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On the Breadth of the Jones Polynomial for Certain Classes of Knots and Links

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ON THE BREADTH OF THE JONES POLYNOMIAL FOR CERTAIN CLASSES OF KNOTS AND LINKS

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ON THE BREADTH OF THE JONES POLYNOMIAL FOR CERTAIN CLASSES OF KNOTS AND LINKS

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Department of Mathematics Western Kentucky University

The problem of finding the crossing number of an arbitrary knot or link is a hard problem in general. Only for very special classes of knots and links can we solve this problem. Often we can only hope to find a lower bound on the crossing number Cr(K) of a knot or a link K by computing the Jones polynomial of K, V (K). The crossing number Cr(K) is bounded from below by the difference between the greatest degree and the smallest degree of the polynomial V(K). However the computation of the Jones polynomial of an arbitrary knot or link is also difficult in general. The goal of this thesis is to find closed formulas for the smallest and largest exponents of the Jones polynomial for certain classes of knots and links. This allows us to find a lower bound on the crossing number for these knots and links very quickly. These formulas for the smallest and largest exponents of the Jones polynomial are constructed from special rational tangles expansions and using these formulas, we can extend these results to for special cases of Montesinos knots and links.

CHAPTER 1

Introduction

The main purpose of this thesis is to develop some techniques that can be used to find the breadth of the Jones polynomial for certain classes of knots and links. Chapter 2 gives an introduction to knot theory. This chapter will be the base upon which all of the thesis is built upon. Section 2.1 will be used to introduce the reader to all of the basic definitions that are used in this thesis. Section 2.2 introduces the Jones polynomial and gives many theorems about the Jones polynomial for different knots and links.

Chapter 3 focuses on knots and links constructed from rational tangles. Section 3.1 is used to introduce the reader to rational tangles and gives some theorems concerning rational tangles that we will use in the next section. Section 3.2 focuses on the Jones polynomial of knots and links constructed from rational tangles and in this section we prove a theorem about the maximum and the minimum exponent of the Jones polynomial of these knots and links.

Chapter 4 will be used to make some conclusions about the breadth of the Jones polynomial for certain Montesinos knots and links. Section 4.1 uses our notation from Chapter 3 to introduce the reader to Montesinos knots and links. Section 4.2 discusses the Jones polynomial of certain Montesinos links and gives some theorems on the maximum and minimum degree of some of these links.

Chapter 5 is used to conclude the thesis and gives ways this research could be extended in order to draw conclusions about the breadth of the Jones polynomial for a general Montesinos knot.

CHAPTER 2

Basic Knot Theory

2.1. A Brief Introduction to Knot Theory

This chapter is an introduction to the basic concepts of knot theory that will be needed for this thesis. The terms defined in this section are standard terms in knot theory, that can be found in any book on knot theory, see for example [**3, 5, 8, 9**]. A *knot* is defined as a simple closed curve in $\mathbb{R}^3$, i.e. a function f: $S^1 \to \mathbb{R}^3$ where f is in C^2 -smooth and has finite arclength. A *link* is a disjoint union of one or more such simple closed curves in $\mathbb{R}^3$. Each closed curve in a link is called a *component* of the link. A *diagram* D of a link L is a regular projection of L, that is the image of a function $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ such that the following criteria are met:

i.) there are only double points, i.e. no triple or quadruple points;

ii.) there are finitely many double points;

iii.) all double points are like the double point shown in Figure 2.1

Double points in our diagram like the one shown in Figure 2.1 are called *crossings*. For examples of knot and link diagrams see Figures 2.2 and 2.3.

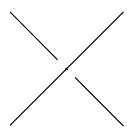

FIGURE 2.1. A crossing in a diagram.

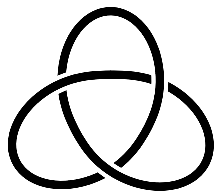

Figure 2.2. The three crossing knot, known as the trefoil.

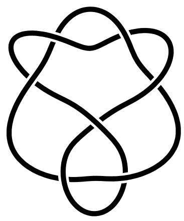

Figure 2.3. A link with 2 components and 7 crossings.

The advantage of a knot diagram is that instead of studying the 3-dimensional knot we can instead study a 2-dimensional diagram. Note that any knot can by drawn with an infinite number of different diagrams, see Figure 2.4. To compensate for this we need to define some type of equivalence for knot diagrams. This equivalence is defined in Figures 2.5, 2.6, and 2.7 by simple moves on diagrams called Reidemeister moves. Two links L and L', with diagrams D and D' respectively, are *equivalent* iff D can be obtained by applying a series of Reidemeister moves to D'.

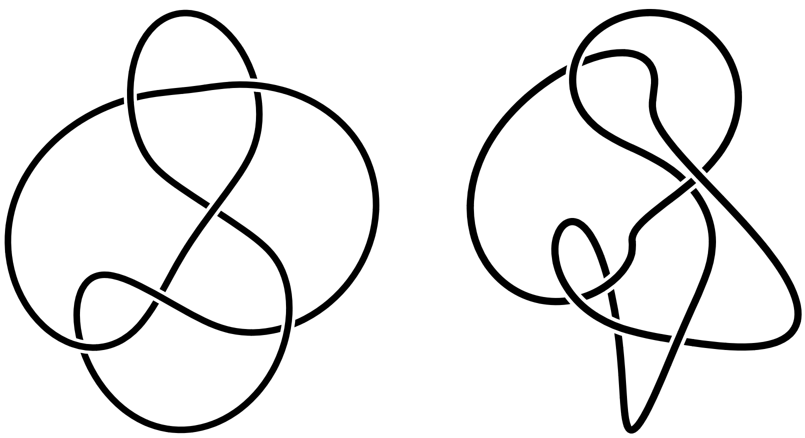

Figure 2.4. These are two different diagrams of the same knot.

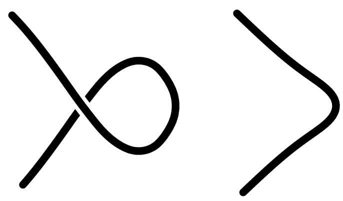

Figure 2.5. Type I Reidemeister Move.

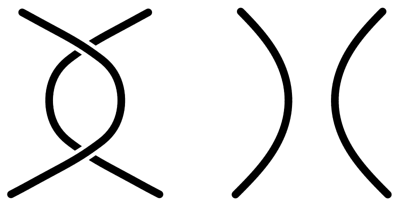

Figure 2.6. Type II Reidemeister Move.

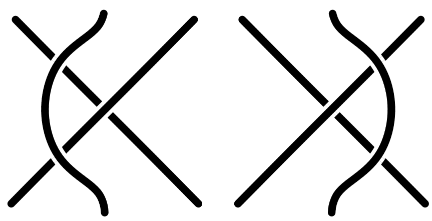

Figure 2.7. Type III Reidemeister Move.

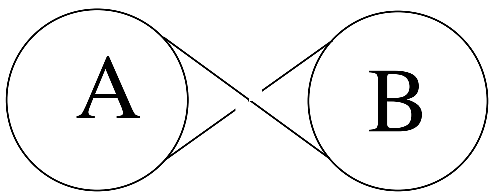

Figure 2.8. A diagram that is not reduced

Each link diagram can be classified as either alternating or non-alternating. A link diagram is called *alternating* if you can walk along each one of its components and at each crossing you alternate between being the over strand and the under strand. If a link diagram is not alternating then it is called *non-alternating*. In Figure 2.4

the diagram shown on the left is alternating while the diagram on the right is non-alternating. A diagram is *minimal* if we cannot perform any Reidemeister moves to lower the number of crossings. Again Figure 2.4 is a good example of this concept in that the diagram to the left is minimal while the diagram on the right is not. We can also classify a diagram as *reduced* if it contains no crossing of the form shown in Figure 2.8 or its reflection. In Figure 2.8 A and B contain the whole diagram away from the crossing.

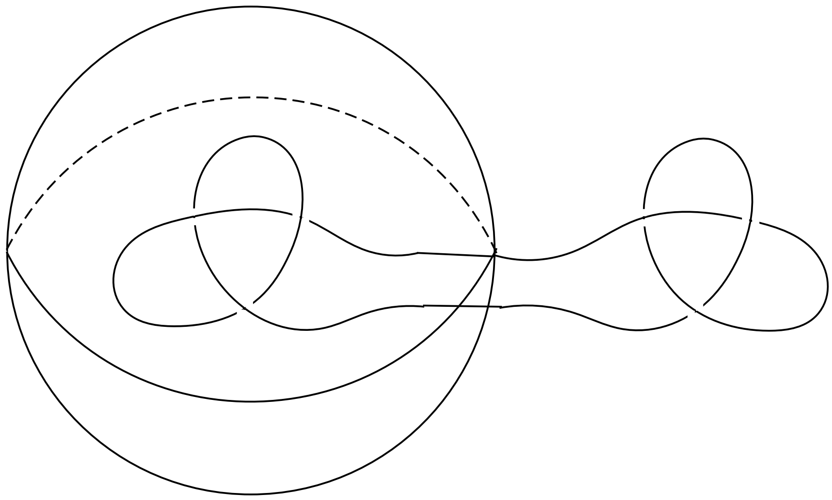

FIGURE 2.9. The connected sum of two trefoil knots denoted L # L where L is the trefoil.

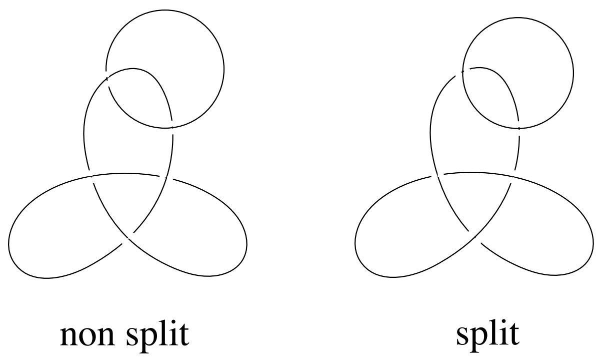

FIGURE 2.10. An example of a non split and a split link

As in many mathematical areas of study, knot theory also has the concept of primeness. To do this we will first see a way to construct a link diagram by putting two diagrams together. A link is *non split* if I cannot pull out one or several components of a link, see Figure 2.10 for an example. Suppose L is a link that is intersected with a 2-sphere in $\mathbb{R}^3$ in exactly two points. Then this sphere separates one component of the link into two arcs. Further the endpoints of either arc can be joined by an arc lying on the sphere, resulting in two links L_1 and L_2. When this can be done we call L a *connected sum* of L_1 and L_2, denoted $L = L_1 \# L_2$, see Figure 2.9. Then a link non split L is *prime* if for every decomposition of L as a connected sum, $L = L_1 \# L_2$, either L_1 or L_2 is the unknot [**8**]. The unknot is just a circle as in Figure 2.15.

One construction that we will see later in this thesis is called a mutation. Suppose that a link L is intersected by a circle in exactly four points. By rotating the portion of L in the circle 180 degrees, but leaving the rest of the link the same we create a *mutation* of L. This concept is illustrated in Figure 2.11. If the intersection points of the link with the circle are not symmetric with respect to the rotation then we need to make appropriate adjustments to the connecting arcs, see Figure 2.11. Another move that can be made on a diagram is called a *flype*. Suppose L is a link that is intersected by a circle in exactly four points, see Figure 2.12. To perform a flype the portion the circle will be turned over about the horizontal axis. Again this move is illustrated by Figure 2.12.

