{N}_a\left(\left\{\frac{\widehat{e}_{\alpha,\alpha}\left(ib,(t-a)^{\overline{\alpha}}\right)-\widehat{e}_{\alpha,\alpha}\left(-ib,(t-a)^{\overline{\alpha}}\right)}{2i}\right\}\right)(s).$$

For simplicity, we call I for the right hand side of the equation above. We just follow the same steps as we did in the previous section and we get

$$\begin{aligned}
I &= \frac{\mathcal{N}_a\left\{\widehat{e}_{\alpha,\alpha}\left(ib,(t-a)^{\overline{\alpha}}\right)\right\}(s)-\mathcal{N}_a\left\{\widehat{e}_{\alpha,\alpha}\left(-ib,(t-a)^{\overline{\alpha}}\right)\right\}(s)}{2i}\\
&= \frac{\dfrac{(1-s)^{a-1}}{(s^{\alpha}-ib)}-\dfrac{(1-s)^{a-1}}{(s^{\alpha}+ib)}}{2i}\\
&= \frac{b\,(1-s)^{a-1}}{s^{2\alpha}+b^2}.
\end{aligned}$$

Thus, we conclude that

$$\mathcal{N}_a\left(\left\{\widehat{\sin}_{\alpha,\alpha}(b,t-a)\right\}\right)(s)=\frac{b\,(1-s)^{a-1}}{s^{2\alpha}+b^2}.$$

Next, we proceed to $\mathcal{N}_a$-transform of cosine fractional function. We apply the method as we did for $\widehat{\sin}_{\alpha,\alpha}(b,t-a)$. So we have

$$\mathcal{N}_a\left(\left\{\widehat{\cos}_{\alpha,\alpha}(b,t-a)\right\}\right)(s)=\mathcal{N}_a\left(\left\{\frac{\widehat{e}_{\alpha,\alpha}\left(ib,(t-a)^{\overline{\alpha}}\right)+\widehat{e}_{\alpha,\alpha}\left(-ib,(t-a)^{\overline{\alpha}}\right)}{2}\right\}\right)(s).$$

We use the linearity property of $\mathcal{N}$-transform to the right hand side of the equation above to obtain

$$=\frac{\mathcal{N}_a\left(\widehat{e}_{\alpha,\alpha}\left(ib,(t-a)^{\overline{\alpha}}\right)\right)(s)+\mathcal{N}_a\left(\widehat{e}_{\alpha,\alpha}\left(-ib,(t-a)^{\overline{\alpha}}\right)\right)(s)}{2}.$$

Using the Lemma 3.2.2, we obtain

$$=\frac{\dfrac{(1-s)^{a-1}}{(s^{\alpha}-ib)}+\dfrac{(1-s)^{a-1}}{(s^{\alpha}+ib)}}{2}$$

which equals to

$$=\frac{s^{\alpha}\,(1-s)^{a-1}}{s^{2\alpha}+b^2}.$$

Thus, we conclude that

$$\mathcal{N}_a\left(\{\widehat{\cos}_{\alpha,\alpha}(b,t-a)\}\right)(s) = \frac{s^{\alpha}\left(1-s\right)^{a-1}}{s^{2\alpha}+b^2}.$$

We continue our proofs with $\mathcal{N}$-transform of fractional hyperbolic functions namely, $\widehat{\cosh}_{\alpha,\alpha}(b,t-a)$ and $\widehat{\sinh}_{\alpha,\alpha}(b,t-a)$. We apply $\mathcal{N}_a$-transform to each side of the definition of fractional hyperbolic cosine function to obtain

$$\mathcal{N}_a\left(\widehat{\cosh}_{\alpha,\alpha}(b,t-a)\right)(s) = \mathcal{N}_a\left(\left\{\frac{\widehat{e}_{\alpha,\alpha}\left(b,(t-a)^{\overline{\alpha}}\right)+\widehat{e}_{\alpha,\alpha}\left(-b,(t-a)^{\overline{\alpha}}\right)}{2}\right\}\right)(s).$$

Using the same steps for the previous proofs above, we conclude

$$\mathcal{N}_a\left(\widehat{\cosh}_{\alpha,\alpha}(b,t-a)\right)(s) = \frac{s^{\alpha}\left(1-s\right)^{a-1}}{s^{2\alpha}-b^2}.$$

Similarly, we follow the same steps to prove the $\mathcal{N}$-transform of fractional hyperbolic sine function, thus we have

$$\mathcal{N}_a\left(\widehat{\sinh}_{\alpha,\alpha}(b,t-a)\right)(s) = \mathcal{N}_a\left(\left\{\frac{\widehat{e}_{\alpha,\alpha}\left(b,(t-a)^{\overline{\alpha}}\right)-\widehat{e}_{\alpha,\alpha}\left(-b,(t-a)^{\overline{\alpha}}\right)}{2}\right\}\right)(s).$$

Finally, we get

$$\mathcal{N}_a\left(\widehat{\sinh}_{\alpha,\alpha}(b,t-a)\right)(s) = \frac{b\left(1-s\right)^{a-1}}{s^{2\alpha}-b^2}.$$

In order to see the method of $\mathcal{N}$-transform to solve fractional equations easier, we proceed to an example.

Example 3.4.1. *Consider the following initial value problem*

$$\begin{aligned} \nabla_0^{\frac{1}{3}}y(t) &= 4, \ \text{for } t = 1,2,\ldots \\ \nabla_0^{-\frac{2}{3}}y(t)_{t=0} &= y(0) = 1. \end{aligned} \tag{3.4}$$

Applying $\mathcal{N}_1$–transform to each side of the equation (3.4), we have

$$\mathcal{N}_1(\nabla_0^{\frac{1}{3}}y(t))(s) = \mathcal{N}_1(4)(s).$$

Using Lemma 3.2.2 , we obtain

$$s^{\frac{1}{3}}\mathcal{N}_0(y(t))(s) - (1-s)^{-1}y(0) = \frac{4}{s}\ .$$

$f(t)$	$F(s)$
$\dfrac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$	$\dfrac{(1-s)^{a-1}}{s^{\alpha}}$
$\dfrac{(t-a+1)^{\overline{\alpha-1}}\,\nu^{-t}}{\Gamma(\alpha)}$	$\dfrac{\nu^{\alpha-1}(1-s)^{a-1}}{(s+\nu-1)^{\alpha}}$
$\widehat{e}_{\alpha,\alpha}(\lambda,(t-a+1)^{\overline{\alpha}})$	$\dfrac{(1-s)^{a-1}}{(s^{\alpha}-\lambda)}$
$\widehat{\sin}_{\alpha,\alpha}(b,t-a+1)$	$\dfrac{b\,(1-s)^{a-1}}{(s^{2\alpha}+b^{2})}$
$\widehat{\cos}_{\alpha,\alpha}(b,t-a+1)$	$\dfrac{s^{\alpha}(1-s)^{a-1}}{(s^{2\alpha}+b^{2})}$
$\widehat{\sinh}_{\alpha,\alpha}(b,t-a+1)$	$\dfrac{b\,(1-s)^{a-1}}{(s^{2\alpha}-b^{2})}$
$\widehat{\cosh}_{\alpha,\alpha}(b,t-a+1)$	$\dfrac{s^{\alpha}(1-s)^{a-1}}{(s^{2\alpha}-b^{2})}$
$\left(\dfrac{1}{1-a^{2}}\right)^{t-a+1}$	$\dfrac{(1-s)^{a-1}}{s-a^{2}}$

TABLE 3.4.1. $\mathcal{N}$-Transform Table

Since $y(0)=1$, the equation above can be written as

$$\mathcal{N}_0(y(t))(s)=\frac{4}{s^{4/3}}+\frac{1}{(1-s)s^{1/3}}.$$

Next, we use (Table 3.4.1*) to obtain*

$$\mathcal{N}_0(y(t))(s)=\frac{4}{\Gamma(4/3)}\mathcal{N}_0(t^{\overline{1/3}})(s)+\frac{1}{\Gamma(1/3)}\mathcal{N}_0((t+1)^{\overline{-2/3}})(s).$$

Apply the inverse $\mathcal{N}_0$–transform to each side of the equation, finally we conclude the solution of the initial value problem in the given form

$$y(t)=\frac{4}{\Gamma(4/3)}t^{\overline{1/3}}+\frac{1}{\Gamma(1/3)}(t+1)^{\overline{-2/3}},$$

where $t=0,\ 1,\ 2,\dots$.

3.5. Solutions of up to First Order Nonhomogeneous Nabla Fractional Difference Equations

Consider the up to first order nonhomogeneous initial value problem.

$$\begin{cases} \nabla_0^{\alpha} y(t) = -ay(t-1) + b \quad \text{for } t = 1, 2, \ldots \\ \nabla_0^{-(1-\alpha)} y(t)\,|_{t=0} = y(0) = c \end{cases} \tag{3.5}$$

where $0 < \alpha < 1$ and $|a| < 1$.

