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Dice Are Blessed or Cursed

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Abstract:

Dice are cursed or blessed; that is, they roll low or high, but they are never fair. They cannot be manufactured with uniform density and geometric precision. This is particularly true of 20-sided dice or D20s. Faces are smaller than 6-sided dice, and manufacturing tolerances are similar. However, some dice are fairer than others. In our studies of plastic-mold dice about 1 in 4 test fair in 3000 rolls. We have used different statistical tests, including chi-square, modified Kolmogorov Smirnov, and double binomial tests. Of these, the method that consistently performed better is the chi-square goodness of fit test. The probability distribution of the χ^2 statistic for a fair die only asymptotically approaches the chi-square distribution. The exact distribution is the multinomial distribution that is computationally intensive for many rolls of dice, especially those with high numbers of faces. The χ^2 statistic for unfair dice asymptotically approaches the noncentral chi-square distribution. Both of these distributions are continuous, but the multinomial distribution is a discrete distribution. Furthermore, χ^2 for dice is a rational number. Ninety-five percent values for the chi-square distribution do not match those from the exact distribution. For up to 100 rolls, they are significantly different. For χ^2, the minimum interval between possible values is 2 divided by the expected value for a fair die. Some exact distributions of fair and unfair dice are presented with exact values of the power of the test.

Background:

According to Hasbro, 50 million copies of Yahtzee are sold each year. Each game includes at least 5 dice. One game uses one-quarter of a billion dice each year. Millions more polyhedral dice are sold for use in role-playing games like Dungeons and Dragons and Pathfinder. These dice are unfair because they cannot be manufactured with precise dimensions and uniform density. Casino dice, which are controlled by Federal and state gaming laws, come the closest. The cubes of these D6 dice are machined to within a few ten-thousandths of an inch, and the material of the pips (spots) has the same density as the body of the die. Six-sided dice are probably the easiest to manufacture precisely.

Every year gamers in tens of thousands gather at gaming conferences worldwide. In 2023, Gen Con, the largest gaming conference in North America, had an estimated 75,000 attendees. At this conference, at least 18 kiosks in the exhibit hall primarily sold dice. One topic consistently discussed at these conferences and on numerous Internet sites is the fairness of dice. Are dice cursed, roll consistently low, or blessed, roll consistently high? At least one company, Precision Play Dice, is going to extreme measures in an attempt to manufacture fairer dice.

Dice can never be manufactured with uniform density and geometric precision, though casino dice come the closest. Therefore, all dice are either blessed or cursed, but some dice are fairer than others. We have tested more than 30 dice by rolling them 3000 times each and calculating the chi-square statistic. For plastic-mold dice, about 1 in 4 test fair for most manufacturers. Metal dice did not fare better in or tests.

The Internet has much information and misinformation on cursed dice, and on methods of testing dice. Campbell and Dolan (2019) investigated float tests of dice and concluded that the method, though apparently popular, is without value. Neagley (2018) and others also drew the same conclusion. Despite actual comparisons between chi-square tests and float tests, float tests remain popular with one having almost a million views on Youtube (Fisher 2015). Campbell and Dolan (2020) revisited dice testing and applied the running chi-square test. They also promoted the value of the running chi-square test which calculates and plots χ^2 after every roll and plots it as a function of the number of rolls. Unfair dice have a linear trend, the steeper the trend, the more unfair the die.

Campbell and Wimsatt (2022) expanded on the running chi-square test by providing dome of the characteristics of the χ^2 statistic for unfair dice.

In an effort to promote fair game mechanics using unfair dice, Campbell and Miller (2023), suggested using dice sums. Their physical rolls and simulations indicated that summing multiple dice usually, but not always, gave fairer results.

Data Analysis:

The χ^2 statistic for unfair dice asymptotically approaches the noncentral chi-square distribution. This distribution is a function of the number of dice rolls, whereas the chi-square distribution is not. The greater the number of rolls, the greater the departure of the noncentral chi-square distribution from the chi-square distribution. Also, as the number of rolls increases, so does the spread of the noncentral chi-square distribution. Figure 1 illustrates this.