Another construction that will be needed in this thesis is the mirror image of a link. Suppose L is a link and suppose L' is a link obtained from L by switching the over and under strands at each crossing, then L' is the *mirror image* of L, see

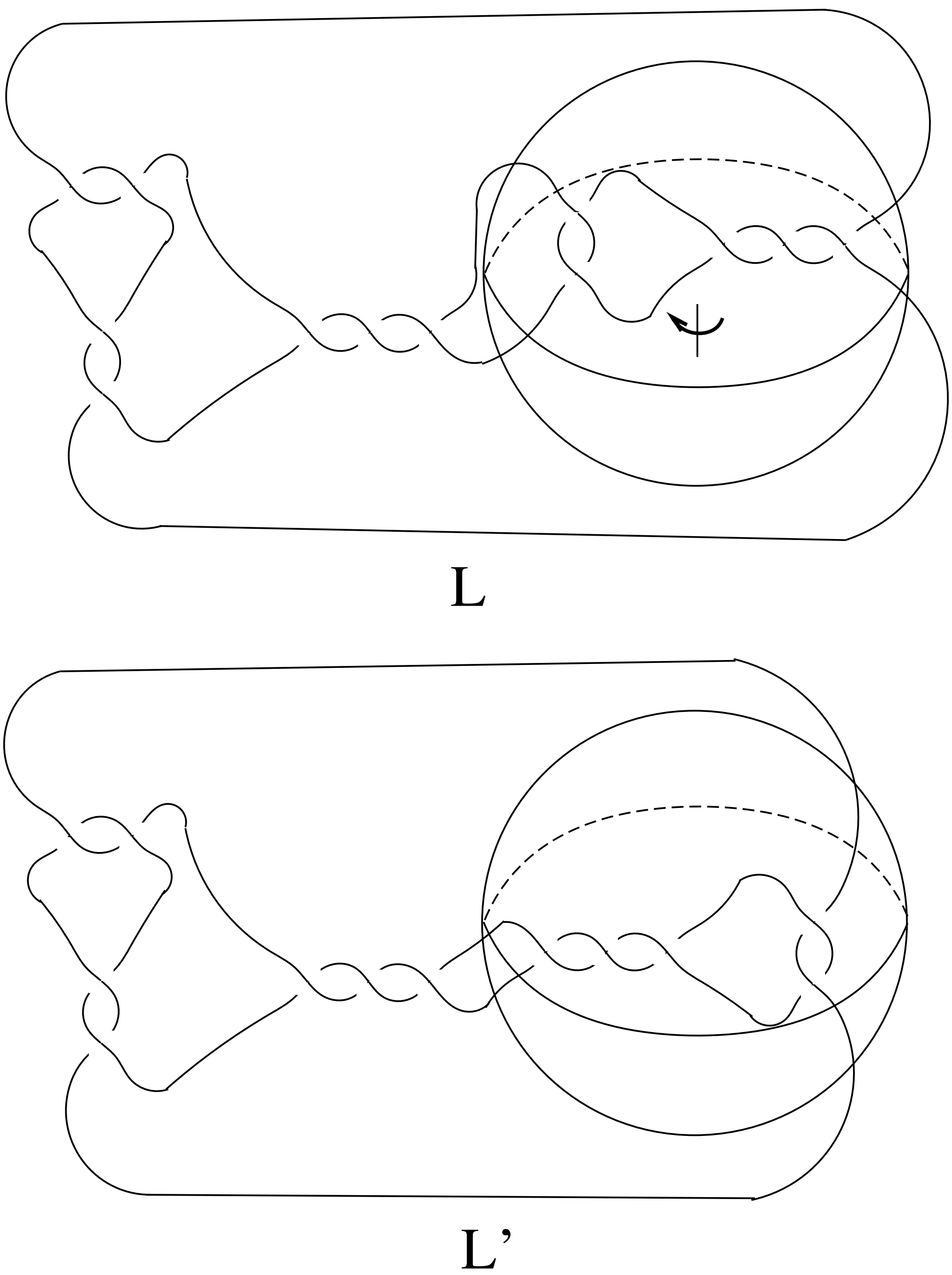

FIGURE 2.11. L and L' are links where L' is a mutation of L.

Figure 2.13. It is important to note that a flype will not change the knot-type while a mirror and mutation might.

A diagram can be oriented in that we can give a direction in which to walk along the diagram for each component of the link. When this is done each crossing will

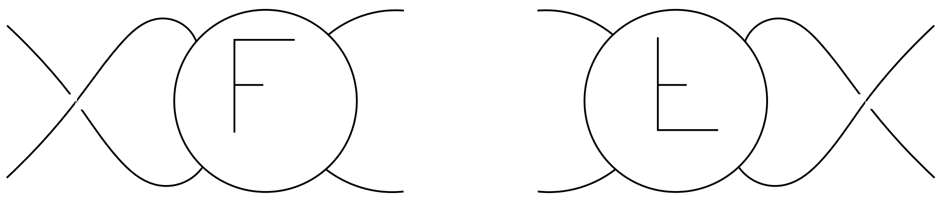

FIGURE 2.12. An illustration of a flype.

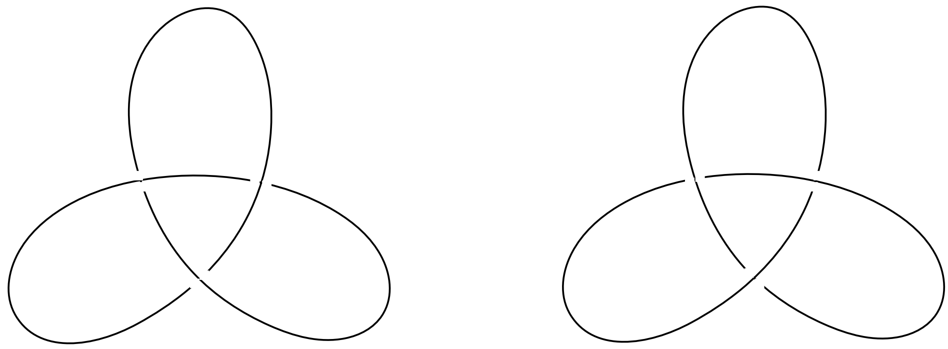

FIGURE 2.13. The trefoil and its mirror image.

now look like one shown in Figure 2.14. The crossing on the left can be called a right hand twist, because if a person puts their thumb and index finger from their right hand on the incoming strands of this crossing and twist their hand to the right the crossing becomes untwisted. Similarly the crossing on the right is called a left hand twist. The standard sign convention is that a right hand twist is positive while a left hand twist is negative. Using this sign convention we can define two new concepts that will be needed in later sections. The *writhe* of a diagram D of an oriented link, denoted $w(\mathrm{D})$, is the sum of the signs of all the crossings of D. Note that $w(\mathrm{D})$ for a knot diagram does not depend on how the diagram is oriented since the opposite orientation will reverse the direction of both strands at each crossing. Our sign convention is invariant when both orientations at a crossing are changed. For a link $w(\mathrm{D})$ depends on the choice of orientation. A similar concept can be defined for a two component oriented link L with components L_1 and L_2. The linking number $lk(L_1, L_2)$ is the half sum of the signs, in a diagram of L, of the crossings at which one strand is from L_1 and the other is from L_2 [**5**].

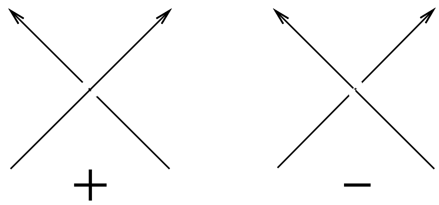

Figure 2.14. A left hand and a right hand crossing.

The last thing we will discuss in this section is the idea of link invariants. A *link invariant* is a property of a link that will not change after applying Type I, II, or III Reidemeister moves to a diagram of L. This implies that regardless which diagram is used the computation of the link invariant will always produce the same answer. This thesis will focus on one important invariant in knot theory called the Jones polynomial. One of the reasons the Jones polynomial is important in knot theory is that it provides a lower bound on the crossing number of a link, see the next section. The crossing number of a link L, denoted Cr(L), is the minimal number of crossings of L over all diagrams of L. Because all diagrams must be considered the crossing number is not easily calculated. For really large links we often only hope to attain a lower bound on the crossing number. The problem of finding a lower bound on the crossing number is the motivation for this thesis. A tool we will use to find a lower bound on the crossing number is the Jones polynomial which we will discuss in the next section.

2.2. The Jones Polynomial

To define the Jones polynomial we need to define another polynomial called the Kauffman bracket. Let U denote the diagram of the unknot as shown below in Figure 2.15. Let D, D_1, D_2 be three link diagrams that are identical outside a small neighborhood. The difference between them is shown in Figure 2.16 where only the part of the diagram that is in the small neighborhood is shown. Then the Kauffman Bracket is defined as follows:

DEFINITION 2.2.1. *Let D be an unoriented link diagram. Then the Kauffman Bracket of D, denoted* $<D>$ *is the polynomial* $<D>$*:* $D \to \mathbb{Z}[A^{-1}, A]$ *characterized by*

$$i.) < U >= 1$$

$$ii.) < D \sqcup U >= (-A^{-2} - A^{2}) < D >$$

$$iii.) < D >= A < D_1 > + A^{-1} < D_2 >$$

Here U denotes the unknot and $D \sqcup U$ *is the disjoint union between a diagram D and an unknotted circle U.*

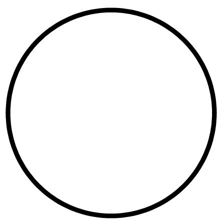

FIGURE 2.15. The unknot U.

Using this definition we can find the Kauffman bracket polynomial of a link diagram. For example the Kauffman Bracket polynomial of the trefoil knot shown

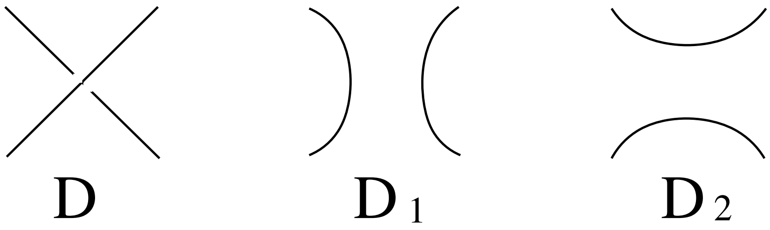

FIGURE 2.16. D, D_1, D_2 respectively

in Figure 2.2 has the value $A^{-7} - A^{-3} - A^5$. The Kauffman bracket is not a link invariant, it is however an invariant under Type II or Type III Reidemeister moves, (It is not invariant under Type I Reidemeister moves.) So to obtain an invariant under all three Reidemeister moves and hence a link invariant we will need modify the Kauffman bracket. [**5**]

DEFINITION 2.2.2. *Let L be an oriented link with diagram D. The Jones polynomial V(L)(t) is a Laurent polynomial, $V : L \to \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$, defined by:*

$$V(L)(t) = (-A)^{-3w(D)} < D >$$

where $A = t^{-\frac{1}{4}}$.

Note that the Jones polynomial, unlike the Kauffman bracket is defined for oriented links because we cannot compute $w(D)$ for an unoriented diagram. It can be shown that V(L) is an invariant under all 3 Reidemeister moves and hence a link invariant [**5**]. Let L_+, L_-, and L_0 be three oriented link diagrams that are identical outside of a small neighborhood. The difference between these diagrams is shown in Figure 2.17 where only the part of the diagram that is in the small neighborhood is shown. An equivalent definition of the Jones polynomial is the following [**5**]:

$$V(unknot) = 1$$

$$and$$

$$t^{-1}V(L_+) - tV(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0). \tag{2.1}$$

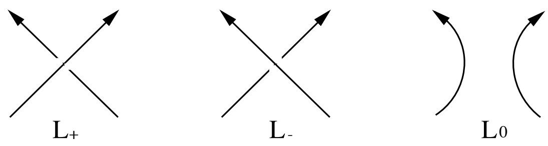

FIGURE 2.17. L_+, L_-, and L_0 are identical except in a small neighborhood. This neighborhood is shown for each diagram respectively

Also we can relate the Jones polynomial of a link to the Jones polynomial of its mirror image by the following theorem:

THEOREM 2.2.3. [**9**] *Let K be a link and K' its mirror image. Then*

$$V(K)(t) = V(K')(t^{-1}). \tag{2.2}$$

Recall that our goal is to relate the Jones polynomial to the crossing number of a link. First consider a polynomial p(t), the *breadth* of p(t), denoted Br(p), is the difference of the maximal exponent of t in p and the minimal exponent of t in p. Now from [**5**] we have the following theorem:

THEOREM 2.2.4. *Let D be a connected diagram of an oriented link L with Jones polynomial V(L). Then*

1.) $Br(V(L)) \leq Cr(L)$;

2.) $Br(V(L)) = Cr(L)$ *iff D is alternating and reduced*;

3.) $Br(V(L)) < Cr(L)$ *if D is non − alternating and a prime diagram*;

So one way of finding a lower bound on the crossing number of link reduces to the problem of finding the maximal and minimal exponent of t of the Jones polynomial of a link. The main focus of this thesis will be to find closed formulas for the values of these exponents for certain classes of knots.

Before moving on to the next section there are more theorems pertaining to the Jones polynomial that will be used in later sections. From [**9**] we have the following two theorems:

THEOREM 2.2.5. *Suppose that* $L = L_1 \# L_2$ *then* $V(L) = V(L_1)V(L_2)$.

THEOREM 2.2.6. *Suppose* L *is a* m*-component oriented link with components* $K_1, K_2, \ldots, K_m$. *Further suppose that L' is the same as L except the orientation of* K_m *has been reversed. Then*

$$V(L') = t^{-3l}V(L)$$

where $l = \sum_{i=1}^{m-1} lk(K_m, K_i)$.

Another theorem that we will use in later sections comes from the relation in Equation 2.1:

THEOREM 2.2.7. [**3**] *Suppose that L is a link and L' is a link that differs from L by only a mutation. Then V(L) = V(L').*

CHAPTER 3

Rational Tangles

This chapter will introduce the basic concepts of rational tangles and their properties. This chapter will also give a formula for the minimal and maximal exponent of t in the Jones polynomial for knots and links arising from rational tangles.

3.1. An Introduction of Rational Tangles

To understand rational tangles we must first start with the question, what is a tangle? A tangle is a 3-ball B that contains two properly embedded disjoint arcs and possibly some embedded closed curves. We denote this by (B, t) where B is the three ball and t is the union of the two arcs and simple closed curves in B. Rational tangles are a special class of tangles. Some examples can be seen in Figure 3.1. A trivial tangle is like the tangle shown in Figure 3.2.