First, apply the $\mathcal{N}_1$-transform to each side of the equation (3.5) to have

$$\mathcal{N}_1(\nabla_0^{\alpha} y(t))(s) = -a\mathcal{N}_1(y(t-1))(s) + b\mathcal{N}_1(1)(s).$$

By using the Lemma 3.2.2 we have

$$s^{\alpha}\mathcal{N}_0(y(t))(s) - (1-s)^{-1}y(0) = -a\mathcal{N}_1(y(t-1))(s) + \frac{b}{s}\cdot$$

Doing some algebra, we get the following

$$\left(\frac{s^{\alpha}}{1-s} + a\right)\mathcal{N}_1(y(t-1))(s) = \frac{y(0)}{(1-s)} + \frac{b}{s}$$

and finally

$$\mathcal{N}_1(y(t-1))(s) = \frac{y(0)}{s^{\alpha}\left(1 + \frac{a(1-s)}{s^{\alpha}}\right)} + \frac{b(1-s)}{ss^{\alpha}\left(1 + \frac{a(1-s)}{s^{\alpha}}\right)}\cdot$$

Expand $\dfrac{1}{1-(-a)(1-s)s^{-\alpha}}$ as a geometric series and we have

$$\mathcal{N}_1(y(t-1))(s) = \left(\frac{y(0)}{s^{\alpha}} + \frac{b(1-s)}{ss^{\alpha}}\right)\left(1 + \frac{(-a)(1-s)}{s^{\alpha}} + \frac{(-a)^2(1-s)}{s^{2\alpha}} + \cdots\right)\cdot$$

Since $y(0) = c$ we obtain

$$\mathcal{N}_1(y(t-1))(s) = \left(c + \frac{b(1-s)}{s}\right)\left(\frac{1}{s^{\alpha}} + \frac{(-a)(1-s)}{s^{2\alpha}} + \frac{(-a)^2(1-s)^2}{s^{3\alpha}} + \cdots\right)\cdot \tag{3.6}$$

By using Lemma 3.2.2 (i), the right hand side of the equation (3.6) yields

$$= c\left\{\frac{\mathcal{N}_1(t^{\overline{\alpha-1}})(s)}{\Gamma(\alpha)} + (-a)(1-s)\frac{\mathcal{N}_1(t^{\overline{2\alpha-1}})(s)}{\Gamma(2\alpha)} + (-a)^2(1-s)^2\frac{\mathcal{N}_1(t^{\overline{3\alpha-1}})(s)}{\Gamma(3\alpha)} + \cdots\right\}$$
$$+ \left[\frac{b(1-s)}{s}\right]\left\{\frac{\mathcal{N}_1(t^{\overline{\alpha-1}})(s)}{\Gamma(\alpha)} + (-a)(1-s)\frac{\mathcal{N}_1(t^{\overline{2\alpha-1}})(s)}{\Gamma(2\alpha)} + (-a)^2(1-s)^2\frac{\mathcal{N}_1(t^{\overline{3\alpha-1}})(s)}{\Gamma(3\alpha)} + \cdots\right\}$$

and apply Lemma 3.2.2 (iv) to have the following form

$$= c\left\{\frac{\mathcal{N}_1(t^{\overline{\alpha-1}})(s)}{\Gamma(\alpha)} + (-a)\frac{\mathcal{N}_2((t-1)^{\overline{2\alpha-1}})(s)}{\Gamma(2\alpha)} + (-a)^2\frac{\mathcal{N}_3((t-2)^{\overline{3\alpha-1}})(s)}{\Gamma(3\alpha)} + \cdots\right\}$$
$$+ \frac{b}{s}\left\{\frac{\mathcal{N}_2((t-1)^{\overline{\alpha-1}})(s)}{\Gamma(\alpha)} + (-a)\frac{N_2((t-2)^{\overline{2\alpha-1}})(s)}{\Gamma(2\alpha)} + (-a)^2\frac{\mathcal{N}_2((t-3)^{\overline{3\alpha-1}})(s)}{\Gamma(3\alpha)} + \cdots\right\}.$$

Again, by employing Lemma 3.2.2 (i) to obtain

$$= c\left\{\frac{\mathcal{N}_1(t^{\overline{\alpha-1}})(s)}{\Gamma(\alpha)} + (-a)\frac{\mathcal{N}_1((t-1)^{\overline{2\alpha-1}})(s)}{\Gamma(2\alpha)} + (-a)^2\frac{\mathcal{N}_1((t-2)^{\overline{3\alpha-1}})(s)}{\Gamma(3\alpha)} + \cdots\right\}$$
$$+ b\left\{\frac{\mathcal{N}_2((t-1)^{\overline{\alpha}})(s)}{\Gamma(\alpha+1)} + (-a)\frac{\mathcal{N}_2((t-2)^{\overline{2\alpha}})(s)}{\Gamma(2\alpha+1)} + (-a)^2\frac{\mathcal{N}_2((t-3)^{\overline{3\alpha}})(s)}{\Gamma(3\nu\alpha+1)} + \cdots\right\}.$$

Similar to the previous proof, it can easily be seen that

$$\mathcal{N}_2((t-1)^{\overline{2\alpha-1}})(s) = \mathcal{N}_1((t-1)^{\overline{2\alpha-1}})(s),\ \mathcal{N}_3((t-2)^{\overline{3\alpha-1}})(s) = \mathcal{N}_1((t-2)^{\overline{3\alpha-1}})(s), \ldots$$

by using the definition of $\mathcal{N}-$transform, we have the following

$$\mathcal{N}_1(y(t-1))(s) = c\left\{\frac{\mathcal{N}_1(t^{\overline{\alpha-1}})(s)}{\Gamma(\alpha)} + (-a)\frac{\mathcal{N}_1((t-1)^{\overline{2\alpha-1}})(s)}{\Gamma(2\alpha)} + (-a)^2\frac{\mathcal{N}_1((t-2)^{\overline{3\alpha-1}})(s)}{\Gamma(3\alpha)} + \cdots\right\}$$
$$+ b\left\{\frac{\mathcal{N}_1((t-1)^{\overline{\alpha}})(s)}{\Gamma(\alpha+1)} + (-a)\frac{\mathcal{N}_1((t-2)^{\overline{2\alpha}})(s)}{\Gamma(2\alpha+1)} + (-a)^2\frac{\mathcal{N}_1((t-3)^{\overline{3\alpha}})(s)}{\Gamma(3\alpha+1)} + \cdots\right\}.$$

Employing the inverse $\mathcal{N}_1-$transform to each side of the equation above to obtain

$$y(t-1) = c\left\{\frac{t^{\overline{\alpha-1}}}{\Gamma(\alpha)} + (-a)\frac{(t-1)^{\overline{2\alpha-1}}}{\Gamma(2\alpha)} + (-a)^2\frac{(t-2)^{\overline{3\alpha-1}}}{\Gamma(3\alpha)} + \cdots\right\}$$
$$+ b\left\{\frac{(t-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)} + (-a)\frac{(t-2)^{\overline{2\alpha}}}{\Gamma(2\alpha+1)} + (-a)^2\frac{(t-3)^{\overline{3\alpha}}}{\Gamma(3\alpha+1)} + \cdots\right\}$$

which can be written as the following form

$$y(t-1) = c\sum_{n=0}^{\infty}\frac{(-a)^n(t-n)^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)} + b\sum_{n=1}^{\infty}\frac{(-a)^{n-1}(t-n)^{\overline{n\alpha}}}{\Gamma(n\alpha+1)}.$$

Shift one unit left to conclude the solution of the initial value problem as the following form

$$y(t) = c\sum_{n=0}^{\infty} \frac{(-a)^n (t-n+1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)} + b\sum_{n=1}^{\infty} \frac{(-a)^{n-1}(t-n+1)^{\overline{n\alpha}}}{\Gamma(n\alpha+1)}.$$

Note that, for $b = 0$, we obtain the discrete Mittag-Leffler function.

Chapter 4

SEQUENTIAL FRACTIONAL DIFFERENCE EQUATIONS

In this chapter, we state and prove some theorems on the solutions of nabla fractional difference equations being linearly independent or linearly dependent. Then, we introduce the Casoration for discrete functions. Finally, we find the solutions of up to second order nabla fractional difference equation considering the characteristic roots of its characteristic equation as distinct and real, same and real, and complex.

4.1. Casoration and Linear Independence

In this section, we define the Casoration for discrete functions. Casoration helps us to examine whether the set of solutions of homogeneous linear fractional equations are linearly independent or dependent. In relation to these solutions, we state and prove some fundamental theorems about the general solution of a nabla fractional difference equation to support our claims.

Definition 4.1.1. *The $n \times n$ matrix of Casorati is given by*

$$c(t) = \begin{bmatrix} \nabla_a^{-(1-\alpha)} y_1(t) & \nabla_a^{-(1-\alpha)} y_2(t) & \cdots & \nabla_a^{-(1-\alpha)} y_n(t) \\ \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(t) & \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_2(t) & \cdots & \nabla_a^{-(1-\alpha)} \nabla^{\alpha} y_n(t) \\ \nabla_a^{-(1-\alpha)} \nabla_a^{(2\alpha)} y_1(t) & \nabla_a^{-(1-\alpha)} \nabla_a^{(2\alpha)} y_2(t) & \cdots & \nabla_a^{-(1-\alpha)} \nabla_a^{(2\alpha)} y_n(t) \\ \vdots & & & \vdots \\ \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_1(t) & \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_2(t) & \cdots & \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_n(t) \end{bmatrix}$$

where $\nabla_a^{((n-1)\alpha)}y(t) = \underbrace{\nabla_a^{\alpha}\nabla_a^{\alpha}\cdots\nabla_a^{\alpha}}_{n-1\ times}y(t)$ *and* $y_1,\ y_2,\ldots,y_n$ *are given functions. The determinant*

$$C\left[y_1,y_2,\ldots,y_n\right] = \begin{vmatrix} \nabla_a^{-(1-\alpha)}y_1(t) & \nabla_a^{-(1-\alpha)}y_2(t) & \cdots & \nabla_a^{-(1-\alpha)}y_n(t) \\ \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_1(t) & \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_2(t) & \cdots & \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_n(t) \\ \nabla_a^{-(1-\alpha)}\nabla_a^{(2\alpha)}y_1(t) & \nabla_a^{-(1-\alpha)}\nabla_a^{(2\alpha)}y_2(t) & \cdots & \nabla_a^{-(1-\alpha)}\nabla_a^{(2\alpha)}y_n(t) \\ \vdots & & & \vdots \\ \nabla_a^{-(1-\alpha)}\nabla_a^{((n-1)\alpha)}y_1(t) & \nabla_a^{-(1-\alpha)}\nabla_a^{((n-1)\alpha)}y_2(t) & \cdots & \nabla_a^{-(1-\alpha)}\nabla_a^{((n-1)\alpha)}y_n(t) \end{vmatrix}$$

is called ***Casoration****.*

Theorem 4.1.2. *Let* $\{y_1,y_2,\ldots,y_n\}$ *be a set of* n *solutions of an up to* $n-th$ *order fractional linear homogeneous nabla difference equation.*

The set is linearly independent $\Longleftrightarrow$ *Casoration is not identically equal to zero on a discrete interval* I*.*

Proof. We prove for the case $n = 2$. Let $y_1(t)$ and $y_2(t)$ be a solution of the following initial value problem

$$\begin{cases} p\nabla_a^{\alpha}\nabla_a^{\alpha}y(t) + q\nabla_a^{\alpha}y(t) + ry(t) = 0, \quad \text{for t=a+1, a+2,...} \\ \nabla_a^{-(1-\alpha)}y(t)\,|_{t=a} = y(a) = 0 \quad \text{and} \quad \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y(a) = 0 \end{cases} \tag{4.1}$$

on a discrete interval I, for $0 < \alpha \leq 1$, and where p, q, r are constants.