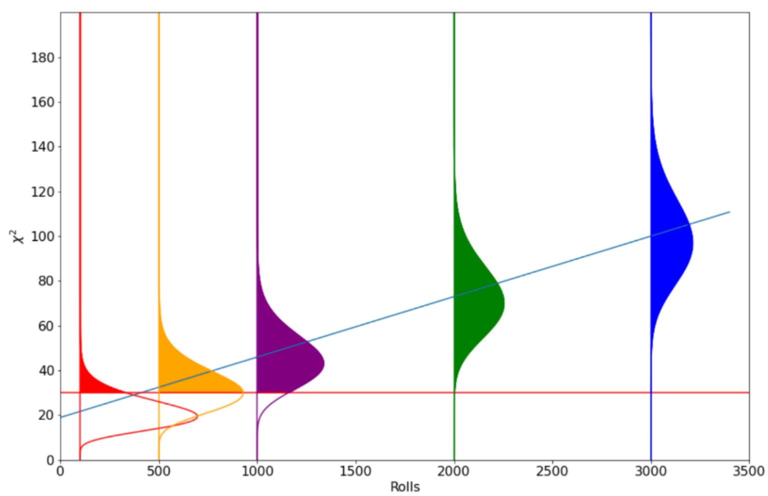

Figure 1. The noncentral chi-square distribution as a function of the number of rolls for $\chi^2 = 100$ after 3000 rolls

Possible Values of the χ^2 Statistic

The χ^2 statistic for a single die is given by Equation 1. For dice, both the numerator and the denominator are either integers or rational fractions so the statistic must also be rational. There are infinitely more irrational numbers than rational ones.

$$\chi^2 = \sum_{i=1}^{s} \frac{(O_i - E)^2}{E} \qquad (1)$$

s = the number of sides of the die

O_i = the number of times side i is rolled

$E = \dfrac{n}{s}$ = expected number of times each side of a fair die is rolled

n = total number of rolls of the die

The minimum spacing between possible values of the χ^2 statistic is given by Equation 2. It is the minimum spacing because the spacing increases for larger values of the statistic. Figure 2 illustrates this fact for a D2 (coin). The points shown are the only possible χ^2 values for the given number of rolls. This minimum spacing occurs only on the low end of χ^2 values. On the high end, spacing is integer multiples of the minimum spacing.

$$\Delta\chi^2_{\min} = \frac{2}{E} \qquad (2)$$

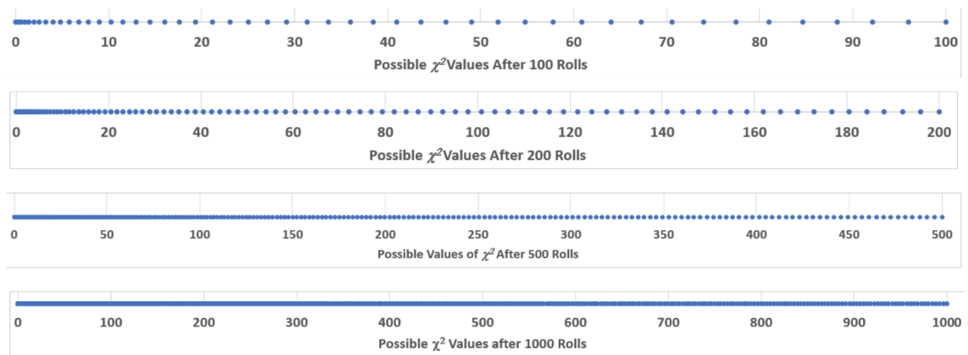

Figure 2. Possible values of χ^2 for a D2 for different numbers of rolls

Asymptotic Approach

The χ^2 statistic only asymptotically approaches the chi-square distribution. Figures 3 and 4 illustrate this. Figure 3 has the exact probability distribution for 9 rolls and 30 rolls of a fair D3. Figure 4 shows the theoretical (chi-square distribution) and the exact and theoretical probability distributions for 100 rolls of a fair D3. The probability spikes indicate the possible values of the χ^2 statistic, whether the dice are fair or not. Between these, there is no possible value of the statistic. The maximum possible value of the statistic for a general die and a general number of rolls is given by Equation 3. For 100 rolls of a D3, the maximum possible value of χ^2 is 200, though the chi-square distribution is defined to infinity.

