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ORTHOGONAL MACROELEMENT SCALING VECTORS AND WAVELETS IN 1-D

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Abstract. We develop a *macroelement* based technique for constructing orthogonal univariate multiwavelets. We illustrate the technique with two examples. In the first example we provide a new construction of the symmetric, orthogonal, continuous scaling vector given in [1]. In the second example (first constructed in [12]) we give a construction of a continuous orthogonal scaling vector with three components. The components of this scaling vector are symmetric or antisymmetric and provide approximation order 3, (equivalently, the components of Ψ are orthogonal to polynomials of degree 2 or less.)

Suppose $\Phi = (\phi_1, \ldots, \phi_n)^T$ is a vector of compactly supported functions $\phi_i \in L^2(\mathbb{R})$, $i = 1, \ldots, n$. We call such a vector Φ a *generator*. Let $T(\Phi) := \{\phi_i(\cdot - k) \mid i = 1, \ldots, n;\ k \in \mathbb{Z}\}$ denote the set of integer translates of the components of Φ. The shift invariant space $S(\Phi)$ generated by Φ is the closure (in $L^2(\mathbb{R})$) of the finite linear span of $T(\Phi)$. If $T(\Phi)$ is an orthonormal basis of $S(\Phi)$ then Φ is said to have *orthonormal shifts*.

A generator $\Phi = (\phi_1, \cdots, \phi_n)^T$ is said to be *refinable* if

$$\Phi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} b[k] \Phi(2x - k) \qquad (x \in \mathbb{R})$$

for some finitely supported sequence $b = (b[k])$ of $n \times n$ matrices. We call a refinable generator with orthonormal shifts a *scaling vector*.

Suppose Φ is a scaling vector. Let $V_k := \{f(\cdot/2^k) \mid f \in S(\Phi)\}$. Then it is known that $(V_k)_{k \in \mathbb{Z}}$ forms a *multiresolution analysis of* $L^2(\mathbb{R})$ (MRA), that is, $(V_k)_{k \in \mathbb{Z}}$ satisfies

(a) $\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$,

(b) $\bigcup_{k \in \mathbb{Z}} V_k$ is dense in $L^2(\mathbb{R})$, and

(c) $\bigcap_{k \in \mathbb{Z}} V_k = \{0\}$.

Let $W_k := V_k \cap V_{k+1}^{\perp}$ denote the orthogonal complement of V_{k+1} in V_k for $k \in \mathbb{Z}$. Then, as we review in Appendix A, there is a generator $\Psi = (\psi_1, \ldots, \psi_n)$ with orthonormal shifts such that $W_0 = S(\Psi)$. In the case

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$n > 1$, the generator Ψ is called a *multiwavelet*. The properties (a), (b), and (c) of multiresolution analysis imply that

$$\{2^{j/2}\psi_i(2^j \cdot -k) \mid i = 1, \ldots, n,\ j, k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

There are a number of properties that scaling vectors with $n > 1$ components may have that are not possible for scaling functions (that is, the $n = 1$ case); for example, it is known that no continuous (orthogonal) scaling function with compact support may be symmetric while there is such scaling vectors do exist for $n \geq 2$. The first such scaling vector was constructed in [1] using self-affine "fractal interpolation functions" (see [2, 3] for more on fractal interpolation functions). Fractal interpolation functions were also used in the construction of scaling vectors in [1, 4, 5]. In fact, any continuous scaling vector consists of piecewise fractal interpolation functions (see [6, 7] and Section 1.1.)

While it is convenient to consider refinable generators as vectors with matrix valued masks, we emphasize that scaling vectors and multiwavelets are used to approximate *scalar* valued functions and applied to *scalar* valued signals. For example, a function $f \in V_0$ may be expressed in the form

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{n} c_i[k]\phi_i(x-k) = \sum_{k\in\mathbb{Z}} c[k]\phi_i(x-k)$$

where $c[k] = (c_1[k], \ldots, c_n[k])$.

Multiwavelets have been applied in a number of signal processing and engineering applications (see, for example, [8, 9, 10]). Multiwavelets also have the advantage of having smaller support than equivalent "uniwavelets". This property makes it easier to adapt orthogonal wavelets to bounded intervals and non-uniform partitions as in [11]. In recent work [13], we construct bivariate orthogonal scaling vectors and wavelets on triangulations by first constructing orthogonal refinable "macroelements". In this paper we develop this approach in the simpler univariate case and illustrate the technique with two examples. In the first example we provide a new construction of the symmetric, orthogonal, continuous scaling vector constructed in [1]. In the second example we construct a continuous orthogonal scaling vector with three components. The components of this scaling vector are symmetric or antisymmetric and provide approximation order 3, (equivalently, the components of Ψ are orthogonal to polynomials of degree 2 or less.) We are not aware of a prior construction of this scaling vector.

Finally, we note that if $f, g \in L^2(\mathbb{R})$ then $\langle f, g\rangle$ denotes the usual inner product of f and g in $L^2(\mathbb{R})$. If $\Phi = (\phi_1, \ldots, \phi_n)^T$ and $\Psi = (\psi_1, \ldots, \psi_m)^T$ are generators, we define

$$\langle \Phi, \Psi\rangle := (\langle \phi_i, \psi_j\rangle) \in \mathbb{R}^{n\times m}.$$

1. Macroelements

We shall say that a linearly independent set of functions $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \subset C([0,1])$ is a C^0 **macroelement on [0,1]** if

(a) $\lambda_0(0) = \lambda_1(1) = 0$,
(b) $\lambda_0(1) = \lambda_1(0) \neq 0$, and
(c) $\lambda_i(0) = \lambda_i(1) = 0$ for $i = 2, \ldots, n$.

and shall say that Λ is an **orthogonal** C^0 **macroelement** if Λ is an orthogonal basis for its span, *i.e.*, $\langle \lambda_i, \lambda_j \rangle = 0$ and $\langle \lambda_i, \lambda_i \rangle \neq 0$ if $0 \leq i \neq j \leq n$. An orthogonal C^0 macroelement Λ is **normalized** if $\|\lambda_0\|^2 + \|\lambda_1\|^2 = 1$ and $\|\lambda_i\| = 1$ for $i = 2, \ldots, n$. Note that there is some assumed ordering of the elements of Λ and we often find it convenient to consider Λ to be the column vector $(\lambda_0, \lambda_1, \ldots, \lambda_n)^T$. Whenever we refer to the span of Λ we consider it as a set and whenever we multiply Λ by a matrix we consider Λ to be a column vector.