We need to show the Casoration of y_1 and y_2, $C\left[y_1,y_2\right] \neq 0$ in order to prove that $y_1(t)$ and $y_2(t)$ are linearly independent. The Casoration is given by

$$C\left[y_1,y_2\right] = \begin{vmatrix} \nabla_a^{-(1-\alpha)}y_1(t) & \nabla_a^{-(1-\alpha)}y_2(t) \\ \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_1(t) & \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_2(t) \end{vmatrix}$$

$$= \nabla_a^{-(1-\alpha)} y_1(t) \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_2(t) - \nabla_a^{-(1-\alpha)} y_2(t) \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(t).$$

To start our proof, first we need to consider the linear system of nabla fractional difference equations using the method of change of variables such that

$$y_1(t) = y(t) \implies \nabla_a^{\alpha} y_1(t) = \nabla_a^{\alpha} y(t) = y_2(t)$$

$$y_2(t) = \nabla_a^{\alpha} y(t) \implies \nabla_a^{\alpha} y_2(t) = \nabla_a^{\alpha} \nabla_a^{\alpha} y(t) = -\frac{q}{p} \nabla_a^{\alpha} y(t) - \frac{r}{p} y(t) = -\frac{q}{p} y_2(t) - \frac{r}{p} y_1(t)$$

Thus, we have the following matrix form

$$\begin{bmatrix} \nabla_a^{\alpha} y_1(t) \\ \\ \nabla_a^{\alpha} y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ -\frac{r}{p} & -\frac{q}{p} \end{bmatrix} \begin{bmatrix} y_1(t) \\ \\ y_2(t) \end{bmatrix}.$$

Also the initial conditions turn into the following form

$$\nabla_a^{-(1-\alpha)} y(t) = \begin{bmatrix} \nabla_a^{-(1-\alpha)} y_1(t) \\ \\ \nabla_a^{-(1-\alpha)} y_2(t) \end{bmatrix} = \begin{bmatrix} \nabla_a^{-(1-\alpha)} y(t) \\ \\ \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y(t) \end{bmatrix}_{t=a} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}. \tag{4.2}$$

($\Longleftarrow$) We prove by contradiction, that is, y_1 and y_2 are linearly dependent, then Casoration is identically equal to zero. If y_1 and y_2 are linearly dependent, then

$$y_2 = k y_1 \tag{4.3}$$

holds for some k. Thus,

$$C\left[y_1, y_2\right] = (\nabla_a^{-(1-\alpha)} y_1(t))(\nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} k y_1(t)) - (\nabla_a^{-(1-\alpha)} k y_1(t))(\nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(t)).$$

Using (4.3), we have

$$C\left[y_1, y_2\right] = k(\nabla_a^{-(1-\alpha)} y_1(t))(\nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(t)) - k(\nabla_a^{-(1-\alpha)} y_1(t))(\nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(t)) = 0.$$

Thus, y_1 and y_2 are linearly dependent.

($\Longrightarrow$)Assume that $y_1 \neq 0$, $y_2 \neq 0$ and let $C\left[y_1, y_2\right](a) = 0$.

Then the system

$$\begin{cases} k_1\nabla_a^{-(1-\alpha)}y_1(a) + k_2\nabla_a^{-(1-\alpha)}y_2(a) = 0 \\ k_1\nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_1(a) + k_2\nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_2(a) = 0 \end{cases} \tag{4.4}$$

can be represented by the matrix form as the following

$$\underbrace{\begin{bmatrix} \nabla_a^{-(1-\alpha)}y_1(a) & \nabla_a^{-(1-\alpha)}y_2(a) \\ \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_1(a) & \nabla_a^{-(1-\alpha)}\nabla_a^{\alpha}y_2(a) \end{bmatrix}}_{[A]}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0. \tag{4.5}$$

By Cramer's Theorem, if $\det A = 0$, then the system (4.4) has nontrivial solution, that is k_1 and k_2 are both nonzero. So, $y(t)$, a solution of linear system of the equation can be represented in this form

$$y(t) = k_1y_1(t) + k_2y_2(t).$$

Since $y(t)$ is a solution, it satisfies the initial value problem (4.1). By the theorem in [**3**], the linear system of equation (4.2) has the trivial solution considering the theorem of existence and uniqueness of a solution. Therefore, we obtain

$$k_1y_1(t) + k_2y_2(t) = 0$$

on an interval I. Since k_1 and k_2 are both nonzero, y_1 and y_2 are linearly dependent. □

We finish our section with a theorem about the general solution of up to the $n-th$ order linear homogeneous fractional nabla difference equation. Furthermore, we use this theorem to find the general solution of second order nabla fractional difference equation in Section 4.2.

THEOREM 4.1.3. *The up to $n-th$ order linear homogeneous fractional nabla difference equation is given as the following form*

$$p_n(t)\nabla_a^{(n\alpha)}y(t) + p_{n-1}(t)\nabla_a^{((n-1)\alpha)}y(t) + \cdots + p_1(t)\nabla_a^{\alpha}y(t) + p_0(t)y(t) = 0 \tag{4.6}$$

for any t in I. Let $y_1(t), y_2(t), \dots, y_n(t)$ be independent solutions of (4.6). Then every solution $y(t)$ of (4.6) can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t), \quad \textit{for some constants } c_1, c_2, \dots, c_n.$$

Proof. Let $y(t)$ be a solution of (4.6). Consider the system of fractional nabla difference and sum equations for a fixed point $t = m$.

$$\begin{cases} c_1 \nabla_a^{-(1-\alpha)} y_1(m) + c_2 \nabla_a^{-(1-\alpha)} y_2(m) + \cdots + c_n \nabla_a^{-(1-\alpha)} y_n(m) = \nabla_a^{-(1-\alpha)} y(m) \\ c_1 \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(m) + c_2 \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_2(m) + \cdots + c_n \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_n(m) = \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y(m) \\ \vdots \\ c_1 \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_1(m) + c_2 \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_2(m) + \cdots + c_n \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_n(m) = \\ \qquad \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y(m) \end{cases}$$

The system above can be represented as matrix form such that

$$\underbrace{\begin{bmatrix} \nabla_a^{-(1-\alpha)} y(m) \\ \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y(m) \\ \vdots \\ \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y(m) \end{bmatrix}}_{[Y]} =$$

$$\underbrace{\begin{bmatrix} \nabla_a^{-(1-\alpha)} y_1(m) & \nabla_a^{-(1-\alpha)} y_2(m) & \cdots & \nabla_a^{-(1-\alpha)} y_n(m) \\ \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_1(m) & \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_2(m) & \cdots & \nabla_a^{-(1-\alpha)} \nabla_a^{\alpha} y_n(m) \\ \vdots & & & \vdots \\ \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_1(m) & \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_2(m) & \cdots & \nabla_a^{-(1-\alpha)} \nabla_a^{((n-1)\alpha)} y_n(m) \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{[C]}.$$

For a given fixed point m, Casoration $C\left[y_{1,}y_2,\ldots,y_n\right]\neq 0$, since the set of solutions is linearly independent. Therefore, $\det A\neq 0$ which means the matrix A is invertible. Apply $[A]^{-1}$ to each side of the system above from the left, we have

$$[C]=[A]^{-1}\ \ [Y].$$

Thus, the system of equations has a unique solution C_1, $C_2,\ldots,C_n$. Note that the solution of (4.6) is uniquely determined by its values at $t=m$, so for all t we conclude the general solution as

$$y(t)=C_1y_1(t)+C_2y_2(t)+\cdots+C_ny_n(t).$$

□

4.2. Up to Second Order Linear Homogeneous Nabla Fractional Equations

In this section, we consider up to second order linear nabla fractional equation and develop its solutions. The second order nabla fractional equation is given by

$$p\nabla_a^\alpha\nabla_a^\alpha y(t)+q\nabla_a^\alpha y(t)+ry(t)=0 \text{ for } t=a+1,a+2,\ldots \tag{4.7}$$

where $0<\alpha<1$ and where p,q,r are constant coefficients. The characteristic equation of (4.7) is given as

$$p\lambda^2+q\lambda+r=0.$$

Assume that λ_1 and λ_2 are the roots of the characteristic equation. By using the fact that any given equation can be represented by its characteristic roots, we have

$$\nabla_a^\alpha\nabla_a^\alpha y\,(t)-(\lambda_1+\lambda_2)\nabla_a^\alpha y(t)+(\lambda_1\lambda_2)y(t)=0. \tag{4.8}$$

We assume the initials

$$\nabla_a^{-(1-\alpha)}y(t)\left|_{t=a}\right.=y(a)=A \text{ and } \nabla_a^{-(1-\alpha)}\nabla_a^\alpha y(t)\left|_{t=a}\right.=B$$

exist that is $A<\infty$ and $B<\infty$.