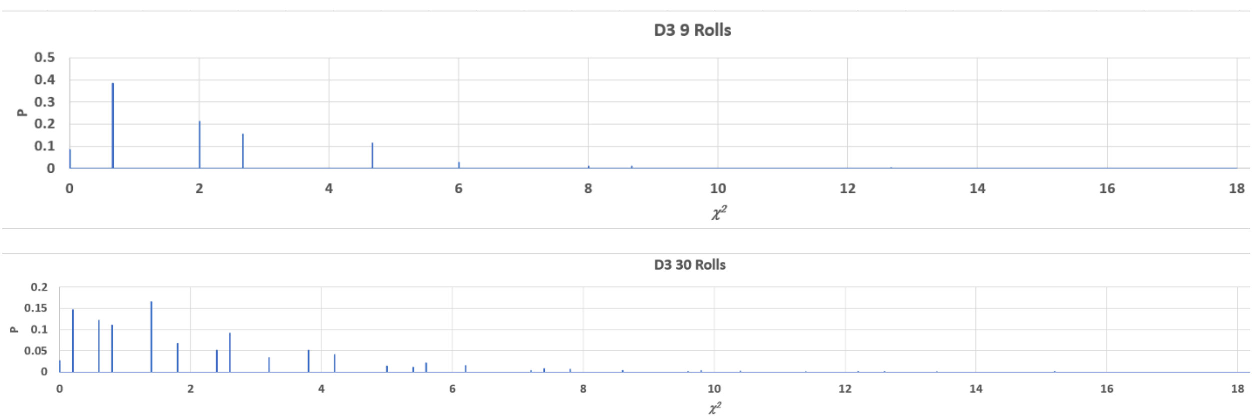

Figure 3. Exact probability distributions for 9 and 30 rolls of a D3

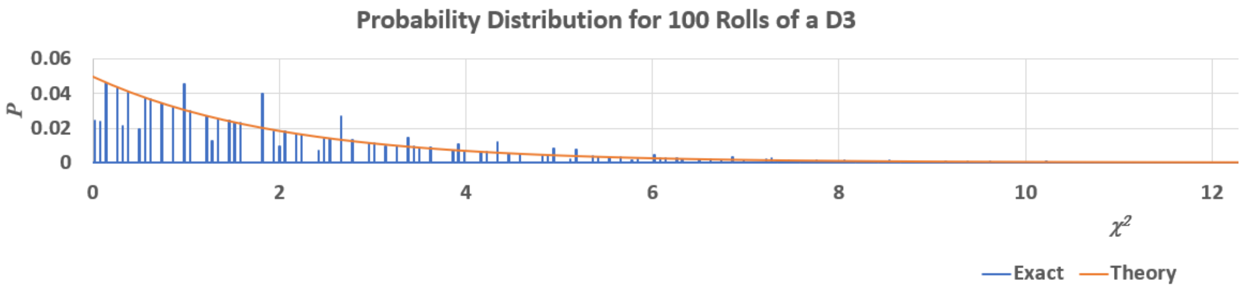

Figure 4. Exact and theoretical probability distribution for 100 rolls of a D3

The maximum value of χ^2 is given in Equation 3. A higher value of the statistic is impossible.

$$\chi^2_{\max} = n(s-1) \qquad (3)$$

n = number of rolls of the die

s = the number of sides of the die

Moving to a die that is actually used in role-playing games, Figure 5 shows the exact and theoretical probability distributions of a fair D4 rolled 100 times. The chi-square distribution is not a good fit to the exact distribution even after 100 rolls. The exact 95 percent value of χ^2 is 7.76 compared to 7.815 predicted by the chi-square distribution. Figure 6 compares 400 rolls of a D4 for a fair, and an unfair die. For the unfair die, the probability of rolling a 1 or a 2 for the unfair die is 0.2, while for a 3 or 4 is 0.3. Some interesting patterns emerge from the exact distribution for the fair die. We do not have an explanation for those patterns, but they are real.

Calculations of the exact distributions for large numbers of rolls, especially for dice with large numbers of sides, are computationally intensive. Some of these calculations required a week on our computers. We observe that dice with higher numbers of sides approach the chi-square distribution more quickly than those with fewer sides. Figures 7, 8, and 9 illustrate this point.