For $\sigma \in \{\{0\}, \{1\}, \{0,1\}\}$ we let $\Lambda^\sigma = \{\lambda \in \Lambda \mid \lambda\big|_\sigma \equiv 0\}$. Thus, $\Lambda^{\{i\}} = \{\lambda_i\}$ for $i = 0, 1$, and $\Lambda^{\{0,1\}} = \{\lambda_2, \ldots, \lambda_n\}$.

If Λ and $\tilde{\Lambda}$ are macroelements such that $\operatorname{span} \Lambda = \operatorname{span} \tilde{\Lambda}$ we say that Λ and $\tilde{\Lambda}$ are **equivalent**. Then C^0 macroelements Λ and $\tilde{\Lambda}$ are equivalent if and only if $\tilde{\Lambda} = U\Lambda$ for some nonsingular square matrix U (note that if Λ and $\tilde{\Lambda}$ are equivalent then they must have the same number of components) having the block form

$$U = \begin{bmatrix} u_{00} & 0 & u_{02} \\ 0 & u_{11} & u_{12} \\ 0 & 0 & u_{22} \end{bmatrix} \tag{1}$$

with respect to the decomposition $\Lambda = (\lambda_0, \lambda_1, \lambda_{[2:n]})$, that is $u_{i,j}[k]$ is $m_i \times m_j$, where $m_0 = m_1 = 1$ and $m_2 = n - 1$.

The following lemma is an immediate consequence of the above definitions.

Lemma 1. *Suppose Λ is a C^0 macroelement. Let $P_{0,1}$ denote the orthogonal projection onto $\operatorname{span} \Lambda^{\{0,1\}}$. Then Λ is equivalent to an orthogonal C^0 macroelement if and only if*

$$\langle \lambda_0, \lambda_1 \rangle = \langle \lambda_0, P_{0,1}\lambda_1 \rangle. \tag{2}$$

1.1. Refinable macroelements and fractal interpolation functions.

A macroelement Λ is **refinable** if there are two matrices $a[0]$ and $a[1]$ such that

$$\Lambda(x) = \sqrt{2}a[i]\Lambda(2x - i) \qquad (i = 0, 1;\ x \in [i/2, (i+1)/2]).$$

Because of the linear independence of Λ, the matrices $a[i]$, $i = 0, 1$, are unique. The pair of matrices $(a[0], a[1])$ is called the **mask** of Λ. Let $\lambda_{[2:n]} = (\lambda_2, \ldots, \lambda_n)$. We find it convenient to express $a[k]$ in the block form $a[k] = (a_{i,j}[k])_{0 \leq i,j \leq 2}$ with respect to the decomposition $\Lambda = (\lambda_0, \lambda_1, \lambda_{[2:n]})$, that is $a_{i,j}[k]$ is $m_i \times m_j$, where $m_0 = m_1 = 1$ and $m_2 = n - 1$. The fact that the components of Λ are continuous, and $\lambda_0(1) = \lambda_1(0) \neq 0$ implies the following conditions on $a[0]$ and $a[1]$:

$$a_{1,1}[0] = a_{0,0}[1] = 1/\sqrt{2}, \quad a_{0,1}[0] = a_{1,0}[1] = 0, \quad a_{2,1}[0] = a_{2,0}[1] = 0,$$

$$\alpha := a_{0,0}[0] = a_{0,1}[1], \quad \beta := a_{1,0}[0] = a_{11}[1], \quad \text{and } \gamma := a_{2,0}[0] = a_{2,1}[1].$$

Then we have

$$a[0] = \begin{bmatrix} \alpha & 0 & a_{02}[0] \\ \beta & 1/\sqrt{2} & a_{12}[0] \\ \gamma & 0 & a_{22}[0] \end{bmatrix} \quad \text{and} \quad a[1] = \begin{bmatrix} 1/\sqrt{2} & \alpha & a_{02}[1] \\ 0 & \beta & a_{12}[1] \\ 0 & \gamma & a_{22}[1] \end{bmatrix} \tag{3}$$

Suppose $\Pi = \{\pi_0, \pi_1, \ldots, \pi_n\}$ is a refinable macroelement with mask $p = (p[0], p[1])$. Let $\Omega = \{\omega_1, \ldots, \omega_m\} \subset C[0,1]$ be such that $\omega(i) = 0$ for $\omega \in \Omega$ and $i = 0, 1$ and such that $\Lambda := \Pi \cup \Omega = \{\pi_0, \ldots, \pi_n, \omega_0, \ldots, \omega_m\}$ is a refinable C^0 macroelement with mask a. Then, using the above ordering of Λ, a must have the block form

$$a[i] = \frac{1}{\sqrt{2}} \begin{bmatrix} p[i] & 0 \\ q[i] & s[i] \end{bmatrix} \qquad (i = 0, 1). \tag{4}$$

Equivalently, Ω satisfies an inhomogeneous refinement equation of the form

$$\Omega(x) = q[i]\Pi(2x - i) + s[i]\Omega(2x - i) \qquad (x \in [i/2, (i+1)/2]). \tag{5}$$

Such equations have been studied in [2, 14, 15, 16]. We include the following lemma for completeness. For $\mathbf{i} = (i_1, \ldots, i_k) \in \{0, 1\}^k$ we define $|\mathbf{i}| = k$ and

$$s[\mathbf{i}] := s[i_1]s[i_2] \cdots s[i_k].$$

Let $\|\cdot\|$ be a norm on the vector space $\mathbb{R}^m$. If A is an $m \times m$ matrix we let $\|A\| := \max_{\|v\|=1} \|Av\|$ denote the matrix norm with respect to this vector norm. If $f \in C([0,1])^{m\times 1}$, we let

$$\|f\|_\infty := \max_{x\in[0,1]} \|f(x)\|.$$

Lemma 2. *Suppose Π is a refinable C^0 macroelement with $n+1$ elements and mask p. Let $m \in \mathbb{N}$ and suppose $q[0], q[1] \in \mathbb{R}^{m\times(n+1)}$ and $s[0], s[1] \in \mathbb{R}^{m\times m}$ are such that*

(a) *$q[0]\Pi(1) = q[1]\Pi(0)$,*
(b) *$q[i]\Pi(i) = 0$ for $i = 0, 1$ and*
(c) *there exists some $K \in \mathbb{N}$ such that*

$$\max_{|\mathbf{i}|=K} \|s[\mathbf{i}]\| < 1. \tag{6}$$

Then there is a unique $\Omega = (\omega_1, \ldots, \omega_m) \in C_0([0,1])^{m\times 1}$ that satisfies (5).