CASE I. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

We assume that $\widehat{e}_{\alpha,\alpha}(\lambda_1, (t-a)^{\overline{\alpha}})$ and $\widehat{e}_{\alpha,\alpha}(\lambda_2, (t-a)^{\overline{\alpha}})$ are solutions of up to the second order linear nabla fractional equation (4.8). We prove that these are solutions of (4.8) for any λ_1, λ_2. In this case, we consider that $a = 0$. By using the fact that $\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ is the solution of $\nabla_0^{\alpha} y(t) = \lambda y(t)$ for $t = 0, 1, 2, \ldots$, we can rewrite the first and second nabla fractional derivative of $\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$. In order to consider $\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ is the solution of equation (4.8), we need to show that it satisfies (4.8). So we have,

$$(\lambda_1)^2 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) - (\lambda_1 + \lambda_2)\lambda_1 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) + \lambda_1\lambda_2 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$$

which equals to

$$(\lambda_1)^2 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) - (\lambda_1)^2 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) - \lambda_2\lambda_1 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) + \lambda_1\lambda_2 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) = 0$$

Thus, $\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ is the solution of equation (4.8).

Similarly, we can find the first and second nabla fractional derivative of $\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}})$ considering it is also a solution of $\nabla_0^{\alpha} y(t) = \lambda y(t)$ for $t = 0, 1, 2, \ldots$. In addition, $\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}})$ satisfies the equation (4.8). By the Theorem 4.1.3, If the set of solutions is linearly independent, then the general solution can be written as a linear combination of these solutions. In order to show whether the set of solutions is linearly independent, we will check Casoration given in the following form

$$C\,[y_1, y_2] = \begin{vmatrix} \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \\ \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \end{vmatrix}.$$

$$= \begin{vmatrix} \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \\ \lambda_1 \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \lambda_2 \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \end{vmatrix}.$$

Calculating the determinant above, we have

$$(\lambda_2 - \lambda_1)\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \neq 0\cdot$$

Because $\lambda_1 \neq \lambda_2$ and we can show that the nabla sum of discrete exponential function is nonzero. Nabla sum of discrete exponential function is given in this form

$$\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) = \nabla_0^{-(1-\alpha)}\sum_{n=0}^{\infty}\frac{\lambda_1^n(t+1)^{\overline{(n+1)\alpha-1}}}{\Gamma\left((n+1)\,\alpha\right)}\cdot \tag{4.9}$$

Nabla sum can move into the infinite sum, so the right hand side of (4.9) yields

$$= \sum_{n=0}^{\infty}\frac{\lambda_1^n\nabla_0^{-(1-\alpha)}(t+1)^{\overline{(n+1)\alpha-1}}}{\Gamma\left((n+1)\,\alpha\right)}\cdot$$

By using Power Rule, Lemma 1.3.1 (iii), we obtain

$$= \sum_{n=0}^{\infty}\frac{\lambda_1^n(t+1)^{\overline{n\alpha}}}{\Gamma\left(n\alpha+1\right)}\cdot$$

Thus, we obtain the discrete Mittag-Leffler function

$$= F_\alpha(\lambda_1, (t+1)^{\overline{\alpha}}).$$

We know that Mittag-Leffler function is nonzero. Therefore, the sum of the discrete fractional exponential function is nonzero. Thus, $C\,[y_1, y_2] \neq 0$ and the set of solutions $\{\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}),\ \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}})\}$ is linearly independent and by the Theorem 4.1.3, we conclude the general solution of (4.8) as the following form

$$y(t) = c_1\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) + c_2\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}).$$

CASE II. If $\lambda_1 = \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Similar to the previous case, we consider that $a = 0$. In addition, we claim that $\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ and $t\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ are solutions of (4.8). Since they are solutions, they satisfy the equation (4.8). In Case I we determined that $\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ is a solution of the equation (4.8). We also need to show that $t\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ is a solution of (4.8). To continue the proof, we use Leibniz rule [**5**]

$$\overset{t}{\nabla}{}_a^\alpha f(t).g(t) = \sum_{n=0}^{t-a} \binom{\alpha}{n} \left[\overset{t-n}{\nabla}{}^{\alpha-n} f(t-n) \right] \left[\nabla^n g(t) \right].$$

For $a = 0$ we have,

$$\overset{t}{\nabla}{}_0^\alpha \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t = \sum_{n=0}^{1} \binom{\alpha}{n} \left[\overset{t-n}{\nabla}{}_0^{\alpha-n} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] \left[\nabla^n t \right]. \tag{4.10}$$

The right hand side of the equation (4.10) can be written as

$$\binom{\alpha}{0} \left[\overset{t}{\nabla}{}_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] t + \binom{\alpha}{1} \left[\overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] \nabla t \tag{4.11}$$

by using the Lemma [**4**],

$$\binom{-\alpha}{k} = \frac{\Gamma(-\alpha+1)}{\Gamma(k+1)\Gamma(-\alpha-k+1)},$$

we obtain

$$\binom{\alpha}{0} = 1, \ \binom{\alpha}{1} = \alpha.$$

Thus, the equation (4.11) has the following form

$$\left[\overset{t}{\nabla}{}_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] t + \alpha \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}).$$

Since $\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ is the solution of $\nabla_0^\alpha y(t) = \lambda y(t)$ for $t = 0, 1, 2, \ldots$ we can rewrite the first nabla fractional derivative of $te^{\wedge}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ as the following form

$$\lambda \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t + \alpha \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}).$$

Next, we consider the second nabla fractional derivative of $t\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ to obtain

$$\lambda^2 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t + \alpha\lambda \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha \overset{t}{\nabla}{}_0^{\alpha} \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}).$$

We need to show that the solution $t\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ satisfies the equation (4.8), so we have the following

$$\lambda^2 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t + \alpha\lambda \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha \overset{t}{\nabla}{}_0^{\alpha} \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$
$$-2\lambda \left[\lambda \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t + \alpha \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] + \lambda^2 t \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = 0.$$

If we prove our following claim, it finishes the proof.

Claim:

$$\overset{t}{\nabla}{}_{0}^{\alpha}\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) = \lambda\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}). \tag{4.12}$$

We start to prove our claim by writing the left side of the equation (4.12) as

$$\nabla\overset{t}{\nabla}{}_{0}^{\alpha-1}\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}). \tag{4.13}$$

Then, we use the Lemma [**37**]

$$\nabla_{a+1}^{-\alpha}\nabla f(t) = \nabla\nabla_{a}^{-\alpha}f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a).$$

Call $\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) = f(t)$. So, for $a=0$, $f(0)=0$ and we conclude

$$\overset{t}{\nabla}{}_{1}^{\alpha-1}\nabla f(t) = \nabla\overset{t}{\nabla}{}_{0}^{\alpha-1}f(t).$$

Thus, (4.13) can be written as

$$\overset{t}{\nabla}{}_{1}^{\alpha-1}\nabla\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}})$$

which can be easily seen as the following form

$$\overset{t}{\nabla}{}_{1}^{\alpha-1}\overset{t-1}{\nabla}{}_{0}^{\alpha}\ \widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}).$$

By subclaim that we prove below, we obtain

$$\overset{t}{\nabla}{}_{1}^{\alpha-1}\lambda\widehat{e}_{\alpha,\alpha}(\lambda,(t-1)^{\overline{\alpha}}).$$

Using the definition of nabla sum, we have

$$\lambda\sum_{s=1}^{t}\frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\hat{e}_{\alpha,\alpha}(\lambda,(s-1)^{\overline{\alpha}}).$$

By using substitution method, we have

$$\lambda\sum_{u=0}^{t-1}\frac{(t-1-\rho(u))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\hat{e}_{\alpha,\alpha}(\lambda,u^{\overline{\alpha}})$$

which equals to

$$\lambda\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}).$$

Thus, we conclude

$$\overset{t}{\nabla}{}_0^{\alpha} \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \;=\; \lambda \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \tag{4.14}$$

<u>Subclaim</u>: $\overset{t-1}{\nabla}{}_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = \lambda \widehat{e}_{\alpha,\alpha}(\lambda, (t-1)^{\overline{\alpha}})$

Consider the first order nabla fractional difference equation with initial condition,

$$\overset{t}{\nabla}{}_0^{\alpha} y(t) = \lambda y(t), \quad t = 0, 1, \ldots \tag{4.15}$$

$$y(0) = 1.$$

It is concluded in the paper [**3**] that $\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ satisfies (4.15). Shift (4.15) one unit left, we have the following

$$\overset{t-1}{\nabla}{}_0^{\alpha} y(t) = \lambda y(t-1), \quad t = 1, 2, \ldots$$

Thus, we obtain

$$\overset{t-1}{\nabla}{}_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = \lambda e^{\wedge}_{\alpha,\alpha}(\lambda, (t-1)^{\overline{\alpha}}).$$

We now return to the proof and by using (4.14), we conclude that

$$\lambda^2 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t + \alpha\lambda \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha\lambda \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$

$$-2\lambda^2 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t - 2\alpha\lambda \overset{t-1}{\nabla}{}_0^{\alpha-1} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$

$$+\lambda^2 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t = 0$$

As a result, $\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t$ satisfies (4.8) and thus $\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t$ is also a solution of (4.8). We know from Theorem 4.1.3 that if the set of solutions are linearly independent, then the general solution can be written as a linear combination of these solutions. So, it is sufficient to show that the set of solutions are linearly independent. By the Theorem 4.1.2, if the Casoration is not identically equal to zero, then the set of solutions are linearly independent.