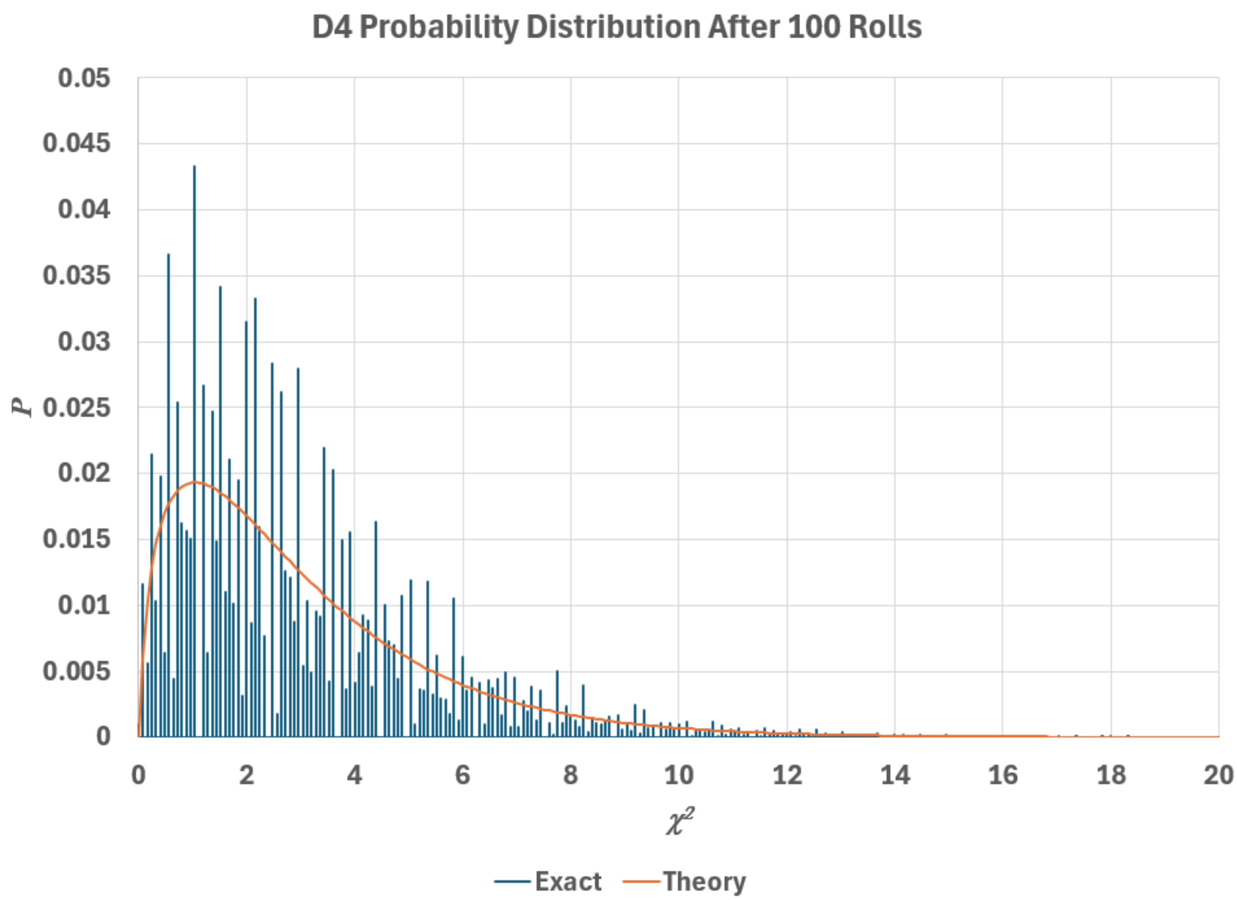

Figure 5. Exact and theoretical probability distributions for a D4 rolled 100 times