Remark 1. The components of Ω fall into a class of functions called "fractal interpolation functions" (or, more specifically, "hidden variable fractal interpolation functions") [2, 17]. Here we provide a proof of Lemma 2 for completeness.

Proof. Let $C_0([0,1])$ be the subspace of $C([0,1])$ consisting of functions that vanish at 0 and 1. Let $\mathcal{F} : C_0([0,1])^{m\times 1} \to C_0([0,1])^{m\times 1}$ be defined, for $f \in C_0([0,1])^{m\times 1}$, by

$$\mathcal{F}(f)(x) = q[i]\Pi(2x-i) + s[i]f(2x-i) \qquad (x \in [i/2,(i+1)/2]). \tag{7}$$

Conditions (a) and (b) imply that $\mathcal{F}$ is well-defined. Suppose (c) holds for $K = 1$, that is, $\rho := \max_{i=0,1} \|s[i]\| < 1$. Then, for $f, g \in C_0([0,1])^{m\times 1}$, we have

$$\begin{aligned}
\|\mathcal{F}(f) - \mathcal{F}(g)\|_\infty &= \max_{i=0,1} \max_{x\in[i/2,(i+1)/2]} \|\mathcal{F}(f)(x) - \mathcal{F}(g)(x)\| \\
&= \max_{i=0,1} \max_{x\in[i/2,(i+1)/2]} \|s[i]f(2x-i) - s[i]g(2x-i)\| \\
&= \max_{i=0,1} \|s[i](f-g)\|_\infty \\
&\le \max_{i=0,1} \|s[i]\|\|(f-g)\|_\infty \\
&= \rho\|f-g\|_\infty
\end{aligned}$$

which shows that $\mathcal{F}$ is a contraction mapping on $C_0([0,1])^{m\times 1}$ with contractivity $\rho < 1$. The Banach Contraction Mapping Principle then implies that there is a unique $\Omega \in C_0([0,1])^{m\times 1}$ such that $\mathcal{F}(\Omega) = \Omega$. If $K > 1$, essentially the same computations applied to $\mathcal{F}^K$ (the composition of $\mathcal{F}$, K times) shows that $\mathcal{F}^K$ is contractive with contractivity $\max_{|\mathbf{i}|=K} \|s[\mathbf{i}]\| < 1$. Because $\mathcal{F}$ is an affine mapping, it then follows that $\mathcal{F}$ has a unique fixed point $\Omega \in C_0([0,1])^{m\times 1}$. □

If $\Lambda = \Pi \cup \Omega$ where Π and Ω are as in Lemma 2 and the components of Λ are linearly independent then Λ is a refinable macroelement. While one may determine the linear independence of Λ from the mask a (see [18]), it will be a simple matter to verify linear independence directly for the macroelements constructed in this paper. In both examples we construct in this paper, we have $m = 1$, and then (6) is equivalent to $\max_{i=0,1} |s[i]| < 1$.

1.2. **Inner Products.** Suppose $q[0]$, $q[1]$, $s[0]$, and $s[1]$ satisfy the hypotheses of Lemma 2 and let $\Omega = (\omega_1, \dots, \omega_m)$ be the unique solution of (5) in $C_0([0,1])^{m\times 1}$. Assuming that $\langle \Pi, \Pi\rangle$ is known, equation 5 may be used to calculate inner products involving the components of Π and Ω:

$$2\langle \Pi, \Omega\rangle = \sum_{i=0,1} p[i]\langle \Pi,\Pi\rangle q[i]^T + \sum_{i=0,1} p[i]\langle \Pi,\Omega\rangle s[i]^T, \text{ and} \tag{8}$$

$$\begin{aligned}
2\langle \Omega, \Omega\rangle = {} & \sum_{i=0,1} q[i]\langle \Pi,\Pi\rangle q[i]^T + \sum_{i=0,1} s[i]\langle \Omega,\Pi\rangle q[i]^T \\
& + \sum_{i=0,1} q[i]\langle \Pi,\Omega\rangle s[i]^T + \sum_{i=0,1} s[i]\langle \Omega,\Omega\rangle s[i]^T
\end{aligned} \tag{9}$$

In the examples constructed in this paper we only consider $m = 1$. In that case $\Omega = \{\omega\}$ and $s[0]$ and $s[1]$ are scalars. Equations 8 and 9 may be solved to get

$$\text{(10)}\quad \langle \Pi, \omega \rangle = (2I - \sum_{i=0,1} s[i]p[i])^{-1} \sum_{i=0,1} p[i]\langle \Pi, \Pi \rangle q[i]^T, \text{ and}$$

$$\text{(11)}\quad \langle \omega, \omega \rangle = (2 - s[i]^2)^{-1} \Big(\sum_{i=0,1} q[i]\langle \Pi, \Pi \rangle q[i]^T + \sum_{i=0,1} s[i]\langle \Omega, \Pi \rangle q[i]^T + \sum_{i=0,1} q[i]\langle \Pi, \Omega \rangle s[i]^T \Big)$$

1.3. C^0 orthogonal generators from C^0 orthogonal macroelements.

Suppose $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a normalized orthogonal C^0 macroelement. Define the generator $\Phi_\Lambda = (\phi_1, \dots, \phi_n)$ where

$$\phi_1(x) := \begin{cases} \lambda_0(x+1) & \text{for } -1 \le x \le 0 \\ \lambda_1(x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\phi_i := \lambda_i$ for $i = 2, \dots, n$. Then $\phi_i \in C(\mathbb{R})$, $i = 1, \dots, n$, and Φ is a refinable generator with orthonormal shifts. It then follows that Φ is refinable with mask b given (in block form) by

$$\text{(12)}\quad \begin{array}{ll} b[-2] = \begin{bmatrix} 0 & a_{02}[0] \\ 0 & 0 \end{bmatrix} & b[-1] = \begin{bmatrix} \alpha & a_{02}[1] \\ 0 & 0 \end{bmatrix} \\ b[0] = \begin{bmatrix} 1/\sqrt{2} & a_{12}[0] \\ 0 & a_{22}[0] \end{bmatrix} & b[1] = \begin{bmatrix} \beta & a_{12}[1] \\ \gamma & a_{22}[1] \end{bmatrix} \end{array}$$

