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On Nullification of Knots and Links

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ON NULLIFICATION OF KNOTS AND LINKS

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ON NULLIFICATION OF KNOTS AND LINKS

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I dedicate this thesis to my parents, Alex and Sheri, for their support and patience. Also, I dedicate this work to my advisor, Claus Ernst, for allowing me the chance to work with him.

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PREFACE

This thesis is submitted in partial fulfilment of the requirements for a Master of Science Degree in Mathematics at Western Kentucky University. It contains work done from January 2010 to April 2012. The results included here are a continuation of joint work with Claus Ernst of Western Kentucky University, Yuanan Diao of the University of North Carolina Charlotte, and Andrezj Stasiak of Centre Intégratif de Génomique (see [13] and [16]). Original to this thesis are the results involving the four genus invariant, the nullification bounds on the rational links and torus links, and the table of nullification numbers for prime knots of 10 crossings and prime links up to 9 crossings.

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ON NULLIFICATION OF KNOTS AND LINKS

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Directed by Dr. Claus Ernst, Dr. John Spraker, and Dr. Uta Ziegler

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Motivated by the action of XER site-specific recombinase on DNA, this thesis will study the topological properties of a type of local crossing change on oriented knots and links called nullification.

One can define a distance between types of knots and links based on the minimum number of nullification moves necessary to change one to the other. Nullification distances form a class of isotopy invariants for oriented knots and links which may help inform potential reaction pathways for enzyme action on DNA. The minimal number of nullification moves to reach a θ-component unlink will be called the θ-nullification number.

This thesis will demonstrate the relationship of the nullification numbers to a variety of knot invariants, and use these to solve the θ-nullification numbers for prime knots up to 10 crossings for any θ. A table of nullification numbers for oriented prime links up to 9 crossings is also presented, but not all cases are solved.

In addition, we examine the families of rational links and torus links for explicit results on nullification. Nullification numbers of torus knots and a subfamily of rational links are solved. In doing so, we obtain an expression for the four genus of said subfamily of rational links, and an expression for the nullity of any torus link.

Chapter 1

INTRODUCTION

1.1 Biological Motivation

Deoxyribonucleic acid (DNA) is the foundation for nearly all life known on Earth. This foundation only acts via a vast array of enzyme intermediaries. Some of these enzymes (e.g. topoisomerase and recombinase) act on their substrate to reconfigure the geometrical/topological arrangement of the DNA helices. Both topoisomerase and recombinase have been observed to help separate (decatenate) newly replicated DNA from its pair [20, 52].

Topoisomerase is well studied, but only recently has recombinase action been seen to participate in the decatenation process [6]. Interestingly, one experiment observes that the recombinase actions to do this proceed in an optimal way (i.e., needing the minimal number of recombinase actions) [20]. For any variety of DNA topology what can be said about the optimal reaction pathways for topoisomerase and recombinase?

To better understand such gross-level theoretical constraints on the results of these enzymes' action, we will follow the common approach of treating DNA as a closed loop in three space. Instead of including the fine detail of the double helical structure one can model the helix by its core curve. Furthermore, as DNA transcription proceeds in a predictable direction, one can assign a natural orientation to this curve.

With this approach one can study enzyme action by investigating a

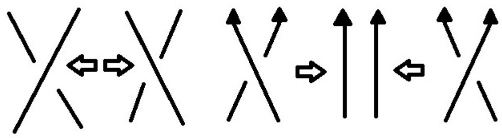


Figure 1.1: (left) topoisomerase action, (right) recombinase action

representative operation on this oriented closed loop. Topoisomerase action can be modeled as a strand passage, and recombinase action with a slightly more involved strand reconfiguration [6] (Figure 1.1). The topological consequences of such operations on curves can be investigated with the large assortment of mathematical machinery found in the field of knot theory. This thesis will be solely concerned with the number of recombinase actions necessary to take a given substrate to reach some product, i.e. the number of recombination operations needed to convert between topologically different curve types.

To demonstrate the mathematical results of pertinence we will need to first develop the prerequisite knot theory. After this we will establish a variety of bounds on the number of recombination operations to move between tabulated curve types. This will lead to definitive answers for minimal operation counts for many curve types. We will then conclude with a table compiling the results of the paper.

1.2 Basic Knot Theory

The following definitions and results of this section can be found in almost every book on knot theory (e.g. [1, 7, 12, 35, 40, 47]). Some of the more advanced knot theory requires additional topological results dealing with algebraic topology, i.e., manifold classification and surgery, homotopy/homology groups, among other topics. Thorough descriptions of these methods are beyond the scope of this thesis, and so when needed we will refer the reader to necessary texts.

1.2.1 3-Dimensional Interpretation

As one might expect, a knot is simply a closed and possibly tangled loop in three dimensional space. A link is a finite set of such loops that can be tangled with each other as well. Our intuition for manipulations of these loops carries over exactly to how we study these loops in mathematics. But as with all mathematical endeavors we must assign precise mathematical notation for the sake of rigor and clarity. In this vein we present the following set of definitions [12].

Homeomorphism A *homeomorphism* is a continuous function between two topological spaces with a continuous inverse function.

Knot A *knot* $K \subset \mathbb{R}^3$ is a subset of points homeomorphic to a circle, S^1.

Link A *link* is a finite disjoint union of knots: $L = K_1 \cup \cdots \cup K_n$. Each knot K_i is a *component* of the link. The number of components of L is called the *multiplicity* of the link, and is denoted $\mu(L)$. A subset of the components of L (embedded in the same way) is called a *sublink* of L.

Homotopy A *homotopy* of a space $X \subset \mathbb{R}^3$ is a continuous map $h : X \times [0,1] \to \mathbb{R}^3$ where h restricted to level 0 ($h_0 : X \times \{0\} \to \mathbb{R}^3$) is the identity map.

Isotopy An *isotopy* is a homotopy where each $h_t : X \times \{t\} \to \mathbb{R}^3$ is one-to-one.

Ambient Isotopy An *ambient isotopy* of a space $X \subset \mathbb{R}^3$ is an isotopy of $\mathbb{R}^3$.

Ambient Isotopic Two knots K_1 and K_2 are *ambient isotopic* if there is an ambient isotopy $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $h(K_1, 0) = h_0(K_1) = K_1$ and $h(K_1, 1) = h_1(K_1) = K_2$.

We see that deforming knots and links can be accomplished mathematically by an ambient isotopy. Why not homotopy or isotopy alone? Homotopy allows a curve to pass through itself in the mapping. Hence we introduce isotopy to restrict the mappings to be one-to-one. But isotopy of the knot's curve could allow an infinite compression to a point. To prevent this behavior we use an isotopy on the entire (ambient) space, thus we have an ambient isotopy. Ambient isotopy is an equivalence relation on knots, and each equivalence class is called a *knot type*. Often the term knot is abused to mean the knot's equivalence class as well as a particular representative.

These definitions of knot and link are very general, but they allow constructions of so-called *wild knots* that are quite unintuitive (see Figure 1.2 for an example). Their tangles can become infinitely small, and so understanding even the simplest manipulations is difficult. To eliminate this concern we restrict our attention to *tame knots*, a knot that is ambient isotopic to a simple closed polygon with finitely many edges in the knot's respective space. *Tame links* are defined with an extension analagous to the above definitions of knot and link. From now on we will only consider tame knots and links under ambient isotopies, but we will omit "tame" and "ambient" for brevity.

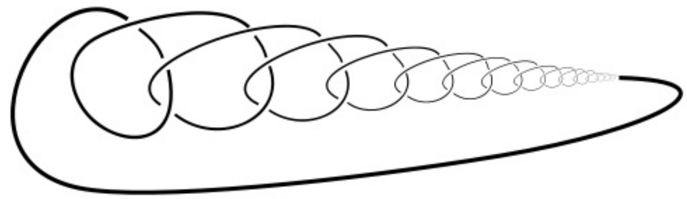


Figure 1.2: A wild knot [50]

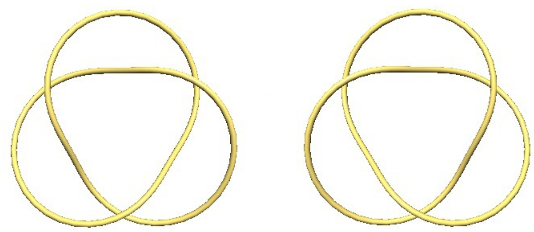


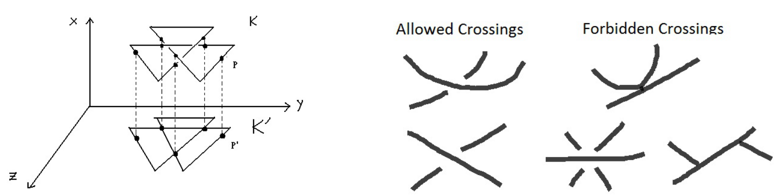
Figure 1.3: A knot and its mirror image

The definitions of knot and link can be extended to include an assignment of *orientation* to each component of a link. This orientation is again quite intuitive. You consistently apply a traversal direction to the mapping into 3-space for each component. A link is equivalent to another link if there exists an isotopy between the two that preserves orientation. If a link can be shown equivalent to itself when all orientations are reversed, the link is called *reversible*, else it is *nonreversible*. A link L with all orientations reversed (the *reverse* of L) is denoted $-L$. If a link can be shown equivalent to its *mirror image* (see Figure 1.3 for an example of a mirror image) it is called *amphicheiral*, else it is *chiral*. The mirror image of a link L is denoted $L!$. If a link L is nonreversible and chiral, but $L = -L!$ then it is called *(-)amphicheiral*.

This establishes the most basic vocabulary for knot theory. With the notion of knot equivalency defined we wish to only consider the distinct sets of knots i.e., those knots that are not isotopic to one another. The question becomes how we establish that two knots are distinct. Using 3-dimensional pictures, however, can be unwieldy. To assist us we will need to develop easier techniques to visualize knots.

1.2.2 Planar Projections

Visualizing knots in less than three dimensions requires the notion of a knot's planar projections. As the name implies this is a projection of the knot onto a 2-dimensional plane (see Figure 1.4(a)), but we have a few restrictions as shown in Figure 1.4(b). These restrictions imply that any projection has only finitely many double-points (crossings) from strings that cross transversely in the projection.



(a) (b)

Figure 1.4: (a) Planar projection of a knot [34], (b) Projection restrictions

If our projection follows these restrictions we will call the projection *regular*. Notice that at each double-point of the projection we redraw the image to denote an "over" or "under" pass. We use the regular projections to more easily visualize the knot without needing three dimensional rendering. But how do isotopies affect such projections? After all, to establish knot equivalency we need to know if they are isotopic to one another. The answer to this was given by Reidemeister[1]. All isotopies are equivalent to a combination of finitely many manipulations of the types as shown in Figure 1.5.

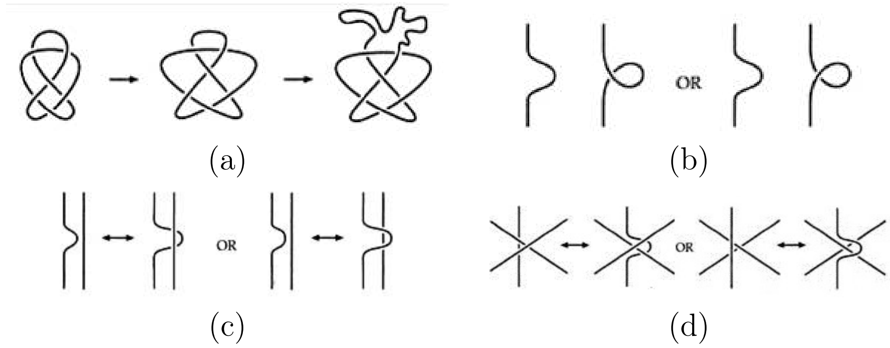


Figure 1.5: (a) segment stretching, shrinking, or deformation (b) Reidemeister Move (RM) type I (c) RM II (d) RM III [1]

The planar projections also have some basic vocabulary associated with them. Hence we give the following definitions. The *unknot* is a knot that has a diagram without any crossings. An *unlink* has a diagram that is a disjoint union of unknots. We can also call the unknot or unlink a *trivial knot/link*. We will denote an unlink with n components as U^n. Given an oriented knot or link diagram, a *positive crossing* is an oriented crossing as shown in the left of Figure 1.6. Similarly, a *negative crossing* of an oriented knot/link diagram is an oriented crossing as shown in the right of Figure 1.6. If a projection is such that every pair of consecutive crossings as we traverse the diagram changes from over to under or vice versa, we call the projection an *alternating projection*. If a knot has an alternating projection then it is an *alternating knot*. If no such projection exists then the knot is *non-alternating*. If an alternating projection does not have any easily removable crossings as shown in Figure 1.7 then the projection is called *reduced*.

One other important concept in knot manipulation is called the *connected sum*. This operation is depicted in Figure 1.8, and is denoted with the binary operator #. This operator leads to the idea of a *prime* knot or link. If a link cannot be isotoped to yield a projection that is a connected sum then we say the link is prime. As all links can be described in terms of the connected sum of prime links, we will primarily focus our attention on distinct prime links.

Now that we have established the methods and vocabulary associated with knot diagrams and the finite number of manipulation types we can return to the question of distinguishing knots.

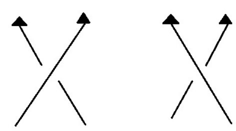


Figure 1.6: +1 or right-handed crossing (on left), -1 or left-handed crossing (on right)

Figure 1.7: Non-reduced knot projections [1]

1.2.3 Invariants of Knots

In order to differentiate knots we need a tool that detects differences in knots despite the existence of an infinite range of possible projections. So we need an object that is insensitive to projection change, or in other words, that remains

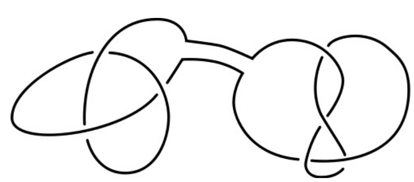


Figure 1.8: $K_1 \# K_2$ [51]

unchanged by ambient isotopy. Equivalently, the computation of this object is independent of the particular knot diagram we are using for that knot type. Such objects are called *knot invariants*, and topologists have created many.

One example is the *crossing number* which is defined as the minimal number of crossings for a knot across all projections. So considering all projections we get a number that represents how few crossings can appear in a diagram. If two knots have a different crossing number then they are different knots. Sadly the converse is not necessarily true. This is the usual dilemma for knot invariants. We can construct several examples of invariants under isotopy that allow us to differentiate knots with different invariant values, but we can usually say much less about knots that have the same value invariant.

If we further consider the signs of the crossings in the diagram we can obtain another invariant called the *linking number*. Given an oriented diagram D for any crossing, c, in D we define

$$\epsilon(c) = \begin{cases} +1 & if\, c\, is\, a\, positive\, crossing \\ \\ -1 & if\, c\, is\, a\, negative\, crossing \end{cases}.$$

Writing $c \in D$ will mean that c is a crossing in D. With this notation we can define the linking number of two component links.

Linking Number Let D be an oriented diagram of a 2-component link $K_1 \cup K_2$, and let D_i denote the component of D corresponding to K_i. The crossings of D are of three types: D_1 with itself, D_2 with itself, and D_1 with D_2. The last group will be denoted $D_1 \cap D_2$. The *linking number* of D_1 with D_2 is defined to be $lk(D_1, D_2) = {}^1\!/\!_2 \sum_{c \in D_1 \cap D_2} \epsilon(c)$.

This can be proven to be independent of the diagram (a knot invariant), and so we can write $lk(K_1, K_2)$ instead of $lk(D_1, D_2)$. For an n-component link, $L = K_1 \cup \cdots \cup K_n$, we can define the linking number similarly as: $lk(L) = \sum_{i<j} lk(K_i, K_j)$.

More advanced invariant constructs involve topological characterization of the surfaces/spaces that can be associated with the knot. A known result in knot theory is that for any knot there exists an orientable surface with the knot as its boundary (a spanning surface) [12, Thm. 5.1.1]. Such a surface is called a *Seifert surface*. For a fixed knot/link type L we can consider the minimal genus of all such surfaces across all projections to yield the knot *3-genus* ($g(L)$) invariant. There exists an algorithm to create Seifert surfaces, but this algorithm isn't necessarily capable of creating the minimal genus surface. The algorithm simultaneously nullifies (nullification is shown in Figure 1.14) all crossings in a diagram, and connects the remaining circles (called *Seifert circles*) by twisted bands that correspond to the crossing. An example of this process for the trefoil knot is shown in Figure 1.9 (the surface is drawn with Jarke van Wijk's program SeifertView [48, 49]). We can even extend the idea of these constructs into 4-dimensions by looking at the properties of spanning surfaces when the surface is smoothly embedded in four space. The minimal genus of such surfaces is the *4-genus* ($g^*(L)$).

Properties of the attached surface lead to other invariants involving algebraic

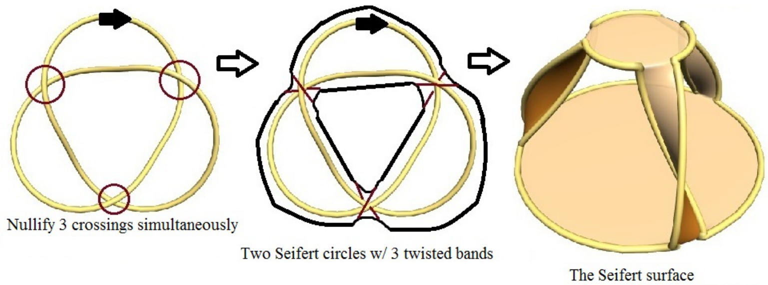


Figure 1.9: Constructing a Seifert surface

structures. Of particular use to this thesis is an invariant called the *Seifert matrix*. Let us construct this invariant following the procedure as described in Cromwell [12, Ch. 6]. As any knot or link is the boundary of an orientable surface, we can lift elements of the homology group to either side of a thickened version of said surface. As the basis elements of the homology group are loops, information about the surface can be obtained by looking at how the loops and their lifts connect. These connections are summarized with their linking numbers placed into a matrix (the Seifert or linking matrix).

More formally, if given a spanning surface F, we have a homeomorphism $b : F \times [-1, 1] \to \mathbb{R}^3$ such that $b(F \times \{0\}) = F$ and $b(F \times \{1\})$ is on the positive side of F. Then any subset $X \subset F$ can be lifted to either side of F. We denote $X^+ = b(X \times \{1\})$ and $X^- = b(X \times \{-1\})$. As mentioned above, we want to see how this mapping links the elements of the homology group. We can then form a mapping $\Theta : H_1(F) \times H_1(F) \to \mathbb{Z}$ where $H_1(F)$ denotes the homology group of the surface. In particular for some elements $a, b \in H_1(F)$, $\Theta(a, b) \to lk(a, b^+)$. We form a matrix of these linking numbers of the basis loops of $H_1(F)$, and will call this the Seifert matrix.

Let's calculate the Seifert matrix for the trefoil shown in Figure 1.9. From the figure we see the Seifert algorithm creates a surface F with two Seifert circles connected by three crossing bands. To establish a basis for $H_1(F)$ we will use what is known as a *Seifert graph.* This is a graph G that is made by letting each Seifert circle be represented by a vertex, and each crossing band by an edge. Next we choose a spanning tree T for G. For each edge $e_i \in (G - T)$, the graph $T \cup e_i$ contains a unique circuit. The set of these circuits form a basis for the homology. For the trefoil our spanning tree is a single edge, and we get two circuits, a and b (see Figure 1.10). Placing a and b and their lifts in their respective positions around F gives us a picture as in Figure 1.11. The orientations for a and b are arbitrary, but once chosen the orientations for a^+ and b^+ must agree with a and b respectively. We find that for our choices $lk(a, a^+) = 1$, $lk(a, b^+) = -1$, $lk(b, b^+) = 1$, and $lk(b, a^+) = 0$. So our Seifert matrix is a matrix of these linking numbers. So our Seifert matrix is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Since a knot can be spanned by several (topologically distinct) surfaces we need to work a little harder to make this matrix a knot invariant. We need the invariant to be insensitive to any surgical changes that preserve the boundary, any isotopy of the surface, and any change of basis in the underlying homology group. The operations that do this are the typical change of basis transformations from linear algebra, and an enlargement/reduction scheme where matrices of the

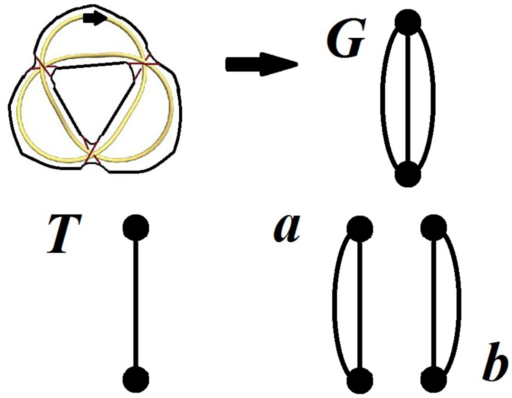


Figure 1.10: Seifert graph for trefoil to find basis loops

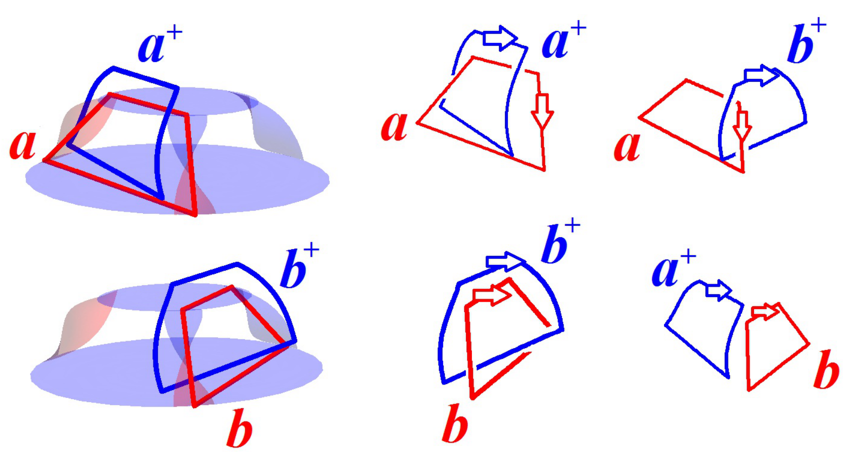


Figure 1.11: Placing basis loops and finding their linking numbers

following type are considered equivalent to a given Seifert matrix M:

$$\begin{pmatrix} & & & * & 0 \\ & M & & \vdots & \vdots \\ & & & * & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \; or \; \begin{pmatrix} & & & 0 & 0 \\ & M & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Here the asterisks are unknown integers. These equivalences are called S-equivalence, and the S-equivalence class of matrices is a knot invariant. For more details on Seifert matrices and S-equivalence see [7, 12, 29, 40]. Furthermore, we can obtain other invariants from these matrices. Given a Seifert matrix, M, for a link, L, the following are knot invariants (M^T denotes the transpose of M):

Signature σ The number of positive eigenvalues minus the number of negative eigenvalues of $M + M^T$: $\sigma(L) = \sigma(M + M^T)$.

Nullity ω The number of zero eigenvalues of $M + M^T$ plus one: $\omega(L) = \omega(M + M^T) + 1$.

The reader might see the tedious nature of computing these invariants. With links we will need to consider all orientation choices of the subcomponents, but we will need these invariants. Luckily we have two other results that will let us get around computing the Seifert matrices and their eigenvalues for all orientation choices. Firstly nullity is insensitive to orientation of the components [28]. Secondly although signature isn't orientation insensitive, it is proven that $\sigma(L) + lk(L)$ is a knot invariant [39]. This means that we can compute the signature for one orientation choice, and then we only need to compute the linking number for the other orientations to find the new signature. So, if we reverse orientations on any number of components in L to get a new link L', then $\sigma(L) + lk(L) = \sigma(L') + lk(L')$. As this sum is independent of orientation then this is an invariant of unoriented links which we will denote by $\xi(L)$.

Other commonly used invariants are the *knot polynomials* [7, 12, 40, 47]. They are simply ways of associating polynomials in one or more variables to a knot. There are several types and approaches for their calculation. Useful for this thesis are the *Q polynomial*, *Jones polynomial*, and *HOMFLYPT polynomial*. They are defined in terms of skein relations. Skein relations represent sets of links that are identical except near one point. The Jones and HOMFLYPT polynomials use the oriented skein relations as in Figure 1.12. The Q polynomial uses the unoriented skein relation as in Figure 1.13. Their definitions are:

Jones Polynomial

$$V(U^n; t) = (-t^{1/2} - t^{-1/2})^{(n-1)}$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = (t^{1/2} - t^{-1/2})V(L_0; t)$$

HOMFLYPT Polynomial

$$P(U^n; v, z) = \left(\frac{v^{-1} - v}{z}\right)^{n-1}$$

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z)$$

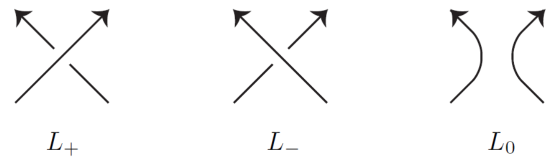


Figure 1.12: An oriented skein triple

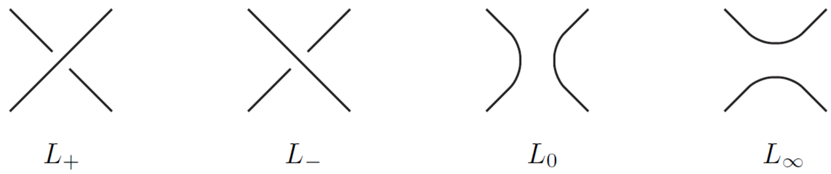


Figure 1.13: An unoriented skein quadruple

Q Polynomial

$$Q(U^n; z) = (2z^{-1} - 1)^{n-1}$$

$$Q(L_+; z) + Q(L_-; z) = z(Q(L_0; z) + Q(L_\infty; z))$$

Of central interest to this thesis are the invariants known as *unknotting number* and *nullification number*. The former is defined as the minimum number of unknotting moves (or crossing changes) across all projections that change a projection of the knot/link into the unknot/unlink. Similarly, the latter is the minimum number of nullification moves (or smoothings) across all projections that change a projection of the knot/link into a trivial link. These moves are depicted in Figure 1.14. The most general definitions of unknotting/nullification number allow ambient isotopy between steps in the unknotting/nullification process. It is these definitions that we will be considering. One can see that the crossing changes in 1.14 are exactly the same as in Figure 1.1. As such our question of the number of

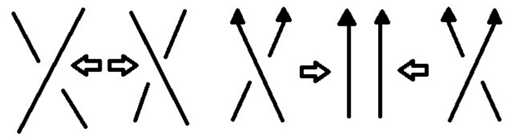


Figure 1.14: (left) unknotting move, (right) nullification move

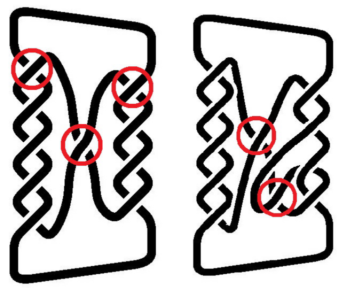


Figure 1.15: Unknotting 10_8 in two moves requires a non-minimal diagram

recombinase moves is now a question of nullification number.

These invariants, even though easy to define, are notoriously difficult to compute. There are knots with just 10 crossings whose unknotting numbers are unknown. The difficulty is that a minimal sequence of unknotting or nullification moves might not occur in the minimal projections in a knot table. For example, consider the knot 10_8. The best one can do in a minimal projection is an unknotting sequence of length three. In a non-minimal diagram, however, we can find a sequence of length two (see Figure 1.15). This is the true unknotting number for 10_8 [4, 42]. Similarly this happens for nullification numbers. Take 10_{22} for example. Any minimal projection of 10_{22} can be shown to need six nullification moves [46]. But a non-minimal projection can be nullified in one step [13](see Figure 1.16).

With the aim of the thesis being the tabulation of nullification numbers, we need to study the nullification number's properties and relationships to other

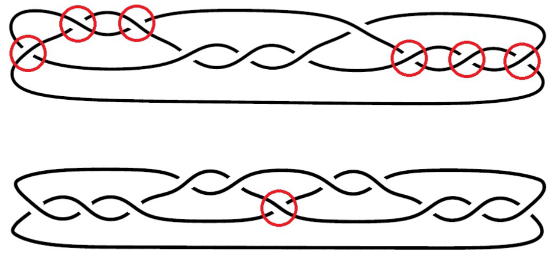


Figure 1.16: Nullifying 10_{22} in one move requires a non-minimal diagram

invariants. What can be said about its value when restricted to a class of knots (e.g. torus knots)? Is it related by some inequality to another known invariant?

Chapter 2

BOUNDS ON NULLIFICATION

In terms of local crossing changes the nullification move, as it is called in this thesis, is much less studied than the unknotting move. However, the nullification operation can be seen in several different perspectives which have frequent appearance in the literature [5, 21, 23, 24, 25, 26, 28, 29, 38, 43, 46].