Casoration of the set of solutions $\{\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})t,\ \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})\}$ is given in this form

$$\begin{vmatrix} \nabla_0^{-(1-\alpha)} t\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \\ \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} t\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \end{vmatrix}$$

which equals to

$$\begin{vmatrix} \nabla_0^{-(1-\alpha)} t\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \\ \nabla_0^{-(1-\alpha)} (\lambda t\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha \overset{t-1}{\nabla}{}_0^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) & \lambda \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \end{vmatrix}.$$

By calculating the determinant, we have

$$-\left[\alpha \nabla_0^{-(1-\alpha)} \overset{t-1}{\nabla}{}_0^{\alpha-1}\ \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})\right] \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}).$$

We use the definition of nabla fractional sum, thus we get

$$-\alpha \nabla_0^{-(1-\alpha)} \left[\sum_{s=0}^{t-1} \frac{(t-1-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, s^{\overline{\alpha}})\right] \nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, s^{\overline{\alpha}}).$$

$\alpha \neq 0$ and $\nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \neq 0$. So it follows, the Casoration is not identically equal to zero. Finally, by the Theorem 4.1.3, the general solution of (4.8) is given in this form

$$y(t) = c_1 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + c_2 t\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$

where c_1, c_2 are constants.

CASE III. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

Consider up to second order linear fractional nabla equation.

$$\nabla_a^{\alpha} \nabla_a^{\alpha} y(t) + b^2 y(t) = 0, \qquad \text{where } t = a+1, a+2, \ldots. \tag{4.16}$$

and $0 < \alpha < 1$.

The equation (4.16) can be transformed into system of nabla fractional difference equations by the change of the variables, such that

$$y_1(t) = y(t) \implies \nabla^\alpha y_1(t) = \nabla^\alpha y(t) = y_2(t)$$

$$y_2(t) = \nabla^\alpha y(t) \implies \nabla^\alpha \nabla^\alpha y(t) = -b^2 y(t) = -b^2 y_1(t)$$

and

$$y_1(t)\,|_{t=a} = 0, \;\; y_2(t)\,|_{t=a} = 0.$$

So, we have the following linear system of fractional difference equations

$$\nabla_a^\alpha Y(t) = AY(t) \begin{bmatrix} \nabla_a^\alpha y_1(t) \\ \nabla_a^\alpha y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b^2 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}. \tag{4.17}$$

The characteristic equation of (4.16) is given as

$$\lambda^2 + b^2 = 0$$

and the roots of the characteristic equation are

$$\lambda_{1,2} = \mp ib.$$

To find the solution of (4.16), we will use Putzer Algorithm.

$$M_0 = I$$

$$M_1 = (A - \lambda_1 I) = \begin{bmatrix} bi & 1 \\ -b^2 & bi \end{bmatrix}$$

and the vector valued function $p(t)$ is defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}.$$

$p(t)$ is the solution of the initial value problem

$$\nabla_a^\alpha y_1(t) = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} y(t), \qquad \text{for } t = 1, 2, \ldots \tag{4.18}$$

$$\nabla_a^{-(1-\alpha)} y(t) \,|_{t=a} = y(a) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, we have the following system of nabla fractional difference equation with initial condition

$$\begin{bmatrix} \nabla_a^\alpha p_1(t) \\ \nabla_a^\alpha p_2(t) \end{bmatrix} = \begin{bmatrix} ib & 0 \\ 1 & -ib \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \qquad \begin{bmatrix} p_1(a) \\ p_2(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.19}$$

The first component $p_1(t)$ of $p(t)$ solves the initial value problem (4.19)

$$\nabla_a^\alpha p_1(t) = (ib)p_1(t), \quad p_1(a) = 1.$$

We proved the generalized form of the nabla exponential function in (Table 3.4.1). So, we have the solution of (4.19) as

$$p_1(t) = (1 - ib)\widehat{e}_{\alpha,\alpha}(ib, (t-a)^{\overline{\alpha}}).$$

Also, the second component $p_2(t)$ of $p(t)$ solves the initial value problem (4.19), thus we have

$$\nabla_a^\alpha p_2(t) = p_1(t) - (ib)\, p_2(t), \;\; p_2(a) = 0. \tag{4.20}$$

In order to find $p_2(t)$, we use $\mathcal{N}$-transform, and with this application we see how $\mathcal{N}$-transform works for discrete functions. Apply $\mathcal{N}_{a+1}$-transform to each side of (4.20) to obtain

$$\mathcal{N}_{a+1}(\nabla_a^\alpha p_2(t))(s) = \mathcal{N}_{a+1}(p_1(t))(s) - (ib)\mathcal{N}_{a+1}(p_2(t))(s).$$

By using Lemma 3.2.2, we get

$$s^{\alpha}\mathcal{N}_a(p_2(t))(s) - (1-s)^{a-1}p_2(a) =$$

$$\mathcal{N}_a(p_1(t))(s) - (1-s)^{a-1}p_1(a) - (ib)\left\{\mathcal{N}_a(p_2(t))(s) - (1-s)^{a-1}p_2(a)\right\}. \tag{4.21}$$

Since $p_1(a) = 1$ and $p_2(a) = 0$, we simplify the equation (4.21) to have

$$(s^{\alpha} + ib)\mathcal{N}_a(p_2(t))(s) = \mathcal{N}_a(p_1(t))(s) - (1-s)^{a-1}.$$

From (Table 3.4.1) , we get

$$\mathcal{N}_a(p_2(t))(s) = \frac{(1-ib)(1-s)^{a-1}}{(s^{\alpha}+ib)(s^{\alpha}-ib)} - \frac{(1-s)^{a-1}}{(s^{\alpha}+ib)}. \tag{4.22}$$

By applying the method of partial fraction decomposition, we can write the equation (5.2) as

$$\mathcal{N}_a(p_2(t))(s) = \frac{(1-ib)(1-s)^{a-1}}{2ib(s^{\alpha}-ib)} - \frac{(1-ib)(1-s)^{a-1}}{2ib(s^{\alpha}+ib)} - \frac{(1-s)^{a-1}}{(s^{\alpha}+ib)}.$$

Again, from the (Table 3.4.1), we have the following

$$p_2(t) = \frac{(1-ib)}{2ib}\left\{\widehat{e}_{\alpha,\alpha}(ib,(t-a)^{\overline{\alpha}}) - \widehat{e}_{\alpha,\alpha}(-ib,(t-a)^{\overline{\alpha}})\right\} - \widehat{e}_{\alpha,\alpha}(-ib,(t-a)^{\overline{\alpha}})$$

which equals to

$$p_2(t) = (1-ib)\widehat{\sin}_{\alpha,\alpha}(b,t-a) - \widehat{e}_{\alpha,\alpha}(-ib,(t-a)^{\overline{\alpha}}).$$

Since, $\Phi(t) = p_1(t)M_0 + p_2(t)M_1$ is a solution of initial value problem (4.16), $\Phi(t)$ can be written as

$$\Phi(t) = \begin{bmatrix} (1-ib)\widehat{e}_{\alpha,\alpha}(ib,(t-a)^{\overline{\alpha}}) & 0 \\ 0 & (1-ib)\widehat{e}_{\alpha,\alpha}(ib,(t-a)^{\overline{\alpha}}) \end{bmatrix}$$

$$+\left\{\left[(1-ib)\widehat{\sin}_{\alpha,\alpha}(b,t-a) - \widehat{e}_{\alpha,\alpha}(-ib,(t-a)^{\overline{\alpha}})\right] \times \begin{bmatrix} bi & 1 \\ -b^2 & bi \end{bmatrix}\right\}.$$

Doing some algebra and using (Table 3.4.1), we obtain

$$\begin{bmatrix} \widehat{\cos}_{\alpha,\alpha}(b,t-a) - b\widehat{\sin}_{\alpha,\alpha}(b,t-a) & \frac{1}{b}\widehat{\sin}_{\alpha,\alpha}(b,t-a) + \widehat{\cos}_{\alpha,\alpha}(b,t-a) \\ -b\widehat{\sin}_{\alpha,\alpha}(b,t-a) - b^2\widehat{\cos}_{\alpha,\alpha}(b,t-a) & \widehat{\cos}_{\alpha,\alpha}(b,t-a) - b\widehat{\sin}_{\alpha,\alpha}(b,t-a) \end{bmatrix}$$

Thus, the solution to (4.17) is given by $Y(t)$ as the following form

$$\begin{bmatrix} \widehat{\cos}_{\alpha,\alpha}(b,t-a) - b\widehat{\sin}_{\alpha,\alpha}(b,t-a) & \frac{1}{b}\widehat{\sin}_{\alpha,\alpha}(b,t-a) + \widehat{\cos}_{\alpha,\alpha}(b,t-a) \\ \nabla^{\alpha}\left(\widehat{\cos}_{\alpha,\alpha}(b,t-a) - b\widehat{\sin}_{\alpha,\alpha}(b,t-a)\right) & \nabla^{\alpha}\left(\frac{1}{b}\widehat{\sin}_{\alpha,\alpha}(b,t-a) + \widehat{\cos}_{\alpha,\alpha}(b,t-a)\right) \end{bmatrix}.$$

Therefore, the general solution to (4.16) has the following form

$$y(t) = c_1(\widehat{\cos}_{\alpha,\alpha}(b,t-a) - b\widehat{\sin}_{\alpha,\alpha}(b,t-a)) + c_2\left(\frac{1}{b}\widehat{\sin}_{\alpha,\alpha}(b,t-a) + \widehat{\cos}_{\alpha,\alpha}(b,t-a)\right)$$

which is equivalent to

$$y(t) = (c_1 + c_2)\widehat{\cos}_{\alpha,\alpha}(b,t-a) + \left(-bc_1 + \frac{c_2}{b}\right)\widehat{\sin}_{\alpha,\alpha}(b,t-a)$$

where c_1, c_2 are constants.