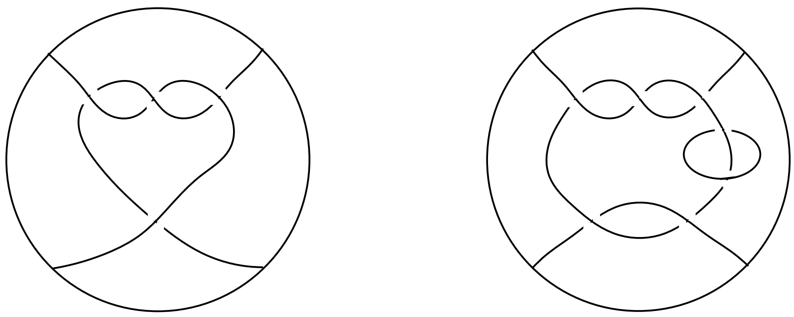

FIGURE 3.1. Two examples of tangles.

A rational tangle is a special class of tangles.

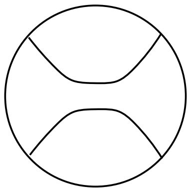

FIGURE 3.2. An example of a trivial tangle.

DEFINITION 3.1.1. *A tangle (B, t) is rational if t is only the disjoint union of two arcs. Furthermore it must be possible to deform (B, t) into the trivial tangle by a deformation that leaves the endpoints of t on the boundary of B.*

Figure 3.3 gives examples of two rational tangles and two non rational tangles. To prove that the tangles on the right are non rational is beyond the scope of this thesis and we refer the reader to a standard text in knot theory [**3, 5, 9**].

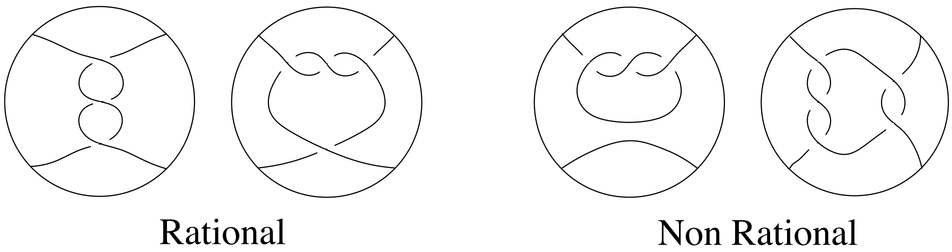

FIGURE 3.3. Two rational tangles and two non rational tangles.

It is a standard convention for rational tangles to think of the endpoints of the two arcs T as the set of points NW, NE, SW, SE as shown in Figure 3.4.

Two rational tangles (B, t) and (B', t') are equivalent if the two arcs in (B, t) can be deformed to the two arcs in (B', t') by a deformation that leaves the four points {NW, NE, SW, SE} as in Figure 3.4 fixed, that is by a deformation that only moves points on the inside of the balls B and B'. Two special rational tangles are given in Figure 3.5

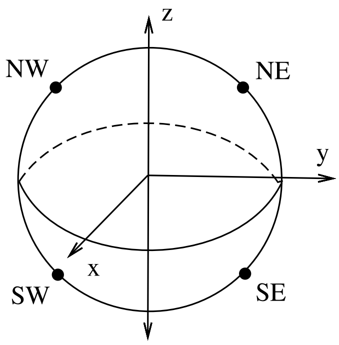

FIGURE 3.4. The 3-Ball B

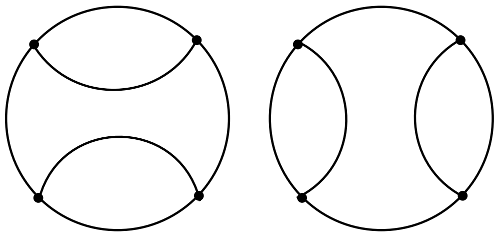

FIGURE 3.5. (0)-Type on the right and (∞)-type on the left.

There is an easy way to think of rational tangles when constructing them. A *vertical twist* occurs when the southern hemisphere is rotated about the z-axis such that SE and SW exchange positions. A *horizontal twist* occurs when the Eastern hemisphere is rotated about the y-axis causing NE and SE to switch positions. To define the terms right twist and left twist see Figure 3.6. For a horizontal twist a left twist is called a *negative twist* while a right twist is called a *positive twist*. For a vertical twist the opposite is true. Now with these terms defined, a *rational tangle* can be defined as a finite alternative sequence of vertical and horizontal twists to a (0)-type or a (∞)-type tangle [**9**]. Therefore we use the notation $< a_1, a_2, \ldots, a_n >$ to define a rational tangle as follows: if n is odd, start with a_1 horizontal twists on a (0)-type followed by a_2 vertical twists, a_3 horizontal twists and so on. Similarly if n is even, start with a_1 vertical twists on a (∞)-type followed by a_2 horizontal twists, a_3

vertical twists and so on. This standard ensures that all rational tangles constructed end with a_n horizontal twists. Figure 3.7 shows an example with n odd and one with n even.

FIGURE 3.6. Right and Left Twists

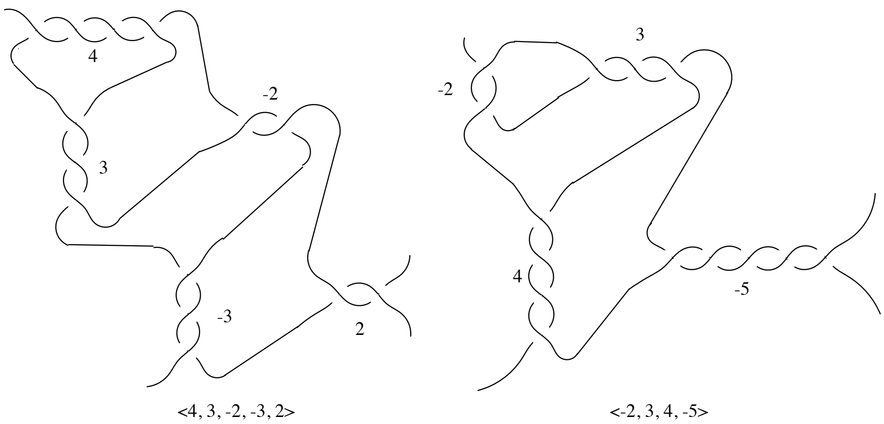

FIGURE 3.7. Two examples of rational tangle construction.

These tangles are called rational tangles because each rational tangle can be represented by a rational number [**9**]. The rational tangle given by $< a_1, a_2, \ldots, a_n >$ can be represented by the rational number $\frac{\beta}{\alpha}$, where $\frac{\beta}{\alpha}$ is obtained as a continued fraction from the vector $< a_1, a_2, \ldots, a_n >$:

$$\frac{\beta}{\alpha} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2}+\cdots+\cfrac{1}{a_2+\cfrac{1}{a_1}}}}$$

To turn tangles into knots and links we use the following construction: the *numerator* of a tangle T = (B, t) denoted N(T), is the knot or link made from joining the points NW to NE and SW to SE by non-intersecting arcs on the boundary of B. The *denominator* of a tangle T, denoted D(T), is made from joining the points NW to SW and NE to SE by non-intersecting arcs on the boundary of B. This is illustrated in Figure 3.8. Let T be a rational tangle, because T ends in a horizontal twist it is easily seen that a Type I Reidemeister applied to D(T) eliminates horizontal twists and we have the following theorem:

LEMMA 3.1.2. [**9**] *Let* $T = < a_1, a_2, \ldots, a_n >$. *Then the link given by D(T) is equivalent to the link N(T') where T'* $= < -a_1, -a_2, \ldots, -a_{n-1} >$.

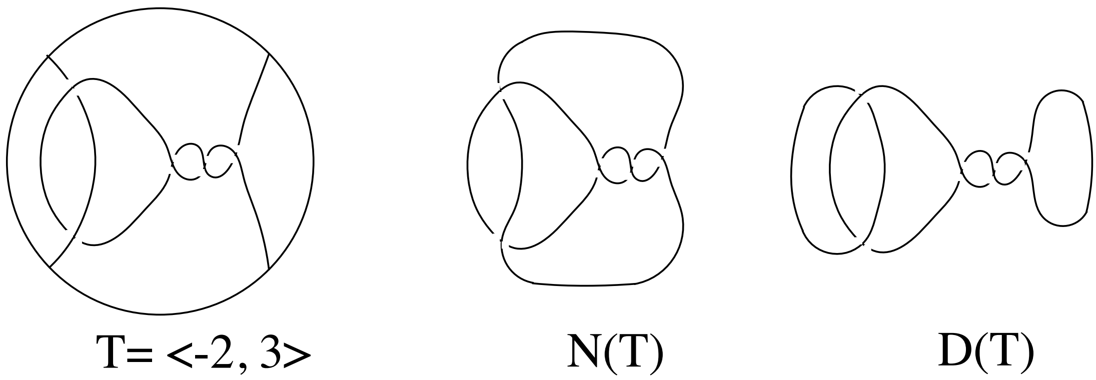

FIGURE 3.8. The numerator and denominator of $< -2, 3 >$.

Recall that equivalence in knots and links has already been defined and applies to the knots constructed from rational tangles. However there is another way to classify knots and links formed from rational tangles based on the continued fraction of the tangle. The following theorem will classify these links:

THEOREM 3.1.3. [**3**] *Let us consider the two links* $L_1 = N(\frac{\beta_1}{\alpha_1})$ *and* $L_2 = N(\frac{\beta_2}{\alpha_2})$. *Then* L_1 *and* L_2 *are equivalent if and only if* $\beta_1 = \beta_2$ *and either* $\alpha_1 \equiv \alpha_2 \; mod \beta_1$ *or* $\alpha_1\alpha_2 \equiv 1 \; mod \beta_1$.

On top of operations such as the numerator and denominator there are other operations that can be applied to tangles. The most important is the sum of two tangles A and B, denoted A + B. A + B can be constructed by connecting the NE point in A to the NW point in B and the SE point in A to the SW point in B by two disjoint arcs in the plane z = 0 as shown in Figure 3.9.

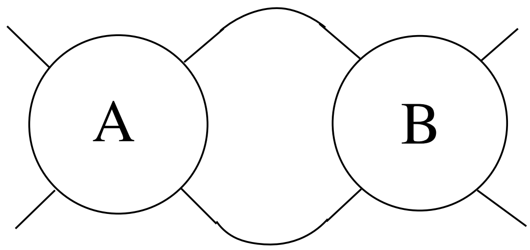

FIGURE 3.9. The sum of tangles A and B

Let A and B be two tangles. A link can be obtained from the sum of these two tangles, A + B, by constructing the numerator of the sum, N(A + B). For certain tangles this link can be oriented as shown in Figure 3.10. A link with this diagram will yield a formula for the Jones Polynomial using the numerators and denominators of the tangle summants A and B.

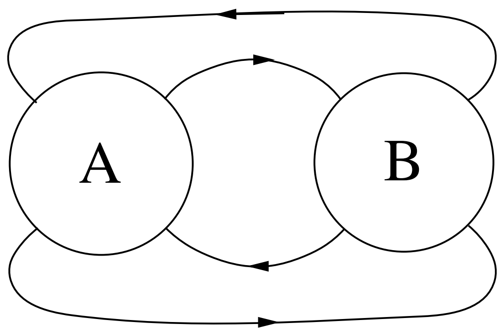

FIGURE 3.10. The orientation used for Theorem 3.1.4.

THEOREM 3.1.4. [**6**] *Let L be a link that can be drawn as the diagram shown in Figure 3.10 where L = N(A + B). Then we can obtain the Jones polynomial of L, V(L), in the following way:*

$$\begin{aligned} V(L) =& \frac{-t}{t^2+t+1}((t^{\frac{1}{2}}+t^{-\frac{1}{2}})(V(N(A))V(N(B))+V(D(A))V(D(B))) \\ &+V(N(A))V(D(B))+V(D(A))V(N(B))) \end{aligned} \tag{3.1}$$

Here the diagrams of N(A), D(A), N(B), and D(B) have the orientation induced by the diagram N(A + B) shown in Figure 3.10.

For the following corollary we will need some new notation. Let p(t) be a polynomial in t then we denote the greatest exponent of t in p to be maxexp(p) and the least exponent of t in p to be minexp(p). With this new notation the following corollary falls directly from Theorem 3.1.4.

COROLLARY 3.1.5. *Let L be a link that can be drawn as in the diagram shown in Figure 3.10 where L = N(A + B). Then the greatest exponent of V(L) is given by*

$$maxexp(V(L)) \leq max \begin{cases} maxexp(V(N(A))V(N(B))) - \frac{1}{2} \\ maxexp(V(D(A))V(D(B))) - \frac{1}{2} \\ maxexp(V(N(A))V(D(B))) - 1 \\ maxexp(V(D(A))V(N(B))) - 1 \end{cases}$$

and the lest exponent of V(L) is given by:

$$minexp(V(L)) \geq min \begin{cases} minexp(V(N(A))V(N(B))) + \frac{1}{2} \\ minexp(V(D(A))V(D(B))) + \frac{1}{2} \\ minexp(V(N(A))V(D(B))) + 1 \\ minexp(V(D(A))V(N(B))) + 1 \end{cases}$$

Note the inequality holds only if there are cancelations of terms involving the highest or lowest exponents. If no such cancelations occur we have equality in Corollary 3.1.5.