2. Examples

In this section we provide two examples to illustrate the macroelement based construction. In the first example we provide a new construction of the symmetric, orthogonal, continuous scaling vector constructed in [1]. In the second example we construct a continuous orthogonal scaling vector with three components. The components of this scaling vector are symmetric or antisymmetric and provide approximation order 3, (equivalently, the components of Ψ are orthogonal to polynomials of degree 2 or less.)

2.1. Example. In this section we present a construction of the orthogonal scaling vector constructed in [1] from the macroelement point of view. First, let $\pi_i \in C([0,1])$ be defined by

$$\pi_0(x) = x \qquad \text{and} \qquad \pi_1(x) = 1 - x \qquad (x \in [0,1])$$

and so $\Pi := \{\pi_0, \pi_1\}$ is the usual linear element on [0,1]. Then

$$\text{(13)}\quad p[0] = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}, \quad p[1] = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \text{ and } \langle \Pi, \Pi \rangle = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}.$$

Let $q[0] = (1,0)$, $q[1] = (0,1)$, and $s[0] = s[1] = t$ for some $t \in (-1,1)$. Then by Lemma 2 there is a unique function $\omega \in C_0([0,1])$ such that ω satisfies (5). Since $s[0] = s[1]$, one may verify that $\omega(1-x)$ also satisfies (5) and so it follows that ω is symmetric about $x = 1/2$. Then Equations 10 and 11 give

$$\langle \pi_0, \omega \rangle = \langle \pi_1, \omega \rangle = \frac{1}{4 - 4\,t} \quad \text{and} \quad \langle \omega, \omega \rangle = \frac{2+t}{6\,(-1+t)^2\,(1+t)}. \tag{14}$$

Let $\Omega := \{\omega\}$ and $\tilde{\Lambda} := \Pi \cup \Omega$. Then $\tilde{\Lambda}$ is refinable with mask $\tilde{a}$ given by (4):

$$\tilde{a}[0] = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & 0 & t \end{bmatrix} \text{ and } \tilde{a}[1] = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & t \end{bmatrix}.$$

By Lemma 1, $\tilde{\Lambda}$ is equivalent to an orthogonal macroelement Λ if and only if Equation 2 holds which, the reader may verify, is equivalent to

$$\langle \pi_0, \pi_1 \rangle = \frac{\langle \pi_0, \omega \rangle \langle \omega, \pi_1 \rangle}{\langle \omega, \omega \rangle} = \frac{\langle \pi_0, \omega \rangle^2}{\langle \omega, \omega \rangle}. \tag{15}$$

Using (14), then (15) reduces to

$$1/6 = \frac{3\,(1+t)}{8\,(2+t)}$$

which may be solved to get $t = -1/5$. Then $\langle \pi_0, \omega \rangle = 5/24$ and $\langle \omega, \omega \rangle = 25/96$. Let

$$\lambda_i := \sqrt{3}(\pi_i - \frac{\langle \pi_i, \omega \rangle}{\langle \omega, \omega \rangle}\omega) = \sqrt{3}(\pi_i - \frac{4}{5}\,\omega) \qquad (i = 0, 1)$$

and

$$\lambda_2 := \frac{\omega}{\sqrt{\langle \omega, \omega \rangle}} = \frac{5}{4\,\sqrt{6}}\,\omega$$

and so $\Lambda := (\lambda_0, \lambda_1, \lambda_2)^T = U\tilde{\Lambda}$ where

$$U = \sqrt{3} \begin{bmatrix} 1 & 0 & -4/5 \\ 0 & 1 & -4/5 \\ 0 & 0 & 5\sqrt{2}/24 \end{bmatrix}$$

Then the mask for Λ may be constructed using $a[i] = U\tilde{a}[i]U^{-1}$ to get

$$a[0] = \begin{bmatrix} \frac{-3}{10\sqrt{2}} & 0 & -\frac{1}{20} \\ \frac{-3}{10\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{9}{20} \\ \frac{4}{5} & 0 & \frac{3}{5\sqrt{2}} \end{bmatrix}, \text{ and } a[1] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{10\sqrt{2}} & \frac{9}{20} \\ 0 & \frac{-3}{10\sqrt{2}} & -\frac{1}{20} \\ 0 & \frac{4}{5} & \frac{3}{5\sqrt{2}} \end{bmatrix}.$$

Then using Equation 12, we have the matrix coefficients

$$b[-2] = \begin{bmatrix} 0 & -\frac{1}{20} \\ 0 & 0 \end{bmatrix}, \qquad b[-1] = \begin{bmatrix} -\frac{3}{10\sqrt{2}} & \frac{9}{20} \\ 0 & 0 \end{bmatrix},$$

$$b[0] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{9}{20} \\ 0 & \frac{3}{5\sqrt{2}} \end{bmatrix}, \text{ and } b[1] = \begin{bmatrix} -\frac{3}{10\sqrt{2}} & -\frac{1}{20} \\ \frac{4}{5} & \frac{3}{5\sqrt{2}} \end{bmatrix}.$$

Figure 1 shows the resulting orthogonal scaling vector.

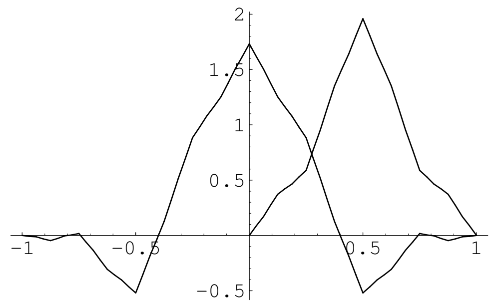

FIGURE 1. ϕ_1 and ϕ_2 in Example 2.1.