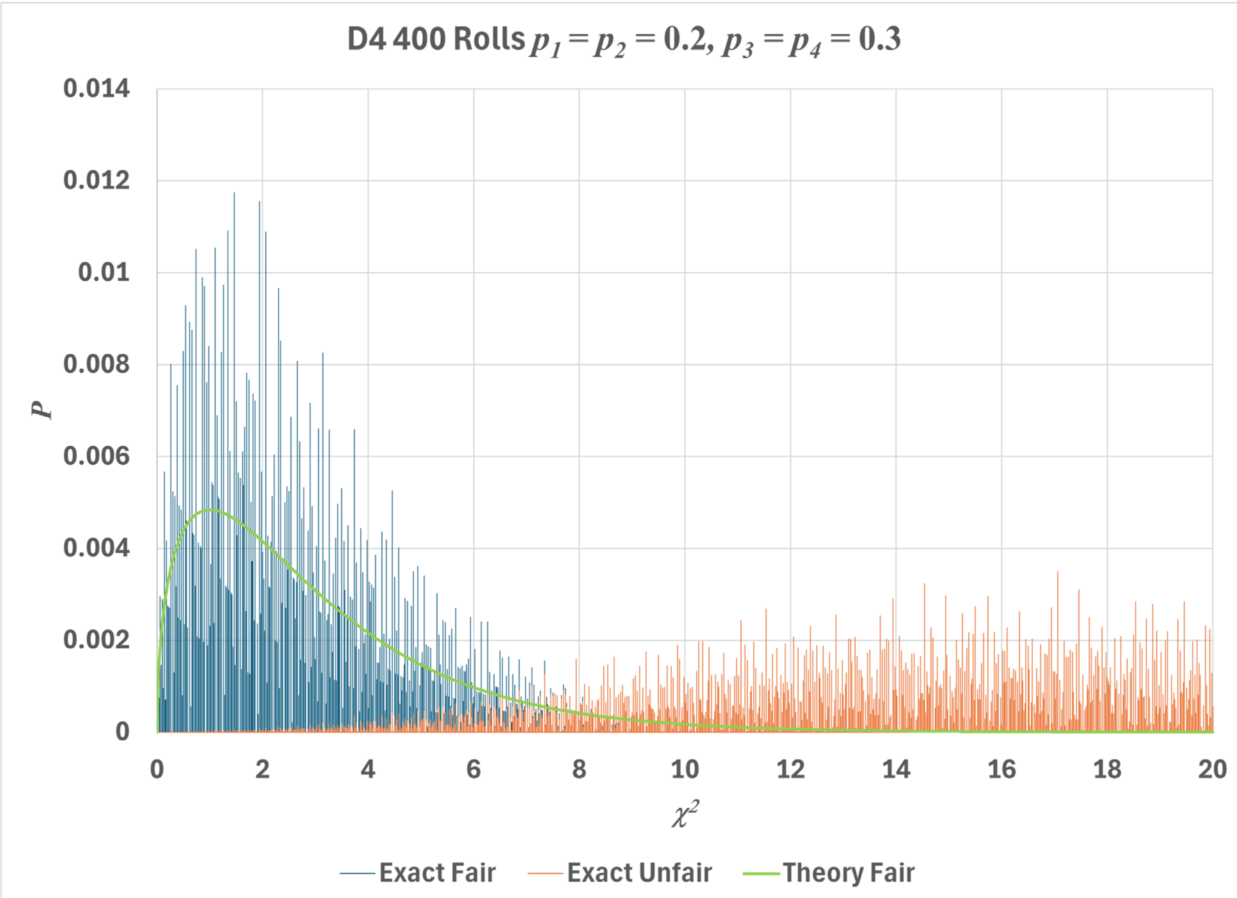

Figure 6. Exact distributions of fair and unfair D4s.

Since the computational effort is so great, why worry about these calculations? Table 1 provides the answer. The exact 95 percent value of χ^2 is given in the third column. The 95 percent value predicted by the chi-square distribution is given in the fourth column. There are significant differences. In addition to this information, the table also provides the exact powers of the chi-square test for several unfair dice. The power of a test ranges from zero to one. The power of a statistical test is the probability that the test will correctly identify an unfair die, given that it is unfair. A value of one means that the test will always correctly identify the die as unfair. A zero value means that the test will never correctly identify the unfair die.

The hypothetical probabilities of the D2 (coin) are extremely unfair. The probability of a 1 (head) is 0.4, while a 2 (tail) is 0.6. After only 100 rolls, there is a better than even chance of identifying the unfair coin. The D4s had hypothetical probabilities of 0.23 for a 1 or 2, and 0.27 for a 3 or a 4. After 100 rolls, there is less than a 1 in 10 chance of identifying the unfair die correctly. Even after 2000 rolls, there is only an 87 percent chance of detecting that the die is unfair. This means that if you rolled the D4 2000 times and then calculated χ^2 and repeated this process over and over, only 87 percent of the time would χ^2 exceed the 95 percent value of 7.812. All of this assumes that the die was not damaged after many thousands of rolls.