Tangles which contain no closed curves can be separated into 3 classes depending on how their arcs are connected, see Figure 3.11. These classes are referred to as the parity of the tangle. For example in Figure 3.12 tangle T_1 is of parity 0, tangle T_2 is of parity 1, and tangle T_3 is of parity ∞.

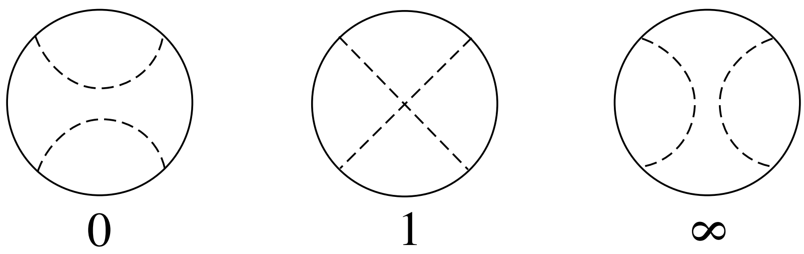

FIGURE 3.11. The 3 parity classes of tangles.

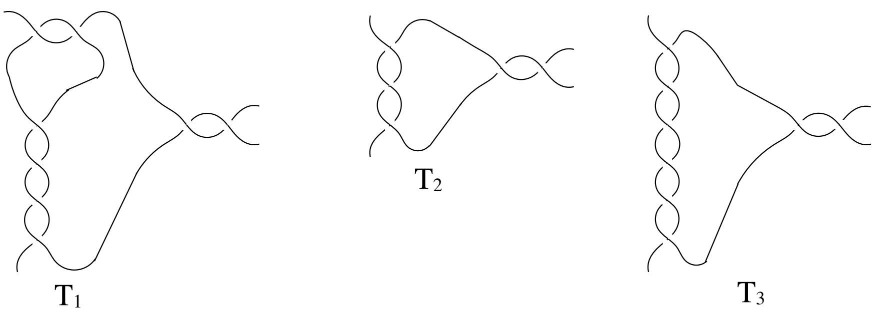

FIGURE 3.12. Three tangles with different parities.

The following two theorems are well known facts about rational tangles [**5, 9**]

THEOREM 3.1.6. *Given a rational tangle* $\frac{\beta}{\alpha}$ *then*

1.) β *is even and* α *is odd if and only if* $\frac{\beta}{\alpha}$ *has parity 0,* *2.) β is odd and α is odd if and only if $\frac{\beta}{\alpha}$ has parity 1,*

3.) β is odd and α is even if and only if $\frac{\beta}{\alpha}$ has parity ∞.

THEOREM 3.1.7. *Given a rational tangle $\frac{\beta}{\alpha} = < a_1, a_2, \ldots, a_n >$ if all a_i are even integers then*

1.) if n is odd $\frac{\beta}{\alpha}$ has parity 0,

2.) if n is even $\frac{\beta}{\alpha}$ has parity ∞.

Moreover if α and β are both odd then $\frac{\beta}{\alpha}$ does not have a continued fraction expansion $< a_1, a_2, \ldots, a_n >$ where all a_i are even.

3.2. The Jones polynomial for special rational tangles

This section is devoted to only one type of rational tangle, namely those that can be written as $< a_1, a_2, \ldots, a_n >$ where each a_i is even. As the section heading suggests this section will explore the Jones polynomial of such tangles. Recall that in order to find the Jones polynomial of a link, that link must have an orientation. To solve this problem for knots and links arising from rational tangles the standard orientation of a tangle as shown in Figure 3.13 will be used.

The next notation that will be given is from [**6**] and is useful in formulating a formula for the Jones polynomial of a rational tangle made up of even twists.

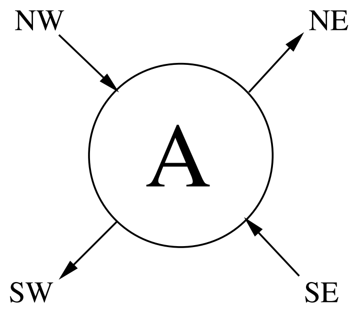

FIGURE 3.13. The standard orientation for tangle A $= < a_1, a_2, \ldots, a_n >$ where each a_i is even.

DEFINITION 3.2.1. *Let $r \in \mathbb{Z}$, then the matrices $M(2r), \bar{M}(2r) \in GL_2(\mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}])$ are defined as:*

$$M(2r) = \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2r}-1)}{t+1} & t^{2r} \\ 1 & 0 \end{pmatrix} \tag{3.2}$$

$$\bar{M}(2r) = \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} & t^{-2r} \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

PROPOSITION 3.2.2. $M(2r) = \bar{M}(-2r)$

PROOF. From Definition 3.2.1 we have the following:

$$\begin{aligned} \bar{M}(-2r) &= \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{-2r}-1)}{t^{-2r}(t+1)} & t^{2r} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{-2r})(1-t^{2r})}{t^{-2r}(t+1)} & t^{2r} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2r}-1)}{t+1} & t^{2r} \\ 1 & 0 \end{pmatrix} \\ &= M(2r) \end{aligned} \tag{3.4}$$

□

The following theorem gives a formula for the Jones polynomial of the links created from rational tangles with only even twists.

THEOREM 3.2.3. *(Proposition 14 [**6**]) Let L be a link such that* $L = N(< a_1, a_2, \ldots, a_n >)$ *where each* a_i *is a nonzero even integer. Then the Jones polynomial of L, V(L), can be computed as a product of matrices:*

$$V(L) = \begin{pmatrix} 1 & 0 \end{pmatrix} \bar{M}(a_1)M(a_2)\ldots\bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n)\begin{pmatrix} 1 \\ -t^{-\frac{1}{2}} - t^{\frac{1}{2}} \end{pmatrix} \text{ if } n \text{ is odd.} \tag{3.5}$$

$$V(L) = \begin{pmatrix} 1 & 0 \end{pmatrix} M(a_1)\bar{M}(a_2)\ldots\bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n)\begin{pmatrix} 1 \\ -t^{-\frac{1}{2}} - t^{\frac{1}{2}} \end{pmatrix} \text{ if } n \text{ is even.} \tag{3.6}$$

From Chapter 2 we know that the Jones polynomial is a Laurent polynomial, so if we can calculate the Jones polynomial using these matrices then the rational expressions in the matrices must simplify to give a Laurent polynomial. The following lemma will be used to simplify these matrices.

LEMMA 3.2.4. *Let* $r \in \mathbb{Z}$*, then the following is true.*

$$\frac{t^{2r} - 1}{t + 1} = t^{2r-1} - t^{2r-2} + t^{2r-3} - + \cdots + t - 1$$

PROOF.

$$\begin{aligned}
&(t+1)(t^{2r-1} - t^{2r-2} + t^{2r-3} - + \cdots + t - 1) \\
&\quad = t^{2r} + t^{2r-1} - t^{2r-1} - t^{2r-2} + t^{2r-2} + \cdots + t - t - 1 \\
&\quad = t^{2r} - 1
\end{aligned}$$

□

In later proofs $M(2r)$ and $\bar{M}(2r)$ will need to be multiplied often and the following lemma gives us some formulas that will be helpful.

Lemma 3.2.5. *Let $r, s \in \mathbb{Z}$, then the following matrix multiplications can be simplified as follows:*

$$M(2r)M(2s) = \begin{pmatrix} t^{2r+2s-1} + - \cdots + t & t^{2r+2s-\frac{1}{2}} - + \cdots - t^{2s+\frac{1}{2}} \\ t^{2s-\frac{1}{2}} - + \cdots - t^{\frac{1}{2}} & t^{2s} \end{pmatrix} \tag{3.7}$$

$$M(2r)\bar{M}(2s) = \begin{pmatrix} t^{2r} - + \cdots - t^{-2s+1} & t^{2r-2s-\frac{1}{2}} - + \cdots - t^{-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix} \tag{3.8}$$

$$\bar{M}(2r)\bar{M}(2s) = \begin{pmatrix} t^{-1} - + \cdots + t^{-2r-2s+1} & -t^{-2s-\frac{1}{2}} - + \cdots + t^{-2r-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix} \tag{3.9}$$

$$\bar{M}(2r)M(2s) = \begin{pmatrix} -t^{2s-1} + - \cdots + t^{-2r} & -t^{2s-\frac{1}{2}} + - \cdots + t^{-2r-2s+\frac{1}{2}} \\ t^{2s-\frac{1}{2}} - + \cdots - t^{\frac{1}{2}} & t^{2s} \end{pmatrix} \tag{3.10}$$

Proof. We will start by proving Equation 3.7 and use Definition 3.2.1 and Lemma 3.2.4.

$$M(2r)M(2s) = \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2r}-1)}{t+1} & t^{2r} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2s}-1)}{t+1} & t^{2s} \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t(t^{2r}-1)(t^{2s}-1)}{(t+1)^2} + t^{2r} & \frac{t^{2s+\frac{1}{2}}(t^{2r}-1)}{t+1} \\ \frac{t^{\frac{1}{2}}(t^{2s}-1)}{t+1} & t^{2s} \end{pmatrix}$$

$$= \begin{pmatrix} t(t^{2r-1} - + \cdots - 1)(t^{2s-1} - + \cdots - 1) + t^{2r} & t^{2s+\frac{1}{2}}(t^{2r-1} - + \cdots - 1) \\ t^{\frac{1}{2}}(t^{2s-1} - + \cdots - 1) & t^{2s} \end{pmatrix}$$

$$= \begin{pmatrix} t^{2r+2s-1} + - \cdots + t & t^{2r+2s-\frac{1}{2}} - + \cdots - t^{2s+\frac{1}{2}} \\ t^{2s-\frac{1}{2}} - + \cdots - t^{\frac{1}{2}} & t^{2s} \end{pmatrix}$$

Next we will show that Equation 3.8 is correct.

$$M(2r)\bar{M}(2s) = \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2r}-1)}{t+1} & t^{2r} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2s}-1)}{t^{2s}(t+1)} & t^{-2s} \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-t(t^{2r}-1)(t^{2s}-1)}{t^{2s}(t+1)^2} + t^{2r} & \frac{t^{-2s+\frac{1}{2}}(t^{2r}-1)}{t+1} \\ \frac{-t^{\frac{1}{2}}(t^{2s}-1)}{t^{2s}(t+1)} & t^{-2s} \end{pmatrix}$$

$$= \begin{pmatrix} -t^{-2s+1}(t^{2r-1} - + \cdots - 1)(t^{2s-1} - + \cdots - 1) + t^{2r} & t^{-2s+\frac{1}{2}}(t^{2r-1} - + \cdots - 1 \\ -t^{-2s+\frac{1}{2}}(t^{2s-1} - + \cdots - 1) & t^{-2s} \end{pmatrix}$$

$$= \begin{pmatrix} t^{2r} - + \cdots - t^{-2s+1} & t^{2r-2s-\frac{1}{2}} - + \cdots - t^{-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix}$$

Next Equation 3.9 will be proved.

$$\bar{M}(2r)\bar{M}(2s) = \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} & t^{-2r} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2s}-1)}{t^{2s}(t+1)} & t^{-2s} \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{t(t^{2r}-1)(t^{2s}-1)}{t^{2r+2s}(t+1)^2} + t^{-2r} & \frac{-t^{-2s+\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} \\ \frac{-t^{\frac{1}{2}}(t^{2s}-1)}{t^{2s}(t+1)} & t^{-2s} \end{pmatrix}$$
$$= \begin{pmatrix} t^{-2r-2s+1}(t^{2r-1} - + \cdots - 1)(t^{2s-1} - + \cdots - 1) + t^{-2r} & -t^{-2r-2s+\frac{1}{2}}(t^{2r-1} - + \cdots - 1) \\ -t^{-2s+\frac{1}{2}}(t^{2s-1} - + \cdots - 1) & t^{-2s} \end{pmatrix}$$
$$= \begin{pmatrix} t^{-1} - + \cdots + t^{-2r-2s+1} & -t^{-2s-\frac{1}{2}} - + \cdots + t^{-2r-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix}$$

Lastly we will show Equation 3.10 is true.

$$\bar{M}(2r)M(2s) = \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} & t^{-2r} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2s}-1)}{t+1} & t^{2s} \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-t(t^{2r}-1)(t^{2s}-1)}{t^{2r}(t+1)^2} + t^{-2r} & \frac{-t^{2s+\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} \\ \frac{t^{\frac{1}{2}}(t^{2s}-1)}{t+1} & t^{2s} \end{pmatrix}$$
$$= \begin{pmatrix} -t^{-2r+1}(t^{2r-1} - + \cdots - 1)(t^{2s-1} - + \cdots - 1) + t^{-2r} & -t^{-2r+2s+\frac{1}{2}}(t^{2r-1} - + \cdots - 1) \\ t^{\frac{1}{2}}(t^{2s-1} - + \cdots - 1) & t^{2s} \end{pmatrix}$$
$$= \begin{pmatrix} -t^{2s-1} + - \cdots - t^{-2r+1} + t^{-2r} & -t^{2s-\frac{1}{2}} + - \cdots + t^{-2r+2s+\frac{1}{2}} \\ t^{2s-\frac{1}{2}} - + \cdots - t^{\frac{1}{2}} & t^{2s} \end{pmatrix}$$

□

Our goal is to investigate the breadth of the Jones polynomial for links made from rational tangles. The next two quantities will take the place of the maximal and minimal exponents of t in the Jones polynomial of a knot made from a special rational tangle.