The most common operation considered in the above literature that is equivalent to nullification is called *coherent band surgery* [5, 23, 24, 25, 26, 29, 43]. In particular, let L be an oriented link, and $b : I \times I \to \mathbb{R}^3$ an embedding such that $b(I \times I) \cap L = b(I \times \partial I)$, where I is a closed interval. Let $L' = (L - b(I \times \partial I)) \cup b(\partial I \times I)$ be another link with compatible orientation to $L - b(I \times I) \cap L$ and $b(\partial I \times I)$. Then L' is said to be the link obtained from L by the (orientation coherent) band surgery along the band b. The equivalence of this operation to the nullification move is easily seen from Figure 2.1.

Another useful equivalent formulation of nullification moves is called a *saddle point transformation* [28, 38]. Consider a surface bounding a loop in the plane. We

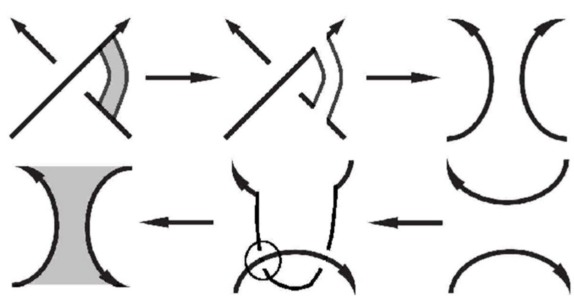


Figure 2.1: Nullification via band surgery (and vice versa)

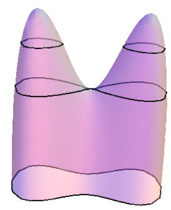


Figure 2.2: A saddle point on a surface bounding a loop

Figure 2.3: Appearance of arcs before/at/after a saddle point singularity

want to deform this surface and consider its subsequent cross-sections as in Figure 2.2 . We see that for the deformation illustrated the cross-section curve at one point has a singularity (self-intersection). Near this point we see that the cross-sections appear as in Figure 2.3. As the surface around the point is vaguely reminiscent of a saddle we call the point a saddle point. The change in the cross-section loops motivates the use of saddle point transformations. In $\mathbb{R}^4$ we can construct surfaces with cross-sections that are knots, and alterations in the surface lead to saddle point transformations in the constituent knot. Comparing the bottom of Figure 2.1 and Figure 2.3 quickly shows that saddle point transformations and nullification moves are equivalent.

A third formulation is called the $SH(2) - move$ [21]. This is a local operation that exchanges arcs of a knot/link as in Figure 2.4. Quickly comparing Figure 2.3 and Figure 2.4 shows the equivalence of this operation with nullification.

The first difficulty in describing the results of nullification moves (or band

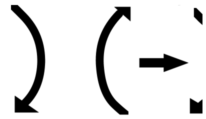


Figure 2.4: A $SH(2)$ move

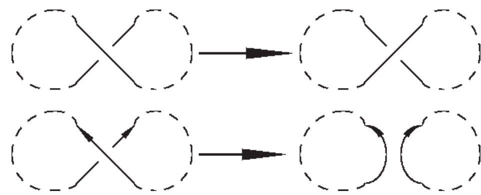


Figure 2.5: Change in the number of components after an unknotting and nullification move

surgeries, or saddle point transformation, or $SH(2)$ moves) is that we always change the number of components in the knot/link. Unknotting does not have this problem as can be seen in Figure 2.5. Knots and their invariants are extensively studied and tabulated, but as we increase the number of components we often have distinct links when orientation is changed on its sublinks. This leads to an explosion of possible products to consider when undergoing nullification moves. Additionally we get the ambiguity that we can trivialize a knot/link by reaching an end product of any unlink. We must define the nullification number to resolve this ambiguity.

Definition 2.1: *A sequence of links $L = L_1 \to L_2 \to \cdots \to L_k = L'$ where each L_i and L_{i+1} are related by a single nullification move is called a* nullification sequence.

The nullification distance, *$d_n(L, L')$, is the minimum length of nullification sequences to reach L' from L.*

The θ-nullification number, *$n_\theta(L)$, is the minimum length of nullification sequences to reach an unlink with θ components.*

The nullification number *is the minimum across all unlink products,*

$n(L) = \min\{n_\theta(L)|\theta \in \mathbb{N}\}$.

For the most part we will favor discussion of the various $n_\theta(L)$ rather than focus on the value of $n(L)$. The reason for this is that if we can explicitly solve for all $n_\theta(L)$ than we can quickly recover the nullification number.

Notice that $d_n(L, L')$ satisfies all the usual conditions to be a metric. Namely:

1. $d_n(L, L') \geq 0$ and $d_n(L, L') = 0$ iff $L = L'$ (nullifications are required to change the knot/link type)

2. $d_n(L, L') = d_n(L', L)$ (nullification moves are reversible)

3. $d_n(L, L') \leq d_n(L, L'') + d_n(L'', L')$ (nullification sequences can be concatenated, but there may be more efficient pathways)

Utilizing the reverse triangle inequality we can immediately say

Proposition 2.1: *Given links L, L', and L'' we have*

$$|d_n(L, L'') - d_n(L', L'')| \leq d_n(L, L')$$

and in particular, letting $L'' = U^\theta$

$$|n_\theta(L) - n_\theta(L')| \leq d_n(L, L')$$

Another quick result is that orientation reversal in all components, or taking a mirror image of the link have no effect on the nullification number of said link. So given a link L we have that

$$n_\theta(L) = n_\theta(-L) = n_\theta(L!) = n_\theta(-L!).$$

The fact that nullification always changes the number of components leads to our first immediately applicable result on the nullification number.

Proposition 2.2: *Given links L and L'*

$$|\mu(L) - \mu(L')| \leq d_n(L, L').$$

Furthermore $|\mu(L) - \mu(L')| \equiv d_n(L, L') \,(mod\, 2)$.

While this proposition isn't surprising or particularly clever it can be used to establish the nullification distance between the unlinks.

Corollary 2.1: $n_\theta(U^\mu) = d_n(U^\mu, U^\theta) = |\mu - \theta|$

Proof: It suffices to show that $n_\theta(U^\mu) \leq |\mu - \theta|$. As band surgeries are reversible, we can without loss of generality consider only where $\mu \geq \theta$. If $\mu = \theta$ we trivially have that U^μ is equivalent to U^θ, and so zero nullification moves are required to change one to the other. If $\mu > \theta$ we can establish a nullification sequence of length $\mu - \theta$: Connect any two components of U^μ with a band surgery corresponding to the usual connected sum. We obtain $U^{\mu-1}$. Continue these connected sums to reach U^θ. ∎

Another straight-forward result gives us an upper bound. Recall that the Seifert

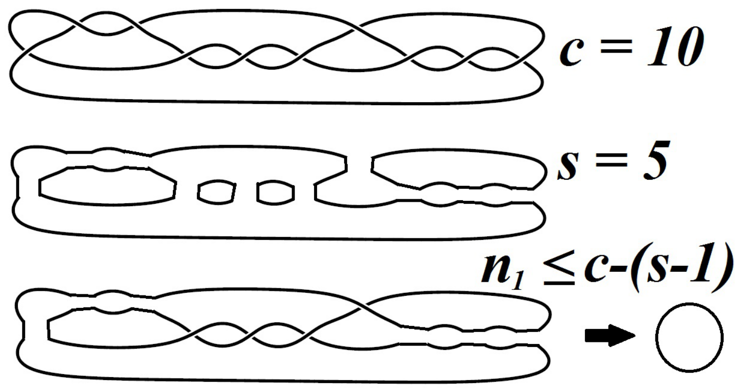


Figure 2.6: An upper bound to n_1 from the Seifert algorithm

algorithm simultaneously nullifies every crossing of a knot diagram, and yields a set of disjoint circles (Seifert circles). We can take these circles and add back just enough crossings to leave a trivial knot after using Reidemeister I moves. The number of crossings we add back is just enough to establish minimal connectivity of the Seifert circles for an unknot (see Figure 2.6).

A slight complication arises for links. If we have a link whose sublinks are completely separate from one another in the link diagram (called a *split link*) then we cannot add crossings to yield an unknot directly. More formally we have the following definition.

Definition 2.2: *Let p be the map that projects a link L to a regular diagram $D \in \mathbb{R}^2$. D is* split *if there is a circle S embedded in $\mathbb{R}^2 - D$ so that there are some components of L whose projections are on each side of S. If a link L has a split diagram then we call L a* split link*. If we denote the two components of $\mathbb{R}^2 - S$ by R_1 and R_2 and let $D_i = R_i \cap D$ then we write $D = D_1 \sqcup D_2$ and say that D_1 and D_2 are the split components of D. Furthermore if we let $L_i = p^{-1}(D_i)$ then we write*

$L = L_1 \sqcup L_2$ and say that L_1 and L_2 are the split components of L.

If we have a split link our procedure can simply be generalized to unknot each split component. This yields the following result.

Proposition 2.3: *Let $D = D_1 \sqcup \cdots \sqcup D_k$ be a diagram of a split link L with k split components. Denote the number of crossings of D by c, and when the Seifert algorithm is applied to D denote the number of Seifert circles by s. Then*

$$n_\theta(L) \leq c - s + k + |k - \theta|.$$

Furthermore $n(L) \leq c - s + k$.

Proof: If D_i decomposes into s_i Seifert circles, we can add back $s_i - 1$ crossings to establish minimal connectivity for an unknot on that split component. This tells us $n_k \leq \sum_{i=1}^{k}(c_i - (s_i - 1)) = c - s + k$. As $d(U^k, U^\theta) = |k - \theta|$ by Corollary 2.1 then $n_\theta(L) \leq d_n(L, U^k) + d_n(U^k, U^\theta) \leq (c - s + k) + |k - \theta|$. As $n(L) \leq n_k(L) \leq c - s + k$ we obtain the second claim. ∎

In this thesis we will be exclusively concerned with non-split links, and so we take from the previous proposition the special case where $k = 1$. While this is a helpful bound, it would be helpful to not rely on a particular projection to check the inequalities for nullification numbers. We can achieve this for alternating links. We have the following from Gabai.

Theorem 2.1: *[17] Applying Seifert's algorithm to an alternating projection of an oriented link yields a minimal genus surface.*

Finding the genus of a surface obtained by Seifert's algorithm is a simple matter. So for alternating links we obtain this nice result.

Corollary 2.2: *For a non-split alternating link L with μ components and genus g we have that*

$$n_\theta(L) \leq 2g + \mu - 2 + \theta.$$

PROOF: Let D be an alternating projection of L. As D is alternating we know that Seifert's algorithm gives a minimal genus surface with genus g. Let c denote the number of crossings in D and s the number of Seifert circles. Letting F denote the surface obtained from the algorithm, a simple calculation of the Euler characteristic yields $-c + s$. As $2 - 2g = \mu + \chi(F)$ we obtain $c - s + \theta = 2g + \mu - 2 + \theta$ and the result follows from the special case of Proposition 2.3 for $k = 1$. ∎

With this result we can look at a knot/link table, and for any alternating knot we merely have to read off the three genus for quick bounds on the nullification number. In fact Proposition 2.2 and Corollary 2.2 give us explicit results for several knots (e.g. n_1 for $3_1, 4_1, 5_2, 6_1, 7_2, 7_4, 8_1, 8_3, 9_2, 9_5, 9_{35}, 10_1, 10_3$ is 2). While this is encouraging, solving n_1 for 13 of 249 prime knots of up to 10 crossings is insufficient for our task.

What other lower bounds can be put on the nullification number? Viewing nullification as a band surgery, we can use a paper from Murasugi that includes a result on links related with a band surgery. From this paper we have the following:

Lemma 2.1: *[38] If a link L and L' are related by a band surgery then*

$|\omega(L) - \omega(L')| + |\sigma(L) + \sigma(L')| = 1.$

This Lemma tells us that a nullification move can only change the signature or nullity by one, but not change both. Note some results regarding nullity and signature: $\sigma(U^m) = 0$, $\omega(U^m) = m$, and for any link L we have $1 \leq \omega(L) \leq \mu(L)$. The previous lemma and these facts give us some simple consequences:

Corollary 2.3: *For any two links L and L'*

$d_n(L, L') \geq |\sigma(L) - \sigma(L')| + |\omega(L) - \omega(L')|.$

Letting $L' = U^\theta$ gives us $n_\theta(L) \geq |\sigma(L)| + |\omega(L) - \theta|$, and if $\mu(L) = 1$ then $n_\theta(L) \geq |\sigma(L)| + \theta - 1.$

From Kauffman and Taylor we have the related Corollary:

Corollary 2.4: *[28] If a link L and L' are related by a band surgery, and if $\omega(L) = \mu(L)$ where $\mu(L) > \mu(L')$, then $\omega(L') = \mu(L')$ and $\sigma(L) = \sigma(L')$.*

In these papers ([28] and [38]) unknotting number is related to signature and nullity as well. Perhaps we can relate nullification number to the unknotting number. As unknotting is so well studied perhaps this approach will prove quite advantageous. It turns out we can achieve any unknotting move via two nullifications. This is demonstrated in Figure 2.7. So if we can undo a link with $u = u(L)$ unknotting moves we can nullify a knot (to a link of the same number components) with at most $2u$ nullification moves. This can be improved slightly by using our nullification moves more efficiently. If we need to unknot two crossings of opposite parity then we can use two nullification steps for those two strand passages (see Figure 2.8).

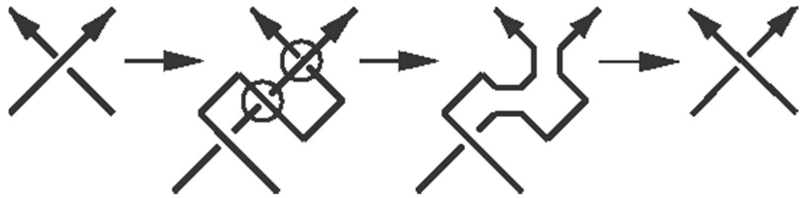


Figure 2.7: Unknotting via nullification of the circled crossings

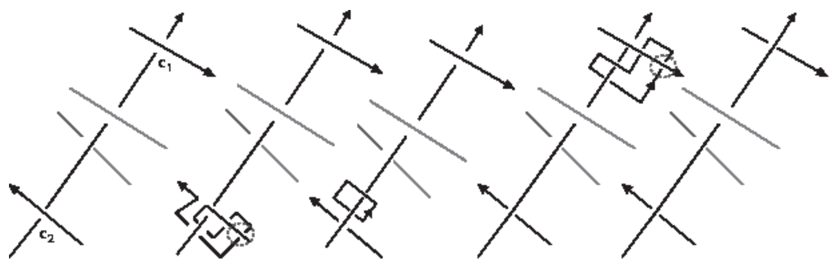


Figure 2.8: Unknotting two opposite parity crossings with nullification moves

Note that to use this move the two crossings that are unknotted have to be connected by some component. By connected we mean that if we unknot a crossing c_1 between links L_1 and L_2 and c_2 between links L_3 and L_4, then at least one of the following must be true:

$$\begin{array}{ll} (1):\ L_1 = L_3 & (2):\ L_1 = L_4 \\ (3):\ L_2 = L_3 & (4):\ L_2 = L_4 \end{array}.$$

So if we have an unknotting sequence that unknots u_+ positive crossings and u_- negative crossings then, if these moves can be mutually connected, we can nullify with at most $2u - 2\min\{u_+, u_-\}$ nullification moves. Putting this into a formal statement:

Proposition 2.4: *Let L be a link with μ components that has an unknotting sequence of length u that unknots u_+ positive and u_- negative with* $\min\{u_-, u_+\}$

connected pairs. Then we have that

$$n_\theta(L) \leq 2u - 2\min\{u_+, u_-\} + |\mu - \theta| \leq 2u + |\mu - \theta|.$$

The more efficient unknotting-via-nullification procedure is quite important in solving nullification numbers. With the coarse bound of $2u$ (and the results prior) we can solve $n_1(L)$ for 111 of the 249 prime knots up to 10 crossings. By looking for connected oppositely signed unknotting moves we can solve $n_1(L)$ for 159 of the 249.

Another well known result in knot theory is that $g^*(L) \leq u(L)$ [38, 28]. We can mimic this result for nullification by using the saddle point interpretation of nullification moves. Before this result we need to describe how surfaces are described in four space.

Following the introduction by Murasugi, let $H_{(a,b)}, H_{[a,b)}, H_{(a,b]}$ and $H_{[a,b]}$ $(-\infty \leq a, b \leq \infty)$ be subspaces of 4-space S^4 defined as follows:

$$H_{(a,b)} = \{x = (x_1, x_2, x_3, x_4) | a < x_4 < b\},$$
$$H_{[a,b)} = \{x | a \leq x_4 < b\},$$
$$H_{(a,b]} = \{x | a < x_4 \leq b\},$$
$$H_{[a,b]} = \{x | a \leq x_4 \leq b\}.$$

In particular, H_a means a hyperplane $H_{[a,a]}$. Consider a smooth tamely embedded orientable surface F in general position in S^4 and cut it by the family of hyperplanes H_t, $-\infty < t < \infty$. We can assume without loss of generality that there are only finite number of t-values that are singular. A singular hyperplane may intersect F in an a maximum or minimum point, called an *extreme point*, or it may intersect F in a *saddle point*. Let a link L be the boundary of F. Then F is said to be in *normal* position in S^4 if F is placed in such a way that

1. $F \subset H_{[-1,n]}$ for a sufficiently large $n > 0$ and $L \subset F \cap H_0$.

2. All minimal points lie on H_{-1}.

3. All maximal points lie on H_n.

4. All saddle points are in $H_{(0,n)}$ and ordered in distinct heights.

Given a surface F, F can be deformed into normal position. In this thesis we will always work under the assumption that the surface has no closed components. From the theory of surfaces [27] we have the next Lemma.

Lemma 2.2: *Given a link L of multiplicity μ bounding a (orientable) surface F with κ components in normal position, let p and q denote the number of extreme points and saddle points in F. Then*

$$2g^*(F) = q - p + 2\kappa - \mu.$$

Each nullification step on a link is a saddle point on the bounding surface. After the nullification sequence we are left with some number of disjoint loops, θ. θ is then the number of extreme points on our bounding surface. As $g^*(L)$ represents the minimal four genus of all bounding surfaces we immediately get the following Theorem.

Theorem 2.2: *Let $F \subset D^4$ be the smoothly embedded orientable bounding surface of a link $L \subset S^3$ constructed using a nullification sequence of length n_θ that takes L to U^θ. If F has κ disconnected components then*

$$2g^*(L) + \theta - 2\kappa + \mu(L) \leq n_\theta(L).$$

As κ isn't necessarily accessible without drawing the nullification sequence this is of limited direct use. If we use the intuitive fact that
$1 \leq \kappa \leq \min\{\mu, \theta\} = 1/2\,(\mu + \theta - |\mu - \theta|)$ we can give a weaker inequality.

Corollary 2.5: *Let a link L of mulitplicity μ have a nullification sequence of length n_θ that takes L to U^θ. We have*

$$2g^*(L) + |\theta - \mu| \leq n_\theta.$$

In particular

$$2g^*(L) + \mu - 1 \leq n_1,$$

or if $\mu = 1$ then

$$2g^*(L) + \theta - 1 \leq n_\theta.$$

This has immediate utility for links that have crossings of only one sign. If a link diagram has only positive (negative) crossings then we call it a positive (negative) diagram. If a link has some projection with a positive (negative) diagram then we call the link positive (negative). Nakamura proves the following Theorem.

Theorem 2.3: *[41] Let L be a positive (negative) link with μ components, and D any non-split positive (negative) diagram of L. Then we have*

$$2g(L) = 2g^*(L) = 2 - \mu - s(D) + c(D),$$

where s is the number of Seifert circles, c is the number of crossings of D.

Combining this with Proposition 2.3 and Theorem 2.2 gives us this Corollary.

Corollary 2.6: *Let L be a positive (negative) link with μ components, and D any non-split positive (negative) diagram of L. Then we have*

$$n_1(L) = c - s + 1,$$

and if $\mu = 1$

$$n_\theta(L) = c - s + \theta,$$

otherwise if $\mu > 1$

$$c - s - \mu + 2 + |\mu - \theta| \leq n_\theta(L).$$

Kauffman and Taylor also relate the four genus and signature.

Theorem 2.4: *[28] Let F be an orientable surface in B^4 spanning $L \subset S^3$ with κ disconnected components. Then*

$$|\sigma(L)| - \mu(L) + \kappa + |\kappa - \omega(L)| \leq 2g^*(F).$$

Combining this with Theorem 2.2 gives us an alternative of Corollary 2.3.

Corollary 2.7: *Let F be an orientable surface with κ disconnected components in B^4 spanning $L \subset S^3$ formed by a nullification sequence of length n_θ that takes L to U^θ. We have:*

$$|\sigma(L)| + \theta - \kappa + |\kappa - \omega(L)| \leq n_\theta.$$

Using the four genus as a lower bound solves $n_1(L)$ for 217 of the 249 prime knots up to 10 crossings.

These most recent results bounded only $n_\theta(L)$, but we can also use a similar approach to bound $d_n(L, L')$. The following is a generalization of [26, Prop. 3.2].

Theorem 2.5: *For two links L and L' we have*

$$|(2g^*(L) - 2g^*(L')) + (\mu(L) - \mu(L'))| \leq d_n(L, L')$$

PROOF: Denote the surface that bounds L with genus $2g^*(L)$ as T. We can without loss of generality assume T is connected and place it in normal position with n_1 saddle points, θ_1 extreme points, and $\mu_1 = \mu(L)$ boundary components. Similarly we have a connected surface B with genus $2g^*(L')$ that bounds L', and placed in normal position with n_2 saddle points, θ_2 extreme points, and $\mu_2 = \mu(L')$ boundary components. If we connect these two surfaces with a surface M that has $d_n = d_n(L, L')$ saddle points, and is bounded by L and L' we get a construction like in Figure 2.9. Let's denote the surface obtained from pasting T to M along L as $T + M$, and similarly $M + B$ the surface obtained by pasting M to B along L'. We

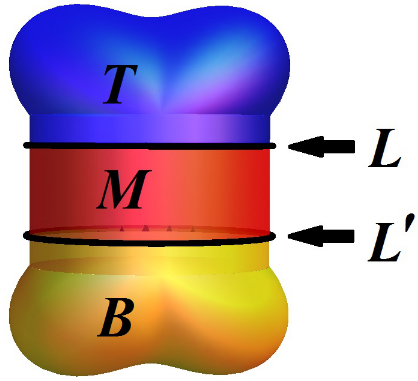


Figure 2.9: Constructing a surface from a nullification sequence between two links L and L'

note that $T + M$ is a surface with boundary L', and $M + B$ a surface with boundary L. So we have that $2g^*(L) \le 2g^*(M+B)$ and $2g^*(L') \le 2g^*(T+M)$. Using Lemma 2.2 we get

$$\begin{gathered} 2g^*(L) \le (d_n + n_2) - \theta_2 + 2 - \mu_1 \\ 2g^*(L) \le 2g^*(L') + d_n + \mu_2 - \mu_1 \quad , \\ 2g^*(L) - 2g^*(L') - \mu_2 + \mu_1 \le d_n \end{gathered}$$

and similarly

$$\begin{gathered} 2g^*(L') \le (d_n + n_1) - \theta_1 + 2 - \mu_2 \\ 2g^*(L') \le 2g^*(L) + d_n + \mu_1 - \mu_2 \quad . \\ 2g^*(L') - 2g^*(L) - \mu_1 + \mu_2 \le d_n \end{gathered}$$

So putting these together we have

$$|(2g^*(L) - 2g^*(L')) + (\mu_1 - \mu_2)| \le d_n.$$

■

Note that this is close to the statement in Corollary 2.5 for $L' = U^\theta$. We

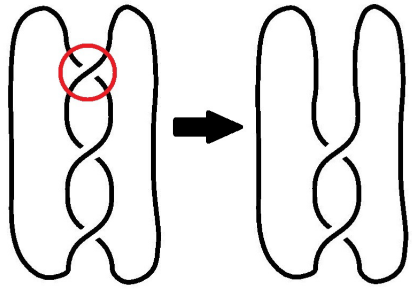


Figure 2.10: Nullifying one crossing in 3_1 yields 2_1^2

might desire to claim that $|(2g^*(L) - 2g^*(L'))| + |(\mu_1 - \mu_2)| \leq d_n$. But consider the link 2_1^2 and the knot 3_1. $g^*(2_1^2) = 0$, $g^*(3_1) = 1$, $\mu(2_1^2) = 2$, and $\mu(3_1) = 1$. If we believed the claim then we'd have $3 \leq d_n(2_1^2, 3_1)$. But nullifying a single crossing in 3_1 yields 2_1^2 (see Figure 2.10). It seems the stronger result in Corollary 2.5 is peculiar to nullification sequences ending in a trivial link.

We'll close this section with a few more results from other authors. One is from Kanenobu [25, 26]. It describes how the knot polynomials of links related by a single band surgery behave. We can use this to potentially show that two links are not related by a band surgery.

Proposition 2.5: *[25, 26] If two links L and L' are related by a single band surgery then*

$$j(L)/j(L') \in \{\pm i, -\sqrt{3}^{\pm 1}\},$$

$$h(L)/h(L') \in \{1, (-2)^{\pm 1}\},$$

$$q(L)/q(L') \in \{\pm 1, \sqrt{5}^{\pm 1}\},$$

where

$$j(L) = V(L; e^{i\pi/3}),$$

$$h(L) = P(L; i, i),$$

$$q(L) = Q(L; (\sqrt{5} - 1)/2).$$

An extension of this describes the behavior of the knot polynomials related by several band surgeries.

Corollary 2.8: *If two links L and L' are related by n band surgeries then:*

$$q(L)/q(L') \in \{\pm\sqrt{5}^{\pm k}, \sqrt{5}^{\pm n} |\, k = 0, 1, \cdots, n-1\},$$

If furthermore we have $\mu(L) \equiv \mu(L') \, (mod\, 2)$ then

$$\frac{j(L)}{j(L')} \in \{\pm(i\sqrt{3})^{\pm k}, \sqrt{3}^{\pm n} |\, k = 0, 1, \cdots, n-1\}$$
$$\frac{h(L)}{h(L')} \in \{(-2)^{\pm k}, 2^{\pm n} |\, k = 0, 1, \cdots, n-1\}$$

otherwise if $\mu(L) \equiv \mu(L') - 1 \, (mod\, 2)$ then

$$\frac{j(L)}{j(L')} \in \{\pm i(i\sqrt{3})^{\pm k}, -\sqrt{3}^{\pm n} |\, k = 0, 1, \cdots, n-1\}$$
$$\frac{h(L)}{h(L')} \in \{(-2)^{\pm k}, -2^{\pm n} |\, k = 0, 1, \cdots, n-1\}$$

Another invariant that can be of use is called the Arf invariant[44]. Its value is either 0, 1, or undefined. The Arf invariant of a link L, Arf(L), is undefined if L isn't proper.

Proper Link Let $L = k_1 \cup k_2 \cup \cdots \cup k_n$ be an oriented link with components k_i for

$1 \leq i \leq n$. L is said to be a proper link if the sum of the linking numbers between k_i and the rest of the components is an even integer for each choice of $i = 1, \cdots, n$. That is, ${}^1\!/_2 \sum_{j=1}^{n} lk(k_i, k_j) \equiv 0\,(mod\,2)$ for all i.

If L is proper then the Arf invariant can be calculated with [35, Thm. 10.6].

$$\text{Arf}(L) = (-\sqrt{2})^{\mu(L)-1}(-1)^{\text{Arf}(L)}.$$

Furthermore, we have the following Lemma.

Lemma 2.3: *[35, Lemma 10.5] Suppose that L and L' are proper oriented links related by a single band surgery. Then*

$$\mathit{Arf}(L) = \mathit{Arf}(L').$$

By transitivity of equality we have that if there is any length nullification sequence, $L \to L_1 \to \cdots \to L_n \to L'$ such that all L, L', L_i are proper links then $\text{Arf}(L) = \text{Arf}(L')$. In particular as U^θ is proper and $\text{Arf}(U^\theta) = 0$, then $d_n(L, U^\theta) = 1$ implies $\text{Arf}(L) = 0$.