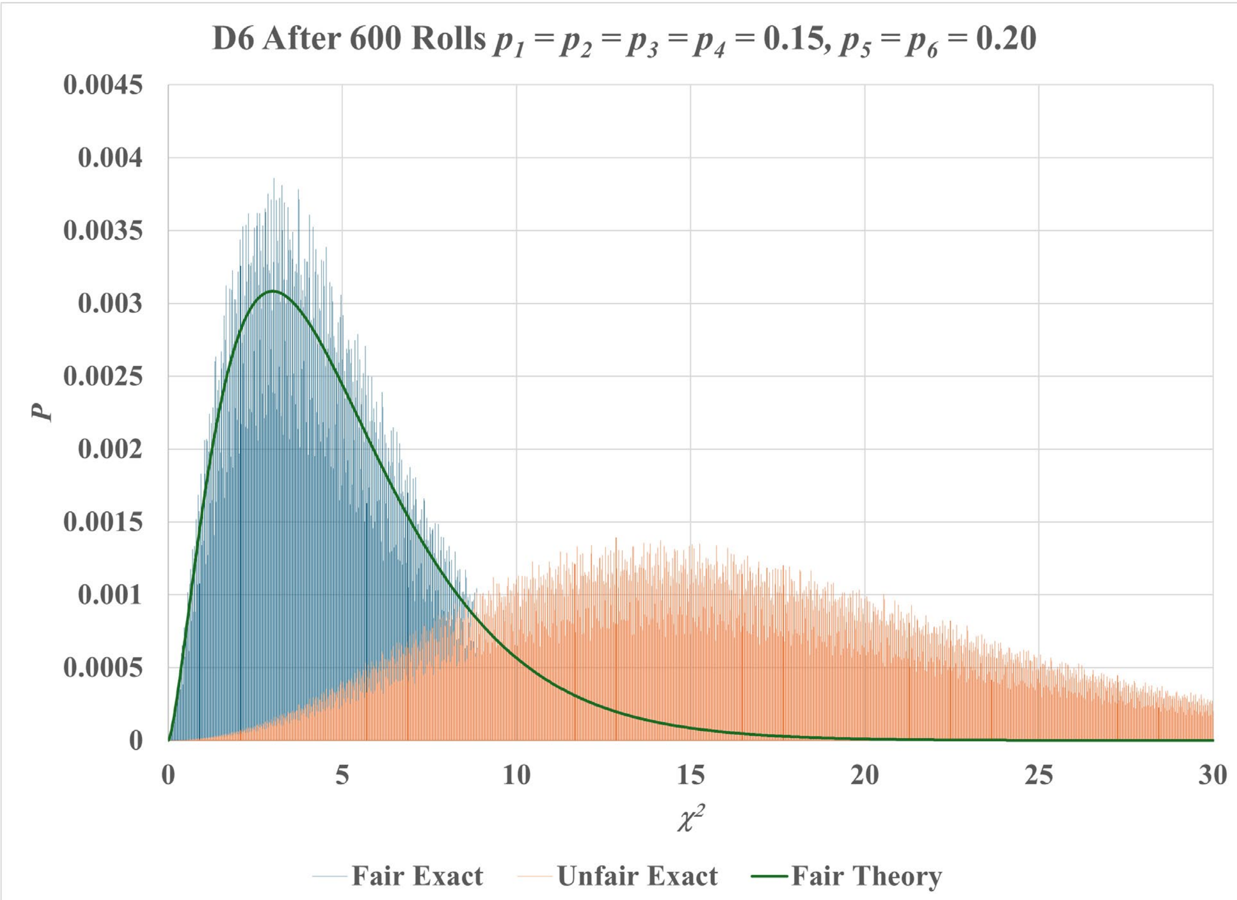

Figure 7. Exact distributions of fair and unfair D6s.