Definition 3.2.6. *Let K be a link such that $K = N(< a_1, a_2, \ldots, a_n >)$. Then define the following two quantities, H(K) representing the highest exponent and L(K)* *representing the lowest exponent:*

$$\begin{aligned}H(K) =&\frac{1}{2}\sum_{i\equiv(n+1)mod2} a_i(sign(a_i)+1)+\frac{1}{2}\sum_{i\equiv nmod2} a_i(sign(a_i)-1)\\ &-\frac{1}{4}\sum_{i=2}^{n}((-1)^n sign(a_i)+1)(-sign(a_{i-1})sign(a_i)+1)\\ &-\frac{1}{2}sign(a_n)(n\ mod2)\end{aligned}$$

and

$$\begin{aligned}L(K) =&-\frac{1}{2}\sum_{i\equiv(n+1)mod2} a_i(sign(a_i)-1)-\frac{1}{2}\sum_{i\equiv nmod2} a_i(sign(a_i)+1)\\ &-\frac{1}{4}\sum_{i=2}^{n}((-1)^n sign(a_i)-1)(-sign(a_{i-1})sign(a_i)+1)\\ &-\frac{1}{2}sign(a_n)(n\ mod2).\end{aligned}$$

The goal is to prove that if p = V(N($< a_1, a_2, \ldots, a_n >$)) where each a_i is even then H(K) = maxexp(p) and L(K) = minexp(p). The next lemma states a concrete but technical relationship for the minimal and maximal exponent of the Jones polynomial of a link created from a rational tangle made up of only even twists.

Lemma 3.2.7. *Let K be a link such that K = N($< a_1, a_2, \ldots, a_n >$) where all a_i are nonzero even integers. For even n let $\dot{\ddot{M}}(a_1) = M(a_1)$ and for odd n let $\dot{\ddot{M}}(a_1) = \bar{M}(a_1)$. Then the 1 × 2 matrix*

$$\begin{pmatrix}1 & 0\end{pmatrix}\dot{\ddot{M}}(a_1)\ldots\bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n)$$

can be written as

$$(c_m t^{L(K)+1}+\cdots+c_p t^{H(K)}, c_q t^{L(K)+\frac{1}{2}}+\cdots+c_g t^g) \tag{3.11}$$

or

$$(c_m t^{L(K)} + \cdots + c_p t^{H(K)-1}, c_q t^q + \cdots + c_k t^{H(K)-\frac{1}{2}}) \tag{3.12}$$

where 3.11 occurs when $a_n > 0$ and 3.12 occurs when $a_n < 0$. Moreover

$$L(K) + 1 \leq g, q \leq H(K) - 1 \tag{3.13}$$

and $|c_m| = |c_g| = |c_p| = |c_q| = |c_k| = 1$.

Proof. To prove this lemma we will use induction on the value of n. Let K be a link such that K = N(A) where A is a rational tangle A = $< a_1, a_2, \ldots, a_n >$ and every a_i is a nonzero even integer. From Theorem 3.2.3 we see that the calculation of Jones polynomial V(K) is different for n odd and n even. Thus we must show the initial case is true for n = 1 and n = 2. When n = 1 then either $a_1 > 0$, or $a_1 < 0$.

Case 1: Let A = $< a_1 >$ where $a_1 = 2r$ where r is a nonzero positive integer. Then

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \bar{M}(2r) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} & t^{-2r} \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{2r}-1)}{t^{2r}(t+1)} & t^{-2r} \end{pmatrix} \\
&= \begin{pmatrix} -t^{-2r+\frac{1}{2}}(t^{2r-1} + - \cdots - 1) & t^{-2r} \end{pmatrix} \\
&= \begin{pmatrix} -t^{-\frac{1}{2}} + - \cdots - t^{-2r+\frac{1}{2}} & t^{-2r} \end{pmatrix} \\
&= \begin{pmatrix} -t^{l+1} + - \cdots - t^{h} & t^{l+\frac{1}{2}} \end{pmatrix}
\end{aligned}$$

where $h = -\frac{1}{2}$ and $l = -2r - \frac{1}{2}$. From Definition 3.2.6 we have the following:

$$\begin{aligned}
H(K) &= \frac{1}{2}a_1(sign(a_1) - 1) - \frac{1}{2}sign(a_1) \\
&= \frac{1}{2}(2r)(0) - \frac{1}{2}(1) \\
&= -\frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned} L(K) &= -\frac{1}{2}a_1(sign(a_1)+1) - \frac{1}{2}sign(a_1) \\ &= -\frac{1}{2}(2r)(2) - \frac{1}{2} \\ &= -2r - \frac{1}{2} \end{aligned}$$

Thus for this case our lemma is satisfied.

<u>Case 2:</u> Let $a_1 = -2r$ where where r is a nonzero positive integer.

$$\begin{aligned} \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \bar{M}(a_1) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{-2r}-1)}{t^{-2r}(t+1)} & t^{2r} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-t^{\frac{1}{2}}(t^{-2r}-1)}{t^{-2r}(t+1)} & t^{2r} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t^{\frac{1}{2}}(t^{2r}-1)}{t+1} & t^{2r} \end{pmatrix} \\ &= \begin{pmatrix} t^{\frac{1}{2}}(t^{2r-1}+-\cdots-1) & t^{2r} \end{pmatrix} \\ &= \begin{pmatrix} t^{2r-\frac{1}{2}}+-\cdots-t^{\frac{1}{2}} & t^{2r} \end{pmatrix} \\ &= \begin{pmatrix} t^{l}+-\cdots-t^{h-1} & t^{h-\frac{1}{2}} \end{pmatrix} \end{aligned}$$

where $h = 2r + \frac{1}{2}$ and $l = \frac{1}{2}$. Again from Definition 3.2.6 we find

$$\begin{aligned} H(K) &= \frac{1}{2}a_1(sign(a_1)-1) - \frac{1}{2}sign(a_1) \\ &= \frac{1}{2}(-2r)(-2) + \frac{1}{2} \\ &= 2r + \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} L(K) &= -\frac{1}{2}a_1(sign(a_1)+1) - \frac{1}{2}sign(a_1) \\ &= -\frac{1}{2}(-2r)(0) + \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Hence the lemma is satisfied for this case as well.

Now that we have shown the claim is true for n = 1, we will need to show it is true for n = 2. Let r and s be nonzero positive integers. For n = 2 we need to prove the following cases:

$$Case1.\ \ a_1 = 2r,\ a_2 = 2s$$

$$Case2.\ \ a_1 = 2r,\ a_2 = -2s$$

$$Case3.\ \ a_1 = -2r,\ a_2 = 2s$$

$$Case4.\ \ a_1 = -2r,\ a_2 = -2s$$

We will be using Lemmas 3.2.4 and 3.2.5 to simplify the matrix multiplications.

<u>Case 1:</u>

$$\begin{aligned}
\begin{pmatrix}1 & 0\end{pmatrix} M(a_1)\bar{M}(a_2) &= \begin{pmatrix}1 & 0\end{pmatrix} M(2r)\bar{M}(2s) \\
&= \begin{pmatrix}1 & 0\end{pmatrix} \cdot \begin{pmatrix} t^{2r} - + \cdots - t^{-2s+1} & t^{2r-2s-\frac{1}{2}} - + \cdots - t^{-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix} \\
&= \begin{pmatrix} t^{2r} - + \cdots - t^{-2s+1} & t^{2r-2s-\frac{1}{2}} - + \cdots - t^{-2s+\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} -t^{l+1} + - \cdots + t^{h} & -t^{l+\frac{1}{2}} + - \cdots + t^{g} \end{pmatrix}
\end{aligned}$$

where $l = -2s$, $h = 2r$ and $g = 2r - 2s - \frac{1}{2}$. Then from Definition 3.2.6 we find

$$\begin{aligned}
H(K) &= \frac{1}{2}(a_1(sign(a_1) + 1)) + \frac{1}{2}(a_2(sign(a_2) - 1)) \\
&- \frac{1}{4}((-1)^2 sign(a_2) + 1)(-sign(a_1)sign(a_2) + 1) - \frac{1}{2}sign(a_2)(2\ mod 2) \\
&= a_1 = 2r
\end{aligned}$$

and similarly one can show

$$L(K) = -a_2 = -2s$$

Note that $g = 2r - 2s - \frac{1}{2}$ and we see that:

$$L(K) + 1 = -2s + 1 \leq -2s - \frac{1}{2} + 2r \leq 2r - 1 = H(K) - 1.$$

So for this case the lemma is true. The other 3 cases will use similar arguments and we will skip some details.

Case 2: Let $a_1 = 2r$ and $a_2 = -2s$, then

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \end{pmatrix} M(a_1)\bar{M}(a_2) &= \begin{pmatrix} 1 & 0 \end{pmatrix} M(2r)\bar{M}(-2s) = \begin{pmatrix} 1 & 0 \end{pmatrix} M(2r)M(2s) \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} t^{2r+2s-1} + - \cdots + t & t^{2r+2s-\frac{1}{2}} - + \cdots - t^{2s+\frac{1}{2}} \\ t^{2s-\frac{1}{2}} - + \cdots - t^{\frac{1}{2}} & t^{2s} \end{pmatrix} \\
&= \begin{pmatrix} t^{2r+2s-1} + - \cdots + t & t^{2r+2s-\frac{1}{2}} - + \cdots - t^{2s+\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} t^{l} - + \cdots + t^{h-1} & -t^{q} + - \cdots + t^{h-\frac{1}{2}} \end{pmatrix}
\end{aligned}$$

where $l = 1$, $h = 2r + 2s$ and $q = 2s + \frac{1}{2}$. Again we will use Definition 3.2.6 and find

$$H(K) = a_1 - a_2 = 2r + 2s$$

and

$$L(K) = -\frac{1}{4}(-2)(2) = 1$$

Also we find that

$$L(K) + 1 \leq q \leq H(K) - 1.$$

Thus for this case our lemma is satisfied.

Case 3: Let $a_1 = -2r$ and $a_2 = 2s$, then we find

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \end{pmatrix} M(a_1)\bar{M}(a_2) &= \begin{pmatrix} 1 & 0 \end{pmatrix} M(-2r)\bar{M}(2s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \bar{M}(2r)\bar{M}(2s) \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} t^{-1} - + \cdots + t^{-2r-2s+1} & -t^{-2s-\frac{1}{2}} + - \cdots + t^{-2r-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix} \\
&= \begin{pmatrix} t^{-1} - + \cdots + t^{-2r-2s+1} & -t^{-2s-\frac{1}{2}} + - \cdots + t^{-2r-2s+\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} t^{l+1} - + \cdots + t^{h} & t^{l+\frac{1}{2}} - + \cdots - t^{g} \end{pmatrix}
\end{aligned}$$

where $l = -2r - 2s$, $h = -1$ and $g = -2s - \frac{1}{2}$. So then by Definition 3.2.6 we have

$$H(K) = -\frac{1}{4}(2)(2) = -1$$

and

$$L(K) = a_1 - a_2 = -2r - 2s$$

and

$$L(K) + 1 \leq g \leq H(K) - 1$$

Thus our lemma is satisfied by this case.

<u>Case 4:</u> Let us consider $a_1 = -2r$ and $a_2 = -2s$, then we have

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \end{pmatrix} M(a_1)\bar{M}(a_2) &= \begin{pmatrix} 1 & 0 \end{pmatrix} M(-2r)\bar{M}(-2s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \bar{M}(2r)M(2s) \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -t^{2s-1} + - \cdots + t^{-2r} & -t^{2s-\frac{1}{2}} + - \cdots + t^{-2r-2s+\frac{1}{2}} \\ t^{2s-\frac{1}{2}} - + \cdots - t^{\frac{1}{2}} & t^{2s} \end{pmatrix} \\
&= \begin{pmatrix} -t^{2s-1} + - \cdots + t^{-2r} & -t^{2s-\frac{1}{2}} + - \cdots + t^{-2r-2s+\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} t^{l} - + \cdots - t^{h-1} & t^{q} - + \cdots - t^{h-\frac{1}{2}} \end{pmatrix}
\end{aligned}$$

where $l = -2r$, $h = 2s$ and $q = -2r - 2s + \frac{1}{2}$. Then by Definition 3.2.6 we find

$$H(K) = -a_2 = 2s$$

and

$$L(K) = a_1 = -2r$$

and

$$L(K) + 1 \leq q \leq H(K) - 1$$

So again the lemma is true for this case. Hence if n = 2 then the lemma is proven to be true.