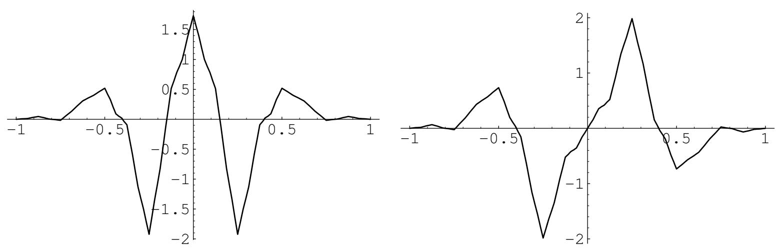

FIGURE 2. ψ_1 and ψ_2 for Example 2.1.

The wavelet coefficients may then be calculated as in Appendix A. The wavelet coefficients are given below and the resulting wavelets are shown in Figure 2

$$g[-2] = \begin{bmatrix} 0 & \frac{1}{20} \\ 0 & \frac{1}{10\sqrt{2}} \end{bmatrix} \qquad g[-1] = \begin{bmatrix} \frac{3}{10\sqrt{2}} & -\frac{9}{20} \\ \frac{3}{10} & -\frac{9}{10\sqrt{2}} \end{bmatrix}$$

$$g[0] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{9}{20} \\ 0 & \frac{9}{10\sqrt{2}} \end{bmatrix} \qquad g[1] = \begin{bmatrix} \frac{3}{10\sqrt{2}} & \frac{1}{20} \\ -\frac{3}{10} & -\frac{1}{10\sqrt{2}} \end{bmatrix}$$

2.2. **Example.** Let π_0, π_1 be as in Example 2.1 and let $\pi_2 \in C[0,1]$ be given by $\pi_2(x) = 4x(1-x)$. Then $\Pi := \{\pi_0, \pi_1, \pi_2\}$ is a quadratic element on [0,1] with

$$p[0] = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1/4 \end{bmatrix}, \quad p[1] = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 1/4 \end{bmatrix}, \tag{16}$$

and

$$\langle \Pi, \Pi \rangle = \begin{bmatrix} 1/3 & 1/6 & 1/3 \\ 1/6 & 1/3 & 1/3 \\ 1/3 & 1/3 & 8/15 \end{bmatrix}. \tag{17}$$

Let $q[0] = (0,0,1)$, $q[1] = (0,0,-1)$, and $s[0] = s[1] = t$. If $|t| < 1$ then, by Lemma 2, there is a unique function $\omega \in C_0([0,1])$ such that ω satisfies (5).

One may then verify that $-\omega(1-x)$ also satisfies (5) and it follows that ω is antisymmetric about $x = 1/2$.

Lemma 3. *Let $\Pi = \{\pi_0, \pi_1, \pi_2\}$, t and ω be as above. If $|t| < 1/2$ then $\omega \in C^1([0,1])$.*

Proof. Let $C_p([0,1]) := \{f \in C([0,1]) \mid f[0] = f[1]\}$, let $E = \{f \in C_0([0,1]) \mid f' \in C_p([0,1])\}$, and let $\mathcal{F}_1 : C_p([0,1]) \to C_p([0,1])$ be the operator obtained by differentiating the right hand side of (7), that is,

$$\mathcal{F}_1(g)(x) = q[i]2\Pi'(2x-i) + 2t/, g(2x-i) \qquad (x \in [i/2, (i+1)/2]).$$

Then, one may verify that $\mathcal{F}$ is well-defined, $\mathcal{F}_1(f') = \mathcal{F}(f)'$ for $f \in E$, and $\mathcal{F}_1$ is contractive on $C_p([0,1])$ in the sup-norm if $|t| < 1/2$. Hence, if $f_0 \in E$ and $f_n := \mathcal{F}(f_{n-1})$ for $n \geq 1$, then f_n and f_n' converge uniformly to some $\omega \in C_0([0,1])$ and $\mu \in C_p([0,1])$, respectively. Then it follows that $\omega' = \mu$ and the result follows. □

Equations 10 and 11 then give

$$(18) \qquad \langle \pi_0, \omega\rangle = -\langle \pi_1, \omega\rangle = \frac{1}{3\,t-6} \quad \text{and} \quad \langle \omega, \omega\rangle = \frac{8}{15\,(1-t^2)}.$$

Let $\Omega := \{\omega\}$ and $\tilde{\Lambda} := \Pi \cup \Omega$. Then $\tilde{\Lambda}$ is refinable with mask $\tilde{a}$ given by (4):

$$\tilde{a}[0] = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & t \end{bmatrix} \text{ and } \tilde{a}[1] = \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & -1 & t \end{bmatrix}.$$

By Lemma 1, $\tilde{\Lambda}$ is equivalent to an orthogonal macroelement Λ if and only if Equation 2 holds which, since π_2, ω form an orthogonal basis of $\tilde{\Lambda}^{\{0,1\}}$, is equivalent to

$$(19) \quad \langle \pi_0, \pi_1\rangle = \frac{\langle \pi_0, \pi_2\rangle\langle \pi_2, \pi_1\rangle}{\langle \pi_2, \pi_2\rangle} + \frac{\langle \pi_0, \omega\rangle\langle \omega, \pi_1\rangle}{\langle \omega, \omega\rangle} = \frac{\langle \pi_0, \pi_2\rangle^2}{\langle \pi_2, \pi_2\rangle} - \frac{\langle \pi_0, \omega\rangle^2}{\langle \omega, \omega\rangle}$$

where the last equality follows by symmetry. Using (18), then (19) reduces to

$$1/6 = \frac{5\,\left(3-4\,t+2\,t^2\right)}{24\,(t-2)^2}$$

which may be solved to get $t = (2 \pm \sqrt{10})/6 \approx 0.86038, -0.19371$. Note that both solutions for t are in $(-1,1)$ so $\omega \in C([0,1])$ in either case. Figure 3 shows the graph of ω for the two solutions for t. In the case $t = (2+\sqrt{10})/6$, the graph of ω is fractal with fractal (box) dimension $d = 1 + \log(2t)/\log 2 \approx 1.78$ (see [19], the proof given there generalizes to cover this case). In the case $t = (2-\sqrt{10})/6$, we have $|t| < 1/2$ and so, by Lemma 3, ω is continuously differentiable on $[0,1]$. We choose the second case $t = (2-\sqrt{10})/6$ for the remainder of this example.