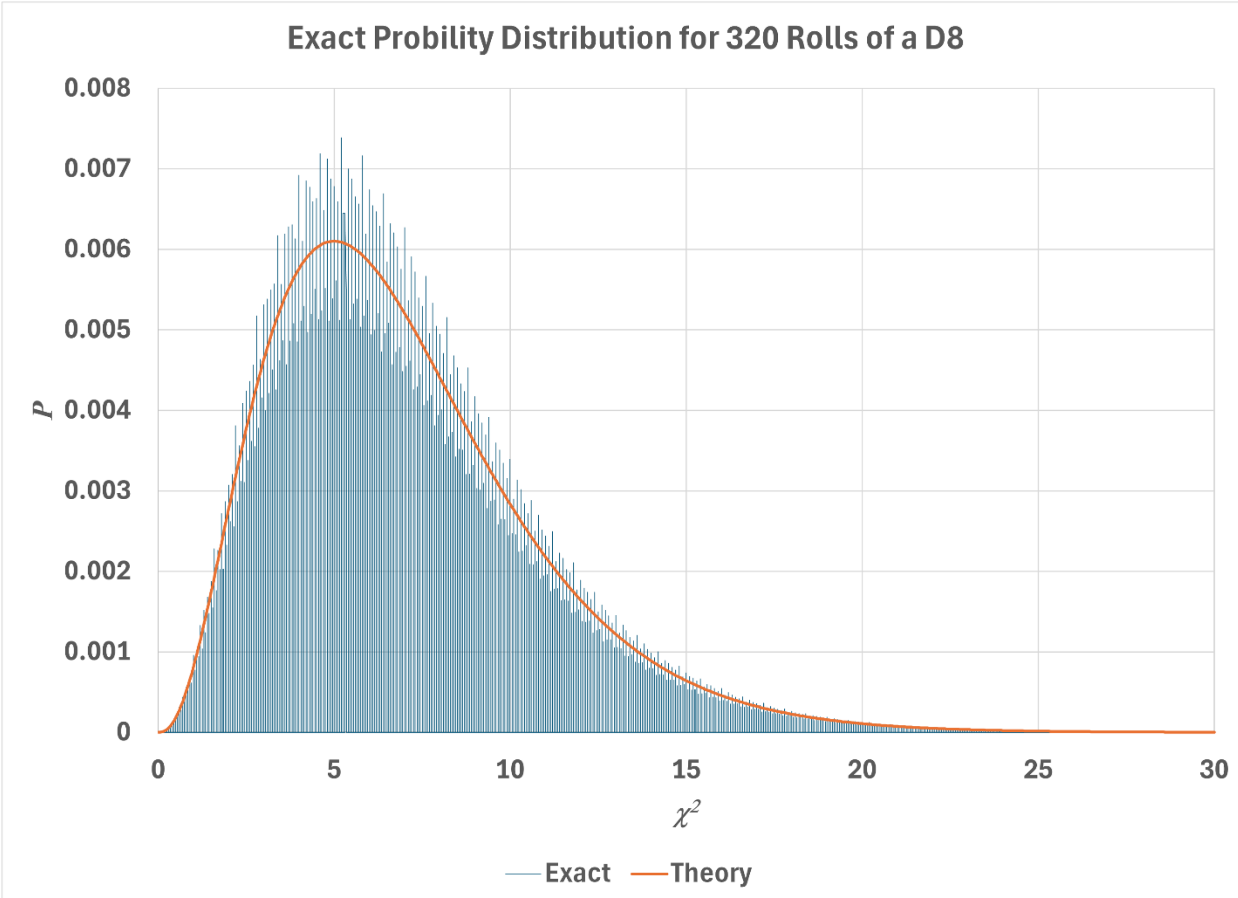

Figure 8. Exact and chi-square probability distribution of a fair D8 rolled 320 times