Now that we have proven the initial case we assume that it is true for n = 1, 2, ...N. We must show that the lemma is true for n = N + 1. Let K be defined as in the lemma and let Q be a link such that Q = N(A') where $A' =< a_1, a_2, \ldots, a_{N-1} >$. Then to show that the lemma is true for n = N + 1 we will have the following cases:

$$Case1.\ \ a_{N-1} > 0,\ a_N = 2r,\ a_{N+1} = 2s$$

$$Case2.\ \ a_{N-1} > 0,\ a_N = 2r,\ a_{N+1} = -2s$$

$$Case3.\ \ a_{N-1} > 0,\ a_N = -2r,\ a_{N+1} = 2s$$

$$Case4.\ \ a_{N-1} > 0,\ a_N = -2r,\ a_{N+1} = -2s$$

$$Case5.\ \ a_{N-1} < 0,\ a_N = 2r,\ a_{N+1} = 2s$$

$$Case6.\ \ a_{N-1} < 0,\ a_N = 2r,\ a_{N+1} = -2s$$

$$Case7.\ \ a_{N-1} < 0,\ a_N = -2r,\ a_{N+1} = 2s$$

$$Case8.\ \ a_{N-1} < 0,\ a_N = -2r,\ a_{N+1} = -2s$$

We will only show the proof for cases 1 and 5 and the arguments for the other cases are similar to those two cases and the arguments used to prove the lemma for n = 2.

<u>Case 1:</u> $a_{N-1} > 0$, $a_N = 2r$ and $a_{N+1} = 2s$. By the induction hypothesis for

Q = $N(< a_1, a_2, \ldots, a_{N-1} >)$ and n = N +1 we have:

$$\begin{aligned}
&\begin{pmatrix}1 & 0\end{pmatrix}\dot{\ddot{M}}(a_1)\ldots\bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n)\\
&\quad = (c_m t^{L(Q)+1} + \cdots + c_p t^{H(Q)}, c_q t^{L(Q)+\frac{1}{2}} + \cdots + c_k t^k) M(2r)\bar{M}(2s)\\
&\quad \textit{where } |c_m| = |c_p| = |c_q| = |c_k| = 1 \textit{ and } L(Q)+1 \le k \le H(Q)-1\\
&\quad = (c_m t^{L(Q)+1} + \cdots + c_p t^{H(Q)}, c_q t^{L(Q)+\frac{1}{2}} + \cdots + c_k t^k)\\
&\qquad\quad \cdot \begin{pmatrix} t^{2r} - + \cdots - t^{-2s+1} & t^{2r-2s-\frac{1}{2}} - + \cdots - t^{-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix}\\
&\quad = \left(-c_m t^{L(Q)-2s+1} + \cdots + c_p t^{H(Q)+2r}, \quad -c_q t^{L(Q)-2s+\frac{1}{2}} + \cdots + c_p t^{H(Q)+2r-2s-\frac{1}{2}}\right)\\
&\quad = \left(-c_m t^{l+1} + \cdots + c_p t^{h}, \quad -c_q t^{l+\frac{1}{2}} + \cdots + c_p t^{g}\right)
\end{aligned}$$

where $l = L(Q) - 2s$, $h = H(Q) + 2r$ and $g = H(Q) + 2r - 2s - \frac{1}{2}$. Next we use Definition 3.2.6 to find

$$H(K) = \begin{cases} H(Q) + \frac{1}{2} + \frac{1}{2}a_N(sign(a_N)+1) + \frac{1}{2}a_{N+1}(sign(a_{N+1})-1) \\ \quad -\frac{1}{4}\sum_{i=N}^{N+1}(-sign(a_i)+1)(-sign(a_{i-1})sign(a_i)+1) - \frac{1}{2} & \text{if N + 1 is odd.} \\ \\ H(Q) + \frac{1}{2}a_N(sign(a_N)+1) + \frac{1}{2}a_{N+1}(sign(a_{N+1})-1) \\ \quad -\frac{1}{4}\sum_{i=N}^{N+1}(sign(a_i)+1)(-sign(a_{i-1})sign(a_i)+1) & \text{if N + 1 is even.} \end{cases}$$

$$=H(Q) + a_N$$

$$=H(Q) + 2r$$

and

$$L(K) = \begin{cases} L(Q) + \frac{1}{2} - \frac{1}{2}a_N(sign(a_N)-1) - \frac{1}{2}a_{N+1}(sign(a_{N+1})+1) \\ \quad -\frac{1}{4}\sum_{i=N}^{N+1}(-sign(a_i)-1)(-sign(a_{i-1})sign(a_i)+1) - \frac{1}{2} & \text{if N + 1 is odd.} \\ \\ L(Q) - \frac{1}{2}a_N(sign(a_N)-1) - \frac{1}{2}a_{N+1}(sign(a_{N+1})+1) \\ \quad -\frac{1}{4}\sum_{i=N}^{N+1}(sign(a_i)-1)(-sign(a_{i-1})sign(a_i)+1) & \text{if N + 1 is even.} \end{cases}$$

$$=L(Q) - a_{N+1}$$

$$=L(Q) - 2s$$

Moreover we still have $|c_m| = |c_p| = |c_q| = |c_k| = 1$ and

$$L(K) + 1 = L(Q) - 2s + 1 \leq H(Q) + 2r - 2s - \frac{1}{2} \leq H(Q) + 2r - 1 = H(K) - 1.$$

Thus we find that the lemma is true for this case.

<u>Case 5:</u> In this case let $a_{N-1} < 0$, $a_N = 2r$ and $a_{N+1} = 2s$. Then similarly we use our induction hypothesis for Q $= N(< a_1, a_2, \ldots, a_{N-1} >)$ and n = N +1 to find the following:

$$\begin{aligned}
&\begin{pmatrix}1 & 0\end{pmatrix}\dot{\bar{M}}(a_1)\ldots\bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n) \\
&\quad = (c_m t^{L(Q)} + \cdots + c_p t^{H(Q)-1}, c_q t^q + \cdots + c_k t^{H(Q)-\frac{1}{2}})M(2r)\bar{M}(2s) \\
&\quad \mathit{where}\ |c_m| = |c_p| = |c_q| = |c_k| = 1\ \mathit{and}\ L(Q) + 1 \leq q \leq H(Q) - 1 \\
&\quad = (c_m t^{L(Q)} + \cdots + c_p t^{H(Q)-1}, c_q t^q + \cdots + c_k t^{H(Q)-\frac{1}{2}}) \\
&\qquad \cdot \begin{pmatrix} t^{2r} - + \cdots - t^{-2s+1} & t^{2r-2s-\frac{1}{2}} - + \cdots - t^{-2s+\frac{1}{2}} \\ -t^{-\frac{1}{2}} + - \cdots + t^{-2s+\frac{1}{2}} & t^{-2s} \end{pmatrix} \\
&\quad = \left(-c_m t^{L(Q)-2s+1} + \cdots + c_p t^{H(Q)+2r-1}, \quad -c_m t^{L(Q)-2s+\frac{1}{2}} + \cdots + c_p t^{H(Q)+2r-2s-\frac{3}{2}}\right) \\
&\quad = \left(-c_m t^{l+1} + \cdots + c_p t^h, \quad -c_m t^{l+\frac{1}{2}} + \cdots + c_p t^g\right)
\end{aligned}$$

where $l = L(Q) - 2s$, $h = H(Q) + 2r - 1$ and $g = H(Q) + 2r - 2s - \frac{3}{2}$. Again we use Definition 3.2.6 to find

$$H(K) = \begin{cases} \begin{array}{c} H(Q) - \frac{1}{2} + \frac{1}{2}a_N(sign(a_N) + 1) + \frac{1}{2}a_{N+1}(sign(a_{N+1}) - 1) \\ -\frac{1}{4}\sum_{i=N}^{N+1}(-sign(a_i) + 1)(-sign(a_{i-1})sign(a_i) + 1) - \frac{1}{2} \end{array} & \text{if N + 1 is odd.} \\ \\ \begin{array}{c} H(Q) + \frac{1}{2}a_N(sign(a_N) + 1) + \frac{1}{2}a_{N+1}(sign(a_{N+1}) - 1) \\ -\frac{1}{4}\sum_{i=N}^{N+1}(sign(a_i) + 1)(-sign(a_{i-1})sign(a_i) + 1) \end{array} & \text{if N + 1 is even.} \end{cases}$$

$$= H(Q) + a_N - 1$$

$$= H(Q) + 2r - 1$$

and

$$L(K) = \begin{cases} L(Q) - \frac{1}{2} - \frac{1}{2}a_N(sign(a_N) - 1) - \frac{1}{2}a_{N+1}(sign(a_{N+1}) + 1) \\ \quad -\frac{1}{4}\sum_{i=N}^{N+1}(-sign(a_i) - 1)(-sign(a_{i-1})sign(a_i) + 1) - \frac{1}{2} & \text{if N + 1 is odd.} \\ \\ L(Q) - \frac{1}{2}a_N(sign(a_N) - 1) - \frac{1}{2}a_{N+1}(sign(a_{N+1}) + 1) \\ \quad -\frac{1}{4}\sum_{i=N}^{N+1}(sign(a_i) - 1)(-sign(a_{i-1})sign(a_i) + 1) & \text{if N + 1 is even.} \end{cases}$$

$$= L(Q) - a_{N+1}$$

$$= L(Q) - 2s$$

Moreover we still have $|c_m| = |c_p| = |c_q| = |c_k| = 1$ and

$$L(K) + 1 = L(Q) - 2s + 1 \leq H(Q) + 2r - 2s - \frac{3}{2} \leq H(Q) + 2r - 2 = H(K) - 1.$$

Hence for this case the lemma is also satisfied. As stated before the other cases can all be proven similarly. □

Using Lemma 3.2.7 we obtain Theorem 3.2.8 below that gives a relation between the formulas we defined in Definition 3.2.6 and the breadth of the Jones polynomial of certain links.

THEOREM 3.2.8. *Let K be a link such that K = N($< a_1, a_2, \ldots, a_n >$) where all a_i are nonzero even integers. Then the greatest exponent of t in V(K) is maxexp(V(K)) = H(K) and the least exponent of t in V(K) is minexp(V(K)) = L(K). Moreover $Br(V(k)) = H(K) - L(K)$ and the coefficients of the maxexp and minexp terms have absolute value one.*

PROOF. Let K be a link constructed as stated in the theorem above. We will again have to consider the two cases, where $a_n > 0$ and $a_n < 0$. Also let $\dot{\ddot{M}}(a_1) = M(a_1)$ when n is even and $\dot{\ddot{M}}(a_1) = \bar{M}(a_1)$ when n is odd.

Case 1: Let $a_n > 0$. Then using Theorem 3.2.3 and Lemma 3.2.7 we can write the

Jones polynomial of K as

$$
\begin{aligned}
V(K) = &\begin{pmatrix}1 & 0\end{pmatrix} \dot{\bar{M}}(a_1)\ldots \bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n)\begin{pmatrix}1\\ -t^{-\frac{1}{2}} - t^{\frac{1}{2}}\end{pmatrix}\\
=&(c_m t^{L(K)+1} + \cdots + c_p t^{H(K)}, c_q t^{L(K)+\frac{1}{2}} + \cdots + c_g t^g)\begin{pmatrix}1\\ -t^{-\frac{1}{2}} - t^{\frac{1}{2}}\end{pmatrix}\\
=&c_m t^{L(K)+1} + \cdots + c_p t^{H(K)} - (c_q t^{L(K)} + \cdots + c_g t^{g+\frac{1}{2}})
\end{aligned}
$$

where $L(K) + 1 \leq g \leq H(K) - 1$. Thus for Case 1 the theorem is true.

<u>Case 2:</u> Let $a_n < 0$. Again by using Theorem 3.2.3 and Lemma 3.2.7 we can write the Jones polynomial of K as

$$
\begin{aligned}
V(K) = &\begin{pmatrix}1 & 0\end{pmatrix} \dot{\bar{M}}(a_1)\ldots \bar{M}(a_{n-2})M(a_{n-1})\bar{M}(a_n)\begin{pmatrix}1\\ -t^{-\frac{1}{2}} - t^{\frac{1}{2}}\end{pmatrix}\\
=&(c_m t^{L(K)} + \cdots + c_p t^{H(K)-1}, c_q t^{q} + \cdots + c_k t^{H(K)-\frac{1}{2}})\begin{pmatrix}1\\ -t^{-\frac{1}{2}} - t^{\frac{1}{2}}\end{pmatrix}\\
=&c_m t^{L(K)} + \cdots + c_p t^{H(K)-1} - (c_q t^{q-\frac{1}{2}} + \cdots + c_k t^{H(K)})
\end{aligned}
$$

where $L(K) + 1 \leq q \leq H(K) - 1$. Hence for this case the theorem is true. $\square$

Later in this thesis it will be beneficial to relate H(K) and L(K) for knots made from rational tangles that are mirror images. This next lemma will do this for us.