The results of this chapter establish the majority of the nullification numbers for the knot/link tables at the end of this thesis. With these most recent results we have $n_1(L)$ solved for 221 of the 249 prime knots up to 10 crossings. 14 of the remaining 28 knots are called ribbon knots, and with the standard presentations as in Kawauchi [29, Appendix F.5] they all have $n_1(L) = 2$. Figure 2.11 shows two examples of ribbon knots reaching U^2 in 1 nullification move. As this move can

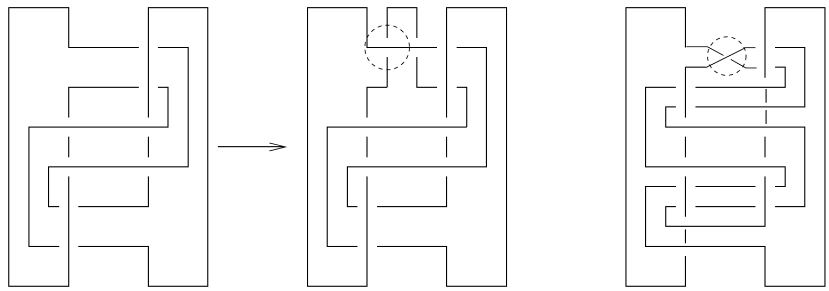


Figure 2.11: Nullifying ribbon knots. 6_1 (left) 8_9 (right)

work in any of the ribbon presentations for knots of 10 or less crossings, this demonstrates that we can reach U in a two nullifications. There are 21 prime ribbon knots of 10 or less: 6_1, 8_8, 8_9, 8_{20}, 9_{27}, 9_{41}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{42}, 10_{48}, 10_{75}, 10_{87}, 10_{99}, 10_{123}, 10_{129}, 10_{137}, 10_{140}, 10_{153}, and 10_{155}.

The remaining $n_1(L)$ values for the final 14 knots can be established by demonstrating an explicit nullification sequence. This is done in Appendix B.2. These results not only yield the $n_1(L)$ values, but (with a few exceptions) use of the triangle inequality can solve all $n_\theta(L)$. These results and more are given in the Appendix.

Chapter 3

NULLIFICATION RESULTS FOR LINK FAMILIES

Instead of exhaustively traversing a table of prime links to prove results on the nullification number, we can instead attempt to establish results on similar types (families) of links. There are several such families, but the easiest to approach are the rational link family and the torus link family. This section will describe these collections of links, and apply the results of the last section to establish more specific nullification bounds. In some cases nullification numbers can be found exactly.

3.1 Rational Links

We will use the definition of rational links used by Cromwell [12, Ch. 4.9, 8]. Let S be a 2-sphere which meets a link L transversely in $2n$ points. The closure of $S - L$ is a $2n$-punctured sphere properly embedded in the link exterior. Wanting S to bound a ball on both sides, we shall think of L as embedded in S^3 (instead of $\mathbb{R}^3$) and let C be a component of $S^3 - S$. The pair $(C, C \cap L)$ is called an n-*tangle*: a ball containing n disjoint properly embedded arcs and a (possibly empty) set of loops. An n-tangle is *trivial* if it is homeomorphic to a cylinder $B \times [0, 1]$ (B represents the disk) containing n parallel straight lines, each connecting the top to the bottom (no loops in a trivial tangle).

Definition 3.1: *A non-trivial link is* rational *if it decomposes into two trivial 2-tangles.*

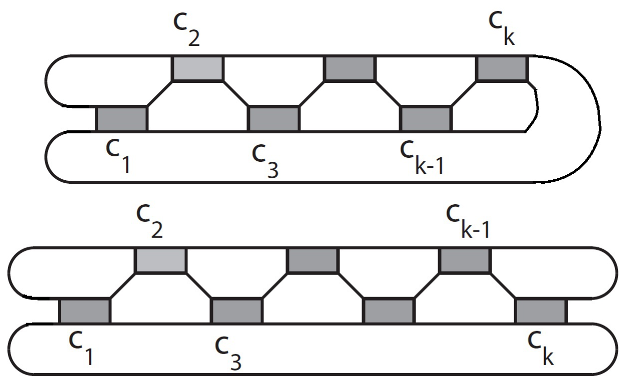


Figure 3.1: Rational link templates for even k (top) and odd k (bottom).

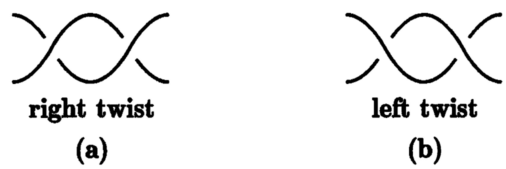


Figure 3.2: Template replacement crossing types [40]

All rational links can be described with a vector of non-zero integers $(c_1, \cdots, c_k)$. Each integer represents a set of crossings in a rational link template. The template used depends on the parity of k (shown in Figure 3.1). The c_i box in these templates represents a set of crossings as given by the integer c_i. If i is odd then a positive c_i represents a right twist crossing set. If i is even then a positive c_i represents a left twist crossing set. Right and left twists are as demonstrated in Figure 3.2. For example, given the vector $(1, 2, 3, 1, 3)$ we get the rational knot in Figure 3.3.

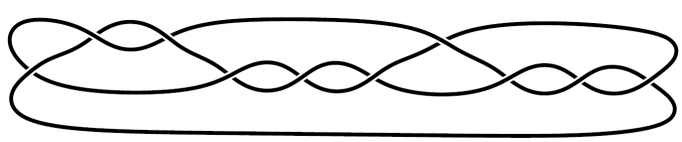


Figure 3.3: An example rational knot

These links are called rational because we can capture the information of this vector into a single rational number. This is done by placing the c_i into a continued fraction, and the value of the continued fraction gives the rational number defining the link. So instead of using the vector $(c_1, \cdots, c_k)$ we have a rational number:

$$\frac{p}{q} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots + \cfrac{1}{c_k}}}}.$$

So for our example knot $(1, 2, 3, 1, 3)$ we get a rational number $49/34$. Based on this number we can classify the rational links and knots. For knots we have

Theorem 3.1: *[40, Thm 9.3.3] Suppose that K and K' are rational knots of type p/q and p'/q', respectively. Then K and K' are equivalent if and only if the following holds:*

$$(1)\, p = p',\, q \equiv q' \,(mod\, p)$$

or

$$(2)\, p = p',\, qq' \equiv 1 \,(mod\, p)$$

Further, the mirror image $K!$ of K is a rational knot of type $p/-q$.

Rational links are known to be reversible, alternating, and have at most two components [40, Ch. 9.3]. Being reversible we have no need to worry about overall

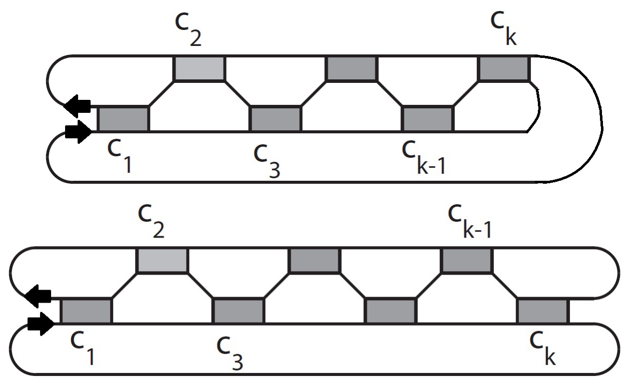


Figure 3.4: Rational link standard orientation choices.

orientation reversal of rational knots/links. For links, however, we do have the additional concern of orientation reversal on a single component.

Theorem 3.2: *[40, Thm 9.4.1] (1) Suppose we reverse the orientation of one component of an (standard) oriented rational link of type p/q, then the (standard) oriented rational link obtained is*

(i) of type $p/(q-p)$ if $q > 0$;

(ii) of type $p/(q+p)$ if $q < 0$.

(2) Two (standard) oriented rational links of type p/q and p'/q' are equivalent with orientation if one of the following cases hold; further, these are sufficient conditions for them to be equivalent.

(i) $p = p'$, $q = q' \,(mod\, 2p)$

or

(ii) $p = p'$, $qq' \,(mod\, 2p)$.

The "standard" orientation assigned to rational links is a matter of convention. Our standard orientation choice will be as in Figure 3.4.

As the link type of the rational link relies on just the fraction, we can expand the continued fraction in different ways to obtain more convenient presentations. For our purposes we wish to consider even vector expansions. That is, a vector expansion of all even entries: $(2a_1, 2a_2, \cdots, 2a_m)$. A fraction p/q can be expanded in such an even expansion as long as p and q aren't both odd [12, Cor. 8.7.3]. p is odd only when the fraction represents a knot [40, Ex. 9.3.7]. However, if we have a rational knot K of type p/q where p and q are odd then we can use Theorem 3.1 to show that K is isotopic to a rational knot of type $p/(q-p)$. This new fraction has an even vector expansion.

So any rational link can be isotoped into a template with all even entries, and the standard orientation choices of Figure 3.4. It can be shown that the surface yielded by the Seifert algorithm on this template is of minimal genus [12, Cor. 8.7.5]. Knowing this let's calculate the genus, signature, and nullity for this link. The number of crossings in this projection is $c = 2\sum_{i=1}^{m} |a_i|$. Applying the Seifert algorithm yields $s = c - m + 1$ Seifert circles (see Figure 3.5 for an example on $(2, 2, 4)$). Keeping in mind that the surface obtained is of minimal genus, we get that $2 - 2g(L) = \mu(L) + (-c + s)$. This implies that $2g(L) = m - \mu + 1$. Using this surface we form the Seifert graph. In general it appears as in Figure 3.6. So the

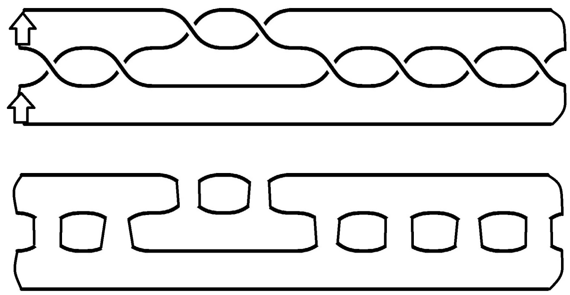


Figure 3.5: Seifert algorithm on an even vector form of a rational link

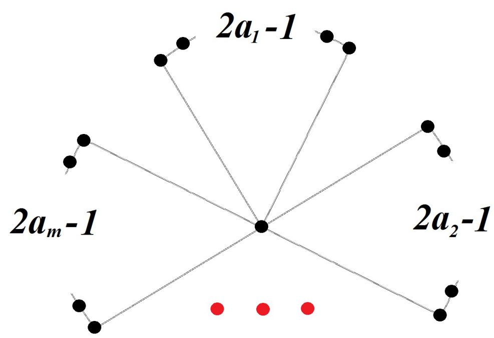


Figure 3.6: Seifert graph for an even vector form of a rational link

Seifert matrix can easily be seen to be the diagonal matrix

$$\begin{bmatrix} a_1 & & & & \\ & -a_2 & & O & \\ & & \ddots & & \\ & O & & (-1)^{m-2}a_{m-1} & \\ & & & & (-1)^{m-1}a_m \end{bmatrix}.$$

As all $a_i \neq 0$ we see immediately that $\omega(L) = 1$, and that $\sigma(L) = \sum_{i=1}^{m} (-1)^{i-1} \frac{a_i}{|a_i|}$ (recall that for a diagonal matrix the eigenvalues appear on the diagonal).

Using these results and the last section, Proposition 2.3 and Corollary 2.3, we have the next Theorem.

Theorem 3.3: *Given a rational link L with even vector expansion* $(2a_1, \cdots, 2a_m)$ *we have that*

$$\left|\sum_{i=1}^{m}(-1)^{i-1}\frac{a_i}{|a_i|}\right| + \theta - 1 \leq n_\theta(L) \leq m + \theta - 1.$$

PROOF: From the previous discussion the Seifert decomposition of L yields $c - m + 1$ Seifert circles where c is the number of crossings of L. Proposition 2.3 gives us that $n_\theta(L) \leq c - s + \theta = m + \theta - 1$. Corollary 2.3 gives us that $|\sigma(L)| + |\omega(L) - \theta| \leq n_\theta(L)$. We've already demonstrated that $\omega(L) = 1$ and that $\sigma(L) = \sum_{i=1}^{m}(-1)^{i-1}\frac{a_i}{|a_i|}$. Our claim is obtained. ∎

This tells us that if we have an alternating even vector expansion then we get an explicit result on $n_\theta(L)$, and via Theorem 2.2 a result on $g^*(L)$:

Corollary 3.1: *If a rational link L has an even vector expansion* $(2a_1, \cdots, 2a_m)$ *where $a_i a_{i+1} < 0$ for $1 \leq i < m$ then*

$$n_\theta(L) = m + \theta - 1$$

Furthermore

$$2g^*(L) = m - \mu + 1$$

PROOF: The result on $n_\theta(L)$ is a clear consequence of the inequality of the previous Theorem. The nullification sequence with this length creates a surface in B^4 with

$m + \theta - 1$ saddle points, θ extreme points, μ boundary curves, and has one connected component. So using Theorem 2.2 we have

$$2g^*(L) \leq m - \mu + 1.$$

From Theorem 2.4 we have that

$$|\sigma| - \mu + 1 = m - \mu + 1 \leq 2g^*(L).$$

We obtain the second claim. ■

We can obtain another result from the band surgery interpretation of the nullification operation. If a knot (not link) can reach U^m in $m-1$ band surgeries then the knot is called a *ribbon knot of m-fusion.* If there is no $k < m$ such that the knot is not a ribbon knot of k-fusion then m is said to have *ribbon fusion number m.* Ribbon knots are known to have signature zero [25]. The vector form of the rational knots that are ribbon are known [9, 15, 33, 36]. They are

$$(-2a, -2, -2b, 2, 2a, 2b),$$

$$(-2b, -2, -2a, 2b, 2, 2a),$$

$$(a_1, a_2, \cdots, a_k, \epsilon, -a_k, \cdots, -a_1)$$

where $\epsilon = \pm 1$ and $a, b,$ a_i are non-zero integers. For each of these vectors we can demonstrate that the nullification distance to the unlink of two components is one. This tells us that for these vectors $n_\theta = 1 + |\theta - 2|$. For the first vector the

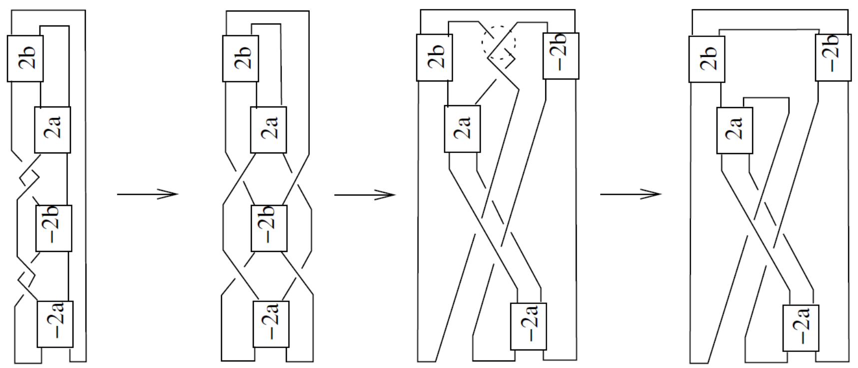


Figure 3.7: Nullifying the $(-2a, -2, -2b, 2, 2a, 2b)$ rational knot

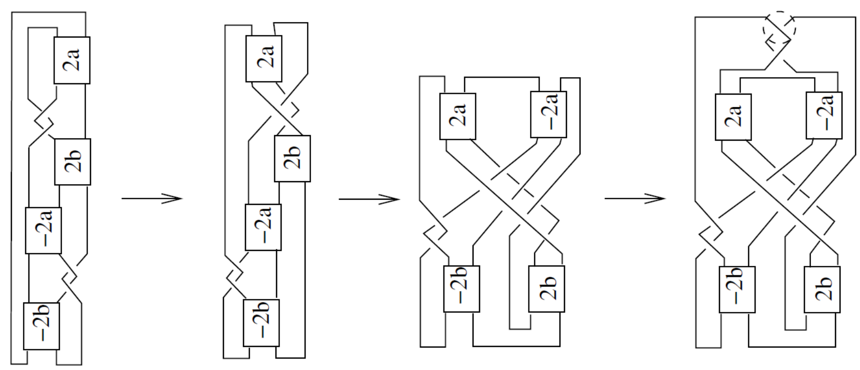


Figure 3.8: Nullifying the $(-2b, -2, -2a, 2b, 2, 2a)$ rational knot

nullification is shown in Figure 3.7. The nullification for the second vector is shown in Figure 3.8. Nullifying the ϵ crossing yields the unlink for the third vector.

As any knot that can reach the two component unlink in one nullification move would be a ribbon knot, then these three vector types represent all the rational knots of nullification number one. Another interesting result we have from Bleiler and Eudave-Muñoz:

Theorem 3.4: *[5] A composite ribbon number one knot has a rational summand.*

This tells us that any composite knot with nullification number one has a rational knot as a summand. While the tables included in this thesis don't consider composite knots/links, this result demonstrates that rational knots are intimately involved in decomposing non-prime links by a single band surgery.

3.2 Torus Links

The family of torus links are a common starting place for investigations in knot theory after alternating links. Their structure, having a variety of symmetries and regularity, makes them particularly amenable to analysis, and as such there is vast literature establishing characteristics of their invariants. We hope to take advantage of this to bound the nullification numbers of the torus links.

To use the results of Chapter 2 we would like to know the four genus, unknotting number, signature and nullity of the torus links. For the standard orientation assignment the four genus and unknotting number are known. This will give us that for the standard orientation assignment, finding the nullification numbers for torus *knots* is a solved problem, and similarly for torus links $n_1(L)$ can be established. It will turn out that for these cases the nullification sequence is the most obvious one.

Much less can be said about non-standard orientations. The four genus result breaks down for these cases, and the unknotting number (an orientation insensitive invariant) is too weak for our purposes. The signature of all torus links can be expressed with a recurrence relation, but we lack explicit formulae for all cases.

As the Seifert matrices are particularly well structured we will prove an expression for the nullity of all torus links, and use this to bound the nullification numbers for classes of torus links that do have explicit expressions for signature.

3.2.1 Notation and Some Nullification Bounds

The family of links known as torus links are links on the surface of a common torus formed with a variety of equivalent constructions. The easiest to visualize is given by Adams [1]. A torus link is defined by two integer values p and q, and is denoted $T(p,q)$. Recall that a torus has two characteristic curves, the meridian and longitude as depicted in Figure 3.9. $T(p,q)$ is formed by placing p points equidistantly along a longitude of the torus and connecting points that are q increments away either clockwise or counter-clockwise depending on your source text (our examples are clockwise). This creates a curve/s that intersect any longitude in p places and any meridian in q places. For example $T(5,2)$ is formed as in Figure 3.10.

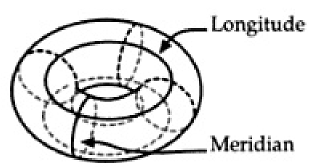


Figure 3.9: Longitude and Meridian of a torus [1]

With this definition it can be shown that we will have a link with $d = gcd(p,q)$ components. So if p and q are relatively prime $T(p,q)$ represents a knot. Another known fact about $T(p,q)$ is that it is isotopic to $T(q,p)$. This mapping can be easily visualized as a map between the meridians of one torus to the longitudes of another, and the longitudes to the meridians. Another way to

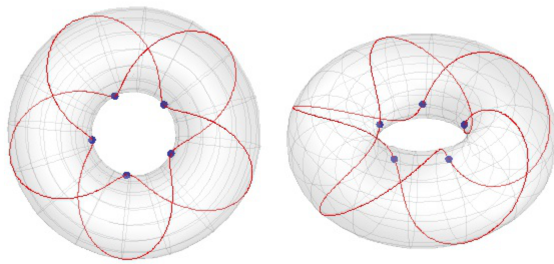


Figure 3.10: $T(5,2)$

understand this is that S^3 can be constructed by the pasting together of two tori. So a knot on one torus has an image on the companion torus that decomposes 3-space. If these descriptions are not sufficient for the reader see [1] for an explicit pictorial demonstration of the correspondence.

To form the regular projection of these knots we will compress the coils to a cylinder and isotope the threads to form what is known as the braid presentation. This process is shown for $T(5,2)$ in Figure 3.11. When we include the connections between the top and bottom parts of the diagram as in the right of Figure 3.11, we call this the braid closure.

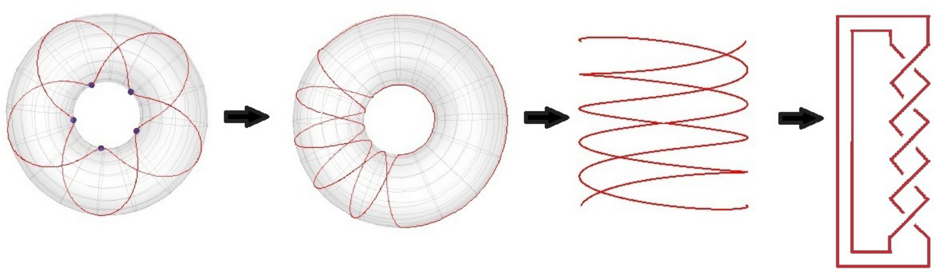


Figure 3.11: $T(5,2)$ being put into braid projection form

So in general for a $T(p,q)$ link we will obtain a presentation as shown in Figure 3.12 that has the closure threads omitted and all threads oriented parallel to one another. If one of the integers is negative then the projection is a mirror image of the standard one. I.e., $T(p,-q) = T(-p,q) = T(p,q)!$. If both are negative then the orientation of all the components are reversed. I.e., $T(-p,-q) = -T(p,q)$. With this in mind we state the following classification theorem [7, 40].

Theorem 3.5: *$T(p,q) = T(p',q')$ iff $(p',q') = (p,q), (q,p), (-p,-q),$ or $(-q,-p)$. Hence $T(p,q)$ is reversible and chiral. Furthermore, if p or $q = 0$ or ± 1 then $T(p,q)$ is the trivial knot/link. Specifically $T(0,0) = \emptyset$, $T(p,0) = U^p$, $T(0,q) = U^q$, $T(1,q) = U$ and $T(p,1) = U$ for $p,q \geq 1$.*

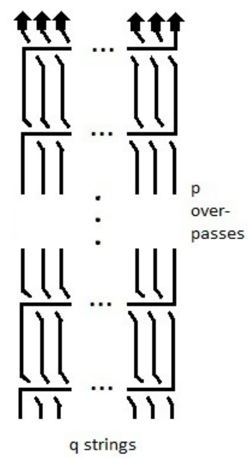


Figure 3.12: The regular projection, omitting the closure threads, of a torus link

What can be said about the nullification numbers for the torus links? Without loss of generality we can establish the convention that $p \geq q > 0$. This will hold for the remainder of the section. In this case the projection in Figure 3.12 is known to be minimal and has $p(q-1)$ crossings [40, Thm. 7.5.4]. As this diagram also has only positive crossings then we can apply Theorem 2.3 to give us $2g(T(p,q)) = 2g^*(T(p,q)) = c - s - \mu + 2$. The Seifert algorithm on the diagram gives $s = q$, and we've already stated the result that $\mu = d = gcd(p,q)$. So our equality becomes $2g^*(T(p,q)) = p(q-1) - q - d + 2$. Using Proposition 2.3 and Corollary 2.6 we obtain:

Theorem 3.6: *For $p, q \in \mathbb{N}$ let F be the smoothly embedded bounding surface of a link $T(p,q)$ constructed using a nullification sequence of length n_θ that takes $T(p,q)$ to U^θ. If F has κ disconnected components then*

$$(p-1)(q-1) + \theta + 1 - 2\kappa \leq n_\theta \leq (p-1)(q-1) + \theta - 1.$$

If $gcd(p,q) = 1$ then

$$n_\theta = (p-1)(q-1) + \theta - 1.$$

Or if $gcd(p,q) > 1$ then

$$n_1 = (p-1)(q-1).$$

As mentioned in the last Chapter, if the value of κ isn't accessible we can use $1 \leq \kappa \leq \min\{\mu, \theta\}$ to possibly weaken the lower bound. I.e.,

$$(p-1)(q-1) + 1 - \mu + |\theta - \mu| \leq n_\theta.$$

Note that there is an obvious nullification sequence of length $(p-1)(q-1) + \theta - 1$. We nullify all but the last overpass with $(p-1)(q-1)$ nullification moves, and then untwist the resultant with Reidemeister I moves into the unknot. To get to the θ component unlink we split the unknot with $\theta - 1$ nullification moves.

Note that the above results are only for presentations with parallel orientations on all components. We can also obtain some information on the nullification numbers if we reverse the orientations on some subset of the components. We will introduce a new notation to represent this. First let $d = gcd(p, q)$ and $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$ where$\lfloor x \rfloor = \{a | a \leq x \, and \, a \in \mathbb{Z}\}$. We introduce the notation that $T_i(p, q)$ represents a link as in Figure 3.12 with i of its components having the opposite orientation from the other $d - i$ components.

We only need to consider the cases where $i \leq \left\lfloor \frac{d}{2} \right\rfloor$ because the torus links are isotopic to the same links with all orientations reversed. Also note that $T_i(p, q)$ can represent several link types depending on how the i reversed-orientation components are selected from the original d components. Clearly our previous notation $T(p, q)$ is equivalent to $T_0(p, q)$. For the remainder of this section when using the letter d we will mean $d = gcd(p, q)$, if using the letters p' and q' it will be implied that $gcd(p', q') = 1$, and when using i it is implied that $i \in \mathbb{Z} \cap \left[0, \left\lfloor \frac{d}{2} \right\rfloor\right]$.

We now wish to consider bounds on the nullification numbers for these links. Reversing the orientations means we no longer have a positive link to use Corollary 2.6. We also don't have an expression for the four genus to use Theorem 2.2 effectively. The other invariants that can help us are the signature and nullity (to use Corollary 2.3) or the unknotting number (to use Proposition 2.4).

The unknotting number is insensitive to orientation choice, and is known for all torus links [32, 31]. Namely

$$u(T_0(p, q)) = \frac{(p-1)(q-1) + d - 1}{2}.$$

Using Proposition 2.4 gives us:

Proposition 3.1: $n_\theta(T_i(p,q)) \leq (p-1)(q-1) + d - 1 + |d - \theta|$

The signature and nullity are discussed in the following sections.

3.2.2 Signature of $T_i(p,q)$

An explicit expression for the signature of all torus links $T_0(p,q)$, much less $T_i(p,q)$, is unknown. For certain values of p and q we do have such an expression for the signature of $T_0(p,q)$. We can use this to find an expression for the other orientation selections in those classes of p and q with the link invariant $\xi(L)$. We will need to know the linking numbers of any orientation selection to do this successfully:

Lemma 3.1: *Let* $lk_i(p,q) = lk(T_i(p,q))$ *then*

$$lk_i(p,q) = \frac{pq}{2d^2}(d^2 + 4i^2 - 4id - d).$$

PROOF: The standard diagram for $T_i(p,q)$ has p overpasses and d components. Each overpass has $q-1$ crossings ($(d-1)q'$ intercomponent crossings and $q'-1$ crossings between a component and itself). A component makes a positive crossing with any component whose orientation is parallel and a negative crossing with any component whose orientation is opposite. Let
$T_i(p,q) = (L_1^+ \cup \cdots \cup L_{d-i}^+) \cup (L_1^- \cup \cdots \cup L_i^-)$. For each L_j^+ we have p' overpasses with $(d-i-1)q'$ positive crossings, and iq' negative crossings. Similarly for each L_j^- we have p' overpasses with $(i-1)q'$ positive crossings, and $(d-i)q'$ negative crossings.

So the total linking number sums over all these components. That is,

$$\begin{aligned} lk_i(p,q) &= \frac{1}{2}\left[(d-i)p'\left((d-i-1)q'-iq'\right)+ip'\left((i-1)\,q'-(d-i)\,q'\right)\right] \\ &= \frac{p'q'}{2}\left[(d-i)(d-2i-1)+i(2i-1-d)\right] \\ &= \frac{pq}{2d^2}\left(d^2-4id+4i^2-d\right) \end{aligned}$$

■

Using the invariance of $\xi(T(p,q))$ we obtain:

Corollary 3.2: *Let $\sigma_i(p,q)=-\sigma(T_i(p,q))$ then. Then*

$$\sigma_i(p,q)=\sigma_0(p,q)+\frac{2pq}{d^2}(i^2-id).$$

PROOF: As

$$\sigma(T_i(p,q))+lk_i(p,q)=\sigma(T_0(p,q))+lk_0(p,q)$$

then

$$\sigma_i(p,q)=\sigma_0(p,q)-lk_0(p,q)+lk_i(p,q).$$

By Lemma 3.1 $lk_i(p,q)=\frac{pq}{2d^2}(d^2+4i^2-4id-d)$ and $lk_0(p,q)=\frac{pq}{2d^2}(d^2-d)$ yielding the claim. ■

So to calculate $\sigma_i(p,q)$ we will need $\sigma_0(p,q)$. We achieve this with the following:

Theorem 3.7: *[19] Suppose that $p,q>0$, then the following recurrence formula holds:*

(I) Assuming $2q < p$, *then*

$$\sigma_0(p,q) = \begin{cases} \sigma_0(p-2q,q) + q^2 - 1 & if\, q\, is\, odd \\ \sigma_0(p-2q,q) + q^2 & otherwise \end{cases}.$$

(II) $\sigma_0(2q,q) = q^2 - 1$.