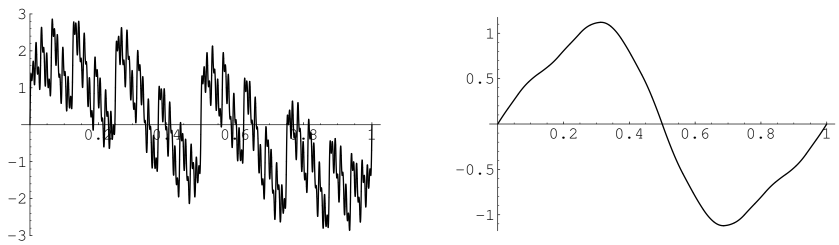

FIGURE 3. The function ω from Example 2.2 for (left) $t = (2+\sqrt{10})/6$ and (right) $t = (2-\sqrt{10})/6$.

Then $\langle \pi_0, \omega \rangle = -\langle \pi_1, \omega \rangle = (-10+\sqrt{10})/45$ and $\langle \omega, \omega \rangle = 48/(55+10\sqrt{10})$. For $j = 0, 1$, let

$$\begin{aligned} \lambda_j &:= \delta(I - P_{\{0,1\}})\pi_j \\ &= \delta(\pi_j - \frac{\langle \pi_2, \pi_j \rangle}{\langle \pi_2, \pi_2 \rangle}\pi_2 - \frac{\langle \pi_j, \omega \rangle}{\langle \omega, \omega \rangle}\omega) \\ &= \delta(\pi_j - (5/8)\pi_2 + (-1)^j \frac{1}{48}(10+\sqrt{10})\,\omega) \end{aligned}$$

where $\delta = \sqrt{6}$ is such that $\|\lambda_0\|^2 + \|\lambda_1\|^2 = 1$. Also, let

$$\lambda_2 = \pi_2/\|\pi_2\| = (\sqrt{15/8})\,\pi_2 \text{ and } \lambda_3 := \frac{\omega}{\sqrt{\langle \omega, \omega \rangle}} = \frac{5}{4\sqrt{6}}\,\omega.$$

Then $\Lambda := (\lambda_0, \lambda_1, \lambda_2)^T = U\tilde{\Lambda}$ where

$$U = \begin{bmatrix} \sqrt{6} & 0 & \frac{-5\sqrt{\frac{3}{2}}}{4} & \frac{10+\sqrt{10}}{8\sqrt{6}} \\ 0 & \sqrt{6} & \frac{-5\sqrt{\frac{3}{2}}}{4} & \frac{-10-\sqrt{10}}{8\sqrt{6}} \\ 0 & 0 & \frac{\sqrt{\frac{15}{2}}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\frac{5\left(11+2\sqrt{10}\right)}{3}}}{4} \end{bmatrix}$$

Then the mask for Λ may be constructed using $a[i] = U\tilde{a}[i]U^{-1}$ to get

$$a[0] = \begin{bmatrix} \frac{-1}{8\sqrt{2}} & 0 & \frac{8-\sqrt{10}}{96} & \frac{11-4\sqrt{10}}{48\sqrt{2}} \\ \frac{-1}{8\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-8+7\sqrt{10}}{96} & \frac{19+4\sqrt{10}}{48\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & 0 & \frac{7}{8\sqrt{2}} & \frac{-\sqrt{\frac{5}{2}}}{8} \\ 0 & 0 & \frac{1+\sqrt{10}}{6} & \frac{-\left(-2+\sqrt{10}\right)}{6\sqrt{2}} \end{bmatrix}$$

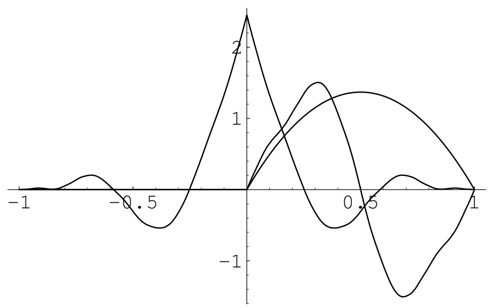

FIGURE 4. ϕ_1, ϕ_2, and ϕ_2 in Example 2.2.

and

$$a[1] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{8\sqrt{2}} & \frac{-8+7\sqrt{10}}{96} & \frac{-(19+4\sqrt{10})}{48\sqrt{2}} \\ 0 & \frac{-1}{8\sqrt{2}} & \frac{8-\sqrt{10}}{96} & \frac{-11+4\sqrt{10}}{48\sqrt{2}} \\ 0 & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{7}{8\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{8} \\ 0 & 0 & \frac{-1-\sqrt{10}}{6} & \frac{-(-2+\sqrt{10})}{6\sqrt{2}} \end{bmatrix}$$

Then using Equation 12, we have the matrix coefficients

$$\begin{aligned} b[-2] &= \begin{bmatrix} 0 & \frac{8-\sqrt{10}}{96} & \frac{11-4\sqrt{10}}{48\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ b[-1] &= \begin{bmatrix} \frac{-1}{8\sqrt{2}} & \frac{-8+7\sqrt{10}}{96} & \frac{-(19+4\sqrt{10})}{48\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ b[0] &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-8+7\sqrt{10}}{96} & \frac{19+4\sqrt{10}}{48\sqrt{2}} \\ 0 & \frac{7}{8\sqrt{2}} & \frac{-\sqrt{\frac{5}{2}}}{8} \\ 0 & \frac{1+\sqrt{10}}{6} & \frac{-(-2+\sqrt{10})}{6\sqrt{2}} \end{bmatrix}, \\ b[1] &= \begin{bmatrix} \frac{-1}{8\sqrt{2}} & \frac{8-\sqrt{10}}{96} & \frac{-11+4\sqrt{10}}{48\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{7}{8\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{8} \\ 0 & \frac{-1-\sqrt{10}}{6} & \frac{-(-2+\sqrt{10})}{6\sqrt{2}} \end{bmatrix}. \end{aligned}$$

Figure 4 shows the components of the orthogonal scaling vector $\Phi = (\phi_1, \phi_2, \phi_3)^T$.