Lemma 3.2.9. *If* $K = N(< a_1, a_2, \ldots, a_n >)$ *and* $\bar{K} = N(< -a_1, -a_2, \ldots, -a_n >)$ *then*

$$H(K) = -L(\bar{K}) \qquad \textit{and} \qquad L(K) = -H(\bar{K})$$

This lemma is proved by noticing that K and $\bar{K}$ are mirror images and applying Theorem 3.2.8 and Theorem 2.2.3.

CHAPTER 4

Montesinos Links

This chapter will focus on the class Montesinos links. The chapter will begin with the definition of Montesinos links and some of their properties then we move to the breadth of the Jones polynomial of this class of knots and links.

4.1. An introduction to Montesinos links

As always with a new concept, this section will start by defining a Montesinos link. This class of knots can be defined in terms of concepts already seen in the previous chapters.

DEFINITION 4.1.1. *Let $A_1, A_2, \ldots A_s$ be a finite number of rational tangles, where $s \geq 3$ and e be a rational tangle such that $e =< a >$, that is e is a rational tangle constructed by a single row of horizontal twists. Then the link L given by $L = N(A_1 + A_2 + \ldots A_s + e)$ is called a Montesinos link; see Figure 4.1*

Note that the restriction s $\geq$ 3 is artificial. For s = 1 or s = 2 we obtain the knots $N(< a_1, a_2, \ldots, a_s >)$ already discussed in Chapter 3 [**4**]. Earlier we have discussed equivalence in links in general and equivalence in links constructed from the numerator of rational tangles. Likewise there is a classification for Montesinos links. Recall that every rational tangle can be represented by a rational number $\frac{\beta}{\alpha}$. The next theorem classifies Montesinos links.

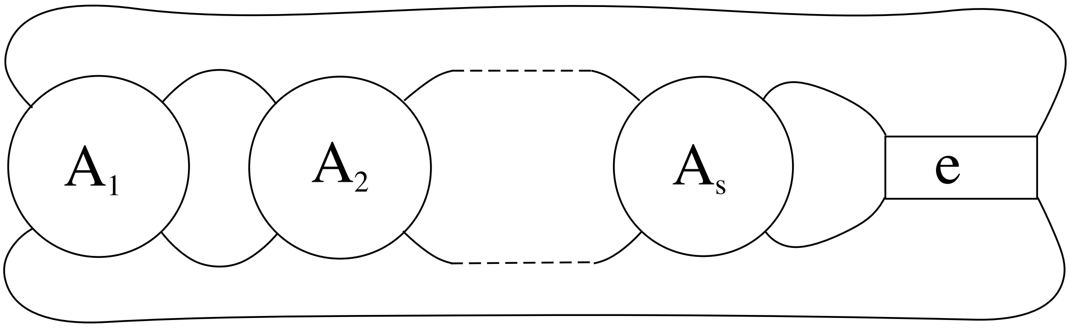

FIGURE 4.1. A Montesinos link.

THEOREM 4.1.2. [**1, 11**] *Let L be a Montesinos link* $L = N(\frac{\beta_1}{\alpha_1}+\frac{\beta_2}{\alpha_2}+\cdots+\frac{\beta_s}{\alpha_s}+e)$ *with* $s \geq 3$ *where* $e \in \mathbb{Z}$ *and* $\frac{\beta_i}{\alpha_i} \in \mathbb{Q}\backslash\mathbb{Z}$ *. Then L can be classified by the ordered set of fractions* $(\frac{\beta_1}{\alpha_1}\ mod 1, \frac{\beta_2}{\alpha_2}\ mod 1, \ldots, \frac{\beta_s}{\alpha_s}\ mod 1)$*, up to cyclic permutations and reversal of order, together with the rational number* $e_0 = e + \sum_{j=1}^{s} \frac{\beta_j}{\alpha_j}$*.*

Recall from Theorem 2.2.7 that a mutation done to a link will not change the Jones polynomial of that link. From this theorem the following lemma can be obtained:

LEMMA 4.1.3. *Let L be a Montesinos link* $L = N(A_1 + A_2 + \ldots A_s)$ *and let L' be a montesinos link* $N(A'_1 + A'_2 + \cdots + A'_s)$ *where the sequence of tangles* $A'_1, A'_2, \ldots, A'_s$ *is a permutation of* $A_1, A_2, \ldots A_s$*. Then V(L) = V(L').*

PROOF. From Theorem 2.2.7 it suffices to show that one can permute two rational tangles in a Montesinos knot by a finite series of mutations. Let L be a Montesinos link as described in Lemma 4.1.3 with two consecutive tangles A_i and A_{i+1} as shown in Figure 4.2.

Figure 4.2 shows a sequence of mutations that can be done to this link to permute the two tangles.

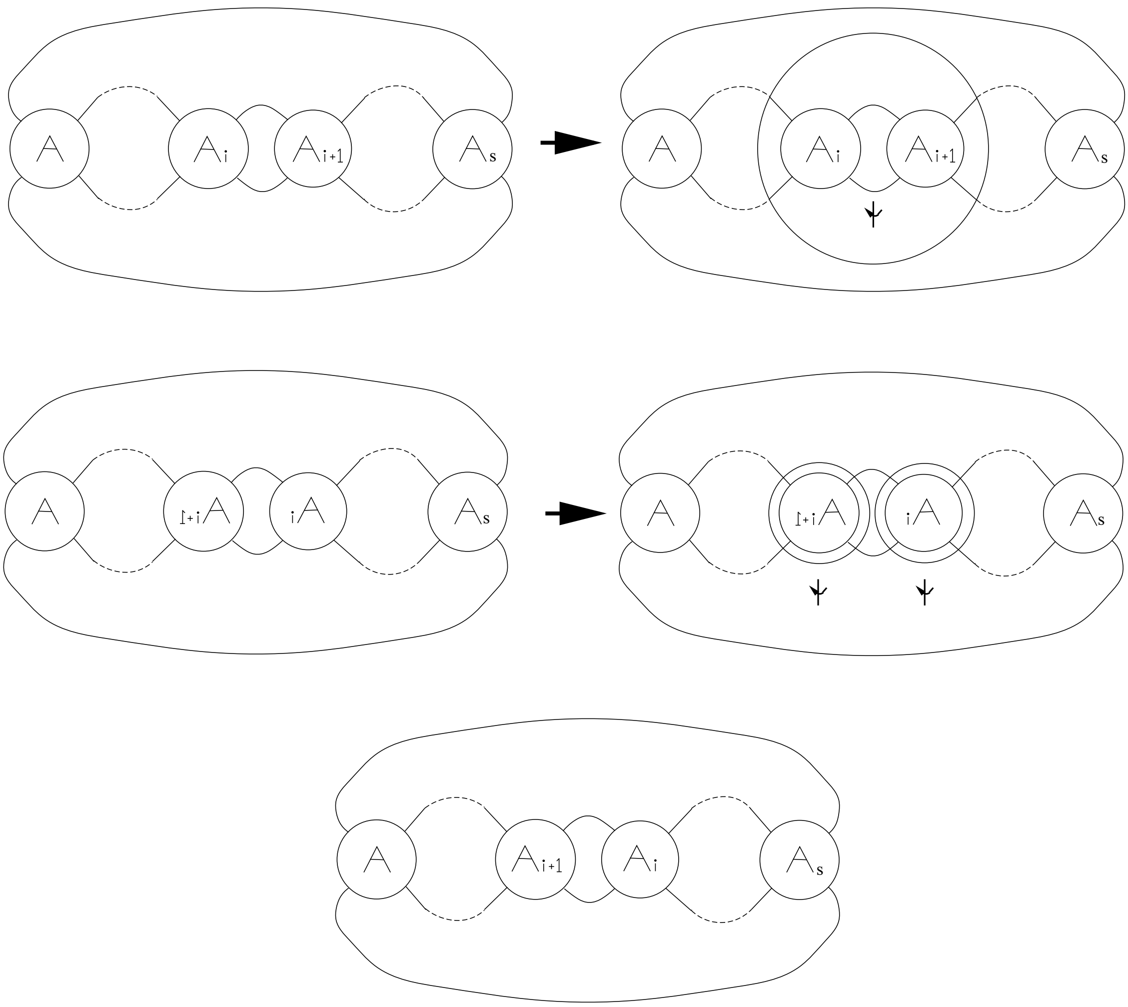

FIGURE 4.2.

□

The next lemma will give us some insight into the orientation of certain Montesinos links. These are the Montesinos links for which we can compute the breadth of the Jones polynomial. However many Montesinos links do not fall in this class; see Chapter 5.

LEMMA 4.1.4. *Let L be a Montesinos link given by $L = N(A_1 + A_2 + \cdots + A_s)$ where each each A_i is a rational tangle $A_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} >$ and each $a_{i,j}$ is even and not zero for $1 \leq j \leq n_i$. Then L can be oriented as shown in Figure 4.3.*

PROOF. By Theorem 3.1.7 each A_i has parity 0 if n_i is odd and parity ∞ otherwise. Thus an orientation like the one shown in Figure 4.3 can be assigned to L. □

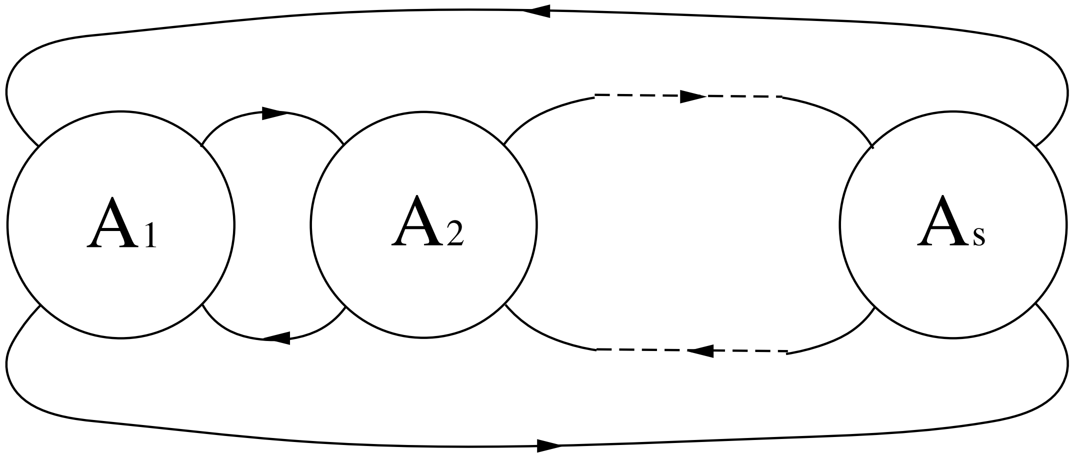

FIGURE 4.3.

4.2. The breadth of the Jones polynomial for Montesinos links

Now that Montesinos links have been defined we can now draw some conclusions about the greatest and least exponent of the Jones polynomial of Montesinos links. We start by proving a couple of lemmas that focus on Montesinos links of the form $N(A_1, A_2, \ldots, A_s)$ where each tangle is constructed from even, nonzero twists. After we have these two lemmas we will use them to prove two more important lemmas and a theorem that focus on links constructed from tangles that end in vertical twists.

For the next group of lemmas we will use the notation H(A) = H(N(A)) and L(A) = L(N(A)) from Definition 3.2.6 where A is a tangle.