(III) Assuming $q \leq p \leq 2q$, *then*

$$\sigma_0(p,q) = \begin{cases} -\sigma_0(2q-p,q) + q^2 - 1 & if\, q\, is\, odd \\ -\sigma_0(2q-p,q) + q^2 - 2 & otherwise \end{cases}.$$

(IV) $\sigma_0(p,q) = \sigma_0(q,p)$, $\sigma_0(p,1) = 0$, $\sigma_0(p,2) = p - 1$.

As mentioned previously for certain classes of p and q we have explicit expressions for the signature. These follow.

Proposition 3.2: *[18] For integers* d, $p > 0$,

$$\sigma_0(dp,d) = \begin{cases} \dfrac{d^2p - 2}{2} & if\, d\, is\, even \\ \dfrac{(d^2-1)p}{2} & otherwise \end{cases}$$

Theorem 3.8: *[53] For* p, q *even and positive* $\sigma_0(p,q) = \dfrac{1}{2}pq - 1$.

3.2.3 Nullity of $T_i(p,q)$

Like the unknotting number the nullity is insensitive to orientation [28]. So we can use the standard presentation $T_0(p,q)$, and the nullity calculated would be valid for all $T_i(p,q)$. Let us use the known Seifert matrix for the diagram in Figure 3.12 to calculate the nullity.

Proposition 3.3: *[40, Prop. 7.3.1] A torus link of type $T_0(p,q)$ has a Seifert matrix of size $(p-1)(q-1) \times (p-1)(q-1)$ which in block form is a matrix of size $(q-1)(q-1)$ as follows*

$$V = \begin{bmatrix} -L & & & & & \\ L & -L & & & O & \\ & L & -L & & & \\ & & \ddots & \ddots & & \\ & O & & \ddots & -L & \\ & & & & L & -L \end{bmatrix}.$$

Here L is a matrix of size $(p-1) \times (p-1)$ in the form

$$L = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

To obtain the nullity of $T(p, q)$ we need the nullity of

$$V + V^T = \begin{bmatrix} -L-U & U & & & & \\ L & -L-U & U & & O & \\ & L & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & O & & L & -L-U & U \\ & & & & L & -L-U \end{bmatrix}$$

where $U = L^T$. As nullity is $dim - rank$ we can row reduce $V + V^T$ to obtain the rank, and hence obtain the nullity. It is straight-forward that $V + V^T$ is row-equivalent to

$$\begin{bmatrix} -U & & & & & -L \\ L & -U & & O & & -L \\ & \ddots & \ddots & & & \vdots \\ & & L & -U & & -L \\ & O & & L & -U & -L \\ & & & & L & -L-U \end{bmatrix}.$$

Now noting that $MU = L$ where M is a $(p-1) \times (p-1)$ invertible matrix such that

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & & & & 0 \\ & -1 & & & 0 \\ & & \ddots & & \vdots \\ & & & -1 & 0 \end{bmatrix}$$

we can further row reduce to

$$\begin{bmatrix} -U & & & & A_0 \\ & -U & & & A_1 \\ & & \ddots & & \vdots \\ & & & -U & A_{q-3} \\ & & & & A_{q-2} - U \end{bmatrix}$$

where we define a sequence of $(p-1) \times (p-1)$ size matrices A_n such that $A_0 = -L$ and $A_n = MA_{n-1} - L$. Solving this recurrence relation we get that $A_n = (\sum_{i=0}^{n} M^i)(-L)$. As U has full rank then the nullity of $V + V^T$ is equal to the nullity of $A_{q-2} - U$. As $U = M^{-1}L$ we get that

$$A_{q-2} - U = (\sum_{i=-1}^{q-2} M^i)(-L) = (M-I)^{-1}M^{-1}(M^q - I)(-L).$$

To simplify the above even further we will need the following elementary lemma.

Lemma 3.2: $M = QDQ^{-1}$ *where*

$$D = \begin{bmatrix} -\alpha_1 & & & 0 \\ & -\alpha_2 & & \\ & & \ddots & \\ 0 & & & -\alpha_{p-1} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \alpha_1^{p-2} & \alpha_2^{p-2} & & \alpha_{p-1}^{p-2} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_1 & \alpha_2 & & \alpha_{p-1} \\ 1 & 1 & & 1 \end{bmatrix}$$

$\alpha_1, \cdots, \alpha_{p-1}$ *being the pth roots of unity excluding one.*

PROOF: Finding the eigenvalues of M we first need to obtain the characteristic polynomial.

$$f_{p-1}(\lambda) = det(M - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & \cdots & 1 & 1 \\ -1 & -\lambda & & & 0 \\ & -1 & -\lambda & & 0 \\ & & \ddots & \ddots & \vdots \\ & & -1 & -\lambda & 0 \\ & & & -1 & -\lambda \end{vmatrix}$$

expanding this determinant with a cofactor expansion along the last column we get

$$f_{p-1}(\lambda) = 1 + (-\lambda) f_{p-2}(\lambda).$$

Noting that $f_1(\lambda) = 1 - \lambda$ the solution to this recurrence is

$$f_{p-1}(\lambda) = \sum_{i=0}^{p-1} (-\lambda)^i = \frac{(-\lambda)^p - 1}{(-\lambda) - 1}.$$

Setting $f_{p-1}(\lambda) = 0$ gives us that the eigenvalues of M are the negatives of the pth roots of unity excluding the root 1 which is cancelled by the denominator. To find the corresponding eigenvector for some eigenvalue $(-\alpha)$ we solve the system

$$\begin{bmatrix} 1+\alpha & 1 & \cdots & 1 & 1 \\ -1 & \alpha & & & 0 \\ & -1 & \alpha & & 0 \\ & & \ddots & \ddots & \vdots \\ & & -1 & \alpha & 0 \\ & & & -1 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_{p-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \vdots \\ \\ 0 \end{bmatrix}.$$

The second through $(p-1)$st rows suggest that letting $x_{p-1} = t$ gives $x_i = \alpha x_{i+1}$ for $1 \leq i \leq p-2$ and

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{p-2} & x_{p-1} \end{bmatrix} = t \cdot \begin{bmatrix} \alpha^{p-2} & \alpha^{p-2} & \cdots & \alpha & 1 \end{bmatrix}.$$

We can check that this is consistent with the first row.

$$\begin{bmatrix} 1+\alpha & 1 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} t\alpha^{p-2} & \cdots & t \end{bmatrix}$$

evaluates to

$$t \sum_{i=0}^{p-1} \alpha^i = t \frac{\alpha^p - 1}{\alpha - 1}.$$

As α is a pth root of unity (excluding the root 1) this sum is zero, and the solution is consistent. ∎

Using the Lemma to rewrite M we can rewrite $A_{q-2} - U$. Firstly we have

$$\begin{aligned}(M-I)^{-1} &= (QDQ^{-1} - QIQ^{-1})^{-1} \\ &= (Q(D-I)Q^{-1})^{-1} \\ &= Q(D-I)^{-1}Q^{-1}\end{aligned}.$$

Secondly

$$\begin{aligned}M^{-1} &= (QDQ^{-1})^{-1} \\ &= QD^{-1}Q^{-1}\end{aligned}.$$

And thirdly

$$\begin{aligned}M^{q} - I &= (QDQ^{-1})^{q} - QIQ^{-1} \\ &= Q(D^{q} - I)Q^{-1}\end{aligned}.$$

All finally yielding

$$\begin{aligned}A_{q-2} - U &= (M-I)^{-1}M^{-1}(M^{q}-I)(-L) \\ &= Q(D-I)^{-1}D^{-1}(D^{q}-I)Q^{-1}(-L)\end{aligned}$$

As $(D-I)^{-1}$, D^{-1}, and $D^{q} - I$ are all diagonal matrices (whose product is also diagonal) then we can write

$$A_{q-2} - U = QD'Q^{-1}(-L)$$

where

$$D' = \begin{bmatrix} \frac{((-\alpha_1)^q - 1)}{\alpha_1(\alpha_1 + 1)} & & \\ & \ddots & \\ & & \frac{((-\alpha_{p-1})^q - 1)}{\alpha_{p-1}(\alpha_{p-1} + 1)} \end{bmatrix}.$$

As U, Q and L are of full rank then the nullity of $V + V^T$ is the nullity of D'. This can be calculated with the number of zeros on the diagonal of D'. There will be a zero on the diagonal when a negative of a pth root of unity that is not on the real axis is also a qth root of unity. Defining $a \equiv b(mod\, r)$ iff $a = b + rl$ for some $l \in \mathbb{Z}$ we require integer solutions in x and y to

$$\pi + \frac{2\pi x}{p} \equiv \frac{2\pi y}{q}(mod\, 2\pi).$$

With a little manipulation this is equivalent to

$$y \equiv \frac{q}{2} + \frac{q}{p}x(mod\, q).$$

This implies that to answer our original question we can just consider the number of integer solutions to $y = \frac{q}{2} + \frac{q}{p}x$ for $x \in [0, p)$ and $y \in [\frac{q}{2}, \frac{3q}{2})$, and discard those that lead to real roots of unity. If p and q are both odd then this will have no solutions. As per usual letting d be the greatest common divisor of p and q, then if q is odd and p is even the solutions are $x \in \{\frac{p}{2d}, 3\frac{p}{2d}, \cdots, (2d-1)\frac{p}{2d}\}$. Otherwise if q is even then regardless of the parity of p we have solutions at $x \in \{0, \frac{p}{d}, 2\frac{p}{d}, \cdots, (d-1)\frac{p}{d}\}$. To eliminate the roots that are real we note first that if p is even then 1 and -1 are

pth roots of unity, and 1 only otherwise. Putting this together we have the following theorem.

Theorem 3.9: *For a torus link of type $T_0(p,q)$ with Seifert matrix V the nullity of $V+V^T$ is zero if p and q are odd, and $d-\epsilon$ otherwise where $\epsilon = \dfrac{3+(-1)^d}{2}$.*

PROOF: From the discussion preceding the Theorem if p and q are odd then no negative of a pth root of unity is a qth root of unity and hence the nullity is zero. Otherwise there are $d = gcd(p,q)$ such solutions including real numbers. To eliminate the real roots we note that if p and q are even (hence d is even) then there are 2 real roots of unity in common between the negative pth and usual qth roots. If exactly one of p or q is odd (hence d is odd) then there is only one real root in common. This leads to the epsilon as defined. ■

We now can give the nullity of the torus links.

Corollary 3.3: *Let $\omega(T_i(p,q)))$ be denoted $\omega_i(p,q)$*

$$\omega_i(p,q) = \begin{cases} 1 & p,q\,odd \\ d+1-\dfrac{(3+(-1)^d)}{2} & otherwise \end{cases}.$$

3.2.4 Results on Nullification of $T_i(p,q)$

Using the last two sections we have the result:

Theorem 3.10: *(1) For any p and q*

$$|\sigma_0(p,q) + \frac{2pq}{d^2}(i^2 - id)| + |\omega_i(p,q) - \theta| \leq n_\theta(T_i(p,q))$$

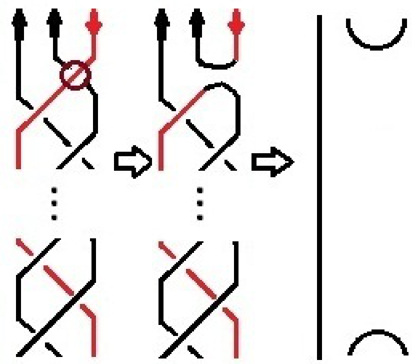


Figure 3.13: Nullifying between antiparallel components removes two components

(2) If d is even

$$\left|\frac{pq}{2d^2}(d^2+4i^2-4id)-1\right|+|d-1-\theta|\leq n_\theta(T_i(p,q)),$$

(3) If d is odd and $q'=1$

$$\left|\frac{p}{2d}(d^2-1+4i^2-4id)\right|+\begin{cases}\theta-1 & p\, is\, odd\\ & \\ |d-\theta| & otherwise\end{cases}\leq n_\theta(T_i(p,d)).$$

PROOF: All results use Corollary 2.3. More specifically (1) uses Corollary 3.2. (2) uses Theorem 3.8 and Corollary 3.3. (3) uses Proposition 3.2 and Corollary 3.3. ∎

In addition to the bounds from the invariants found above we could also use a common sense nullification sequence. Take for example $T_1(3p,3)$ as shown in Figure 3.13.

So if we have a component with a reversed orientation we can with one nullification remove two components from the torus link. If we have i components opposite orientation then we can use i nullification moves and remove $2i$ components from the torus link. This procedure gives us:

Proposition 3.4: *For any* $\gamma \in \mathbb{N}$

$$n_\theta(T_i(p,q)) \leq i + n_\gamma(T_0((d-2i)p',(d-2i)q')) + |\theta - i - \gamma|,$$

In particular for $i = \left\lfloor \frac{d}{2} \right\rfloor$

$$n_\theta(T_i(p,q)) \leq i + \begin{cases} n_\gamma(T_0(p',q')) + |\theta - i - \gamma| & if\, d\, is\, odd \\ |\theta - i| & otherwise \end{cases}.$$

Proof: Use i nullification moves on $T_i(p,q)$ to yield $U^i \sqcup T_0((d-2i)p',(d-2i)q')$. Then perform a nullification sequence on the torus link to yield $U^i \sqcup U^\gamma = U^{i+\gamma}$ in $i + n_\gamma$ moves. As $d_n(U^\theta, U^{i+\gamma}) = |\theta - i - \gamma|$ we have the first claim. If d is odd then $i = \lfloor d/2 \rfloor$ moves leaves the torus knot. If d is even then the $i = d/2$ nullification moves completely unlink the link. ■

Let's summarize some nullification numbers of specific classes of torus links. Unless otherwise stated lower bounds are from Theorem 3.10.

Case $d = 1$ As $gcd(p,q) = 1$ this is a torus knot and so $i = 0$ is the only orientation choice. We can use Theorem 3.6 to explicitly solve all nullification numbers. Namely

$$n_\theta(T_0(p,q)) = (p-1)(q-1) + \theta - 1.$$

Case $d = 2$

Subcase $i = 0$ Using the obvious nullification sequence to U^θ of length $(p-1)(q-1)+\theta-1$ for the upper bound:

$$\frac{1}{2}pq + \theta - 2 \leq n_\theta(T_0(p,q)) \leq (p-1)(q-1) + \theta - 1.$$

In particular for $q = 2$ we get the equality $n_\theta(T_0(p,2)) = p + \theta - 2$. Alternatively we can use the remark after Theorem 3.6 for the lower bound, and get

$$(p-1)(q-1) - 1 + |\theta - 2| \leq n_\theta(T_0(p,q)) \leq (p-1)(q-1) + \theta - 1.$$

As mentioned in Theorem 3.6 we get equality for $\theta = 1$. Namely, $n_1(T_0(p,q)) = (p-1)(q-1)$.

Subcase $i = 1$ Having a clear sequence to U^θ of length θ, we have

$$1 + |\theta - 1| = \theta \leq n_\theta(T_1(p,q)) \leq \theta,$$

and thus $n_\theta(T_1(p,q)) = \theta$.

Case $d \equiv 0 (mod\, 2)$, $i = \dfrac{d}{2}$ The immediate nullification sequence to U^θ is of length $\dfrac{d}{2} + |\theta - \dfrac{d}{2}|$, and so we have

$$1 + |d - 1 - \theta| \leq n_\theta(T_i(p, q)) \leq \frac{d}{2} + \left|\theta - \frac{d}{2}\right|.$$

We get equality if $\theta \leq \dfrac{d}{2}$, or if $d = 2$.

Case $d \equiv 1 (mod\, 2)$, $i = \left\lfloor \dfrac{d}{2} \right\rfloor = \dfrac{d-1}{2}$, $q = 1$

Subcase $p \equiv 0 (mod\, 2)$ A nullification sequence this time is of length $\dfrac{d-1}{2} + \left|\dfrac{d-1}{2} + 1 - \theta\right|$ to U^θ. This yields

$$|d - \theta| \leq n_\theta(T_i(p, d)) \leq \frac{d-1}{2} + \left|\frac{d+1}{2} - \theta\right|.$$

We get equality if $\theta \leq \dfrac{d+1}{2}$, or if $d = 1$.

Subcase $p \equiv 1 (mod\, 2)$ As before we have a nullification sequence of length $\dfrac{d-1}{2} + \left|\dfrac{d-1}{2} + 1 - \theta\right|$ to U^θ. This gives

$$\theta - 1 \leq n_\theta(T_i(p, d)) \leq \frac{d-1}{2} + \left|\frac{d+1}{2} - \theta\right|.$$

We get equality if $\theta \geq \dfrac{d+1}{2}$, or if $d = 1$.

Chapter 4

RESULTS AND CONCLUSIONS

As nullification requires an orientation assignment to the link of interest we need a consistent reference to differentiate among different isotopy classes of links with the various orientation possibilities. The difficulty is that for μ components we have to consider 2^μ possible orientation choices and 2 choices for whether or not we take the mirror image. Our task of tabulating nullification numbers went from looking at 379 knots and links to potentially 2412 of them. There are surely some of these choices that lead to isotopic links.

Describing the symmetries and isotopy classes amongst these choices is not yet completed (see for instance [3, 8]). Doll and Hoste [14] have made an oriented link table, but don't completely address when L, $-L$, $L!$, and $-L!$ are distinct. Corinne Cerf [10] tabulates which orientation and mirroring choices lead to isotopic links for alternating prime knots and links (prime knots up to 10 crossings and prime links up to 9 crossings).

Without this task completed we cannot give a non-redundant table of the nullification distances between links. As $d_n(L, L')$ may be different from $d_n(L, -L')$ or $d_n(L, L'!)$ for two links L and L' we would need a table that shows if $L' = -L'$ or $L' = L'!$ to know if we need more rows and columns for these cases. Furthermore, as other authors are completing nullification distance tables with similar methods (for example [24, 22]), we won't tabulate these distances here. For a table of $\theta-$nullification numbers, however, we don't require such a table. As the

$\theta-$nullification number is insensitive to overall orientation reversal and mirroring we can use the entries in the Doll and Hoste table to give a unique listing for each link.

We will denote a link of μ components with its tabulated name and a set of μ "+"s and "-"s to denote which components have reversed orientations. The ordering of the vector is done by using the default planar diagram (PD) code of the link provided in the KnotTheory database. For details on the planar diagram encoding see [2]. The natural ordering of the arcs of a component tells us the ordering of the "+"s and "-"s.

For example, link L6a1 in the KnotTheory database has a planar diagram presentation of $X_{6,1,7,2}$, $X_{10,3,11,4}$, $X_{12,8,5,7}$, $X_{8,12,9,11}$, $X_{2,5,3,6}$, $X_{4,9,1,10}$. This default orientation of the two components will be denoted $\overset{++}{L6a}1$. One component uses arcs 1 through 4, and the second component uses arcs 5 through 12. We will denote the link with a reveral of orientation on the component that uses arcs 5-12 we get $\overset{+-}{L6a}1$. Switching the orientation of just the first component would be denoted $\overset{-+}{L6a}1$.

Since reversing the orientation on any of the components has straight forward consequences on the PD code this notation can easily be implemented with a computer system. To visualize these cases we can use the DrawMorseLink command in the KnotTheory package for Mathematica. Calling this command for $\overset{++}{L6a}1$ yields the link diagram in Figure 4.1. Note that due to a design quirk the DrawMorseLink command actually draws the mirror image of the original PD code. So the link diagram drawn is actually $\overset{++}{L6a}1!$. Drawing $\overset{+-}{L6a}1$ gives us the link in Figure 4.2.

Using the methods of Doll and Hoste [14] we can classify the various

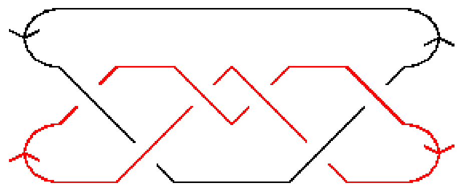


Figure 4.1: Link diagram of $\overset{++}{L6a1}$ from DrawMorseLink

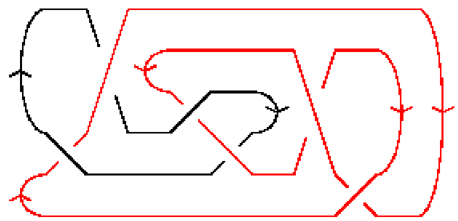


Figure 4.2: Link diagram of $\overset{+-}{L6a1}$ from DrawMorseLink

orientation choices for the links into unique sets of links for our table. Just as in [14] we will ignore the issue of distinctness of mirror images and/or complete orientation reversals in a link. This means we can without loss of generality set the orientation of the first component to be a $+$, and use the non-mirrored link. One problem arises. Without knowing the diagrams used for the notation of [14] there are two ambiguities in our tabulation. Doll and Hoste, using their own notation and ordering of the components, classify 9^3_{12} as not isotopic to $9^3_{12}+-+$ and 9^3_{21} as not isotopic to $9^3_{21}+-+$. Without orientation or mirroring considered 9^3_{12} is the link $L9a53$ in our notation and 9^3_{21} is $L9n27$. The question is which of the orientation choices on $L9a53$ or $L9n27$ corresponds to $9^3_{12}+-+$ and $9^3_{21}+-+$. We will ignore the answer to this, and instead just give separate entries for $\overset{+++}{L9a53}$, $\overset{+-+}{L9a53}$, $\overset{++-}{L9a53}$, $\overset{+++}{L9n27}$, $\overset{+-+}{L9n27}$, and $\overset{++-}{L9n27}$. There will be two redundancies from this.

With these notations and ambiguities in mind we give a table of invariants that are used in the computation of the n_θ values in Appendix A. When these

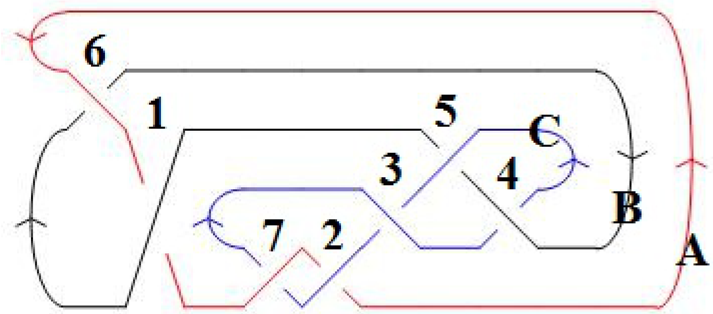


Figure 4.3: Labeled MorseLink presentation of $\overset{+++}{L7a7}$

invariants are insufficient we can find an oppositely signed unknotting sequence or an explicit nullification sequence to establish the values of $n_\theta(L)$. Sequences that we found are given in Appendix B.1 and Appendix B.2.

Let's look at an example to demonstrate how to interpret an entry in the unknotting sequence table of Appendix B.1. Take the three component link $\overset{+++}{L7a7}$. Our table says we have an unknotting sequence $+\{3,4\} - \{1,2\}$. This means we can unknot the 1st, 2nd, 3rd, and 4th crossings to obtain the unlink with three components. Where we note that the 3rd and 4th are positive and the 1st and 2nd are negative.

$\overset{+++}{L7a7}$ has a PD code of $X_{6,1,7,2}$, $X_{10,3,11,4}$, $X_{14,12,9,11}$, $X_{8,14,5,13}$, $X_{12,8,13,7}$, $X_{2,5,3,6}$, $X_{4,9,1,10}$. If we label the components A, B, and C based on the range of arcs used for the component (one component uses arcs 1-4, another 5-8, and the last 9-14), and label the crossings as they are ordered in the PD code then we obtain the (improperly mirrored) MorseLink image in Figure 4.3. Unknotting the necessary crossings we obtain the presentation in Figure 4.4.

How does this translate to a nullification number bound? Unknotting the

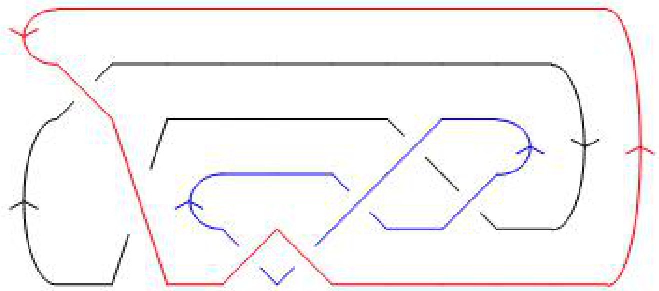


Figure 4.4: Unlinking $\overset{+++}{L7a7}$

link yields a trivial link of 3 components. So we are bounding $n_3(\overset{+++}{L7a7})$. We note that crossings 1, 2, 3, and 4 are between components A and B, A and C, C and C, B and C. So crossings 1 and 4 form a connected unknotting pair, and similarly crossings 2 and 3. So Proposition 2.4 would tell us $n_3(\overset{+++}{L7a7}) \leq 2 \cdot 4 - 2 \cdot 2 = 4$. As the lower bound is known from the link invariants we can say that $n_3(\overset{+++}{L7a7}) = 4$. This is indicated in the table with $n_3 \to 4$.

Consideration of such unknotting sequences and the invariants leave $n_1(L)$ unsolved for 14 prime knots. Furthermore $n_2(L)$ is left unsolved for 4 prime knots. The nullification sequences for 9_{37} and 9_{48} to U^2 are given in [16]. In [26] Kanenobu tabulated whether a two component link can reach the unknot in one nullification move. We will use these results without including the descriptions of the nullification sequence. The other necessary sequences are drawn in Appendix B.2.

The end results of these efforts are finally summarized in a table of n_θ numbers for $1 \leq \theta \leq 5$ in Appendix C. How complete is this table? There are 542 links (2 are duplications). This means 2710 n_θ values to isolate. All n_θ values for the 249 knots are found. The eliminates 1245 cases. Of the remaining 1465 values to find, 576 are not completely established. Where there is uncertainty we average about 2.3941 possibilities for the missing n_θ value. In addition to these results we find that of the 542 links we can establish an expression for $n_\theta(L)$ for $\theta > 5$ for all but 98 cases.

APPENDIX A

LINK INVARIANT TABLE

We tabulate here the knots and links with their relevant invariant values for calculating $n_\theta(L)$. The unknotting numbers, four genera, and three genera for the knots are from KnotInfo [37]. The unknotting numbers for the two component links are from Kohn [30]. The three and four genera for the two component links are from Kanenobu [26]. As the unknotting numbers, four genus, and three genus aren't tabulated for links with more than two components, we tabulate potential values found during a computer search for the nullification numbers.

The invariants besides the unknotting number, four genus, and three genus are found by straight-forward computation. More specifically we use the KnotTheory package for Mathematica to obtain the PD code, braid notation (not mentioned in the text, see for instance [12, Ch. 10]), HOMFLYPT polynomial, Q polynomial and Jones polynomial. We can use the braid notation to calculate the Seifert matrix and its associated signature and nullity with a program by Julia Collins [11] transcribed into Mathematica by the author. The details of unknown unknotting numbers, genera, and Seifert circle decompositions are found via computer code written by the author.

The column headers are consistent with the notation in the text, but for clarity we have, in order: link name, signature, nullity, four genus, unknotting number, three genus, HOMFLYPT polynomial value at (i, i), Jones polynomial value at $e^{i\pi/3}$, Q polynomial value at $1/2(\sqrt{5}-1)$, and the number of crossings minus the number of Seifert circles in the diagram described by the relevant reoriented PD code $(c - s)$. Any uncertainty in the invariant values is reflected with an interval $[a, b]$ that represents that their values can be any integer in this interval.