Chapter 5

PARAMETER ESTIMATIONS OF SIGMOIDAL MODELS

5.1. Parameter Estimation with Fractional Gompertz and Logistic Curves

In life and health sciences, there is an urgent need for the advancement and the widespread use of predictive and quantitative methods to improve delivery of health care and decrease economical and ethical costs. Although there are many ongoing developments in cancer research, there is still a lot to be known about its causes and treatments. Cancer is a class of diseases characterized by unregulated cell growth. There are many different kinds of cancers which can develop in almost any organ or tissue, such as the lung, colon, breast, skin, bones, or nerve tissue. Although there are many of treatment methods, such as surgery, radiotherapy and chemotherapy, the medical doctors should consider one significant parameter: Time. If researchers know which treatment method will lead to a better outcome in advance, the treatment will be easier and more successful. For a treatment to have a better outcome, mathematical models which simulate the rate of given tumor growth data need to be developed. Mathematical models provide theoretical insight into the underlying processes and improve the analysis and interpretation of experimental measurements and observations in biological and biomedical phenomena. Therefore, recent collaboration with mathematicians gives more insight on the scientific innovations for the clinical trials of the cancer research and the development of new treatments.

Tumor growth gives a special relationship between tumor size and time so it can be best described by sigmoidal curves. There are many approaches used in modeling growth behavior in biosciences. Gompertz, Logistic, Richards and Weibull curves are the ones which we will consider in our project. The aim of our project is to develop discrete fractional models of tumor growth for a given data set and to estimate parameters of these models in order to have better data fitting. We use discrete fractional calculus because we think the discrete counterpart of this mathematical theory will give a better and more accurate outcome. Discrete fractional calculus was first introduced by Kenneth S. Miller and Bertram Ross in 1988. More recently, the theory of nabla and delta fractional calculus have been developed [**6, 23, 24, 25**].

In [**5**], the following Gompertz fractional difference equation has been introduced with the $\Delta-$operator:

$$\Delta^{\alpha} y(t-\alpha+1) = (b-1)y(t) + a \tag{5.1}$$

where a, b are parameters and $\alpha \in (0,1]$ is the order of the fractional difference equation. α can also be considered as the third parameter. The graph of solution of equation (5.1) is a Gompertz curve and represents a sigmoid function. In this study, we model Gompertz and Logistic curves with $\alpha-$ order $\nabla-$ difference equations. In this project, we prefer to use $\nabla-$operator instead of Δ operator since the $\Delta^{-\alpha}$ maps functions defined on positive integers to functions defined on non-integers. This nature of the operator forces us to use a fractional delay difference equation. However, we will not have delay equations if we use $\nabla-$operator. Therefore, we claim that nabla fractional calculus will give us better data fitting than delta fractional calculus.

In order to estimate parameters for discrete nabla fractional Gompertz and Logistic curves, we use Mathematica. Then we compared continuous, discrete, continuous fractional and discrete fractional forms of these sigmoidal curves by using the tumor growth data for twenty-eight control mice. These control mice had inoculated tumors but did not receive any subsequent treatment. Tumor size was measured at 14HALO (hours after light on) daily until day 17. For these data, we collaborated with Dr. William J.M. Hrushesky who gave us permission to use his published data obtained in Medical Chronobiology Laboratory, University of South Carolina [**38**].

In addition, we used statistical computation techniques such as residual sum of squares and cross-validation to access and compare fitting and predictive performance of these models. Cross-validation method is a statistical method to show that our parameters serve for the best prediction of the tumor growth. We refer to this method in Section 5.3. At the end of our project, by interpreting these outcomes in a manner of biomedical science, we hope that our results will enhance time dependent cancer therapeutic study.

5.2. A Technique for Estimating the Performance of a Predictive Model: Cross-Validation

Cross-validation is one of the approaches to estimate the performance of a statistical model. This approach was first introduced in the 1930s [**19**]. Mosteller and Turkey [**20**], and then other scientists further developed the idea. A clear statement of cross-validation, which is similar to the current version of k-fold cross-validation, first appeared in [**21**]. In the 1970s, both Stone [**22**] and Geisser [**18**] employed the cross-validation method by choosing proper model parameters to estimate the performance of the model. Cross-validation is widely accepted in the data mining and machine learning community, and serves as a standard procedure for performance estimation and model selection.

In this chapter, we consider k-fold cross-validation. In k-fold cross-validation, the data is partitioned into k subsets. One of the k subsets is chosen for testing the model, namely validation set, and the remaining $k-1$ subsets are used as training data so it is called training set. The k-fold cross-validation process repeats k times. The advantage of this method is that all observations are used for both training and validation, and each observation is used for validation exactly once.

In our study, we use tumor growth data of 28 control mice for 17 days [**38**]. Therefore, we have $k = 17$ independent observations as a training set. In order to present an example, we consider rat id 140. In (Table 5.2.1), we give the experimental values $y_1, y_2, \ldots, y_{17}$. By using FindFit in Mathematica, we obtain the parameters and then we use these parameters in fractional Gompertz and Logistic curves, we have our observed values, which is the training set for cross-validation method. However, we divided these experimental values into 1000 since it is more convenient to obtain our parameters. We repeat this program 17 times, so we call it 17–fold cross validation. Each time we leave one experimental data, so our training set $G_1 = 16$. (Table 5.2.2) helps us to visualize the training set G, and validation set T for a random one (id 140) among 28 mice.

This statistical method is considered for both Gompertz and Logistic curves. First, we consider the Gompertz model which is given as the following form

$$\hat{y}(t) = ae^{-e(b-ct)}. \tag{5.2}$$

Take log of each side of the equation (5.2) to obtain

$$\hat{Y}(t) = \ln a - e^{b}(e^{-c})^{t},$$

where $\hat{Y}(t) = \log \hat{y}(t)$.

Using FindFit in Mathematica, we recalculate our parameters a, b, c and we do this process 17 times. Then, we use the same parameters in discrete fractional Gompertz

Days	*id* 140 *Experimental Values*
*Day*1	$y_1 = 23.275$
*Day*2	$y_2 = 62.953$
*Day*3	$y_3 = 112.665$
*Day*4	$y_4 = 124.712$
*Day*5	$y_5 = 215.730$
*Day*6	$y_6 = 325.260$
*Day*7	$y_7 = 285.120$
*Day*8	$y_8 = 354.760$
*Day*9	$y_9 = 218.295$
*Day*10	$y_{10} = 406.575$
*Day*11	$y_{11} = 481.665$
*Day*12	$y_{12} = 555.270$
*Day*13	$y_{13} = 643.552$
*Day*14	$y_{14} = 666.000$
*Day*15	$y_{15} = 893.000$
*Day*16	$y_{16} = 1050.000$
*Day*17	$y_{17} = 1209.600$

TABLE 5.2.1. Experimental Values Table of Rat id 140

curve. The discrete fractional Gompertz curve is given as

$$\check{Y}(t) = \ln a - e^b \sum_{n=0}^{\infty} (-c)^n \frac{(t-n+1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n+1)\alpha)}. \tag{5.3}$$

After estimating α, we set a, b, c and α in $\check{Y}(t)$ for each iteration. Finally, we use square residual sum method by considering y_i as observed value and $\check{Y}(t)$ as predicted value, therefore we have

$$e_i = \left(y_i - \check{Y}(t)\right)^2, \quad \text{for } 1 \le i \le 17.$$

The minimum square residual sum among 17 training set is our best data fitting for an exact value of α. We do the same steps for Logistic curve and we will compare the results in the following section.

5.3. Graphical Results and Comparisons

In this section, we visualize our claims by demonstrating table and graphs. First, we give continuous, discrete, continuous fractional and discrete fractional types of Gompertz and Logistic curves. In the previous section, using FindFit, we find our parameters a, b, c for continuous type of these sigmoidal curves. We use these parameters for discrete, continuous fractional and discrete fractional types of Gompertz and Logistic curves to estimate the range of α. Then, we compare our square residual sum results for these 4 types of Gompertz curve in (Table 5.3.1) and we follow the same route for the Logistic curve as shown in (Table 5.3.2). As a result, we state which sigmoidal curve serves better data fitting. In addition, we present the mean of data and we examine the minimum square residual sum and the value of α among the mean of continuous, discrete, continuous fractional and discrete fractional types of these sigmoidal models. Finally, we show the results of cross-validation for each control mice in (Table 5.3.1) and in (Table 5.3.2). Consider the continuous, discrete, continuous fractional and discrete fractional types of Gompertz curve.

$$Y(t) = \ln a - e^b (e^{-c})^t. \qquad (continuous)$$

$$Y(t) = \ln a - e^b (1-c)^t. \qquad (discrete)$$

$$Y(t) = \ln a - e^b \sum_{n=0}^{\infty} (-c)^n \frac{t^{(n+1)\alpha - 1}}{\Gamma((n+1)\alpha)}. \qquad (continuous\ fractional)$$

$$Y(t) = \ln a - e^b \sum_{n=0}^{\infty} (-c)^n \frac{(t-n+1)^{\overline{(n+1)\alpha - 1}}}{\Gamma((n+1)\alpha)}. \qquad (discrete\ fractional)$$

where $0 < \alpha < 1$.

We use the tumor growth data for 28 control mice and we did the data fitting for these four types of Gompertz curves above . As shown in (Table 5.3.1) we compare our square residual sum and the bold one indicates the minimum square residual sum. It is clearly seen that 21 of 28 mice have the better data fitting. Therefore, we can conclude that the other types (discrete, continuous fractional, discrete fractional) work better than the continuous type for Gompertz curve. Also, we obtain a range of α. For continuous fractional type, we concluded $0.9999 < \alpha < 0.99998$ and for discrete fractional type, we had $0.99941 < \alpha < 0.99998$. In some cases, we observe up to 5.44% better data fitting in fractional curves when they are compared to continuous case. Similarly, consider the continuous, discrete, continuous fractional and discrete fractional types of Logistic curve.

$$y(t) = \frac{a}{1 + e^b (e^{-c})^t}. \qquad (continuous)$$

$$y(t) = \frac{a}{1 + e^b (1-c)^t}. \qquad (discrete)$$

$$y(t) = \frac{a}{1 + e^b \sum_{n=0}^{\infty} (-c)^n \frac{t^{(n+1)\alpha - 1}}{\Gamma((n+1)\alpha)}}. \qquad (continuous\ fractional)$$

$$y(t) = \frac{a}{1 + e^b \sum_{n=0}^{\infty} (-c)^n \frac{(t-n+1)^{\overline{(n+1)\alpha - 1}}}{\Gamma((n+1)\alpha)}}. \qquad (discrete\ fractional)$$

where $0 < \alpha < 1$.