The wavelet coefficients may then be calculated as in Appendix A. The wavelet coefficients are given below and the resulting wavelets are shown in Figure 5.

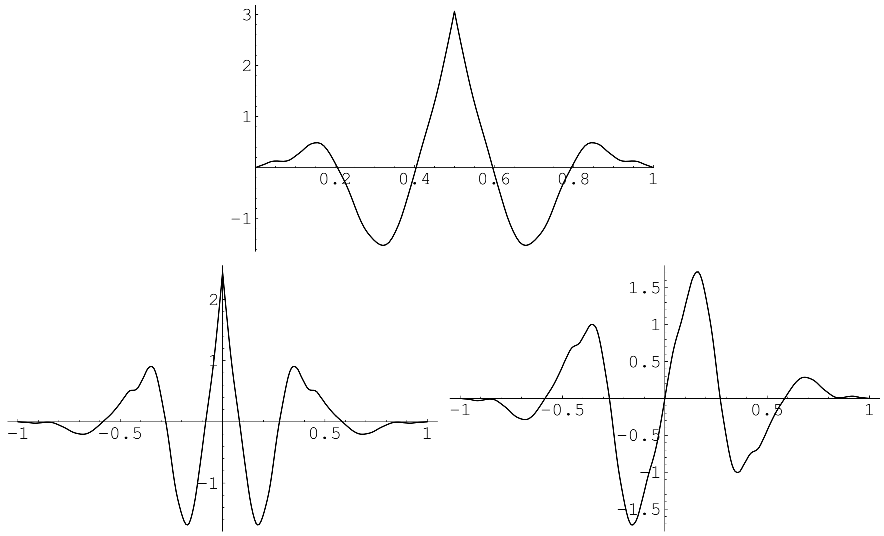

FIGURE 5. ψ_1, ψ_2, and ψ_3 for Example 2.2.

$$
\begin{aligned}
g[-2] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{10}-8}{96} & -\frac{11-4\sqrt{10}}{48\sqrt{2}} \\ 0 & \frac{\sqrt{10}-8}{48\sqrt{2}} & \frac{4\sqrt{10}-11}{48} \end{bmatrix} \\
g[-1] &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{8\sqrt{2}} & \frac{8-7\sqrt{10}}{96} & \frac{19+4\sqrt{10}}{48\sqrt{2}} \\ \frac{1}{8} & \frac{8-7\sqrt{10}}{48\sqrt{2}} & \frac{19+4\sqrt{10}}{48} \end{bmatrix} \\
g[0] &= \begin{bmatrix} 0 & -\frac{\sqrt{5}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{8-7\sqrt{10}}{96} & -\frac{19+4\sqrt{10}}{48\sqrt{2}} \\ 0 & \frac{7\sqrt{10}-8}{48\sqrt{2}} & \frac{19+4\sqrt{10}}{48} \end{bmatrix} \\
g[1] &= \begin{bmatrix} \frac{5}{4\sqrt{2}} & -\frac{\sqrt{5}}{8\sqrt{2}} & -\frac{3}{8\sqrt{2}} \\ \frac{1}{8\sqrt{2}} & \frac{\sqrt{10}-8}{96} & \frac{11-4\sqrt{10}}{48\sqrt{2}} \\ -\frac{1}{8} & \frac{8-\sqrt{10}}{48\sqrt{2}} & \frac{4\sqrt{10}-11}{48} \end{bmatrix}
\end{aligned}
$$

APPENDIX A. WAVELET CONSTRUCTION

Suppose $\Phi = (\phi_1, \ldots, \phi_n)$ is a refinable generator with orthonormal shifts. Let $h = (h[k])$ be the mask for Φ, that is,

$$\Phi(t) = \sqrt{2} \sum_k h[k] \Phi(2t - k) \qquad (t \in \mathbb{R}).$$

In order to find a wavelet one must find wavelet coefficients $g[k] \in \mathbb{R}^{n\times n}$, for $k \in \mathbb{Z}$, such that if Ψ is given by

$$\Psi(t) = \sqrt{2}\sum_k g[k]\Phi(2t-k) \qquad (t \in \mathbb{R}) \tag{20}$$

then Ψ is a generator for W_0 such that Ψ has orthonormal shifts.

When $n = 1$ one can use the formula $g[k] = (-1)^{1-k}h[1-k]$ to find such a g. Here we sketch a procedure for constructing g in the case $n > 1$ (the multiwavelet case) ([20, 21]). First, we introduce the *polyphase* form of h and g:

$$b[k] = (h[2k] \quad h[2k+1]) \text{ and } \ell[k] = (g[2k] \quad g[2k+1]) \qquad (k \in \mathbb{Z}) \tag{21}$$

One can show that the orthonormality of the shifts of Φ implies that $B(z) := \sum_k b[k]z^k$ is *paraunitary*, that is, $B(z)B(1/z)^T = I$ where I denotes the $n \times n$ identity matrix. Furthermore, letting $L(z) = \sum_k \ell[k]z^k$, one can show that Ψ is a generator for W_0 with orthonormal shifts if and only if the square matrix

$$U(z) = \begin{bmatrix} B(z) \\ L(z) \end{bmatrix} \tag{22}$$

is paraunitary, that is iff $L(z)$ is paraunitary and $B(z)L(1/z)^T = 0$.

Recall that a matrix $P \in \mathbb{R}^{n\times n}$ is an orthogonal projection matrix if $P^2 = P$ and $P^T = P$. If M is a subspace of $\mathbb{R}^n$ then P is the orthogonal projection onto M (that is, Px gives the point in M closest to x for all $x \in \mathbb{R}^n$) if and only if $P = XX^T$ for a matrix X whose columns form an orthonormal basis of M. If $y \in M^\perp := \{x \in \mathbb{R}^n \mid x^Tm = 0, \forall m \in M\}$, then one may show $Py = 0$. If $P \in \mathbb{R}^{n\times n}$ is an orthogonal projection, then one may easily verify that $U_P(z) := I - P + Pz$ is paraunitary.

We need the following lemma concerning a canonical factorization of paraunitary matrices ([21]). We include a constructive proof for completeness.