Table A.1: Link Invariant Table

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{3_1}$	2	1	1	1	1	-2	$-i\sqrt{3}$	-1	1	1
$\overset{+}{4_1}$	0	1	1	1	1	-2	-1	$-\sqrt{5}$	1	1
$\overset{+}{5_1}$	4	1	2	2	2	1	-1	$\sqrt{5}$	1	3
$\overset{+}{5_2}$	2	1	1	1	1	1	-1	-1	0	1
$\overset{+}{6_1}$	0	1	0	1	1	1	$i\sqrt{3}$	1	0	1
$\overset{+}{6_2}$	2	1	1	1	2	1	1	1	1	3
$\overset{+}{6_3}$	0	1	1	1	2	1	1	-1	1	3
$\overset{+}{7_1}$	6	1	3	3	3	1	-1	-1	0	5
$\overset{+}{7_2}$	2	1	1	1	1	-2	1	1	1	1
$\overset{+}{7_3}$	-4	1	2	2	2	-2	1	-1	1	3
$\overset{+}{7_4}$	-2	1	1	2	1	1	$-i\sqrt{3}$	$\sqrt{5}$	0	1
$\overset{+}{7_5}$	4	1	2	2	2	1	-1	-1	0	3
$\overset{+}{7_6}$	2	1	1	1	2	1	-1	1	1	3
$\overset{+}{7_7}$	0	1	1	1	2	1	$-i\sqrt{3}$	1	1	3
$\overset{+}{8_1}$	0	1	1	1	1	-2	1	-1	1	1
$\overset{+}{8_2}$	4	1	2	2	3	1	-1	-1	0	5
$\overset{+}{8_3}$	0	1	1	2	1	1	-1	-1	0	1
$\overset{+}{8_4}$	2	1	1	2	2	-2	-1	1	1	3
$\overset{+}{8_5}$	-4	1	2	2	3	-2	$i\sqrt{3}$	1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{8}_6$	2	1	1	2	2	1	1	-1	0	3
$\overset{+}{8}_7$	-2	1	1	1	3	1	1	-1	0	5
$\overset{+}{8}_8$	0	1	0	2	2	1	1	$\sqrt{5}$	0	3
$\overset{+}{8}_9$	0	1	0	1	3	1	1	$-\sqrt{5}$	0	5
$\overset{+}{8}_{10}$	-2	1	1	2	3	-2	$i\sqrt{3}$	-1	1	5
$\overset{+}{8}_{11}$	2	1	1	1	2	-2	$-i\sqrt{3}$	-1	1	3
$\overset{+}{8}_{12}$	0	1	1	2	2	1	-1	1	1	3
$\overset{+}{8}_{13}$	0	1	1	1	2	-2	-1	1	1	3
$\overset{+}{8}_{14}$	2	1	1	1	2	1	-1	1	0	3
$\overset{+}{8}_{15}$	4	1	2	2	2	-2	$i\sqrt{3}$	-1	0	3
$\overset{+}{8}_{16}$	2	1	1	2	3	1	1	$\sqrt{5}$	1	5
$\overset{+}{8}_{17}$	0	1	1	1	3	1	1	-1	1	5
$\overset{+}{8}_{18}$	0	1	1	2	3	4	3	$\sqrt{5}$	1	5
$\overset{+}{8}_{19}$	-6	1	3	3	3	-2	$-i\sqrt{3}$	-1	1	5
$\overset{+}{8}_{20}$	0	1	0	1	2	-2	$-i\sqrt{3}$	1	0	3
$\overset{+}{8}_{21}$	2	1	1	1	2	-2	$-i\sqrt{3}$	$-\sqrt{5}$	0	3
$\overset{+}{9}_1$	8	1	4	4	4	-2	$i\sqrt{3}$	1	0	7
$\overset{+}{9}_2$	2	1	1	1	1	1	$-i\sqrt{3}$	$-\sqrt{5}$	0	1
$\overset{+}{9}_3$	-6	1	3	3	3	1	-1	1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{9}_{4}$	4	1	2	2	2	1	$-i\sqrt{3}$	1	1	3
$\overset{+}{9}_{5}$	-2	1	1	2	1	1	1	-1	0	1
$\overset{+}{9}_{6}$	6	1	3	3	3	-2	$i\sqrt{3}$	-1	1	5
$\overset{+}{9}_{7}$	4	1	2	2	2	1	-1	1	1	3
$\overset{+}{9}_{8}$	2	1	1	2	2	1	-1	1	0	3
$\overset{+}{9}_{9}$	6	1	3	3	3	1	-1	1	0	5
$\overset{+}{9}_{10}$	-4	1	2	3	2	1	$i\sqrt{3}$	-1	0	3
$\overset{+}{9}_{11}$	-4	1	2	2	3	1	$-i\sqrt{3}$	-1	0	5
$\overset{+}{9}_{12}$	2	1	1	1	2	-2	1	$-\sqrt{5}$	1	3
$\overset{+}{9}_{13}$	-4	1	2	3	2	-2	1	-1	1	3
$\overset{+}{9}_{14}$	0	1	1	1	2	-2	1	-1	1	3
$\overset{+}{9}_{15}$	-2	1	1	2	2	1	$-i\sqrt{3}$	1	0	3
$\overset{+}{9}_{16}$	-6	1	3	3	3	-2	$-i\sqrt{3}$	1	0	5
$\overset{+}{9}_{17}$	2	1	1	2	3	1	$i\sqrt{3}$	1	0	5
$\overset{+}{9}_{18}$	4	1	2	2	2	1	-1	1	0	3
$\overset{+}{9}_{19}$	0	1	1	1	2	1	-1	1	0	3
$\overset{+}{9}_{20}$	4	1	2	2	3	1	-1	1	0	5
$\overset{+}{9}_{21}$	-2	1	1	1	2	-2	-1	-1	1	3
$\overset{+}{9}_{22}$	-2	1	1	1	3	-2	-1	-1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{9}_{23}$	4	1	2	2	2	-2	$i\sqrt{3}$	$\sqrt{5}$	1	3
$\overset{+}{9}_{24}$	0	1	1	1	3	-2	$-i\sqrt{3}$	$-\sqrt{5}$	1	5
$\overset{+}{9}_{25}$	2	1	1	2	2	-2	1	-1	0	3
$\overset{+}{9}_{26}$	-2	1	1	1	3	1	1	-1	0	5
$\overset{+}{9}_{27}$	0	1	0	1	3	1	1	1	0	5
$\overset{+}{9}_{28}$	2	1	1	1	3	-2	$-i\sqrt{3}$	1	1	5
$\overset{+}{9}_{29}$	2	1	1	2	3	-2	$-i\sqrt{3}$	1	1	5
$\overset{+}{9}_{30}$	0	1	1	1	3	-2	-1	-1	1	5
$\overset{+}{9}_{31}$	2	1	1	2	3	1	-1	$\sqrt{5}$	0	5
$\overset{+}{9}_{32}$	-2	1	1	2	3	1	1	1	1	5
$\overset{+}{9}_{33}$	0	1	1	1	3	1	1	1	1	5
$\overset{+}{9}_{34}$	0	1	1	1	3	1	$i\sqrt{3}$	1	1	5
$\overset{+}{9}_{35}$	2	1	1	3	1	-2	3	-1	1	1
$\overset{+}{9}_{36}$	-4	1	2	2	3	-2	1	-1	1	5
$\overset{+}{9}_{37}$	0	1	1	2	2	-2	-3	$-\sqrt{5}$	1	3
$\overset{+}{9}_{38}$	4	1	2	3	2	-2	$i\sqrt{3}$	-1	0	3
$\overset{+}{9}_{39}$	-2	1	1	1	2	-2	-1	$-\sqrt{5}$	0	3
$\overset{+}{9}_{40}$	2	1	1	2	3	4	$i\sqrt{3}$	5	1	5
$\overset{+}{9}_{41}$	0	1	0	2	2	-2	1	1	0	3

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{9}_{42}$	-2	1	1	1	2	-2	-1	-1	0	5
$\overset{+}{9}_{43}$	-4	1	2	2	3	-2	1	-1	1	5
$\overset{+}{9}_{44}$	0	1	1	1	2	-2	-1	-1	0	5
$\overset{+}{9}_{45}$	2	1	1	1	2	-2	1	-1	0	5
$\overset{+}{9}_{46}$	0	1	0	2	1	1	3	1	0	3
$\overset{+}{9}_{47}$	-2	1	1	2	3	1	3	-1	1	5
$\overset{+}{9}_{48}$	-2	1	1	2	2	1	-3	-1	1	3
$\overset{+}{9}_{49}$	-4	1	2	3	2	-2	1	-5	0	3
$\overset{+}{10}_{1}$	0	1	1	1	1	1	-1	-1	0	1
$\overset{+}{10}_{2}$	6	1	3	3	4	-2	1	-1	0	7
$\overset{+}{10}_{3}$	0	1	0	2	1	1	1	$\sqrt{5}$	0	1
$\overset{+}{10}_{4}$	2	1	1	2	2	1	$-i\sqrt{3}$	-1	1	3
$\overset{+}{10}_{5}$	-4	1	2	2	4	-2	$i\sqrt{3}$	-1	0	7
$\overset{+}{10}_{6}$	4	1	2	3	3	-2	1	-1	1	5
$\overset{+}{10}_{7}$	2	1	1	1	2	1	-1	-1	1	3
$\overset{+}{10}_{8}$	4	1	2	2	3	1	-1	1	1	5
$\overset{+}{10}_{9}$	-2	1	1	1	4	-2	$i\sqrt{3}$	1	0	7
$\overset{+}{10}_{10}$	0	1	1	1	2	1	$-i\sqrt{3}$	$-\sqrt{5}$	1	3
$\overset{+}{10}_{11}$	2	1	1	$[2,3]$	2	-2	-1	-1	1	3

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{12}$	-2	1	1	2	3	1	1	-1	0	5
$\overset{+}{10}_{13}$	0	1	1	2	2	-2	-1	-1	1	3
$\overset{+}{10}_{14}$	4	1	2	2	3	1	$i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{15}$	-2	1	1	2	3	-2	-1	-1	1	5
$\overset{+}{10}_{16}$	-2	1	1	2	2	1	1	-1	0	3
$\overset{+}{10}_{17}$	0	1	1	1	4	-2	-1	1	0	7
$\overset{+}{10}_{18}$	2	1	1	1	2	1	-1	$-\sqrt{5}$	0	3
$\overset{+}{10}_{19}$	2	1	1	2	3	1	$i\sqrt{3}$	1	1	5
$\overset{+}{10}_{20}$	2	1	1	2	2	1	1	$\sqrt{5}$	1	3
$\overset{+}{10}_{21}$	4	1	2	2	3	1	$i\sqrt{3}$	$\sqrt{5}$	1	5
$\overset{+}{10}_{22}$	0	1	0	2	3	1	1	1	0	5
$\overset{+}{10}_{23}$	-2	1	1	1	3	1	1	1	1	5
$\overset{+}{10}_{24}$	2	1	1	2	2	1	-1	$\sqrt{5}$	0	3
$\overset{+}{10}_{25}$	4	1	2	2	3	1	-1	$\sqrt{5}$	0	5
$\overset{+}{10}_{26}$	0	1	1	1	3	1	1	1	1	5
$\overset{+}{10}_{27}$	2	1	1	1	3	1	1	1	0	5
$\overset{+}{10}_{28}$	0	1	1	2	2	-2	-1	-1	1	3
$\overset{+}{10}_{29}$	2	1	1	2	3	1	$i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{30}$	2	1	1	1	2	-2	-1	-1	1	3

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{31}$	0	1	1	1	2	1	$i\sqrt{3}$	-1	0	3
$\overset{+}{10}_{32}$	0	1	1	1	3	-2	$-i\sqrt{3}$	1	1	5
$\overset{+}{10}_{33}$	0	1	1	1	2	1	-1	$-\sqrt{5}$	0	3
$\overset{+}{10}_{34}$	0	1	1	2	2	1	1	-1	1	3
$\overset{+}{10}_{35}$	0	1	0	2	2	1	1	1	0	3
$\overset{+}{10}_{36}$	2	1	1	2	2	1	$i\sqrt{3}$	1	1	3
$\overset{+}{10}_{37}$	0	1	1	2	2	-2	-1	-1	1	3
$\overset{+}{10}_{38}$	2	1	1	2	2	-2	1	1	1	3
$\overset{+}{10}_{39}$	4	1	2	2	3	-2	1	1	1	5
$\overset{+}{10}_{40}$	-2	1	1	2	3	-2	$i\sqrt{3}$	$\sqrt{5}$	1	5
$\overset{+}{10}_{41}$	2	1	1	2	3	1	1	1	0	5
$\overset{+}{10}_{42}$	0	1	0	1	3	1	$i\sqrt{3}$	1	0	5
$\overset{+}{10}_{43}$	0	1	1	2	3	1	1	-1	0	5
$\overset{+}{10}_{44}$	2	1	1	1	3	1	-1	1	0	5
$\overset{+}{10}_{45}$	0	1	1	2	3	1	-1	1	0	5
$\overset{+}{10}_{46}$	-6	1	3	3	4	1	-1	1	0	7
$\overset{+}{10}_{47}$	-4	1	2	$[2,3]$	4	1	-1	1	0	7
$\overset{+}{10}_{48}$	0	1	0	2	4	1	1	1	0	7
$\overset{+}{10}_{49}$	6	1	3	3	3	1	1	1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{50}$	-4	1	2	2	3	1	-1	-1	1	5
$\overset{+}{10}_{51}$	-2	1	1	$[2,3]$	3	1	-1	-1	1	5
$\overset{+}{10}_{52}$	-2	1	1	2	3	1	1	1	1	5
$\overset{+}{10}_{53}$	4	1	2	3	2	1	1	-1	0	3
$\overset{+}{10}_{54}$	-2	1	1	$[2,3]$	3	1	1	-1	0	5
$\overset{+}{10}_{55}$	4	1	2	2	2	1	1	1	1	3
$\overset{+}{10}_{56}$	-4	1	2	2	3	1	-1	$-\sqrt{5}$	0	5
$\overset{+}{10}_{57}$	-2	1	1	2	3	1	-1	1	0	5
$\overset{+}{10}_{58}$	0	1	1	2	2	-2	-1	$-\sqrt{5}$	0	3
$\overset{+}{10}_{59}$	-2	1	1	1	3	-2	$i\sqrt{3}$	$-\sqrt{5}$	1	5
$\overset{+}{10}_{60}$	0	1	1	1	3	-2	1	$-\sqrt{5}$	1	5
$\overset{+}{10}_{61}$	-4	1	2	$[2,3]$	3	4	$-i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{62}$	-4	1	2	2	4	-2	$i\sqrt{3}$	$\sqrt{5}$	1	7
$\overset{+}{10}_{63}$	4	1	2	2	2	4	$-i\sqrt{3}$	-1	0	3
$\overset{+}{10}_{64}$	-2	1	1	2	4	-2	$i\sqrt{3}$	1	1	7
$\overset{+}{10}_{65}$	-2	1	1	2	3	4	$-i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{66}$	6	1	3	3	3	-2	$i\sqrt{3}$	$\sqrt{5}$	1	5
$\overset{+}{10}_{67}$	2	1	1	2	2	1	$i\sqrt{3}$	-1	0	3
$\overset{+}{10}_{68}$	0	1	1	2	2	1	$i\sqrt{3}$	-1	0	3

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{69}$	-2	1	1	2	3	1	$-i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{70}$	-2	1	1	2	3	-2	-1	-1	1	5
$\overset{+}{10}_{71}$	0	1	1	1	3	-2	-1	-1	1	5
$\overset{+}{10}_{72}$	-4	1	2	2	3	-2	1	-1	0	5
$\overset{+}{10}_{73}$	2	1	1	1	3	-2	1	-1	1	5
$\overset{+}{10}_{74}$	2	1	1	2	2	1	-3	-1	0	3
$\overset{+}{10}_{75}$	0	1	0	2	3	1	3	1	0	5
$\overset{+}{10}_{76}$	-4	1	2	$[2,3]$	3	-2	$i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{77}$	-2	1	1	$[2,3]$	3	-2	$i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{78}$	4	1	2	2	3	-2	$i\sqrt{3}$	1	1	5
$\overset{+}{10}_{79}$	0	1	1	$[2,3]$	4	1	1	1	1	7
$\overset{+}{10}_{80}$	6	1	3	3	3	1	1	1	0	5
$\overset{+}{10}_{81}$	0	1	1	2	3	1	1	$-\sqrt{5}$	1	5
$\overset{+}{10}_{82}$	2	1	1	1	4	-2	$-i\sqrt{3}$	-1	0	7
$\overset{+}{10}_{83}$	-2	1	1	2	3	1	1	-1	1	5
$\overset{+}{10}_{84}$	-2	1	1	1	3	-2	$i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{85}$	4	1	2	2	4	-2	$-i\sqrt{3}$	-1	0	7
$\overset{+}{10}_{86}$	0	1	1	2	3	1	1	$\sqrt{5}$	1	5
$\overset{+}{10}_{87}$	0	1	0	2	3	-2	$i\sqrt{3}$	1	0	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{88}$	0	1	1	1	3	1	-1	1	1	5
$\overset{+}{10}_{89}$	2	1	1	2	3	1	$i\sqrt{3}$	1	1	5
$\overset{+}{10}_{90}$	0	1	1	2	3	-2	-1	-1	1	5
$\overset{+}{10}_{91}$	0	1	1	1	4	1	1	-1	0	7
$\overset{+}{10}_{92}$	-4	1	2	2	3	1	-1	1	0	5
$\overset{+}{10}_{93}$	2	1	1	2	3	-2	-1	-1	1	5
$\overset{+}{10}_{94}$	-2	1	1	2	4	1	1	1	0	7
$\overset{+}{10}_{95}$	-2	1	1	1	3	1	-1	1	1	5
$\overset{+}{10}_{96}$	0	1	1	2	3	-2	$-i\sqrt{3}$	-1	1	5
$\overset{+}{10}_{97}$	-2	1	1	2	2	-2	$-i\sqrt{3}$	-1	0	3
$\overset{+}{10}_{98}$	4	1	2	2	3	4	-3	1	0	5
$\overset{+}{10}_{99}$	0	1	0	2	4	4	3	1	0	7
$\overset{+}{10}_{100}$	4	1	2	$[2,3]$	4	1	-1	$\sqrt{5}$	0	7
$\overset{+}{10}_{101}$	-4	1	2	3	2	1	1	$\sqrt{5}$	1	3
$\overset{+}{10}_{102}$	0	1	1	1	3	1	1	-1	0	5
$\overset{+}{10}_{103}$	2	1	1	3	3	-2	$-i\sqrt{3}$	-5	1	5
$\overset{+}{10}_{104}$	0	1	1	1	4	-2	-1	-1	1	7
$\overset{+}{10}_{105}$	-2	1	1	2	3	1	-1	1	1	5
$\overset{+}{10}_{106}$	-2	1	1	2	4	-2	$i\sqrt{3}$	$\sqrt{5}$	1	7

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{107}$	0	1	1	1	3	1	$i\sqrt{3}$	-1	1	5
$\overset{+}{10}_{108}$	-2	1	1	2	3	1	$-i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{109}$	0	1	1	2	4	1	1	$\sqrt{5}$	1	7
$\overset{+}{10}_{110}$	2	1	1	2	3	1	1	-1	1	5
$\overset{+}{10}_{111}$	-4	1	2	2	3	1	-1	-1	1	5
$\overset{+}{10}_{112}$	2	1	1	2	4	-2	$-i\sqrt{3}$	-1	0	7
$\overset{+}{10}_{113}$	-2	1	1	1	3	-2	$i\sqrt{3}$	1	0	5
$\overset{+}{10}_{114}$	0	1	1	1	3	-2	$-i\sqrt{3}$	-1	1	5
$\overset{+}{10}_{115}$	0	1	1	2	3	4	1	1	1	5
$\overset{+}{10}_{116}$	2	1	1	2	4	1	1	$\sqrt{5}$	0	7
$\overset{+}{10}_{117}$	-2	1	1	2	3	1	-1	-1	0	5
$\overset{+}{10}_{118}$	0	1	1	1	4	1	1	-1	0	7
$\overset{+}{10}_{119}$	0	1	1	1	3	-2	-1	1	1	5
$\overset{+}{10}_{120}$	4	1	2	3	2	1	$i\sqrt{3}$	$\sqrt{5}$	0	3
$\overset{+}{10}_{121}$	2	1	1	2	3	1	-1	$\sqrt{5}$	1	5
$\overset{+}{10}_{122}$	0	1	1	2	3	-2	$i\sqrt{3}$	$\sqrt{5}$	0	5
$\overset{+}{10}_{123}$	0	1	0	2	4	1	1	1	0	7
$\overset{+}{10}_{124}$	-8	1	4	4	4	1	1	1	0	7
$\overset{+}{10}_{125}$	-2	1	1	2	3	1	1	1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{126}$	2	1	1	2	3	1	-1	1	1	5
$\overset{+}{10}_{127}$	4	1	2	2	3	1	-1	1	1	5
$\overset{+}{10}_{128}$	-6	1	3	3	3	1	1	1	1	5
$\overset{+}{10}_{129}$	0	1	0	1	2	1	1	$-\sqrt{5}$	0	3
$\overset{+}{10}_{130}$	0	1	1	2	2	1	-1	-1	0	3
$\overset{+}{10}_{131}$	2	1	1	1	2	1	-1	1	0	3
$\overset{+}{10}_{132}$	0	1	1	1	2	1	-1	$-\sqrt{5}$	1	3
$\overset{+}{10}_{133}$	2	1	1	1	2	1	-1	1	1	5
$\overset{+}{10}_{134}$	-6	1	3	3	3	1	1	-1	0	5
$\overset{+}{10}_{135}$	0	1	1	2	2	1	1	-1	1	5
$\overset{+}{10}_{136}$	-2	1	1	1	2	-2	$i\sqrt{3}$	$-\sqrt{5}$	0	5
$\overset{+}{10}_{137}$	0	1	0	1	2	-2	1	$-\sqrt{5}$	0	5
$\overset{+}{10}_{138}$	-2	1	1	2	3	-2	1	$-\sqrt{5}$	1	5
$\overset{+}{10}_{139}$	-6	1	4	4	4	-2	$-i\sqrt{3}$	-1	1	7
$\overset{+}{10}_{140}$	0	1	0	2	2	4	$i\sqrt{3}$	1	0	5
$\overset{+}{10}_{141}$	0	1	1	1	3	-2	$-i\sqrt{3}$	1	1	5
$\overset{+}{10}_{142}$	-6	1	3	3	3	4	$i\sqrt{3}$	$\sqrt{5}$	0	5
$\overset{+}{10}_{143}$	2	1	1	1	3	-2	$-i\sqrt{3}$	-1	1	5
$\overset{+}{10}_{144}$	2	1	1	2	2	4	$i\sqrt{3}$	1	0	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{145}$	2	1	2	2	2	1	$-i\sqrt{3}$	-1	1	3
$\overset{+}{10}_{146}$	0	1	1	1	2	1	$i\sqrt{3}$	-1	0	5
$\overset{+}{10}_{147}$	-2	1	1	1	2	-2	$i\sqrt{3}$	-1	1	5
$\overset{+}{10}_{148}$	2	1	1	2	3	1	-1	1	0	5
$\overset{+}{10}_{149}$	4	1	2	2	3	1	-1	1	0	5
$\overset{+}{10}_{150}$	-4	1	2	2	3	1	-1	1	1	5
$\overset{+}{10}_{151}$	-2	1	1	2	3	1	-1	-1	1	5
$\overset{+}{10}_{152}$	6	1	4	4	4	1	1	1	1	7
$\overset{+}{10}_{153}$	0	1	0	2	3	1	1	1	0	5
$\overset{+}{10}_{154}$	-4	1	3	3	3	1	1	-1	1	5
$\overset{+}{10}_{155}$	0	1	0	2	3	1	1	5	0	7
$\overset{+}{10}_{156}$	2	1	1	1	3	1	1	$-\sqrt{5}$	1	5
$\overset{+}{10}_{157}$	-4	1	2	2	3	-2	1	1	0	7
$\overset{+}{10}_{158}$	0	1	1	2	3	1	$-i\sqrt{3}$	$-\sqrt{5}$	1	5
$\overset{+}{10}_{159}$	2	1	1	1	3	-2	$-i\sqrt{3}$	1	0	7
$\overset{+}{10}_{160}$	-4	1	2	2	3	1	$-i\sqrt{3}$	1	1	5
$\overset{+}{10}_{161}$	4	1	3	3	3	1	-1	$\sqrt{5}$	1	5
$\overset{+}{10}_{162}$	2	1	1	2	2	1	1	$-\sqrt{5}$	1	5
$\overset{+}{10}_{163}$	-2	1	1	2	3	4	$-i\sqrt{3}$	1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+}{10}_{164}$	0	1	1	1	2	1	$i\sqrt{3}$	$-\sqrt{5}$	1	5
$\overset{+}{10}_{165}$	-2	1	1	2	2	1	$-i\sqrt{3}$	1	0	5
$\overset{++}{\mathrm{L2a1}}$	-1	1	0	1	0	1	$-i$	-1	x	0
$\overset{++}{\mathrm{L4a1}}$	1	1	0	2	0	-2	$-i$	1	0	0
$\overset{+-}{\mathrm{L4a1}}$	-3	1	1	2	1	1	$-i$	1	1	2
$\overset{++}{\mathrm{L5a1}}$	1	1	1	1	1	1	i	-1	1	2
$\overset{++}{\mathrm{L6a1}}$	1	1	1	2	1	1	$\sqrt{3}$	-1	1	2
$\overset{+-}{\mathrm{L6a1}}$	-3	1	1	2	1	-2	$\sqrt{3}$	-1	0	2
$\overset{++}{\mathrm{L6a2}}$	3	1	1	3	1	1	i	$-\sqrt{5}$	x	2
$\overset{++}{\mathrm{L6a3}}$	5	1	2	3	2	-2	$\sqrt{3}$	1	x	4
$\overset{+-}{\mathrm{L6a3}}$	-1	1	0	3	0	1	$-\sqrt{3}$	1	x	0
$\overset{+++}{\mathrm{L6a4}}$	0	1	$[0,1]$	2	1	1	1	1	1	3
$\overset{+++}{\mathrm{L6a5}}$	2	1	0	2	0	1	$i\sqrt{3}$	-1	0	1
$\overset{++-}{\mathrm{L6a5}}$	-2	1	$[0,1]$	$[2,3]$	1	-2	$i\sqrt{3}$	-1	1	3
$\overset{+-+}{\mathrm{L6a5}}$	-2	1	$[0,1]$	$[2,3]$	1	-2	$i\sqrt{3}$	-1	1	3
$\overset{+-}{\mathrm{L6a5}}$	-2	1	$[0,1]$	$[2,3]$	1	-2	$i\sqrt{3}$	-1	1	3
$\overset{+++}{\mathrm{L6n1}}$	0	1	$[0,1]$	$[2,3]$	$[0,1]$	1	1	1	0	3
$\overset{++-}{\mathrm{L6n1}}$	4	1	1	3	1	-2	1	1	1	3
$\overset{+-+}{\mathrm{L6n1}}$	0	1	$[0,1]$	$[2,3]$	$[0,1]$	1	1	1	0	3

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+-}{\text{L6n1}}$	0	1	$[0,1]$	$[2,3]$	$[0,1]$	1	1	1	0	3
$\overset{++}{\text{L7a1}}$	-1	1	1	2	2	-2	$-\sqrt{3}$	1	1	4
$\overset{++}{\text{L7a2}}$	3	1	1	3	1	1	$-i$	$-\sqrt{5}$	0	2
$\overset{+-}{\text{L7a2}}$	-1	1	1	3	2	1	$-i$	$-\sqrt{5}$	1	4
$\overset{++}{\text{L7a3}}$	-3	1	1	2	2	1	$-i$	1	1	4
$\overset{++}{\text{L7a4}}$	-1	1	0	2	1	-2	i	1	0	2
$\overset{++}{\text{L7a5}}$	1	1	0	1	1	1	$\sqrt{3}$	-1	x	2
$\overset{+-}{\text{L7a5}}$	-1	1	0	1	2	-2	$-\sqrt{3}$	-1	x	4
$\overset{++}{\text{L7a6}}$	-3	1	1	2	2	-2	i	1	x	4
$\overset{+-}{\text{L7a6}}$	-1	1	0	2	1	1	$-i$	1	x	2
$\overset{+++}{\text{L7a7}}$	0	1	$[0,1]$	$[2,3]$	1	-2	-1	$-\sqrt{5}$	1	3
$\overset{++-}{\text{L7a7}}$	0	1	$[0,1]$	$[2,3]$	1	-2	-1	$-\sqrt{5}$	1	3
$\overset{+-+}{\text{L7a7}}$	0	1	$[0,1]$	$[2,3]$	1	-2	-1	$-\sqrt{5}$	1	3
$\overset{+-}{\text{L7a7}}$	-4	1	1	3	1	1	-1	$-\sqrt{5}$	0	3
$\overset{++}{\text{L7n1}}$	5	1	2	3	2	1	$-i$	1	1	4
$\overset{+-}{\text{L7n1}}$	1	1	0	3	1	1	$-i$	1	0	4
$\overset{++}{\text{L7n2}}$	1	1	0	1	1	1	i	-1	0	4
$\overset{++}{\text{L8a1}}$	1	1	1	2	2	-2	$-i$	$\sqrt{5}$	1	4
$\overset{++}{\text{L8a2}}$	-1	1	1	1	2	1	$-i$	-1	1	4