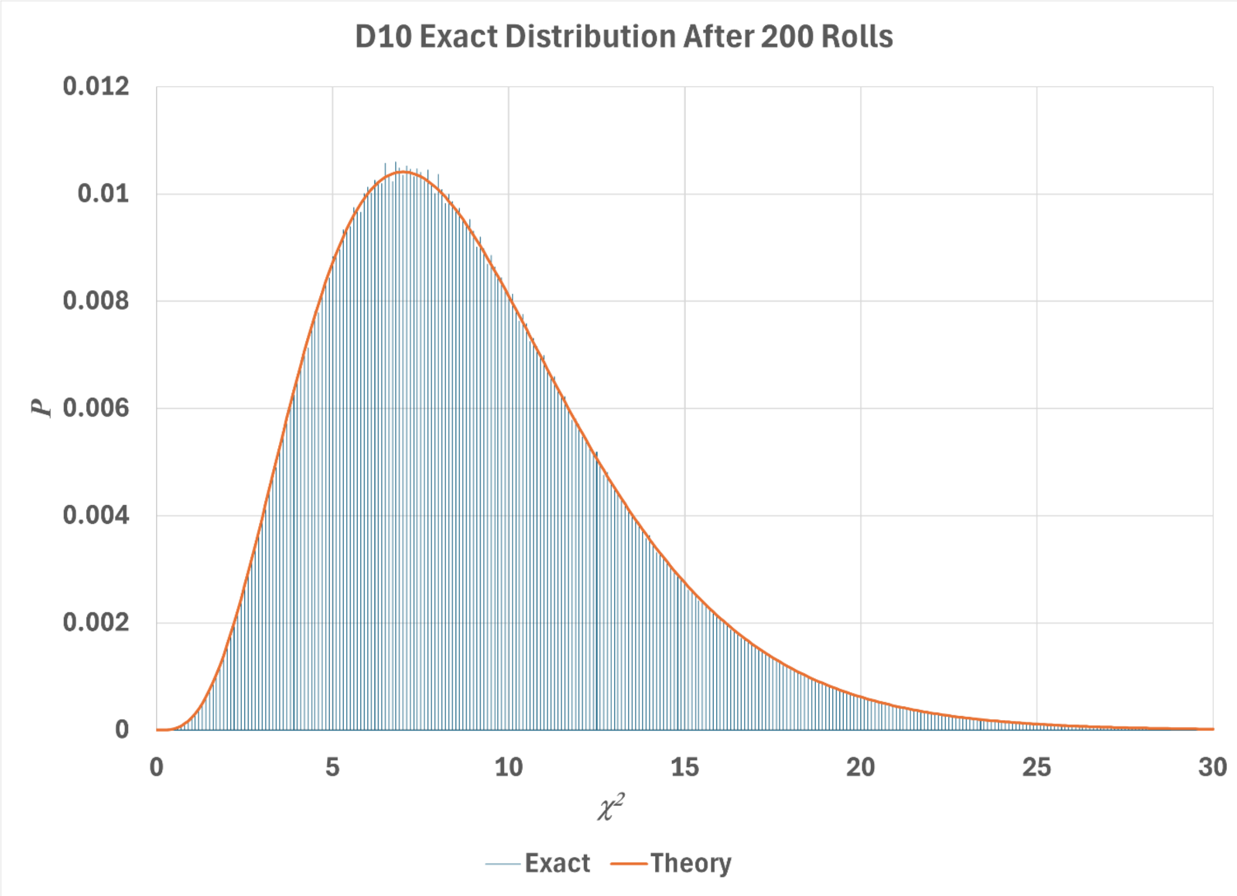

Figure 9. Exact and fair probability distribution of a D10

Table 1. Deviations of the 95% value of χ^2 from that predicted by the chi-square distribution

Die	Rolls	χ^2_{95}	Chi-Sq	P_{low}	P_{hi}	Power
D2	100	4	3.841	0.4	0.6	0.543
D2	200	4	3.841	0.4	0.6	0.981
D2	400	4	3.841	0.4	0.6	0.981
D2	1000	3.844	3.841	0.4	0.6	0.999996
D4	100	7.76	7.815	0.23	0.27	0.097
D4	200	7.8	7.815	0.23	0.27	0.138
D4	400	7.8	7.815	0.23	0.27	0.238
D4	2000	7.812	7.815	0.23	0.27	0.866
D4	4000	7.814	7.815	0.23	0.27	0.994
D6	120	11	11.0705	0.15	0.2	0.182
D6	240	11.05	11.0705	0.15	0.2	0.348
D8	320	14.05	14.067			

Summary

The running chi-square test was the best of the three methods we tried for testing dice. It almost always had better power than the modified Kolmogorov-Smirnov and the double binomial tests. A die must be very unfair, almost loaded, to test unfair after 100 rolls. The power of the test for a given die increases with the number of rolls. The 95 percent value of χ^2 derived from the exact multinomial distribution is different from that predicted by the chi-square distribution. The difference is more apparent for fewer rolls of the die.

The distributions of χ^2 for fair dice asymptotically approach the chi-square distribution. For unfair dice, they asymptotically approach the noncentral chi-square distribution. The exact distributions of χ^2 are always the multinomial distribution whether the dice are fair or unfair.

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