By following the same method that we used in Gompertz curve, which is based on data fitting for Logistic curves above, we can demonstrate (Table 5.3.2). Then we compare our square residual sum, we conclude that 19 of 28 mice have the minimum square residual sum. Again, we have the same result for Logistic curve: The other 3 types (discrete, continuous fractional, discrete fractional) work better than the continuous case. Also, we obtain a range of α. For continuous fractional type, we concluded that $0.9999 < \alpha < 0.99998$ and for discrete fractional type, we had $0.99955 < \alpha < 0.99998$. Note that we had better data fitting up to 0.01% in fractional curves when it is compared to continuous case. As shown in (Table 5.3.3), among 28 mice, 22 of them have better data fitting in Logistic curve. Thus, we conclude that **Logistic curve has better data fitting than Gompertz curve.**

Our last comparison is about mean of data as shown in (Table 5.3.4). We plotted all the data for Gompertz curve, we obtained the minimum square residual sum in continuous fractional type with $\alpha = 0.99996$. Also, we followed the same steps for Logistic curve and we had the minimum square residual sum in continuous fractional type with $\alpha = 0.99989$, as well. As shown in (Figure 5.3.1), the red line indicates the mean of data.

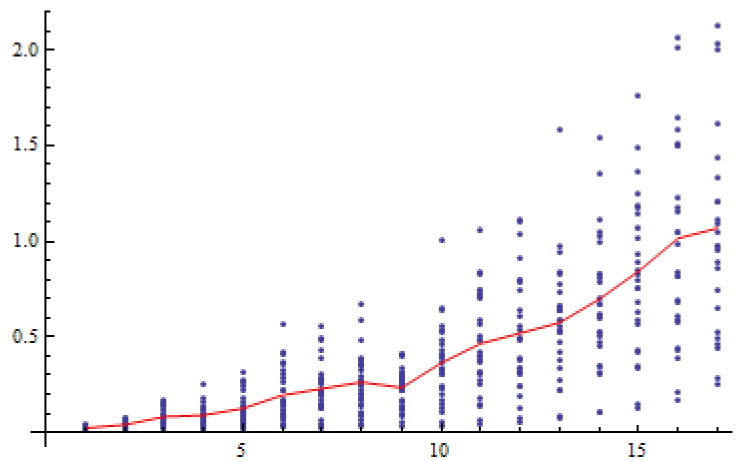

FIGURE 5.3.1. The Graph of Mean

Training Set	*Selected Observations*	*Test Set*	*Selected Observations*
G_1	$\{\check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_1	$\{\check{Y}_1\}$
G_2	$\{\check{Y}_1, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_2	$\{\check{Y}_2\}$
G_3	$\{\check{Y}_1, \check{Y}_2, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_3	$\{\check{Y}_3\}$
G_4	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_4	$\{\check{Y}_4\}$
G_5	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_5	$\{\check{Y}_5\}$
G_6	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_6	$\{\check{Y}_6\}$
G_7	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_7	$\{\check{Y}_7\}$
G_8	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_8	$\{\check{Y}_8\}$
G_9	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_9	$\{\check{Y}_9\}$
G_{10}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_{10}	$\{\check{Y}_{10}\}$
G_{11}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_{11}	$\{\check{Y}_{11}\}$
G_{12}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_{12}	$\{\check{Y}_{12}\}$
G_{13}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_{13}	$\{\check{Y}_{13}\}$
G_{14}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{15}, \check{Y}_{16}, \check{Y}_{17}\}$	T_{14}	$\{\check{Y}_{14}\}$
G_{15}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{16}, \check{Y}_{17}\}$	T_{15}	$\{\check{Y}_{15}\}$
G_{16}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{15}, \check{Y}_{15}, \check{Y}_{17}\}$	T_{16}	$\{\check{Y}_{16}\}$
G_{17}	$\{\check{Y}_1, \check{Y}_2, \check{Y}_3, \check{Y}_4, \check{Y}_5, \check{Y}_6, \check{Y}_7, \check{Y}_8, \check{Y}_9, \check{Y}_{10}, \check{Y}_{11}, \check{Y}_{12}, \check{Y}_{13}, \check{Y}_{14}, \check{Y}_{15}, \check{Y}_{16}\}$	T_{17}	$\{\check{Y}_{17}\}$

TABLE 5.2.2. Seventeen-fold cross-validation

id#	*Continuous* ($\alpha = 1$)	*Discrete* ($\alpha = 1$)	*Continuous Fractional*	*Discrete Fractional*	*Cross Validation*
21	**.09320807348**	**.09320807348**	.09320888453 $\alpha = 0.99998$	.09320888858 $\alpha = 0.99998$	.2207892335 $\alpha = 0.99998$
22	.3334771319	.3526360961	**.3334575178** $\alpha =$ **0.99993**	.3530311101 $\alpha = 0.99998$	.5214368910 $\alpha = 0.99998$
23	.1000671774	.09980773376	.1000676287 $\alpha = 0.99998$	**.09979266879** $\alpha =$ **0.99993**	.1532421764 $\alpha = 0.99998$
26	.01504232035	**.01504232032**	.01504253965 $\alpha = 0.99998$	.0154251431 $\alpha = 0.99998$	.02351250813 $\alpha = 0.9979$
27	.05130023627	.05130023802	**.05129960785** $\alpha =$ **0.99995**	.05129989001 $\alpha = 0.99996$	.06801248089 $\alpha = 0.99558$
28	.09931601289	.09880263381	.09928736635 $\alpha = 0.99993$	**.09877841550** $\alpha =$ **0.99994**	.1337456556 $\alpha = 0.99959$
29	.006923511052	.006860205304	.006923537032 $\alpha = 0.99998$	**.006858511228** $\alpha =$ **0.9999**	.01613621271 $\alpha = 0.99906$
30	.08160721616	**.08160721610**	.08160739012 $\alpha = 0.99998$	.08160739711 $\alpha = 0.99998$	.1360715953 $\alpha = 0.9222$
31	.08094058644	**.08094056469**	.08094155711 $\alpha = 0.99998$	.08094179820 $\alpha = 0.99998$	.1102760978 $\alpha = 0.9906$
32	.009827763484	.009782246634	.009827870273 $\alpha = 0.99998$	**.009779888886** $\alpha =$ **0.99994**	.01453112764 $\alpha = 0.99968$
33	.049822386468	.04977585660	.04981980492 $\alpha = 0.99995$	**.04977301896** $\alpha =$ **0.99996**	.06762074528 $\alpha = 0.99975$
34	**.3197937942**	.3197937956	.3197943089 $\alpha = 0.99998$	.3197942931 $\alpha = 0.99998$	.8865530616 $\alpha = 0.99998$
35	**.04155372344**	.04155372350	.04155427470 $\alpha = 0.99998$	.04155427770 $\alpha = 0.99998$	.1053246814 $\alpha = 0.99998$
136	**.02069668479**	.02069668480	.02069676294 $\alpha = 0.99998$	.02069675329 $\alpha = 0.99998$	.03282944107 $\alpha = 0.99962$
137	**.2605491124**	**.2605491124**	.2605500776 $\alpha = 0.99998$	2605502035 $\alpha = 0.99998$	.3833529677 $\alpha = 0.9433$
138	.1698807616	.1689739478	.1698590173 $\alpha = 0.99994$	**.1689272298** $\alpha =$ **0.99992**	.3082635821 $\alpha = 0.99997$
139	**.1444468940**	.1444468941	.1444475246 $\alpha = 0.99998$	.1444474669 $\alpha = 0.99998$	.2919893664 $\alpha = 0.99998$
140	.05353688898	.05339484131	.05352726755 $\alpha = 0.99998$	**.05338370310** $\alpha =$ **0.99993**	.06674413808 $\alpha = 0.99971$
141	.02861370688	.02861370722	**.02861367403** $\alpha =$ **0.99998**	.02861368016 $\alpha = 0.99998$	.03905906798 $\alpha = 0.99583$
142	.003715603052	**.003657090481**	.003713752094 $\alpha = 0.9999$	.003657168955 $\alpha = 0.99998$	.005161409916 $\alpha = 0.99936$
143	.1279857712	.1279857720	.1279818439 $\alpha = 0.9999$	**.1279762340** $\alpha =$ **0.99941**	.2004571352 $\alpha = 0.99998$
144	**.2024468349**	**.2024468349**	.2024473515 $\alpha = 0.99998$	.2024473387 $\alpha = 0.99998$	.2781483417 $\alpha = 0.99213$
145	.2808836510	.2808836504	**.2808836341** $\alpha =$ **0.99998**	.2808836470 $\alpha = 0.99998$	.4565218530 $\alpha = 0.99998$
146	.2426379750	.2419522531	.2425953572 $\alpha = 0.99993$	**.2419516991** $\alpha =$ **0.99998**	.3914378973 $\alpha = 0.99998$
147	.03264913983	.03264913977	.03264852011 $\alpha = 0.9999$	**.03264789880** $\alpha =$ **0.99967**	.06370510263 $\alpha = 0.9964$
148	.1113379847	**.1113379834**	.1113385662 $\alpha = 0.99998$	.1113384961 $\alpha = 0.99998$	2.025038476 $\alpha = 0.9987$
149	.1394980991	**.1394980990**	.1394988266 $\alpha = 0.99998$	.1394988321 $\alpha = 0.99998$	.6069317878 $\alpha = 0.99998$
150	.3029550047	.3029550052	**.3029542604** $\alpha =$ **0.99991**	.3029547992 $\alpha = 0.99995$	.4128975419 $\alpha = 0.99551$