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{++}{\mathrm{L8a3}}$	1	1	0	3	1	1	$-i$	-1	0	2
$\overset{+-}{\mathrm{L8a3}}$	-3	1	1	3	2	1	$-i$	-1	1	4
$\overset{++}{\mathrm{L8a4}}$	1	1	1	1	2	1	i	-1	1	4
$\overset{++}{\mathrm{L8a5}}$	-1	1	1	3	2	1	i	-1	1	4
$\overset{+-}{\mathrm{L8a5}}$	-5	1	2	3	2	1	i	-1	0	4
$\overset{++}{\mathrm{L8a6}}$	1	1	0	2	1	-2	i	$\sqrt{5}$	0	2
$\overset{+-}{\mathrm{L8a6}}$	-3	1	1	2	1	1	i	$\sqrt{5}$	1	2
$\overset{++}{\mathrm{L8a7}}$	3	1	1	3	1	1	$-3i$	1	0	2
$\overset{+-}{\mathrm{L8a7}}$	-1	1	1	3	2	1	$-3i$	1	1	4
$\overset{++}{\mathrm{L8a8}}$	-1	1	0	2	2	-2	$-\sqrt{3}$	$\sqrt{5}$	x	4
$\overset{+-}{\mathrm{L8a8}}$	-3	1	1	2	2	-2	$\sqrt{3}$	$\sqrt{5}$	x	4
$\overset{++}{\mathrm{L8a9}}$	1	1	0	2	2	-2	$-i$	1	x	4
$\overset{++}{\mathrm{L8a10}}$	3	1	1	3	1	1	$-i$	1	x	2
$\overset{+-}{\mathrm{L8a10}}$	-3	1	1	3	2	-2	i	1	x	4
$\overset{++}{\mathrm{L8a11}}$	5	1	2	3	2	1	$-i$	-1	x	4
$\overset{+-}{\mathrm{L8a11}}$	-1	1	0	3	1	1	i	-1	x	2
$\overset{++}{\mathrm{L8a12}}$	5	1	2	4	2	1	$-i$	1	1	4
$\overset{+-}{\mathrm{L8a12}}$	-3	1	1	4	1	1	$-i$	1	1	2
$\overset{++}{\mathrm{L8a13}}$	3	1	1	4	1	-2	$\sqrt{3}$	1	0	2

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+-}{\text{L8a13}}$	-5	1	2	4	2	-2	$\sqrt{3}$	1	0	4
$\overset{++}{\text{L8a14}}$	7	1	3	4	3	1	$-i$	-1	0	6
$\overset{+-}{\text{L8a14}}$	-1	1	0	4	0	-2	$-i$	-1	0	0
$\overset{+++}{\text{L8a15}}$	2	1	$[0,1]$	$[2,3]$	1	-2	-1	-1	1	3
$\overset{++-}{\text{L8a15}}$	-2	1	$[0,1]$	$[2,3]$	1	1	-1	-1	0	3
$\overset{+-+}{\text{L8a15}}$	-2	1	$[0,1]$	$[2,3]$	1	1	-1	-1	0	3
$\overset{+\text{--}}{\text{L8a15}}$	-2	1	$[0,1]$	$[2,3]$	1	1	-1	-1	0	3
$\overset{+++}{\text{L8a16}}$	-2	1	$[0,1]$	$[2,3]$	2	1	1	-1	x	5
$\overset{++-}{\text{L8a16}}$	0	1	$[0,1]$	$[2,3]$	1	-2	-1	-1	x	3
$\overset{+++}{\text{L8a17}}$	4	1	1	3	1	-2	1	-1	x	3
$\overset{++-}{\text{L8a17}}$	-2	1	$[0,1]$	$[2,4]$	1	1	-1	-1	x	3
$\overset{+-+}{\text{L8a17}}$	-2	1	$[0,1]$	$[2,4]$	1	1	-1	-1	x	3
$\overset{+\text{--}}{\text{L8a17}}$	0	1	$[0,2]$	$[2,4]$	2	1	1	-1	x	5
$\overset{+++}{\text{L8a18}}$	-4	1	$[1,2]$	$[3,4]$	2	1	-1	$\sqrt{5}$	x	5
$\overset{++-}{\text{L8a18}}$	2	1	0	2	0	1	1	$\sqrt{5}$	x	1
$\overset{+-+}{\text{L8a18}}$	-2	1	$[0,1]$	$[2,4]$	1	1	1	$\sqrt{5}$	x	3
$\overset{+\text{--}}{\text{L8a18}}$	-4	1	$[1,2]$	$[3,4]$	2	1	-1	$\sqrt{5}$	x	5
$\overset{+++}{\text{L8a19}}$	0	1	$[0,1]$	2	2	4	3	1	x	5
$\overset{++-}{\text{L8a19}}$	2	1	0	2	1	1	-3	1	x	3

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{\text{+++}}{\text{L8a20}}$	0	1	$[0,1]$	$[2,4]$	1	1	-1	-1	1	3
$\overset{\text{++-}}{\text{L8a20}}$	4	1	1	3	1	1	-1	-1	0	3
$\overset{\text{+-+}}{\text{L8a20}}$	0	1	$[0,2]$	$[2,4]$	2	-2	-1	-1	1	5
$\overset{\text{++++}}{\text{L8a21}}$	3	1	0	3	0	-2	$-i$	-1	0	2
$\overset{\text{+++-}}{\text{L8a21}}$	-1	1	$[0,1]$	$[3,4]$	1	1	$-i$	-1	1	4
$\overset{\text{++-+}}{\text{L8a21}}$	-1	1	$[0,1]$	$[3,4]$	1	1	$-i$	-1	1	4
$\overset{\text{++–}}{\text{L8a21}}$	-1	1	$[0,1]$	$[3,4]$	1	1	$-i$	-1	1	4
$\overset{\text{+-++}}{\text{L8a21}}$	-1	1	$[0,1]$	$[3,4]$	1	1	$-i$	-1	1	4
$\overset{\text{+-+-}}{\text{L8a21}}$	-1	1	$[0,1]$	$[3,4]$	1	1	$-i$	-1	1	4
$\overset{\text{+–+}}{\text{L8a21}}$	-5	1	1	4	1	-2	$-i$	-1	0	4
$\overset{\text{+—}}{\text{L8a21}}$	-1	1	$[0,1]$	$[3,4]$	1	1	$-i$	-1	1	4
$\overset{\text{++}}{\text{L8n1}}$	3	1	1	3	2	1	$-\sqrt{3}$	-1	1	4
$\overset{\text{+-}}{\text{L8n1}}$	-1	1	0	3	1	1	$-\sqrt{3}$	-1	0	4
$\overset{\text{++}}{\text{L8n2}}$	-1	1	0	1	1	1	$-i$	-1	0	4
$\overset{\text{+++}}{\text{L8n3}}$	6	1	2	4	2	1	-1	1	x	5
$\overset{\text{++-}}{\text{L8n3}}$	0	1	$[0,1]$	$[2,4]$	$[0,1]$	1	1	1	x	3
$\overset{\text{+-+}}{\text{L8n3}}$	0	1	$[0,1]$	$[2,4]$	$[0,1]$	1	1	1	x	3
$\overset{\text{+–}}{\text{L8n3}}$	2	1	$[0,2]$	$[2,4]$	$[1,2]$	-2	-1	1	x	5
$\overset{\text{+++}}{\text{L8n4}}$	2	1	$[0,2]$	$[2,4]$	$[1,2]$	-2	$-i\sqrt{3}$	-1	x	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{\text{++-}}{\text{L8n4}}$	0	1	$[0,1]$	$[2,4]$	$[0,1]$	1	$i\sqrt{3}$	-1	x	3
$\overset{\text{+-+}}{\text{L8n4}}$	-4	1	1	3	1	1	$i\sqrt{3}$	-1	x	3
$\overset{\text{+--}}{\text{L8n4}}$	2	1	$[0,2]$	$[2,4]$	$[1,2]$	-2	$-i\sqrt{3}$	-1	x	5
$\overset{\text{+++}}{\text{L8n5}}$	2	1	$[0,1]$	2	$[1,2]$	1	-1	1	0	5
$\overset{\text{++-}}{\text{L8n5}}$	-2	1	$[0,1]$	2	1	1	-1	1	1	3
$\overset{\text{+++}}{\text{L8n6}}$	3	2	1	3	1	1	$-i\sqrt{3}$	$\sqrt{5}$	1	3
$\overset{\text{++-}}{\text{L8n6}}$	-1	2	$[0,1]$	$[1,4]$	1	1	$-i\sqrt{3}$	$\sqrt{5}$	0	3
$\overset{\text{+-+}}{\text{L8n6}}$	-5	2	2	4	2	-2	$-i\sqrt{3}$	$\sqrt{5}$	1	5
$\overset{\text{+--}}{\text{L8n6}}$	-1	2	$[0,1]$	$[1,4]$	1	1	$-i\sqrt{3}$	$\sqrt{5}$	0	3
$\overset{\text{++++}}{\text{L8n7}}$	-1	1	0	$[3,4]$	0	-2	i	1	0	2
$\overset{\text{+++-}}{\text{L8n7}}$	3	1	$[0,1]$	$[3,4]$	1	1	i	1	1	4
$\overset{\text{++-+}}{\text{L8n7}}$	3	1	$[0,1]$	$[3,4]$	1	1	i	1	1	4
$\overset{\text{++--}}{\text{L8n7}}$	3	1	$[0,1]$	$[3,4]$	1	1	i	1	1	4
$\overset{\text{+-++}}{\text{L8n7}}$	-1	1	$[0,1]$	$[3,4]$	$[0,1]$	-2	i	1	0	4
$\overset{\text{+-+-}}{\text{L8n7}}$	-1	1	$[0,1]$	$[3,4]$	$[0,1]$	-2	i	1	0	4
$\overset{\text{+--+}}{\text{L8n7}}$	3	1	$[0,1]$	$[3,4]$	1	1	i	1	1	4
$\overset{\text{+---}}{\text{L8n7}}$	-1	1	$[0,1]$	$[3,4]$	$[0,1]$	-2	i	1	0	4
$\overset{\text{++++}}{\text{L8n8}}$	0	2	0	$[2,4]$	0	-2	$-\sqrt{3}$	$\sqrt{5}$	0	2
$\overset{\text{+++-}}{\text{L8n8}}$	0	2	$[0,1]$	$[2,4]$	$[0,1]$	-2	$-\sqrt{3}$	$\sqrt{5}$	0	4

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{++-+}{\text{L8n8}}$	0	2	$[0,1]$	$[2,4]$	$[0,1]$	-2	$-\sqrt{3}$	$\sqrt{5}$	0	4
$\overset{++-}{\text{L8n8}}$	4	2	1	4	1	1	$-\sqrt{3}$	$\sqrt{5}$	1	4
$\overset{++}{\text{L9a1}}$	-1	1	1	2	2	1	$-i$	1	1	4
$\overset{++}{\text{L9a2}}$	-3	1	1	$[2,3]$	3	1	$-i$	$-\sqrt{5}$	0	6
$\overset{++}{\text{L9a3}}$	-1	1	0	2	2	1	$-i$	1	0	4
$\overset{++}{\text{L9a4}}$	-3	1	1	2	2	-2	$\sqrt{3}$	-1	0	4
$\overset{++}{\text{L9a5}}$	-1	1	1	3	2	-2	i	-1	1	4
$\overset{+-}{\text{L9a5}}$	-5	1	2	3	2	1	i	-1	0	4
$\overset{++}{\text{L9a6}}$	5	1	2	4	2	-2	i	1	0	4
$\overset{+-}{\text{L9a6}}$	1	1	1	4	3	1	i	1	1	6
$\overset{++}{\text{L9a7}}$	3	1	1	3	1	1	$-\sqrt{3}$	1	1	2
$\overset{+-}{\text{L9a7}}$	-1	1	0	3	2	-2	$-\sqrt{3}$	1	0	4
$\overset{++}{\text{L9a8}}$	-1	1	1	2	2	1	$\sqrt{3}$	-1	1	4
$\overset{++}{\text{L9a9}}$	1	1	0	2	3	-2	$-\sqrt{3}$	-1	0	6
$\overset{++}{\text{L9a10}}$	-1	1	1	$[2,3]$	2	4	$\sqrt{3}$	-1	1	4
$\overset{++}{\text{L9a11}}$	1	1	1	3	2	1	$-i$	-1	1	4
$\overset{+-}{\text{L9a11}}$	-3	1	1	3	2	1	$-i$	-1	0	4
$\overset{++}{\text{L9a12}}$	5	1	2	4	2	1	$-\sqrt{3}$	1	1	4
$\overset{+-}{\text{L9a12}}$	1	1	0	4	3	-2	$-\sqrt{3}$	1	0	6

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
L9a13^{++}	3	1	1	3	1	-2	$-i$	1	0	2
L9a13^{+-}	-1	1	1	3	2	1	$-i$	1	1	4
L9a14^{++}	-5	1	2	3	2	-2	$\sqrt{3}$	1	0	6
L9a15^{++}	-3	1	1	$[2,3]$	2	1	$-i$	$-\sqrt{5}$	1	4
L9a16^{++}	3	1	1	3	2	1	$-i$	1	1	4
L9a16^{+-}	-1	1	0	3	2	1	$-i$	1	0	4
L9a17^{++}	-3	1	1	$[2,3]$	2	1	$-i$	$-\sqrt{5}$	0	4
L9a18^{++}	-1	1	1	2	1	1	$-\sqrt{3}$	1	1	2
L9a19^{++}	1	1	1	2	2	1	$\sqrt{3}$	$\sqrt{5}$	1	4
L9a19^{+-}	-3	1	1	2	2	-2	$\sqrt{3}$	$\sqrt{5}$	0	4
L9a20^{++}	1	1	0	2	3	-2	$-\sqrt{3}$	1	x	6
L9a20^{+-}	-1	1	$[0,1]$	2	2	1	$\sqrt{3}$	1	x	4
L9a21^{++}	1	1	0	1	3	1	i	$-\sqrt{5}$	x	6
L9a21^{+-}	-1	1	$[0,1]$	1	2	1	$-i$	$-\sqrt{5}$	x	4
L9a22^{++}	3	1	1	2	3	1	i	1	x	6
L9a22^{+-}	1	1	1	2	2	1	$-i$	1	x	4
L9a23^{++}	1	1	0	4	2	-2	$-i$	1	x	4
L9a23^{+-}	-5	1	2	4	2	1	i	1	x	4
L9a24^{++}	3	1	1	2	2	-2	$-i$	-1	x	4

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+-}{\text{L9a24}}$	1	1	0	2	3	1	i	-1	x	6
$\overset{++}{\text{L9a25}}$	1	1	0	2	1	1	$-i$	1	x	2
$\overset{+-}{\text{L9a25}}$	-1	1	0	2	2	1	i	1	x	4
$\overset{++}{\text{L9a26}}$	-3	1	1	2	2	1	$-i$	1	x	4
$\overset{+-}{\text{L9a26}}$	-1	1	0	2	2	-2	i	1	x	4
$\overset{++}{\text{L9a27}}$	-1	1	0	1	2	-2	$-i$	$\sqrt{5}$	x	4
$\overset{+-}{\text{L9a27}}$	1	1	0	1	2	1	i	$\sqrt{5}$	x	4
$\overset{++}{\text{L9a28}}$	3	1	1	4	3	1	i	1	x	6
$\overset{+-}{\text{L9a28}}$	-3	1	1	4	2	-2	$-i$	1	x	2
$\overset{++}{\text{L9a29}}$	5	1	2	3	3	1	i	1	x	6
$\overset{+-}{\text{L9a29}}$	3	1	1	3	2	-2	$-i$	1	x	4
$\overset{++}{\text{L9a30}}$	3	1	1	$[2,3]$	2	1	$-\sqrt{3}$	$-\sqrt{5}$	x	4
$\overset{+-}{\text{L9a30}}$	1	1	$[0,1]$	$[2,3]$	1	1	$\sqrt{3}$	$-\sqrt{5}$	x	2
$\overset{++}{\text{L9a31}}$	1	1	0	2	3	-2	$-\sqrt{3}$	1	x	6
$\overset{+-}{\text{L9a31}}$	3	1	1	2	2	-2	$\sqrt{3}$	1	x	4
$\overset{++}{\text{L9a32}}$	3	1	1	4	1	-2	$-i$	5	x	2
$\overset{+-}{\text{L9a32}}$	-3	1	1	4	3	-2	i	5	x	6
$\overset{++}{\text{L9a33}}$	1	1	0	3	2	-2	$-3i$	1	x	4
$\overset{+-}{\text{L9a33}}$	-5	1	2	3	2	4	$3i$	1	x	4

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{++}{\text{L9a34}}$	3	1	1	2	2	1	$-\sqrt{3}$	1	1	4
$\overset{+-}{\text{L9a34}}$	-1	1	0	2	2	-2	$-\sqrt{3}$	1	0	4
$\overset{++}{\text{L9a35}}$	-1	1	1	2	2	1	i	$\sqrt{5}$	1	4
$\overset{++}{\text{L9a36}}$	-5	1	2	3	3	1	$-i$	$-\sqrt{5}$	0	6
$\overset{+-}{\text{L9a36}}$	-1	1	1	3	1	1	$-i$	$-\sqrt{5}$	1	2
$\overset{++}{\text{L9a37}}$	-3	1	1	2	2	-2	i	1	0	4
$\overset{+-}{\text{L9a37}}$	1	1	1	2	2	1	i	1	1	4
$\overset{++}{\text{L9a38}}$	-1	1	0	1	3	1	$-i$	-1	0	6
$\overset{+-}{\text{L9a38}}$	-1	1	0	1	1	-2	$-i$	-1	0	2
$\overset{++}{\text{L9a39}}$	3	1	1	2	3	1	i	-1	0	6
$\overset{+-}{\text{L9a39}}$	-1	1	1	2	1	1	i	-1	1	2
$\overset{++}{\text{L9a40}}$	3	1	1	2	2	1	$\sqrt{3}$	1	1	4
$\overset{++}{\text{L9a41}}$	3	1	1	2	3	4	$3i$	1	1	6
$\overset{+-}{\text{L9a41}}$	-1	1	0	2	1	-2	$3i$	1	0	2
$\overset{++}{\text{L9a42}}$	-1	1	1	2	3	1	$-i$	1	1	6
$\overset{+-}{\text{L9a42}}$	-1	1	1	2	2	-2	$-i$	1	1	4
$\overset{+++}{\text{L9a43}}$	4	1	1	3	1	1	1	-1	0	3
$\overset{++-}{\text{L9a43}}$	0	1	$[0,2]$	$[2,4]$	2	1	1	-1	1	5
$\overset{+-+}{\text{L9a43}}$	0	1	$[0,2]$	$[2,4]$	2	1	1	-1	1	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
L9a43^{+-}	0	1	$[0,2]$	$[2,4]$	2	1	1	-1	1	5
L9a44^{+++}	-2	1	$[0,2]$	$[2,4]$	2	1	1	1	1	5
L9a44^{++-}	-2	1	$[0,2]$	$[2,4]$	2	1	1	1	1	5
L9a44^{+-+}	-2	1	$[0,2]$	$[2,4]$	2	1	1	1	1	5
L9a44^{+-}	-6	1	2	4	2	1	1	1	0	5
L9a45^{+++}	0	1	$[0,1]$	$[2,3]$	1	1	$-i\sqrt{3}$	1	0	3
L9a45^{++-}	0	1	$[0,1]$	$[2,3]$	1	1	$-i\sqrt{3}$	1	0	3
L9a45^{+-+}	0	1	$[0,1]$	$[2,3]$	1	1	$-i\sqrt{3}$	1	0	3
L9a45^{+-}	-4	1	1	3	1	-2	$-i\sqrt{3}$	1	1	3
L9a46^{+++}	0	1	$[0,1]$	2	2	1	1	1	x	5
L9a46^{++-}	-2	1	$[0,1]$	2	2	1	-1	1	x	5
L9a47^{+++}	2	1	$[0,1]$	$[2,3]$	1	-2	-1	-1	x	3
L9a47^{++-}	0	1	$[0,2]$	$[2,3]$	2	1	1	-1	x	5
L9a47^{+-+}	0	1	$[0,2]$	$[2,3]$	2	1	1	-1	x	5
L9a47^{+-}	-2	1	$[0,1]$	$[2,3]$	2	1	-1	-1	x	5
L9a48^{+++}	-2	1	$[0,2]$	$[2,4]$	2	1	$-i\sqrt{3}$	1	x	5
L9a48^{++-}	0	1	$[0,1]$	$[2,4]$	1	1	$i\sqrt{3}$	1	x	3
L9a48^{+-+}	0	1	$[0,1]$	$[2,4]$	1	1	$i\sqrt{3}$	1	x	3
L9a48^{+-}	-6	1	2	4	2	-2	$-i\sqrt{3}$	1	x	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+++}{\text{L9a49}}$	0	1	$[0,1]$	$[2,4]$	1	1	-1	1	x	3
$\overset{++-}{\text{L9a49}}$	2	1	$[0,2]$	$[2,4]$	2	1	1	1	x	5
$\overset{+-+}{\text{L9a49}}$	2	1	$[0,2]$	$[2,4]$	2	1	1	1	x	5
$\overset{+-}{\text{L9a49}}$	-4	1	1	3	1	1	-1	1	x	3
$\overset{+++}{\text{L9a50}}$	-2	1	$[0,2]$	$[2,3]$	2	1	1	1	x	5
$\overset{++-}{\text{L9a50}}$	0	1	$[0,1]$	$[2,3]$	1	-2	-1	1	x	3
$\overset{+-+}{\text{L9a50}}$	-4	1	1	3	2	1	-1	1	x	5
$\overset{+-}{\text{L9a50}}$	-2	1	$[0,2]$	$[2,3]$	2	1	1	1	x	5
$\overset{+++}{\text{L9a51}}$	-2	1	$[0,2]$	$[2,4]$	2	4	$-i\sqrt{3}$	$\sqrt{5}$	x	5
$\overset{++-}{\text{L9a51}}$	4	1	1	3	1	-2	$i\sqrt{3}$	$\sqrt{5}$	x	3
$\overset{+-+}{\text{L9a51}}$	0	1	$[0,2]$	$[2,4]$	2	1	$i\sqrt{3}$	$\sqrt{5}$	x	5
$\overset{+-}{\text{L9a51}}$	-2	1	$[0,2]$	$[2,4]$	2	4	$-i\sqrt{3}$	$\sqrt{5}$	x	5
$\overset{+++}{\text{L9a52}}$	2	1	$[0,1]$	$[2,3]$	1	1	$i\sqrt{3}$	-1	0	3
$\overset{+-+}{\text{L9a52}}$	-2	1	$[0,1]$	$[2,4]$	2	-2	$i\sqrt{3}$	-1	1	5
$\overset{+++}{\text{L9a53}}$	0	1	$[0,1]$	2	2	1	1	1	1	5
$\overset{++-}{\text{L9a53}}$	0	1	$[0,1]$	2	2	1	1	1	1	5
$\overset{+-+}{\text{L9a53}}$	0	1	$[0,1]$	2	2	1	1	1	1	5
$\overset{+++}{\text{L9a54}}$	-2	1	$[0,2]$	$[2,3]$	2	-2	$i\sqrt{3}$	-1	1	5
$\overset{++++}{\text{L9a55}}$	1	1	$[0,1]$	$[3,4]$	1	1	$\sqrt{3}$	-1	1	4

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{\text{+++-}}{\text{L9a55}}$	1	1	$[0,1]$	$[3,4]$	1	1	$\sqrt{3}$	-1	1	4
$\overset{\text{++-+}}{\text{L9a55}}$	1	1	$[0,1]$	$[3,4]$	1	1	$\sqrt{3}$	-1	1	4
$\overset{\text{++–}}{\text{L9a55}}$	-3	1	$[0,1]$	$[3,4]$	1	-2	$\sqrt{3}$	-1	0	4
$\overset{\text{+-++}}{\text{L9a55}}$	-3	1	$[0,1]$	$[3,4]$	1	-2	$\sqrt{3}$	-1	0	4
$\overset{\text{+-+-}}{\text{L9a55}}$	1	1	$[0,1]$	$[3,4]$	1	1	$\sqrt{3}$	-1	1	4
$\overset{\text{+–+}}{\text{L9a55}}$	-3	1	$[0,1]$	$[3,4]$	1	-2	$\sqrt{3}$	-1	0	4
$\overset{\text{+—}}{\text{L9a55}}$	-3	1	$[0,1]$	$[3,4]$	1	-2	$\sqrt{3}$	-1	0	4
$\overset{\text{++}}{\text{L9n1}}$	5	1	2	3	2	1	$-\sqrt{3}$	-1	1	4
$\overset{\text{+-}}{\text{L9n1}}$	1	1	0	3	1	-2	$-\sqrt{3}$	-1	0	4
$\overset{\text{++}}{\text{L9n2}}$	1	1	0	2	1	-2	$-i$	1	0	4
$\overset{\text{++}}{\text{L9n3}}$	1	1	1	1	1	1	i	-1	1	4
$\overset{\text{++}}{\text{L9n4}}$	7	1	3	4	3	-2	i	1	0	6
$\overset{\text{+-}}{\text{L9n4}}$	3	1	1	4	2	1	i	1	1	6
$\overset{\text{++}}{\text{L9n5}}$	3	1	1	2	2	1	i	1	1	6
$\overset{\text{++}}{\text{L9n6}}$	3	1	1	2	2	-2	$\sqrt{3}$	1	0	6
$\overset{\text{++}}{\text{L9n7}}$	5	1	2	3	2	-2	i	$-\sqrt{5}$	0	4
$\overset{\text{+-}}{\text{L9n7}}$	1	1	1	3	1	1	i	$-\sqrt{5}$	1	4
$\overset{\text{++}}{\text{L9n8}}$	3	1	1	2	2	-2	$\sqrt{3}$	1	1	4
$\overset{\text{++}}{\text{L9n9}}$	3	1	2	3	2	-2	i	1	1	4

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+-}{\text{L9n9}}$	-1	1	0	3	2	1	i	1	0	4
$\overset{++}{\text{L9n10}}$	1	1	1	2	2	-2	$-i$	-1	1	4
$\overset{+-}{\text{L9n10}}$	-3	1	1	2	2	1	$-i$	-1	0	4
$\overset{++}{\text{L9n11}}$	3	1	1	2	2	-2	$-i$	$\sqrt{5}$	1	4
$\overset{+-}{\text{L9n11}}$	-1	1	0	2	2	1	$-i$	$\sqrt{5}$	0	4
$\overset{++}{\text{L9n12}}$	-1	1	0	4	2	-2	i	1	0	4
$\overset{+-}{\text{L9n12}}$	-5	1	3	4	3	1	i	1	1	6
$\overset{++}{\text{L9n13}}$	1	1	0	1	2	4	$3i$	-1	x	6
$\overset{+-}{\text{L9n13}}$	-1	1	0	1	1	1	$-3i$	-1	x	4
$\overset{++}{\text{L9n14}}$	1	1	0	2	2	1	$-i$	$\sqrt{5}$	x	6
$\overset{+-}{\text{L9n14}}$	-1	1	0	2	1	-2	i	$\sqrt{5}$	x	4
$\overset{++}{\text{L9n15}}$	7	1	3	4	3	1	$-i$	-1	x	6
$\overset{+-}{\text{L9n15}}$	1	1	0	4	1	-2	i	-1	x	4
$\overset{++}{\text{L9n16}}$	1	1	0	4	2	1	i	1	x	6
$\overset{+-}{\text{L9n16}}$	-5	1	2	4	2	-2	$-i$	1	x	4
$\overset{++}{\text{L9n17}}$	3	1	1	2	2	1	i	-1	x	6
$\overset{+-}{\text{L9n17}}$	1	1	0	2	1	-2	$-i$	-1	x	4
$\overset{++}{\text{L9n18}}$	6	2	3	4	3	4	$-3i$	$-\sqrt{5}$	1	6
$\overset{+-}{\text{L9n18}}$	-2	2	1	4	1	1	$-3i$	$-\sqrt{5}$	1	2

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
L9n19^{++}	4	2	2	4	2	-2	$\sqrt{3}$	5	1	6
L9n20^{+++}	-2	1	$[0,1]$	$[2,4]$	1	1	-1	-1	0	3
L9n20^{++-}	2	1	$[0,2]$	$[2,4]$	2	1	-1	-1	1	5
L9n20^{+-+}	-2	1	$[0,2]$	$[2,4]$	$[1,2]$	1	-1	-1	0	5
L9n20$^{+–}$	-2	1	$[0,2]$	$[2,4]$	$[1,2]$	1	-1	-1	0	5
L9n21^{+++}	0	1	$[0,1]$	$[2,4]$	$[0,1]$	1	1	1	0	3
L9n21^{++-}	4	1	2	4	2	1	1	1	1	5
L9n21^{+-+}	0	1	$[0,2]$	$[2,4]$	$[0,2]$	1	1	1	0	5
L9n21$^{+–}$	0	1	$[0,2]$	$[2,4]$	$[0,2]$	1	1	1	0	5
L9n22^{+++}	0	1	$[0,1]$	$[2,4]$	1	1	-1	$\sqrt{5}$	0	3
L9n22^{++-}	-4	1	$[1,2]$	$[3,4]$	2	1	-1	$\sqrt{5}$	1	5
L9n22^{+-+}	-4	1	$[1,2]$	$[3,4]$	2	1	-1	$\sqrt{5}$	1	5
L9n22$^{+–}$	-4	1	$[1,2]$	$[3,4]$	2	1	-1	$\sqrt{5}$	1	5
L9n23^{+++}	4	1	1	3	2	1	$i\sqrt{3}$	-1	x	5
L9n23^{++-}	2	1	$[0,2]$	$[2,3]$	$[1,2]$	-2	$-i\sqrt{3}$	-1	x	5
L9n23^{+-+}	2	1	$[0,2]$	$[2,3]$	$[1,2]$	-2	$-i\sqrt{3}$	-1	x	5
L9n23$^{+–}$	0	1	$[0,1]$	$[2,3]$	$[1,2]$	-2	$i\sqrt{3}$	-1	x	5
L9n24^{+++}	0	1	$[0,2]$	$[2,3]$	$[1,2]$	-2	-1	$-\sqrt{5}$	x	5
L9n24^{++-}	2	1	$[0,1]$	$[2,3]$	$[1,2]$	-2	1	$-\sqrt{5}$	x	5

Table A.1 – Continued

Link	σ	ω	g^*	u	g	h	j	q	Arf	$c-s$
$\overset{+-+}{\mathrm{L9n24}}$	-2	1	$[0,1]$	$[2,3]$	2	1	1	$-\sqrt{5}$	x	5
$\overset{+-}{\mathrm{L9n24}}$	0	1	$[0,2]$	$[2,3]$	$[1,2]$	-2	-1	$-\sqrt{5}$	x	5
$\overset{+++}{\mathrm{L9n25}}$	0	1	$[0,1]$	2	$[1,2]$	1	1	1	0	5
$\overset{+++}{\mathrm{L9n26}}$	0	1	$[0,1]$	$[2,3]$	1	1	1	1	0	3
$\overset{+-+}{\mathrm{L9n26}}$	-4	1	1	3	2	-2	1	1	1	5
$\overset{+++}{\mathrm{L9n27}}$	1	2	$[0,1]$	1	$[0,2]$	-2	$-i\sqrt{3}$	$-\sqrt{5}$	0	5
$\overset{++-}{\mathrm{L9n27}}$	1	2	$[0,1]$	1	$[0,2]$	-2	$-i\sqrt{3}$	$-\sqrt{5}$	0	5
$\overset{+-+}{\mathrm{L9n27}}$	1	2	$[0,1]$	1	$[0,2]$	-2	$-i\sqrt{3}$	$-\sqrt{5}$	0	5
$\overset{+++}{\mathrm{L9n28}}$	-2	1	$[0,1]$	$[2,3]$	2	1	1	-1	1	5
$\overset{++-}{\mathrm{L9n28}}$	2	1	$[0,2]$	$[2,3]$	$[1,2]$	-2	1	-1	0	5

APPENDIX B

SPECIFIC NULLIFICATION SEQUENCES

B.1 Signed Unknotting Sequences

As mentioned in Chapter 2 the $n_\theta(L)$ for many links can be solved with the discovery of a connected opposite sign unknotting sequence (see Figure 2.8). As these aren't obvious from an invariant table we list ones we found from minimal diagrams here.