Lemma 4. *Suppose* $A(z) = \sum_{k=0}^N a[k]z^k$, *where* $a[k] \in \mathbb{R}^{m\times n}$ *satisfies* $A(z)A(1/z)^T = I$ *for* $z \neq 0$. *Then, there exist orthogonal projection matrices* $P_k \in \mathbb{R}^{n\times n}$, $k = 1,\dots,N$, *such that*

$$A(z) = A(1)\prod_{k=0}^N (I - P_k + P_k z).$$

Proof. Since $A(z)A(1/z) = I$ it follows that $a[0]a[N]^T = a[N]a[0]^T = 0$. Let P_N be the orthogonal projection onto the span of the columns of $a[N]^T$. Then $(I - P_N)a[N]^T = 0$ and, since the columns of $a[0]$ are perpendicular to those $a[N]^T$, we have $P_N a[0] = 0$.

Hence,

$$A(z)(I - P_N + P_N z^{-1}) = A_1(z) = \sum_{k=0}^{N-1} a_1[k]z^k$$

is a matrix polynomial of degree at most $N-1$. Iterating this procedure until we get a polynomial of degree 0 and using the fact that $(I-P+Pz^{-1})|_{z=1} = I$ we get

$$A(z)(I - P_N + P_N z^{-1}) \cdots (I - P_0 + P_0 z^{-1}) = A(1).$$

The lemma then follows from $(I - P_k + P_k z^{-1})^{-1} = (I - P_k + P_k z)$. □

Now back to the construction of Ψ. By possibly shifting the components of Φ, we may assume, without loss of generality, that $h[k] = 0$ for $k < 0$. By Lemma 4, $B(z) = B(1)V(z)$ where $V(z) = \prod_{ki=0}^{N}(I - P_k + P_k z)$ is a $2n \times 2n$ paraunitary matrix with polynomial entries such that $V(1) = I$. Note that $B(1)B(1)^T = I$ so the rows of $B(1)$ are orthonormal. Then one can complete the rows of $B(1)$ to get an orthonormal basis of $\mathbb{R}^{2n}$, or, equivalently, to complete $B(1)$ to a $2n \times 2n$ orthogonal matrix

$$U(1) := \begin{bmatrix} B(1) \\ L(1) \end{bmatrix}.$$

One may easily verify that $L(z) := L(1)V(z)$ completes $B(z)$ to a square paraunitary matrix $U(z) = U(1)V(z)$. We may then find $g = (g[k])$ using (21) and it follows that Ψ defined through (20) is a wavelet for Φ, that is, $S(\Psi) = W_0$ and Ψ has orthonormal shifts.

References

[1] J. Geronimo, D. Hardin, and P. Massopust, "Fractal functions and wavelet expansions based on several scaling functions", *Journal of Approximation Theory* **78:3** 373–401 (1994)

[2] M. F. Barnsley, Fractal functions and interpolations, Constr. Approx. **2**, 303–329, (1986)

[3] Barnsley, M. F., Harrington, A. N., "The calculus of fractal interpolation functions", *Journ. of Approx. Theory* **57**, 14–34 (1989)

[4] G. Donovan, J. Geronimo, D. Hardin, and P. Massopust, "Construction of Orthogonal Wavelets Using Fractal Interpolation Functions", *SIAM Journ. Math. Analysis* **27**, 1158–1192 (1996)

[5] D. Hardin, B. Kessler, and P. Massopust, "A multiresolution analysis based on fractal functions", *Journal Approximation Theory*, **71:1**, 104–120 (1992)

[6] Micchelli, C. A., *Mathematical Aspects of Geometric Modeling*, SIAM, Philadelphia **65** (1994)

[7] D. Hardin, "Wavelets are piecewise fractal interpolation functions", *Fractals in Multimedia*, edited by M. Barnsley, Dietmar Saupe, and E. Vrscay, The IMA Volumes in Mathematics and its Applications **132**, 121–136 (2002)

[8] V. Strela, P. N. Heller, G. Strang, P. Topiwala, and C. Heil, "The application of multiwavelet filter banks to image processing", IEEE Trans. Image Processing, 8, 548–563, (1999)

[9] K. Attakitmongcol, D. Hardin, and D. M. Wilkes, "Multiwavelet Prefilters II: Optimal orthogonal prefilters", IEEE Trans. Image Proc. 10, 1476–1487, (2002)

[10] Jian-ao Lian and Charles K. Chui, "Analysis-ready Multi-wavelets (Armlets) for Processing Scalar-valued Signals", to appear IEEE Sig. Proc. Letters, (2003)

[11] G. C. Donovan, J. S. Geronimo, and D. P. Hardin, "Squeezable Orthogonal Bases: Accuracy and Smoothness" *SIAM J. Num. Analysis* **40**:3, 1077–1099 (2002)

[12] G. Donovan, J. Geronimo, and D. Hardin, "Fractal Functions, Splines, Intertwining Multiresolution Analysis, and Wavelets", *Wavelet Application in Signal and Image Processing*, Laine & Unser, editors, SPIE Conf. Proc., San Diego, 238–243 (1994)

[13] D. Hardin and B. Kessler, Macroelement wavelets on triangulations, in preparation (2002).

[14] T. Dinsenbacher and D. Hardin, "Multivariate nonhomogeneous refinement equations", *J. Fourier Anal. and Appl.* **5**, 589–597 (1999)

[15] Jia, R.-Q., Jiang, Q., Shen, Z., "Distributional solutions of nonhomogeneous discrete and continuous refinement equations", *SIAM Journ. Math. Anal.*,**32**:2 pp. 420-434, (2000)

[16] Zhou, D.-X., Strang, G., "Inhomogeneuos refinement equations", Journal of Fourier Analysis and Applications

[17] M. Barnsley, J. Elton, D. Hardin, and P. R. Massopust, (1989) "Hidden variable fractal interpolation functions", *SIAM J. Math. Analysis* **20**:5, 1218–1242.

[18] Hogan, T. A., Jia, R.-Q., "Dependence relations among the shifts of a multivariate refinable distribution", Constructive Approximation, to appear.

[19] D. Hardin and P. Massopust, "The capacity for a class of fractal functions", *Comm. Math. Phys.*,**105**, 455–460 (1986).

[20] G. Strang and V. Strela, Short wavelets and matrix dilation equations, IEEE Trans. SP, **43**, pp. 108–115 (1995)

[21] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Simon and Schuster, (1993)

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