TABLE 5.3.1. Data Analysis for Gompertz Curve

id#	*Continuous* ($\alpha = 1$)	*Discrete* ($\alpha = 1$)	*Continuous Fractional*	*Discrete Fractional*	*Cross Validation*
21	.07720652926	**.07720652925**	.07720928346 $\alpha = 0.99998$	.07721014491 $\alpha = 0.99998$	.1627499337 $\alpha = 0.99085$
22	.3258675033	**.3258675032**	.3258699296 $\alpha = 0.99998$	.3258699296 $\alpha = 0.99998$	.5278740791 $\alpha = 0.99203$
23	.09475605652	**.09474826414**	.09475734260 $\alpha = 0.99998$	.09474977198 $\alpha = 0.99998$	.1478667342 $\alpha = 0.99499$
26	**.01325261841**	**.01325261841**	.01325289039 $\alpha = 0.99998$	.013252893291 $\alpha = 0.99998$	.02040913168 $\alpha = 0.99637$
27	.05247847498	.05247847533	**.05247699452** $\alpha = \mathbf{0.9999}$	.05247709649 $\alpha = 0.99987$	.07871369483 $\alpha = 0.99898$
28	.09270204942	**.09269992014**	.09270298592 $\alpha = 0.99998$	.09270103236 $\alpha = 0.99998$	.1239232970 $\alpha = 0.99973$
29	.006022851218	**.006022041787**	.006023443946 $\alpha = 0.99998$	.006022746136 $\alpha = 0.99998$	.1535358385 $\alpha = 0.99998$
30	**.07976180993**	.07976181000	.07976194294 $\alpha = 0.99998$	.07976199099 $\alpha = 0.99998$	.1283931650 $\alpha = 0.97403$
31	.07471839928	**.07471837868**	.07472871314 $\alpha = 0.99998$	.07473636197 $\alpha = 0.99998$	.1098503228 $\alpha = 0.99972$
32	.008850530144	**.008848677223**	.008850727738 $\alpha = 0.99998$	.008848923819 $\alpha = 0.99998$	.01210642602 $\alpha = 0.99998$
33	**.04950751166**	.04950751171	.04950755291 $\alpha = 0.99998$	.04950759522 $\alpha = 0.99998$	.07332516252 $\alpha = 0.99926$
34	.3125897050	**.3125897049**	.3125907150 $\alpha = 0.99998$	.3125909360 $\alpha = 0.99998$	.7278245829 $\alpha = 0.96914$
35	**.03150005505**	.03150005506	.03150150833 $\alpha = 0.99998$	.03150195893 $\alpha = 0.99998$	.07327269633 $\alpha = 0.99676$
136	.02091431694	.02091431701	**.02091409925** $\alpha = \mathbf{0.9999}$	.02091415007 $\alpha = 0.99993$	.02948826642 $\alpha = 0.99998$
137	**.2552786642**	**.2552786642**	.2552798771 $\alpha = 0.99998$	.2552803705 $\alpha = 0.99998$	.3765101063 $\alpha = 0.97436$
138	.1511995277	**.1511871712**	.1512054896 $\alpha = 0.99998$	.1511934465 $\alpha = 0.99998$	.2661900260 $\alpha = 0.99998$
139	**.1389586529**	.1389586533	.1389594712 $\alpha = 0.99998$	.1389596259 $\alpha = 0.99998$	.2768472093 $\alpha = 0.9821$
140	.05156900468	.05156691145	.05156662982 $\alpha = 0.9999$	**.05156530341** $\alpha = \mathbf{0.99978}$	.06323411271 $\alpha = 0.99998$
141	.02903359507	.02903359504	**.02903328519** $\alpha = \mathbf{0.99991}$	.02903335218 $\alpha = 0.99993$	.04179439341 $\alpha = 0.99851$
142	.003026763981	**.003026069415**	.003027114096 $\alpha = 0.99998$	.003026455572 $\alpha = 0.99998$	.1504494057 $\alpha = 0.57145$
143	.1301209917	.1301209917	.1301175368 $\alpha = 0.9999$	**.1301131682** $\alpha = \mathbf{0.99955}$	.2003043765 $\alpha = 0.98872$
144	**.1942945348**	.1942945349	.1942958863 $\alpha = 0.99998$	.1942961157 $\alpha = 0.99998$	.2895154726 $\alpha = 0.98774$
145	.2850180223	.2850180223	**.2850144884** $\alpha = \mathbf{0.9999}$	.2850148061 $\alpha = 0.9999$	.4489403977 $\alpha = 0.98795$
146	.2326031700	**.2325981230**	.2326041047 $\alpha = 0.99998$	.2325994137 $\alpha = 0.99998$	.3904706165 $\alpha = 0.99063$
147	.03703599137	.03703599146	.03703265619 $\alpha = 0.9999$	**.03702966546** $\alpha = \mathbf{0.99971}$	.08261403356 $\alpha = 0.99998$
148	**.1077952924**	.1077952925	.1077958616 $\alpha = 0.99998$	.1077959516 $\alpha = 0.99998$	.1601409647 $\alpha = 0.9972$
149	**.1242478059**	**.1242478059**	.1242503947 $\alpha = 0.99998$	.1242514016 $\alpha = 0.99998$	.4630737389 $\alpha = 0.96316$
150	.3004093946	.3004093948	**.3004077982** $\alpha = \mathbf{0.99991}$	**.3004077982** $\alpha = \mathbf{0.99995}$	.4056782620 $\alpha = 0.99892$

TABLE 5.3.2. Data Analysis for Logistic Curve

id#	*Gompertz Curve*	*Logistic Curve*
21	.09320807348	**.07720652925**
22	.3334575178 α = 0.99993	**.3258675032**
23	.09979266879 α = 0.99993	**.09474826414**
26	.01504232032	**.01325261841**
27	**.05129960785** α = **0.99995**	.05247699452 α = 0.9999
28	.09877841550 α = 0.99994	**.09269992014**
29	.006858511228 α = 0.9999	**.006022041787**
30	.08160721610	**.07976180993**
31	.08094056469	**.07471837868**
32	.009779888886 α = 0.99994	**.008848677223**
33	.04977301896 α = 0.99996	**.04950751166**
34	.3197937942	**.3125897049**
35	.04155372344	**.03150005505**
136	**.02069668479**	.02091409925 α = 0.9999
137	.2605491124	**.2552786642**
138	.1689272298 α = 0.99992	**.1511871712**
139	.1444468940	**.1389586529**
140	.05338370310 α = 0.99993	**.05156530341** α = **0.99978**
141	**.02861367403** α = **0.99998**	.02903328519 α = 0.99991
142	.003657090481	**.003026069415**
143	**.1279762340** α = **0.99941**	.1301131682 α = 0.99955
144	.2024468349	**.1942945348**
145	**.2808836341** α = **0.99998**	.2850144884 α = 0.9999
146	.2419516991 α = 0.99998	**.2325981230**
147	**.03264789880** α = **0.99967**	.03702966546 α = 0.99971
148	.1113379834	**.1077952924**
149	.1394980990	**.1242478059**
150	.3029542604 α = 0.99991	**.3004077982** α = **0.99991**

TABLE 5.3.3. Gompertz vs.Logistic

	Gompertz Curve	*Logistic Curve*
Continuous (α=1)	0.01467378953	0.01511114565
Discrete (α=1)	0.01467353864	0.01511114569
Continuous Fractional	**0.01467346239** $\alpha = 0.99996$	**0.01511043545** $\alpha = 0.99989$
Discrete Fractional	0.01467360877 $\alpha = 0.99997$	0.01511068031 $\alpha = 0.99993$

TABLE 5.3.4. Gompertz and Logistic Curve Mean Table

Chapter 6

CONCLUSION AND FUTURE WORK

Discrete fractional calculus is an extended form of discrete calculus. More particularly, discrete calculus considers integer order, but fractional calculus enhance the order to include all positive real numbers. In this mathematical theory, there are still many open questions waiting to be studied. In this thesis, we continued to develop nabla fractional calculus. We also showed our developments by demonstrating the graphs of $t^{\alpha}, t^{\overline{\alpha}}, t^{\underline{\alpha}}$. We were interested in the following sigmoidal curves: Gompertz and Logistic. In order to estimate the parameters of Gompertz and Logistic curves, we used Mathematica. After obtaining these parameters, we compared continuous, discrete, continuous fractional and discrete fractional type of these sigmoidal curves. As a result, we concluded that the discrete version of these curves have better data fitting. In addition, we used some statistical methods such as square residual sum and $k-$fold cross validation because making a prediction will enlighten our time dependent cancer therapeutic study. On the other hand, in Chapter 2, we focused on completely monotonic functions on discrete domain using nabla operator. Furthermore, we proved some basic theorems of this concept. Then, by using $\mathcal{N}$-transform, we proved some important results and then we established $\mathcal{N}$-transform table. This table is a great tool for us to find the solutions of up to first or second order of nabla fractional difference equation. Finally, in Chapter 4, we proved some basic theorems about nabla fractional calculus. We proved that, if the set of solutions of up to $n-th$ order nabla fractional equation is linearly independent, then the Casoration is not identically equal to zero. Then, we considered up to second order linear nabla fractional equation and we examined the solutions of the equation by considering the characteristic roots of the characteristic equation case by case.

For the future work, there are still some open questions to be considered. First, we will concentrate on completely monotonic functions on discrete domains. From the literature, we see some papers which claim that there is a relationship between the concept of complete monotonicity and the stability of Mittag-Leffler function. We plan to enhance this idea to the stability of discrete Mittag-Leffler function using complete monotonicity of this special function on discrete domain. Also, in Section 4.2, we stated the discrete fractional exponential function is nonzero. Our goal is to prove this claim, but this work requires some effort. On the other hand, our project will continue next year. Richards and Weibull models will be considered in the same route. At the end of the project, among four main sigmoidal curves, it will be stated which model works best or has the best data fitting.

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