Table B.1: Signed Unknotting Sequences

Link	$\pm\ u$ Seq	Result	Link	$\pm\ u$ Seq	Result
$\overset{+}{8}_4$	$+\{1\}-\{4\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{8}_6$	$+\{3\}-\{5\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{8}_{10}$	$+\{3\}-\{1\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{8}_{12}$	$+\{5\}-\{3\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{8}_{16}$	$+\{1\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{8}_{18}$	$+\{1\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{9}_8$	$+\{4\}-\{1\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{9}_{15}$	$+\{6\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{9}_{17}$	$+\{3\}-\{7\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{9}_{25}$	$+\{4\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{9}_{29}$	$+\{1\}-\{4\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{9}_{31}$	$+\{3\}-\{6\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{9}_{32}$	$+\{5\}-\{7\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{9}_{37}$	$+\{3,8\}$	$\mathrm{n}_1 \to 4$
$\overset{+}{9}_{40}$	$+\{3\}-\{2\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{9}_{47}$	$+\{1\}-\{3\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{9}_{48}$	$+\{2,3\}$	$\mathrm{n}_1 \to 4$	$\overset{+}{10}_6$	$+\{3\}-\{5,6\}$	$\mathrm{n}_1 \to 4$
$\overset{+}{10}_{15}$	$+\{5\}-\{1\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{16}$	$+\{1\}-\{3\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{19}$	$+\{4\}-\{2\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{20}$	$+\{3\}-\{5\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{24}$	$+\{3\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{28}$	$+\{3\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{29}$	$+\{3\}-\{10\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{34}$	$+\{4\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{36}$	$+\{3\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{37}$	$+\{2\}-\{6\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{38}$	$+\{5\}-\{2\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{40}$	$+\{5\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{41}$	$+\{3\}-\{7\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{43}$	$+\{1\}-\{7\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{45}$	$+\{1\}-\{6\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{47}$	$+\{3,6\}-\{1\}$	$\mathrm{n}_1 \to 4$
$\overset{+}{10}_{52}$	$+\{1\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{57}$	$+\{6\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{61}$	$+\{1,2\}-\{5\}$	$\mathrm{n}_1 \to 4$	$\overset{+}{10}_{64}$	$+\{1\}-\{6\}$	$\mathrm{n}_1 \to 2$

Table B.1 – Continued

Link	$\pm\ u$ Seq	Result	Link	$\pm\ u$ Seq	Result
$\overset{+}{10}_{65}$	$+\{3\}-\{1\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{67}$	$+\{3\}-\{7\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{68}$	$+\{1\}-\{4\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{69}$	$+\{6\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{74}$	$-\{1,5\}$	$\mathrm{n}_1 \to 4$	$\overset{+}{10}_{76}$	$+\{2,5\}-\{3\}$	$\mathrm{n}_1 \to 4$
$\overset{+}{10}_{81}$	$+\{1\}-\{6\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{83}$	$+\{3\}-\{10\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{86}$	$+\{1\}-\{3\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{89}$	$+\{1\}-\{7\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{90}$	$+\{3\}-\{4\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{93}$	$+\{2\}-\{4\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{94}$	$+\{1\}-\{9\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{96}$	$+\{3\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{100}$	$+\{1\}-\{4,5\}$	$\mathrm{n}_1 \to 4$	$\overset{+}{10}_{103}$	$+\{1,2\}-\{3,5\}$	$\mathrm{n}_1 \to 4$
$\overset{+}{10}_{105}$	$+\{3\}-\{6\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{106}$	$+\{2\}-\{5\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{108}$	$+\{2\}-\{4\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{109}$	$+\{1\}-\{3\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{110}$	$+\{3\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{112}$	$+\{1\}-\{6\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{115}$	$+\{1\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{116}$	$+\{1\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{117}$	$+\{6\}-\{1\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{121}$	$+\{3\}-\{7\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{122}$	$+\{2\}-\{9\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{125}$	$+\{4\}-\{1\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{130}$	$+\{1\}-\{5\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{135}$	$+\{6\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{138}$	$+\{1\}-\{9\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{144}$	$+\{5\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{151}$	$+\{6\}-\{1\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{158}$	$+\{3\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+}{10}_{162}$	$+\{2\}-\{3\}$	$\mathrm{n}_1 \to 2$	$\overset{+}{10}_{163}$	$+\{1\}-\{2\}$	$\mathrm{n}_1 \to 2$
$\overset{+++}{\mathrm{L6a4}}$	$+\{2\}-\{4\}$	$\mathrm{n}_3 \to 2$	$\overset{++-}{\mathrm{L6a5}}$	$+\{2,3\}-\{1\}$	$\mathrm{n}_3 \to 4$

Table B.1 – Continued

Link	$\pm$ u Seq	Result	Link	$\pm$ u Seq	Result
$\overset{\text{+-+}}{\text{L6a5}}$	$+\{1,3\}-\{2\}$	$\mathrm{n}_3 \to 4$	$\overset{+-}{\text{L6a5}}$	$+\{1,2\}-\{3\}$	$\mathrm{n}_3 \to 4$
$\overset{++}{\text{L7a1}}$	$+\{4\}-\{2\}$	$\mathrm{n}_2 \to 2$	$\overset{++}{\text{L7a3}}$	$+\{3,4\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L7a4}}$	$+\{2\}-\{1\}$	$\mathrm{n}_2 \to 2$	$\overset{\text{+-}}{\text{L7a5}}$	$+\{2\}$	$\mathrm{n}_2 \to 2$
$\overset{++}{\text{L7a6}}$	$+\{2,3\}$	$\mathrm{n}_2 \to 4$	$\overset{\text{+-}}{\text{L7a6}}$	$+\{3\}-\{2\}$	$\mathrm{n}_2 \to 2$
$\overset{+++}{\text{L7a7}}$	$+\{3,4\}-\{1,2\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{++-}}{\text{L7a7}}$	$+\{2,3\}-\{1,4\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+-+}}{\text{L7a7}}$	$+\{1,3\}-\{2,4\}$	$\mathrm{n}_3 \to 4$	$\overset{++}{\text{L7n2}}$	$-\{4\}$	$\mathrm{n}_2 \to 2$
$\overset{++}{\text{L8a1}}$	$+\{4\}-\{2\}$	$\mathrm{n}_2 \to 2$	$\overset{\text{+-}}{\text{L8a3}}$	$+\{1,2\}-\{4\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L8a8}}$	$+\{3\}-\{6\}$	$\mathrm{n}_2 \to 2$	$\overset{\text{+-}}{\text{L8a8}}$	$+\{1,5\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L8a9}}$	$+\{3\}-\{6\}$	$\mathrm{n}_2 \to 2$	$\overset{\text{++-}}{\text{L8a15}}$	$+\{2,3\}-\{1\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+-+}}{\text{L8a15}}$	$+\{1,3\}-\{2\}$	$\mathrm{n}_3 \to 4$	$\overset{+-}{\text{L8a15}}$	$+\{1,2\}-\{3\}$	$\mathrm{n}_3 \to 4$
$\overset{+++}{\text{L8a16}}$	$+\{2,3\}-\{1\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{++-}}{\text{L8a16}}$	$+\{3\}-\{1,2\}$	$\mathrm{n}_3 \to 4$
$\overset{+-}{\text{L8a17}}$	$+\{1,2\}-\{3,4\}$	$\mathrm{n}_3 \to 4$	$\overset{+++}{\text{L8a18}}$	$+\{2,3,4\}-\{1\}$	$\mathrm{n}_3 \to 6$
$\overset{\text{+-+}}{\text{L8a18}}$	$+\{1,2\}-\{3,4\}$	$\mathrm{n}_3 \to 4$	$\overset{+-}{\text{L8a18}}$	$+\{1,3,4\}-\{2\}$	$\mathrm{n}_3 \to 6$
$\overset{+++}{\text{L8a19}}$	$+\{2\}-\{1\}$	$\mathrm{n}_3 \to 2$	$\overset{\text{++-}}{\text{L8a19}}$	$-\{1,2\}$	$\mathrm{n}_3 \to 4$
$\overset{+++}{\text{L8a20}}$	$+\{3,4\}-\{1,2\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+-+}}{\text{L8a20}}$	$+\{1,2\}-\{3,4\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+++-}}{\text{L8a21}}$	$+\{3,6\}-\{1,4\}$	$\mathrm{n}_4 \to 4$	$\overset{\text{++-+}}{\text{L8a21}}$	$+\{3,4\}-\{1,6\}$	$\mathrm{n}_4 \to 4$
$\overset{++-}{\text{L8a21}}$	$+\{4,6\}-\{1,3\}$	$\mathrm{n}_4 \to 4$	$\overset{\text{+-++}}{\text{L8a21}}$	$+\{1,6\}-\{3,4\}$	$\mathrm{n}_4 \to 4$
$\overset{\text{+-+-}}{\text{L8a21}}$	$+\{1,3\}-\{4,6\}$	$\mathrm{n}_4 \to 4$	$\overset{+—}{\text{L8a21}}$	$+\{1,4\}-\{3,6\}$	$\mathrm{n}_4 \to 4$
$\overset{++}{\text{L8n1}}$	$+\{6\}-\{1,3\}$	$\mathrm{n}_2 \to 4$	$\overset{++}{\text{L8n2}}$	$+\{6\}$	$\mathrm{n}_2 \to 2$

Table B.1 – Continued

Link	$\pm$ u Seq	Result	Link	$\pm$ u Seq	Result
$\overset{+-}{\text{L8n3}}$	$+\{1,4\}-\{2,5\}$	$\mathrm{n}_3 \to 4$	$\overset{++-}{\text{L8n4}}$	$+\{2,5\}-\{1,8\}$	$\mathrm{n}_3 \to 4$
$\overset{+++}{\text{L8n5}}$	$-\{2,5\}$	$\mathrm{n}_3 \to 4$	$\overset{++-}{\text{L8n5}}$	$+\{2,5\}$	$\mathrm{n}_3 \to 4$
$\overset{+++-}{\text{L8n7}}$	$+\{4\}-\{1,3,6\}$	$\mathrm{n}_4 \to 6$	$\overset{++-+}{\text{L8n7}}$	$+\{6\}-\{1,3,4\}$	$\mathrm{n}_4 \to 6$
$\overset{++-}{\text{L8n7}}$	$+\{3\}-\{1,4,6\}$	$\mathrm{n}_4 \to 6$	$\overset{+-+}{\text{L8n7}}$	$+\{1\}-\{3,4,6\}$	$\mathrm{n}_4 \to 6$
$\overset{++}{\text{L9a1}}$	$+\{5\}-\{1\}$	$\mathrm{n}_2 \to 2$	$\overset{++}{\text{L9a2}}$	$+\{4,5\}-\{1\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a3}}$	$+\{4\}-\{2\}$	$\mathrm{n}_2 \to 2$	$\overset{++}{\text{L9a4}}$	$+\{3,4\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a9}}$	$+\{7\}-\{1\}$	$\mathrm{n}_2 \to 2$	$\overset{+-}{\text{L9a11}}$	$+\{1,2\}-\{5\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a15}}$	$+\{2,3\}-\{1\}$	$\mathrm{n}_2 \to 4$	$\overset{++}{\text{L9a17}}$	$+\{2,3\}-\{1\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a18}}$	$+\{2\}-\{1\}$	$\mathrm{n}_2 \to 2$	$\overset{+-}{\text{L9a19}}$	$+\{1,3\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a20}}$	$+\{7\}-\{5\}$	$\mathrm{n}_2 \to 2$	$\overset{++}{\text{L9a22}}$	$-\{6,7\}$	$\mathrm{n}_2 \to 4$
$\overset{+-}{\text{L9a22}}$	$+\{6\}-\{7\}$	$\mathrm{n}_2 \to 2$	$\overset{++}{\text{L9a24}}$	$-\{2,4\}$	$\mathrm{n}_2 \to 4$
$\overset{+-}{\text{L9a24}}$	$+\{2\}-\{4\}$	$\mathrm{n}_2 \to 2$	$\overset{+-}{\text{L9a25}}$	$+\{2\}-\{4\}$	$\mathrm{n}_2 \to 2$
$\overset{++}{\text{L9a26}}$	$+\{3,4\}$	$\mathrm{n}_2 \to 4$	$\overset{+-}{\text{L9a26}}$	$+\{3\}-\{4\}$	$\mathrm{n}_2 \to 2$
$\overset{++}{\text{L9a27}}$	$+\{6\}$	$\mathrm{n}_2 \to 2$	$\overset{+-}{\text{L9a27}}$	$+\{3\}-\{7\}$	$\mathrm{n}_2 \to 2$
$\overset{++}{\text{L9a29}}$	$-\{1,5,8\}$	$\mathrm{n}_2 \to 6$	$\overset{+-}{\text{L9a29}}$	$+\{1,2\}-\{5,6\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a30}}$	$+\{6\}-\{1,2\}$	$\mathrm{n}_2 \to 4$	$\overset{++}{\text{L9a31}}$	$+\{6\}-\{2\}$	$\mathrm{n}_2 \to 2$
$\overset{+-}{\text{L9a31}}$	$-\{2,6\}$	$\mathrm{n}_2 \to 4$	$\overset{++}{\text{L9a34}}$	$-\{2,3\}$	$\mathrm{n}_2 \to 4$
$\overset{++}{\text{L9a35}}$	$+\{3\}-\{2\}$	$\mathrm{n}_2 \to 2$	$\overset{++}{\text{L9a36}}$	$+\{2,3,4\}$	$\mathrm{n}_2 \to 6$
$\overset{++}{\text{L9a37}}$	$+\{2,6\}$	$\mathrm{n}_2 \to 4$	$\overset{++}{\text{L9a38}}$	$+\{6\}$	$\mathrm{n}_2 \to 2$

Table B.1 – Continued

Link	$\pm\ u$ Seq	Result	Link	$\pm\ u$ Seq	Result
$\overset{\text{+-}}{\text{L9a38}}$	$+\{6\}$	$\mathrm{n}_2 \to 2$	$\overset{\text{++}}{\text{L9a39}}$	$-\{1,5\}$	$\mathrm{n}_2 \to 4$
$\overset{\text{++}}{\text{L9a40}}$	$+\{6,7\} - \{1,2\}$	$\mathrm{n}_2 \to 4$	$\overset{\text{++}}{\text{L9a41}}$	$-\{1,2\}$	$\mathrm{n}_2 \to 4$
$\overset{\text{++}}{\text{L9a42}}$	$+\{2\} - \{1\}$	$\mathrm{n}_2 \to 2$	$\overset{\text{+-}}{\text{L9a42}}$	$+\{1\} - \{2\}$	$\mathrm{n}_2 \to 2$
$\overset{\text{++-}}{\text{L9a43}}$	$+\{2,3\} - \{1,5\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+-+}}{\text{L9a43}}$	$+\{1,3\} - \{2,5\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+–}}{\text{L9a43}}$	$+\{1,2\} - \{3,5\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+++}}{\text{L9a44}}$	$+\{3,6\} - \{1,2\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{++-}}{\text{L9a44}}$	$+\{2,3\} - \{1,6\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+-+}}{\text{L9a44}}$	$+\{1,3\} - \{2,6\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+++}}{\text{L9a46}}$	$+\{6\} - \{5\}$	$\mathrm{n}_3 \to 2$	$\overset{\text{++-}}{\text{L9a46}}$	$+\{5,6\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{++-}}{\text{L9a47}}$	$+\{2,3\} - \{1,6\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+-+}}{\text{L9a47}}$	$+\{1,3\} - \{2,6\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+–}}{\text{L9a47}}$	$+\{1,2\} - \{4\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+++}}{\text{L9a48}}$	$+\{4,5\} - \{1,2\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{+++}}{\text{L9a49}}$	$+\{3,4\} - \{1,2\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+++}}{\text{L9a50}}$	$+\{2,3\} - \{1\}$	$\mathrm{n}_3 \to 4$
$\overset{\text{++-}}{\text{L9a50}}$	$+\{3\} - \{1,2\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+-+}}{\text{L9a50}}$	$+\{1,2,3\}$	$\mathrm{n}_3 \to 6$
$\overset{\text{+–}}{\text{L9a50}}$	$+\{1,3\} - \{2\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{+++}}{\text{L9a53}}$	$+\{5\} - \{1\}$	$\mathrm{n}_3 \to 2$
$\overset{\text{++-}}{\text{L9a53}}$	$+\{5\} - \{1\}$	$\mathrm{n}_3 \to 2$	$\overset{\text{+-+}}{\text{L9a53}}$	$+\{1\} - \{5\}$	$\mathrm{n}_3 \to 2$
$\overset{\text{+++}}{\text{L9a54}}$	$+\{2,3\} - \{1\}$	$\mathrm{n}_3 \to 4$	$\overset{\text{++–}}{\text{L9a55}}$	$+\{3,4,6\} - \{1\}$	$\mathrm{n}_4 \to 6$
$\overset{\text{+-++}}{\text{L9a55}}$	$+\{1,3,6\} - \{4\}$	$\mathrm{n}_4 \to 6$	$\overset{\text{+–+}}{\text{L9a55}}$	$+\{1,4,6\} - \{3\}$	$\mathrm{n}_4 \to 6$
$\overset{\text{+—}}{\text{L9a55}}$	$+\{1,3,4\} - \{6\}$	$\mathrm{n}_4 \to 6$	$\overset{\text{+-}}{\text{L9n4}}$	$+\{1,3\} - \{4,6\}$	$\mathrm{n}_2 \to 4$
$\overset{\text{++}}{\text{L9n5}}$	$-\{4,6\}$	$\mathrm{n}_2 \to 4$	$\overset{\text{++}}{\text{L9n6}}$	$-\{2,4\}$	$\mathrm{n}_2 \to 4$
$\overset{\text{++}}{\text{L9n8}}$	$-\{2,4\}$	$\mathrm{n}_2 \to 4$	$\overset{\text{+-}}{\text{L9n10}}$	$+\{1,2\}$	$\mathrm{n}_2 \to 4$
$\overset{\text{++}}{\text{L9n11}}$	$-\{1,2\}$	$\mathrm{n}_2 \to 4$	$\overset{\text{++}}{\text{L9n14}}$	$+\{2\} - \{1\}$	$\mathrm{n}_2 \to 2$

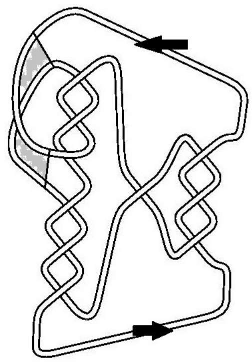
Table B.1 – Continued

Link	$\pm$ u Seq	Result	Link	$\pm$ u Seq	Result
$\overset{++}{\text{L9n17}}$	$-\{1,4\}$	$\text{n}_2 \to 4$	$\overset{+-}{\text{L9n17}}$	$+\{1\}-\{4\}$	$\text{n}_2 \to 2$
$\overset{++-}{\text{L9n22}}$	$+\{2,3,5\}-\{1\}$	$\text{n}_3 \to 6$	$\overset{+-+}{\text{L9n22}}$	$+\{1,3,5\}-\{2\}$	$\text{n}_3 \to 6$
$\overset{+-}{\text{L9n22}}$	$+\{1,2,5\}-\{3\}$	$\text{n}_3 \to 6$	$\overset{+++}{\text{L9n23}}$	$-\{1,4,6\}$	$\text{n}_3 \to 6$
$\overset{++-}{\text{L9n23}}$	$+\{2,4\}-\{1,7\}$	$\text{n}_3 \to 4$	$\overset{+-+}{\text{L9n23}}$	$+\{1,2\}-\{4,7\}$	$\text{n}_3 \to 4$
$\overset{+-}{\text{L9n23}}$	$+\{1,4\}-\{6\}$	$\text{n}_3 \to 4$	$\overset{+++}{\text{L9n24}}$	$+\{4,7\}-\{1,2\}$	$\text{n}_3 \to 4$
$\overset{+-+}{\text{L9n24}}$	$+\{1,4\}-\{6\}$	$\text{n}_3 \to 4$	$\overset{+-}{\text{L9n24}}$	$+\{1,7\}-\{2,4\}$	$\text{n}_3 \to 4$
$\overset{+++}{\text{L9n25}}$	$+\{7\}-\{5\}$	$\text{n}_3 \to 2$	$\overset{+-+}{\text{L9n26}}$	$+\{1,2,6\}$	$\text{n}_3 \to 6$
$\overset{+++}{\text{L9n27}}$	$-\{2\}$	$\text{n}_3 \to 2$	$\overset{++-}{\text{L9n27}}$	$-\{2\}$	$\text{n}_3 \to 2$
$\overset{+-+}{\text{L9n27}}$	$-\{2\}$	$\text{n}_3 \to 2$	$\overset{+++}{\text{L9n28}}$	$+\{4,6\}-\{2\}$	$\text{n}_3 \to 4$

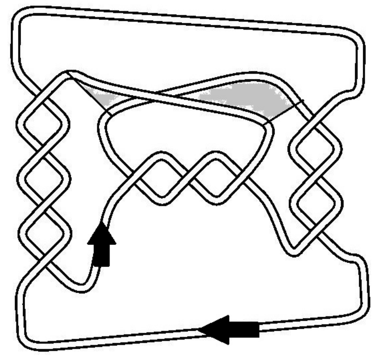
B.2 Some Nullification Sequence Diagrams

We include here a table of nullification sequences establishing the $n_1(L)$ values for the unsolved 14 prime knots, and the $n_2(L)$ values for knots 10_{74} and 10_{103}. These sequences were found by hand and with the aid of Rob Scharein and his program KnotPlot [45]. We demonstrate nullification pathways by drawing band surgeries that yield a link with known nullification distance to an unlink.

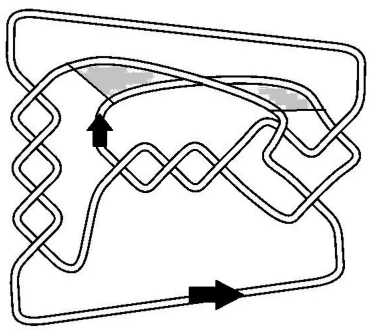
Table B.2: Some Found Nullification Sequences



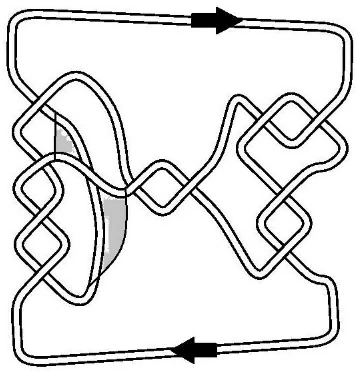
$\overset{+}{10}_4 \rightarrow L\overset{-+}{7a}5$



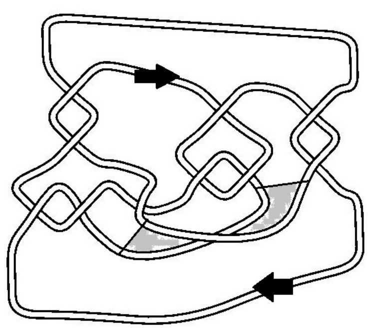
$\overset{+}{10}_{11} \rightarrow L\overset{-+}{9n}17$



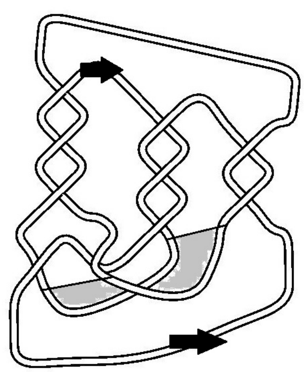
$\overset{+}{10}_{12} \rightarrow L\overset{++}{7n}2!$



$\overset{+}{10}_{13} \rightarrow L\overset{+-}{2a}1$

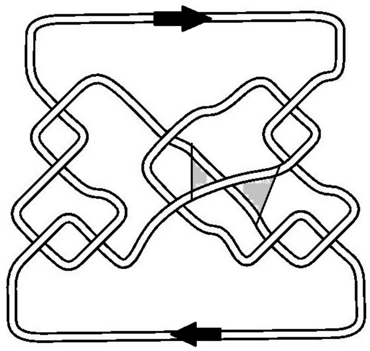


$\overset{+}{10}_{51} \rightarrow L\overset{+-}{2a}1$

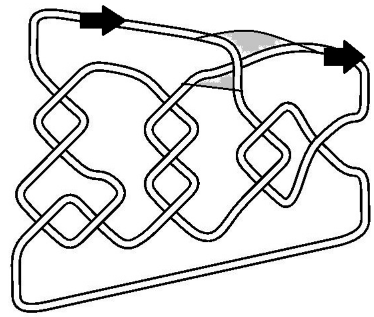


$\overset{+}{10}_{54} \rightarrow L\overset{+-}{7a}6$

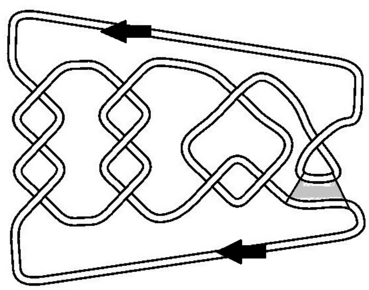
Table B.2 – Continued



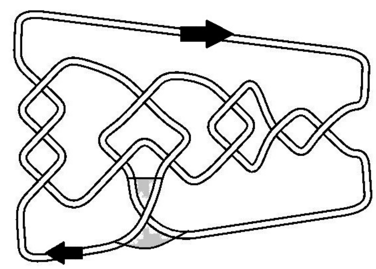
$1\overset{+}{0}_{58} \rightarrow L\overset{++}{2a}1$



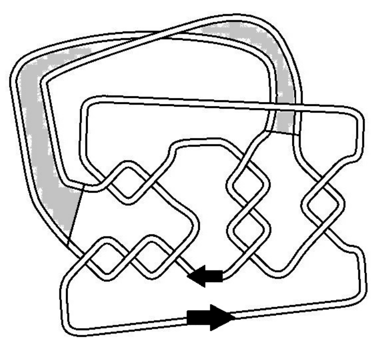
$1\overset{+}{0}_{70} \rightarrow L\overset{++}{8a}8!$



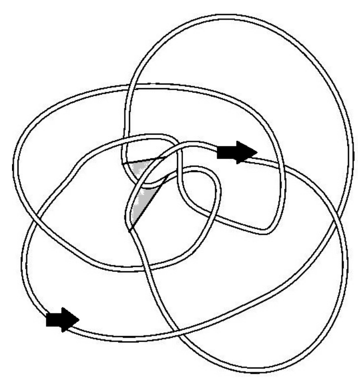
$1\overset{+}{0}_{74} \rightarrow L\overset{-+}{9a}41!$



$1\overset{+}{0}_{77} \rightarrow L\overset{++}{7n}2!$

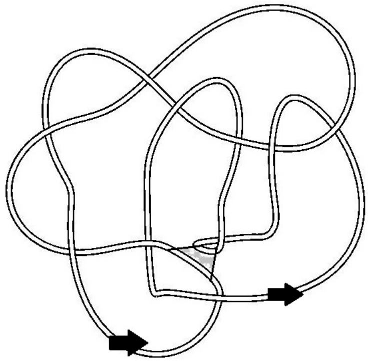


$1\overset{+}{0}_{79} \rightarrow L\overset{+-}{7a}5$

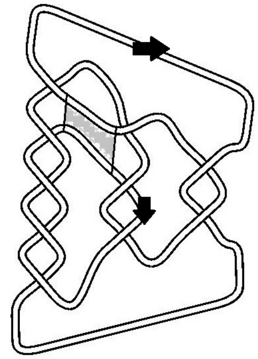


$1\overset{+}{0}_{97} \rightarrow L\overset{++}{4a}1!$

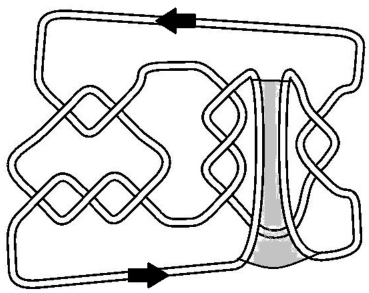
Table B.2 – Continued



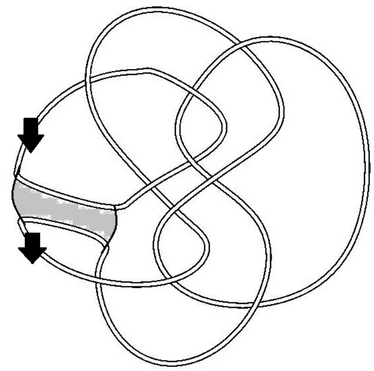
$\overset{+}{10}_{103} \rightarrow \overset{++}{L9a27}$



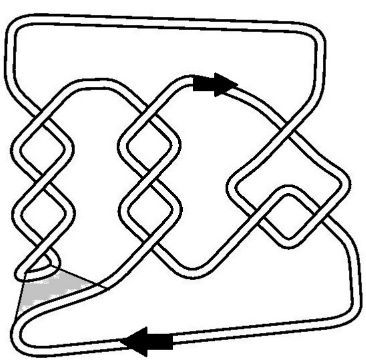
$\overset{+}{10}_{126} \rightarrow \overset{+-}{L2a1}$



$\overset{+}{10}_{148} \rightarrow \overset{++}{L7n2}$



$\overset{+}{10}_{165} \rightarrow \overset{++}{L4a1}!$



$\overset{+-}{L9a41} \rightarrow \overset{+}{6}_{1}$

APPENDIX C

$\theta-$NULLIFICATION NUMBER TABLE

We give the values or possible range of values of the nullification numbers for $1 \leq \theta \leq 5$, and note if all $n_{\theta>5}(L)$ are solved. An entry for $n_\theta(L)$ that is of the form $\{a, b\}$ means that $n_\theta(L) \in \{a, a+2, \cdots, b-2, b\}$. We also note what methods yielded the result with a symbol code. We head the column with an M_θ to represent the method used for the n_θ bounds. The lower bound codes are:

$\neq$: Using Proposition 2.2.

σ: Using Corollary 2.3.

g^*: Using Corollary 2.5.

j, h, or q: Using Corollary 2.8 with the Jones, HOMFLYPT or Q polynomial constraints, respectively, to rule out lower values.

a: Using Lemma 2.3 to rule out a distance one nullification sequence to an unlink.

The upper bound codes are:

s: Using Proposition 2.3.

u: Using Proposition 2.4 without opposite parity sequences as in Figure 2.7.

$u\pm$: Using Proposition 2.4 with opposite parity sequences as in Figure 2.8.

r : Is a ribbon knot and can be nullified as in Figure 2.11.

Δ: Using the triangle inequality from another n_θ value.

d : Using an explicit nullification sequence found for the diagram. These are either listed in Appendix B, found in [26], or [16].

This table was assembled with computer code written by the author in Mathematica.

Table C.1: θ-Nullification Number Table

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{3}_1$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{4}_1$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{5}_1$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{5}_2$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{6}_1$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{6}_2$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{6}_3$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{7}_1$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{7}_2$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{7}_3$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{7}_4$	2	3	4	5	6	$\theta+1$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{7}_5$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{7}_6$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{7}_7$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{8}_1$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{8}_2$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_3$	2	3	4	5	6	$\theta+1$	g^*,s	g^*,s	g^*,s	g^*,s	g^*,s	g^*,Δ
$\overset{+}{8}_4$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{8}_5$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_6$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{8}_7$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_8$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{8}_9$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{8}_{10}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{8}_{11}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_{12}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{8}_{13}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{8}_{14}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_{15}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_{16}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{8}_{17}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{8}_{18}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{8}_{19}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{8}_{20}$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{8}_{21}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{1}$	8	9	10	11	12	$\theta+7$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{9}_2$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_3$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_4$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_5$	2	3	4	5	6	$\theta+1$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{9}_6$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_7$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_8$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_9$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{10}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{9}_{11}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{12}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{13}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{9}_{14}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{15}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{16}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{17}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{18}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{19}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{20}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{21}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{22}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{23}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{24}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{25}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{9}_{26}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{27}$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{28}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{29}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{30}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{31}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{32}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{33}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{34}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{35}$	2	3	4	5	6	$\theta+1$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{9}_{36}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{37}$	4	3	4	5	6	$\theta+1$	j,u$\pm$	g^*,d	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{9}_{38}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{9}_{39}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{40}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{41}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{42}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{43}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{44}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{9}_{45}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{9}_{46}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{47}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{48}$	4	3	4	5	6	$\theta+1$	j,u$\pm$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{9}_{49}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_1$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_2$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_3$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_4$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_5$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_6$	4	5	6	7	8	$\theta+3$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_7$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_8$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_9$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{10}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{11}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{12}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{13}$	2	3	4	5	6	$\theta+1$	g^*,d	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{14}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{15}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{16}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{17}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{18}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{19}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{20}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{21}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{22}$	2	1	2	3	4	$\theta-1$	$\neq$,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{23}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{24}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{25}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{26}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{27}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{28}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{29}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{30}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{31}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{32}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{33}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{34}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{35}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{36}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{37}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{38}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{39}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{40}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{41}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{42}$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{43}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{44}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{45}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{46}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{47}$	4	5	6	7	8	$\theta+3$	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{48}$	2	1	2	3	4	$\theta-1$	$\neq$,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{49}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{50}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{51}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{52}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{53}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{10}_{54}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{55}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{56}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{57}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{58}$	2	3	4	5	6	$\theta+1$	g^*,d	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{59}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{60}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{61}$	4	5	6	7	8	$\theta+3$	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{62}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{63}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{64}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{65}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{66}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{67}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{68}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{69}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{70}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{71}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{72}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{73}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{74}$	4	3	4	5	6	$\theta+1$	j,u$\pm$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{75}$	2	1	2	3	4	$\theta-1$	$\neq$,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{76}$	4	5	6	7	8	$\theta+3$	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{77}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{78}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{79}$	2	3	4	5	6	$\theta+1$	g^*,d	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{80}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{81}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{82}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{83}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{84}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{85}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{86}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{87}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{88}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{89}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{90}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{91}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{92}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{93}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{94}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{95}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{96}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{97}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{98}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{99}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{100}$	4	5	6	7	8	$\theta+3$	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{101}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+}{10}_{102}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{103}$	4	3	4	5	6	$\theta+1$	q,u	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{104}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{105}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{106}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{107}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{108}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{109}$	2	3	4	5	6	$\theta+1$	g^*,u±	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{110}$	2	3	4	5	6	$\theta+1$	σ,u±	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{111}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{112}$	2	3	4	5	6	$\theta+1$	σ,u±	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{113}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{114}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{115}$	2	3	4	5	6	$\theta+1$	g^*,u±	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{116}$	2	3	4	5	6	$\theta+1$	σ,u±	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{117}$	2	3	4	5	6	$\theta+1$	σ,u±	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{118}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{119}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{120}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{121}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{122}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{123}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{124}$	8	9	10	11	12	$\theta+7$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{125}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{126}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{127}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{128}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{129}$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{130}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{131}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{132}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{133}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{134}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{135}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{136}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{137}$	2	1	2	3	4	$\theta-1$	$\neq$,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{138}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{139}$	8	9	10	11	12	$\theta+7$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{140}$	2	1	2	3	4	$\theta-1$	$\neq,\Delta$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{141}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{142}$	6	7	8	9	10	$\theta+5$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{143}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{144}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{145}$	4	5	6	7	8	$\theta+3$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{146}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{147}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{148}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{149}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{150}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{151}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{152}$	8	9	10	11	12	$\theta+7$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{153}$	2	1	2	3	4	$\theta-1$	$\neq$,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{154}$	6	7	8	9	10	$\theta+5$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{155}$	2	1	2	3	4	$\theta-1$	$\neq$,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{156}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+}{10}_{157}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{158}$	2	3	4	5	6	$\theta+1$	g^*,u$\pm$	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ	g^*,Δ
$\overset{+}{10}_{159}$	2	3	4	5	6	$\theta+1$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{160}$	4	5	6	7	8	$\theta+3$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+}{10}_{161}$	6	7	8	9	10	$\theta+5$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{162}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{163}$	2	3	4	5	6	$\theta+1$	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+}{10}_{164}$	2	3	4	5	6	$\theta+1$	g^*,u	g^*,u	g^*,u	g^*,u	g^*,u	g^*,Δ
$\overset{+}{10}_{165}$	2	3	4	5	6	$\theta+1$	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L2a1}}$	1	2	4	5	5	θ	σ,s	$\neq$,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{++}{\text{L4a1}}$	1	2	3	4	5	θ	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L4a1}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L5a1}}$	3	2	3	4	5	θ	g^*,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L6a1}}$	3	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,s	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L6a1}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L6a2}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{++}{\text{L6a3}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L6a3}}$	1	2	3	4	5	θ	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+++}{\text{L6a4}}$	$\{2,4\}$	3	2	3	4	$\theta-1$	g^*,s	a,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L6a5}}$	2	3	4	5	6	$\theta+1$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{++-}{\text{L6a5}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+-+}{\text{L6a5}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L6a5}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L6n1}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,u	σ,u	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++-}{\text{L6n1}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-+}{\text{L6n1}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,u	σ,u	
$\overset{+--}{\text{L6n1}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,u	σ,u	
$\overset{++}{\text{L7a1}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L7a2}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L7a2}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{++}{\text{L7a3}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L7a4}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L7a5}}$	3	2	3	4	5	θ	j,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L7a5}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L7a6}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L7a6}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+++}{\text{L7a7}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	a,s	j,s	j,Δ	j,Δ	
$\overset{++-}{\text{L7a7}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	a,s	j,s	j,Δ	j,Δ	
$\overset{+-+}{\text{L7a7}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	a,s	j,u	j,Δ	j,Δ	
$\overset{+-}{\text{L7a7}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L7n1}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L7n1}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L7n2}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L8a1}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L8a2}}$	3	2	3	4	5	θ	g^*,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L8a3}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L8a3}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L8a4}}$	3	2	3	4	5	θ	g^*,u	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L8a5}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L8a5}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L8a6}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L8a6}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L8a7}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L8a7}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{++}{\text{L8a8}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L8a8}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L8a9}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L8a10}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L8a10}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$		σ,s	σ,u	σ,u	σ,u	σ,u	
$\overset{++}{\text{L8a11}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-}{\mathrm{L8a11}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\mathrm{L8a12}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\mathrm{L8a12}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{++}{\mathrm{L8a13}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\mathrm{L8a13}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{++}{\mathrm{L8a14}}$	7	8	9	10	11	$\theta+6$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\mathrm{L8a14}}$	1	2	3	4	5	θ	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+++}{\mathrm{L8a15}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{++-}{\mathrm{L8a15}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-+}{\mathrm{L8a15}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\mathrm{L8a15}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\mathrm{L8a16}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++-}{\text{L8a16}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	j,s	j,u$\pm$	j,Δ	j,Δ	
$\overset{+++}{\text{L8a17}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{++-}{\text{L8a17}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+-+}{\text{L8a17}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+\text{–}}{\text{L8a17}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	q,Δ	q,u	q,Δ	q,Δ	
$\overset{+++}{\text{L8a18}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L8a18}}$	2	3	4	5	6	$\theta+1$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-+}{\text{L8a18}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+\text{–}}{\text{L8a18}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L8a19}}$	$\{2,4\}$	$\{1,3\}$	2	3	4	$\theta-1$	g^*,Δ	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L8a19}}$	4	$\{3,5\}$	4	5	6	$\theta+1$	j,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L8a20}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	a,s	j,u$\pm$	j,Δ	j,Δ	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++-}{\text{L8a20}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-+}{\text{L8a20}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	a,Δ	j,u	j,Δ	j,Δ	
$\overset{++++}{\text{L8a21}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u	σ,u	σ,Δ
$\overset{+++-}{\text{L8a21}}$	$\{3,5\}$	$\{2,6\}$	$\{3,5\}$	4	5	θ	g^*,s	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{++-+}{\text{L8a21}}$	$\{3,5\}$	$\{2,6\}$	$\{3,5\}$	4	5	θ	g^*,s	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{++--}{\text{L8a21}}$	$\{3,5\}$	$\{2,6\}$	$\{3,5\}$	4	5	θ	g^*,s	σ,s	σ,Δ	σ,u	σ,Δ	σ,Δ
$\overset{+-++}{\text{L8a21}}$	$\{3,5\}$	$\{2,6\}$	$\{3,5\}$	4	5	θ	g^*,s	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{+-+-}{\text{L8a21}}$	$\{3,5\}$	$\{2,6\}$	$\{3,5\}$	4	5	θ	g^*,s	σ,s	σ,Δ	σ,u	σ,Δ	σ,Δ
$\overset{+--+}{\text{L8a21}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,u	σ,u	σ,Δ
$\overset{+---}{\text{L8a21}}$	$\{3,5\}$	$\{2,6\}$	$\{3,5\}$	4	5	θ	g^*,s	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{++}{\text{L8n1}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L8n1}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L8n2}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+++}{\text{L8n3}}$	6	7	8	9	10	$\theta+5$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{++-}{\text{L8n3}}$	$\{2,4\}$	$\{1,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+-+}{\text{L8n3}}$	$\{2,4\}$	$\{1,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+-}{\text{L8n3}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L8n4}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,u	σ,u	
$\overset{++-}{\text{L8n4}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	q,s	q,u$\pm$	q,Δ	q,Δ	
$\overset{+-+}{\text{L8n4}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L8n4}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,u	σ,u	
$\overset{+++}{\text{L8n5}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L8n5}}$	$\{2,4\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L8n6}}$	4	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,u	σ,u	σ,u	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++-}{\text{L8n6}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		σ,s	σ,s	σ,u±	σ,s	σ,s	
$\overset{+-+}{\text{L8n6}}$	6	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$	$\{8,10\}$		σ,s	σ,s	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L8n6}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		σ,s	σ,s	σ,u±	σ,s	σ,s	
$\overset{++++}{\text{L8n7}}$	3	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+++-}{\text{L8n7}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u±	σ,Δ	σ,Δ
$\overset{++-+}{\text{L8n7}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u±	σ,Δ	σ,Δ
$\overset{++-}{\text{L8n7}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u	σ,Δ	σ,Δ
$\overset{+-++}{\text{L8n7}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u±	σ,s	
$\overset{+-+-}{\text{L8n7}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u±	σ,u	
$\overset{+-+}{\text{L8n7}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u±	σ,Δ	σ,Δ
$\overset{+-}{\text{L8n7}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u±	σ,s	
$\overset{++++}{\text{L8n8}}$	3	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$		g^*,s	g^*,s	σ,s	σ,u±	σ,s	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+++-}{\text{L8n8}}$	$\{3,5\}$	$\{2,6\}$	$\{1,7\}$	$\{2,4\}$	$\{3,9\}$		g^*,s	g^*,s	σ,s	σ,u$\pm$	σ,s	
$\overset{++-+}{\text{L8n8}}$	$\{3,5\}$	$\{2,6\}$	$\{1,7\}$	$\{2,4\}$	$\{3,9\}$		g^*,s	g^*,s	σ,s	σ,u$\pm$	σ,s	
$\overset{++-}{\text{L8n8}}$	5	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$		σ,s	σ,s	σ,s	σ,u	σ,u	
$\overset{++}{\text{L9a1}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a2}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,Δ	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a3}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a4}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a5}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9a5}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a6}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L9a6}}$	$\{3,7\}$	$\{2,4\}$	$\{3,9\}$	$\{4,10\}$	$\{5,11\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a7}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-}{\text{L9a7}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a8}}$	$\{3,5\}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,u	σ,u	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a9}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a10}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a11}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9a11}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a12}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L9a12}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a13}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L9a13}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a14}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a15}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L9a16}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$		σ,s	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9a16}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a17}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a18}}$	3	2	3	4	5	θ	g^*,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a19}}$	$\{3,5\}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,u	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9a19}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a20}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a20}}$	$\{3,5\}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		j,u	σ,u	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a21}}$	3	2	3	4	5	θ	q,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a21}}$	3	2	3	4	5	θ	q,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a22}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a22}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L9a23}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a23}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{++}{\text{L9a24}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a24}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a25}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a25}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a26}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a26}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a27}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a27}}$	1	2	3	4	5	θ	σ,d	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a28}}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$	$\{7,11\}$		σ,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9a28}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$		σ,s	σ,s	σ,s	σ,s	σ,s	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L9a29}}$	$\{5,7\}$	6	7	8	9	$\theta+4$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a29}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a30}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a30}}$	3	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		j,s	σ,s	σ,s	σ,s	σ,s	
$\overset{++}{\text{L9a31}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a31}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a32}}$	3	4	5	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{+-}{\text{L9a32}}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$	$\{7,11\}$		σ,s	σ,u$\pm$	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a33}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		j,s	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9a33}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a34}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a34}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L9a35}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9a36}}$	$\{5,7\}$	6	7	8	9	$\theta+4$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a36}}$	3	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,s	σ,s	σ,s	σ,s	σ,s	
$\overset{++}{\text{L9a37}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a37}}$	$\{3,5\}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,u	σ,u	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a38}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a38}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a39}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a39}}$	3	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,s	σ,u	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9a40}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9a41}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9a41}}$	3	2	3	4	5	θ	j,s	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L9a42}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a42}}$	3	2	3	4	5	θ	g^*,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9a43}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{++-}{\text{L9a43}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	a,Δ	q,u	q,Δ	q,Δ	
$\overset{+-+}{\text{L9a43}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	a,Δ	q,u$\pm$	q,Δ	q,Δ	
$\overset{+--}{\text{L9a43}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	a,Δ	q,u$\pm$	q,Δ	q,Δ	
$\overset{+++}{\text{L9a44}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9a44}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-+}{\text{L9a44}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+--}{\text{L9a44}}$	6	7	8	9	10	$\theta+5$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+++}{\text{L9a45}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{++-}{\text{L9a45}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-+}{\text{L9a45}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+-}{\text{L9a45}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+++}{\text{L9a46}}$	$\{2,4\}$	$\{1,3\}$	2	3	4	$\theta-1$	g^*,Δ	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9a46}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9a47}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{++-}{\text{L9a47}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	q,Δ	q,u	q,Δ	q,Δ	
$\overset{+-+}{\text{L9a47}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	q,Δ	q,u$\pm$	q,Δ	q,Δ	
$\overset{+-}{\text{L9a47}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9a48}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9a48}}$	$\{2,4\}$	$\{1,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+-+}{\text{L9a48}}$	$\{2,4\}$	$\{1,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+-}{\text{L9a48}}$	6	7	8	9	10	$\theta+5$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+++}{\text{L9a49}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	j,s	j,u±	j,Δ	j,Δ	
$\overset{++-}{\text{L9a49}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u±	σ,s	σ,s	
$\overset{+-+}{\text{L9a49}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u±	σ,u	σ,u	
$\overset{+-}{\text{L9a49}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+++}{\text{L9a50}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u±	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9a50}}$	$\{2,4\}$	$\{3,5\}$	4	5	6		g^*,s	j,s	j,u±	j,Δ	j,Δ	
$\overset{+-+}{\text{L9a50}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u±	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9a50}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u±	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9a51}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u±	σ,u	σ,u	
$\overset{++-}{\text{L9a51}}$	4	5	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-+}{\text{L9a51}}$	$\{2,6\}$	$\{1,7\}$	$\{2,4\}$	$\{3,9\}$	$\{4,10\}$		g^*,s	σ,s	σ,u±	σ,u	σ,u	
$\overset{+-}{\text{L9a51}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u±	σ,s	σ,s	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+++}{\text{L9a52}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{+-+}{\text{L9a52}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+++}{\text{L9a53}}$	$\{2,4\}$	3	2	3	4	$\theta-1$	g^*,Δ	a,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9a53}}$	$\{2,4\}$	3	2	3	4	$\theta-1$	g^*,Δ	a,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-+}{\text{L9a53}}$	$\{2,4\}$	3	2	3	4	$\theta-1$	g^*,Δ	a,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9a54}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++++}{\text{L9a55}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u$\pm$	σ,s	
$\overset{+++-}{\text{L9a55}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u$\pm$	σ,s	
$\overset{++-+}{\text{L9a55}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u$\pm$	σ,s	
$\overset{++--}{\text{L9a55}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{+-++}{\text{L9a55}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{+-+-}{\text{L9a55}}$	$\{3,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$		g^*,s	σ,s	σ,s	σ,u$\pm$	σ,s	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-+}{\text{L9a55}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{+--}{\text{L9a55}}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	6	7	$\theta+2$	σ,s	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ
$\overset{++}{\text{L9n1}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n1}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n2}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n3}}$	3	2	3	4	5	θ	g^*,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9n4}}$	7	8	9	10	11	$\theta+6$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n4}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n5}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9n6}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9n7}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n7}}$	$\{3,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$		g^*,s	σ,u$\pm$	σ,u	σ,u	σ,u	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{++}{\text{L9n8}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9n9}}$	5	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$		g^*,s	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9n9}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n10}}$	$\{3,5\}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		g^*,u	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9n10}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9n11}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n11}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n12}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9n12}}$	7	$\{6,8\}$	$\{7,9\}$	$\{8,10\}$	$\{9,11\}$		g^*,s	σ,u	σ,u	σ,u	σ,u	
$\overset{++}{\text{L9n13}}$	3	2	3	4	5	θ	j,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n13}}$	3	2	3	4	5	θ	j,u	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{++}{\text{L9n14}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-}{\text{L9n14}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n15}}$	7	8	9	10	11	$\theta+6$	σ,s	σ,u	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n15}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n16}}$	1	2	3	4	5	θ	σ,d	σ,Δ	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9n16}}$	5	6	7	8	9	$\theta+4$	σ,s	σ,s	σ,s	σ,s	σ,s	σ,Δ
$\overset{++}{\text{L9n17}}$	$\{3,5\}$	4	5	6	7	$\theta+2$	σ,u	σ,u$\pm$	σ,u	σ,u	σ,u	σ,Δ
$\overset{+-}{\text{L9n17}}$	1	2	3	4	5	θ	σ,d	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ	σ,Δ
$\overset{++}{\text{L9n18}}$	7	$\{6,8\}$	$\{7,9\}$	$\{8,10\}$	$\{9,11\}$		σ,s	σ,u	σ,u	σ,u	σ,u	
$\overset{+-}{\text{L9n18}}$	3	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{++}{\text{L9n19}}$	$\{5,7\}$	$\{4,8\}$	$\{5,9\}$	$\{6,10\}$	$\{7,11\}$		σ,s	σ,u	σ,u	σ,u	σ,u	
$\overset{+++}{\text{L9n20}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		σ,s	σ,s	σ,s	σ,s	σ,s	
$\overset{++-}{\text{L9n20}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,u	σ,u	

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-+}{\text{L9n20}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+-}{\text{L9n20}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+++}{\text{L9n21}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{++-}{\text{L9n21}}$	6	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$	$\{8,10\}$		g^*,s	σ,s	σ,u	σ,u	σ,u	
$\overset{+-+}{\text{L9n21}}$	$\{2,6\}$	$\{1,7\}$	$\{2,4\}$	$\{3,9\}$	$\{4,10\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+-}{\text{L9n21}}$	$\{2,6\}$	$\{1,7\}$	$\{2,4\}$	$\{3,9\}$	$\{4,10\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+++}{\text{L9n22}}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$		g^*,s	j,s	j,s	j,s	j,s	
$\overset{++-}{\text{L9n22}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+-+}{\text{L9n22}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9n22}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9n23}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9n23}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+-+}{\text{L9n23}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9n23}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	q,Δ	q,u$\pm$	q,Δ	q,Δ	
$\overset{+++}{\text{L9n24}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	j,Δ	j,u	j,Δ	j,Δ	
$\overset{++-}{\text{L9n24}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+-+}{\text{L9n24}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-}{\text{L9n24}}$	$\{2,6\}$	$\{3,5\}$	4	5	6		g^*,s	j,Δ	j,u$\pm$	j,Δ	j,Δ	
$\overset{+++}{\text{L9n25}}$	$\{2,4\}$	$\{1,3\}$	2	3	4	$\theta-1$	g^*,Δ	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9n26}}$	$\{2,4\}$	$\{1,5\}$	$\{2,4\}$	$\{3,7\}$	$\{4,8\}$		g^*,s	σ,s	σ,u$\pm$	σ,s	σ,s	
$\overset{+-+}{\text{L9n26}}$	$\{4,6\}$	$\{5,7\}$	6	7	8	$\theta+3$	σ,s	σ,s	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+++}{\text{L9n27}}$	$\{2,4\}$	$\{1,3\}$	2	3	4	$\theta-1$	σ,Δ	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9n27}}$	$\{2,4\}$	$\{1,3\}$	2	3	4	$\theta-1$	σ,Δ	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{+-+}{\text{L9n27}}$	$\{2,4\}$	$\{1,3\}$	2	3	4	$\theta-1$	σ,Δ	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ

Table C.1 – Continued

Link	n_1	n_2	n_3	n_4	n_5	$n_{\theta>5}$	M_1	M_2	M_3	M_4	M_5	$M_{>5}$
$\overset{+++}{\text{L9n28}}$	$\{2,6\}$	$\{3,5\}$	4	5	6	$\theta+1$	σ,s	σ,Δ	σ,u$\pm$	σ,Δ	σ,Δ	σ,Δ
$\overset{++-}{\text{L9n28}}$	$\{2,6\}$	$\{3,7\}$	$\{4,6\}$	$\{5,9\}$	$\{6,10\}$		σ,s	σ,s	σ,u$\pm$	σ,u	σ,u	

APPENDIX D

NOTATIONS USED

$-L$ The reverse of a link L.

$L!$ The mirror image of a link L.

Arf(L) The Arf invariant of a link L [44].

B^k The k-dimensional ball. I.e., the interior of the k-dimensional sphere, S^k.

$d_n(L, L')$ The nullification distance between two links (Definition 2.1).

$g(L)$ The minimum genus of any connected orientable surface that spans an oriented link L.

$g^*(L)$ The 4-ball genus, or four genus of a link L. The minimum genus of any orientable surfaces spanning L in B^4. For a proper embedding, L is regarded as a subset of $S^3 = \partial B^4$.

$h(L)$ $P(L; i, i)$

$j(L)$ $V(L; e^{i\pi/3})$

$lk(L)$ The linking number of L.

$\mu(L)$ The number of components of L.

$n_\theta(L)$ The nullification distance between a link and U^θ.

$\omega(L)$ The nullity of a link L. Given a Seifert matrix, M, of the link this is the nullity of $M + M^T$ plus one.

$P(L; v, z)$ The HOMFLYPT polynomial of a link in variables v and z.

$q(L)$ $Q(L; {}^1\!/\!{}_2(\sqrt{5} - 1)$

$Q(L; z)$ The Q polynomial of a link in the variable z.

$\mathbb{R}^k$ The k-dimensional real vector space.

S^k The k-dimensional sphere.

$\sigma(L)$ The signature of a link. Given a Seifert matrix, M, of the link this is the signature of $M + M^T$.

$T_i(p, q)$ The p, q torus link with i of its components having reversed orientation. Without a subcript we imply that $i = 0$.

U^k The unlink with k components.

$u(L)$ The unknotting/unlinking number of a link. That is, the minimum number of crossing changes to unlink L.

$V(L; t)$ The Jones polynomial of a link in the variable t.

$\xi(L)$ The link invariant where $\xi(L) = \sigma(L) + lk(L)$.

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