LEMMA 4.2.1. *Let K be a Montesinos link given by* $K = N(A_1 + A_2 + \cdots + A_s)$ *and each each* A_i *is a rational tangle* $A_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} >$ *such that each* $a_{i,j}$ *is even and not zero. Further suppose that each* $a_{i,n_i} > 0$ *and K has the orientation as shown in Figure 4.3. Then the greatest exponent of V(K) is given by:*

$$maxexp(V(K)) = \sum_{i=1}^{s} H(A_i) + \frac{s-1}{2}$$

and the least exponent of V(L) is given by:

$$minexp(V(K)) = \sum_{i=1}^{s} L(A_i) + \frac{s-1}{2}$$

PROOF. We will prove this lemma by induction on the number of tangles s. First we must prove some preliminary facts about the maxexp and minexp of the Jones polynomial for tangles of the form A_i as stated above. These facts will be used throughout this proof. By Theorem 3.2.8 we know the following:

$$maxexp(V(N(A_i))) = H(A_i) \qquad and \qquad minexp(V(N(A_i))) = L(A_i) \tag{4.1}$$

Now suppose that $\bar{A}_i =< -a_{i,1}, -a_{i,2}, \ldots, -a_{i,n_i-1} >$. From Lemma 3.1.2 we know that $D(A_i) = N(\bar{A}_i)$, and thus

$$maxexp(V(D(A_i))) = H(\bar{A}_i) \qquad and \qquad minexp(V(D(A_i))) = L(\bar{A}_i)$$

Then by Definition 3.2.6 we find

$$\begin{aligned}
H(\bar{A}_i) &= \frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} -a_{i,j}(sign(-a_{i,j})+1) + \frac{1}{2}\sum_{j\equiv (n_i-1) \ mod2}^{n_i-1} -a_{i,j}(sign(-a_{i,j})-1) \\
&\quad - \frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i-1}sign(-a_{i,j})+1)(-sign(-a_{i,j-1})sign(-a_{i,j})+1) \\
&\quad - \frac{1}{2}sign(-a_{i,n_i-1})((n_i-1) \ mod2) \\
&= \frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})-1) + \frac{1}{2}\sum_{j\equiv (n_i-1) \ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1) \\
&\quad - \frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1) \\
&\quad + \frac{1}{2}sign(a_{i,n_i-1})((n_i-1) \ mod2)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
L(\bar{A}_i) &= -\frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} -a_{i,j}(sign(-a_{i,j})-1) - \frac{1}{2}\sum_{j\equiv (n_i-1) \ mod2}^{n_i-1} -a_{i,j}(sign(-a_{i,j})+1) \\
&\quad - \frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i-1}sign(-a_{i,j})-1)(-sign(-a_{i,j-1})sign(-a_{i,j})+1) \\
&\quad - \frac{1}{2}sign(-a_{i,n_i-1})((n_i-1) \ mod2) \\
&= -\frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})+1) - \frac{1}{2}\sum_{j\equiv (n_i-1) \ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})-1) \\
&\quad - \frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j})-1)(-sign(a_{i,j-1})sign(a_{i,j})+1) \\
&\quad + \frac{1}{2}sign(a_{i,n_i-1})((n_i-1) \ mod2)
\end{aligned} \tag{4.3}$$

Then for $a_{i,n_i-1} > 0$ we find the following:

$$\begin{aligned}
\frac{1}{2}a_{i,n_i}(sign(a_{i,n_i})-1) &= 0 \\
-\frac{1}{2}a_{i,n_i}(sign(a_{i,n_i})+1) &= -a_{i,n_i} \\
\frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2) &= \begin{cases} \frac{1}{2} & \text{if } n_i \text{ is even} \\ 0 & \text{if } n_i \text{ is odd} \end{cases} \\
-\frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) &= \begin{cases} 0 & \text{if } n_i \text{ is even} \\ -\frac{1}{2} & \text{if } n_i \text{ is odd} \end{cases} \\
-\frac{1}{4}((-1)^{n_i}sign(a_{i,n_i})-1)(-sign(a_{i,n_i-1})sign(i,a_{i,n_i})+1) &= 0 \\
-\frac{1}{4}((-1)^{n_i}sign(a_{i,n_i})+1)(-sign(a_{i,n_i-1})sign(i,a_{i,n_i})+1) &= 0
\end{aligned}$$

Thus we can use Equations 4.2 and 4.3 to find the following for $a_{i,n_i-1} > 0$:

$$\begin{aligned}
H(\bar{A}_i) - \frac{1}{2} =& \frac{1}{2}\sum_{j\equiv n_i\ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})-1) + \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1) \\
&- \frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1) \\
&+ \frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2) - \frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2) \\
&- \frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) \\
=& \frac{1}{2}\sum_{j\equiv n_i\ mod2}^{n_i} a_{i,j}(sign(a_{i,j})-1) + \frac{1}{2}\sum_{j\equiv(n_i+1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1) \\
&- \frac{1}{4}\sum_{j=2}^{n_i}((-1)^{n_i}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1) \\
&- \frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) \\
=& H(A_i). \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
L(\bar{A}_i) - a_{i,n_i} - \frac{1}{2} =& - \frac{1}{2} \sum_{j \equiv n_i \ mod 2}^{n_i - 2} a_{i,j}(sign(a_{i,j}) + 1) - \frac{1}{2} \sum_{j \equiv (n_i - 1) \ mod 2}^{n_i - 1} a_{i,j}(sign(a_{i,j}) - 1) \\
&- \frac{1}{4} \sum_{j=2}^{n_i - 1} ((-1)^{n_i} sign(a_{i,j}) - 1)(-sign(a_{i,j-1}) sign(a_{i,j}) + 1) \\
&+ \frac{1}{2} sign(a_{i,n_i - 1})((n_i - 1) \ mod 2) - \frac{1}{2} sign(a_{i,n_i - 1})((n_i - 1) \ mod 2) \\
&- \frac{1}{2} sign(a_{i,n_i})(n_i \ mod 2) - \frac{1}{2} a_{i,n_i}(sign(a_{i,n_i}) + 1) \\
=& - \frac{1}{2} \sum_{j \equiv n_i \ mod 2}^{n_i} a_{i,j}(sign(a_{i,j}) + 1) - \frac{1}{2} \sum_{j \equiv (n_i + 1) \ mod 2}^{n_i - 1} a_{i,j}(sign(a_{i,j}) - 1) \\
&- \frac{1}{4} \sum_{j=2}^{n_i} ((-1)^{n_i} sign(a_{i,j}) - 1)(-sign(a_{i,j-1}) sign(a_{i,j}) + 1) \\
&- \frac{1}{2} sign(a_{i,n_i})(n_i \ mod 2) \\
=& L(A_i) \qquad (4.5)
\end{aligned}$$

Also for $a_{i,n_i - 1} < 0$ we find the following:

$$\begin{aligned}
\frac{1}{2} a_{i,n_i}(sign(a_{i,n_i}) - 1) &= 0 \\
-\frac{1}{2} a_{i,n_i}(sign(a_{i,n_i}) + 1) &= -a_{i,n_i} \\
\frac{1}{2} sign(a_{i,n_i - 1})((n_i - 1) \ mod 2) &= \begin{cases} -\frac{1}{2} & \text{if } n_i \text{ is even} \\ 0 & \text{if } n_i \text{ is odd} \end{cases} \\
-\frac{1}{2} sign(a_{i,n_i})(n_i \ mod 2) &= \begin{cases} 0 & \text{if } n_i \text{ is even} \\ -\frac{1}{2} & \text{if } n_i \text{ is odd} \end{cases} \\
-\frac{1}{4}((-1)^{n_i} sign(a_{i,n_i}) + 1)(-sign(a_{i,n_i - 1}) sign(i, a_{i,n_i}) + 1) &= \begin{cases} -1 & \text{if } n_i \text{ is even} \\ 0 & \text{if } n_i \text{ is odd} \end{cases} \\
-\frac{1}{4}((-1)^{n_i} sign(a_{i,n_i}) - 1)(-sign(a_{i,n_i - 1}) sign(i, a_{i,n_i}) + 1) &= \begin{cases} 0 & \text{if } n_i \text{ is even} \\ 1 & \text{if } n_i \text{ is odd} \end{cases}
\end{aligned}$$

Similar to Equation 4.4 and 4.5 and we can use Equations 4.2 and 4.3 to find the following for $a_{i,n_i-1} < 0$.

$$H(\bar{A}_i) - \frac{1}{2} = H(A_i). \tag{4.6}$$

and

$$L(\bar{A}_i) - a_{i,n_i} + \frac{1}{2} = L(A_i). \tag{4.7}$$

Therefore we can use Equations 4.4, 4.5, 4.6, 4.7 and the fact that $a_{i,n_i} \geq 2$ to find

$$maxexp(V(D(A_i))) = H(A_i) + \frac{1}{2} \tag{4.8}$$

$$minexp(V(D(A_i))) \geq L(A_i) + \frac{3}{2} \tag{4.9}$$

Now that we have Equations 4.1, 4.8 and 4.9 we need to prove the initial cases for our induction s = 1 and s = 2. If s = 1 we are done because of Theorem 3.2.8. Let s = 2 then by Corollary 3.1.5 we find the greatest exponent of V(K) is

$$maxexp(V(K)) \leq max \begin{cases} maxexp(V(N(A_1))V(N(A_2))) - \frac{1}{2} = H(A_1) + H(A_2) - \frac{1}{2} \\ maxexp(V(D(A_1))V(D(A_2))) - \frac{1}{2} = H(A_1) + H(A_2) + \frac{1}{2} \\ maxexp(V(N(A_1))V(D(A_2))) - 1 = H(A_1) + H(A_2) - \frac{1}{2} \\ maxexp(V(D(A_1))V(N(A_2))) - 1 = H(A_1) + H(A_2) - \frac{1}{2} \end{cases}$$
$$= H(A_1) + H(A_2) + \frac{1}{2}$$

and the least exponent of V(K) is

$$minexp(V(K)) \geq min \begin{cases} minexp(V(N(A_1))V(N(A_2))) + \frac{1}{2} = L(A_1) + L(A_2) + \frac{1}{2} \\ minexp(V(D(A_1))V(D(A_2))) + \frac{1}{2} \geq L(A_1) + L(A_2) + \frac{7}{2} \\ minexp(V(N(A_1))V(D(A_2))) + 1 \geq L(A_1) + L(A_2) + \frac{5}{2} \\ minexp(V(D(A_1))V(N(A_2))) + 1 \geq L(A_1) + L(A_2) + \frac{5}{2} \end{cases}$$
$$= L(A_1) + L(A_2) + \frac{1}{2}$$

We must have equality with both of these because there can be no cancelations of terms involving the greatest and least exponent (Because K is alternating we know that the breadth of V(K) is Cr(K) and Cr(K) = $H(A_1) + H(A_2) - L(A_1) - L(A_2)$). Therefore when s = 2 the lemma is true.

Now assume that the lemma is true for s = k - 1. We must show that it is true for s = k. Let A = $(A_1 + A_2 + \cdots + A_{k-1})$, then we can observe that L = $N(A + A_k)$ and A is a tangle. By our inductive assumption we know that

$$maxexp(V(N(A))) = \sum_{i=1}^{k-1} H(A_i) + \frac{k-2}{2}$$
$$minexp(V(N(A))) = \sum_{i=1}^{k-1} L(A_i) + \frac{k-2}{2}$$

and

$$maxexp(V(N(A_k))) = H(A_k) \qquad and \qquad minexp(V(N(A_k))) = L(A_k).$$

Next we can note that $D(A) = (D(A_1)\#D(A_2)\#\ldots\#D(A_{k-1}))$ and thus by Theorem 2.2.5 we know V(D(A)) = $(V(D(A_1))V(D(A_2))\ldots V(D(A_{k-1})))$. So now by Equations 4.8 and 4.9 we will find the following where $\bar{A_i} =< -a_{i,1}, -a_{i,2}, \ldots, -a_{i,n_i-1} >$:

$$maxexp(V(D(A))) = \sum_{i=1}^{k-1} H(A_i) + \frac{k-1}{2}$$
$$maxexp(V(D(A_k))) = H(A_k) + \frac{1}{2}$$
$$minexp(V(D(A))) = \sum_{i=1}^{k-1} L(\bar{A_i}) \geq \sum_{i=1}^{k-1} L(A_i) + \frac{3}{2}(k-1)$$
$$minexp(V(D(A_k))) = L(\bar{A_k}) \geq L(A_k) + \frac{3}{2}$$

Therefore by Corollary 3.1.5 we have

$$maxexp(V(K)) \leq max \begin{cases} maxexp(V(N(A))V(N(A_k))) - \frac{1}{2} = \sum_{i=1}^{k} H(A_i) + \frac{k-3}{2} \\ maxexp(V(D(A))V(D(A_k))) - \frac{1}{2} = \sum_{i=1}^{k} H(A_i) + \frac{k-1}{2} \\ maxexp(V(N(A))V(D(A_k))) - 1 = \sum_{i=1}^{k} H(A_i) + \frac{k-3}{2} \\ maxexp(V(D(A))V(N(A_k))) - 1 = \sum_{i=1}^{k} H(A_i) + \frac{k-3}{2} \end{cases}$$
$$= \sum_{i=1}^{k} H(A_i) + \frac{k-1}{2}$$

and

$$minexp(V(K)) \geq min \begin{cases} minexp(V(N(A))V(N(A_k))) + \frac{1}{2} = \sum_{i=1}^{k} L(A_i) + \frac{k-1}{2} \\ minexp(V(D(A))V(D(A_k))) + \frac{1}{2} \geq \sum_{i=1}^{k} L(A_i) + \frac{3k+1}{2} \\ minexp(V(N(A))V(D(A_k))) + 1 \geq \sum_{i=1}^{k} L(A_i) + \frac{k+3}{2} \\ minexp(V(D(A))V(N(A_k))) + 1 \geq \sum_{i=1}^{k} L(A_i) + \frac{3k+2}{2} \end{cases}$$
$$= \sum_{i=1}^{k} L(A_i) + \frac{k-1}{2}$$

Again we must have equality with both of these because there are no cancelations of terms involving the greatest and least exponent using the same crossing number argument as before. Thus we find our lemma is true. □

To demonstrate how this first lemma works let us consider the knot K $= N(\frac{3}{2} + \frac{54}{25} + \frac{532}{87})$. Note that we can find the Jones polynomial of K to be

$$V(K) = \frac{1}{t^9} - \frac{3}{t^8} + \frac{8}{t^7} - \frac{16}{t^6} + \frac{26}{t^5} - \frac{38}{t^4} + \frac{49}{t^3} - \frac{58}{t^2} + \frac{65}{t} - 68$$
$$+ 67t - 62t^2 + 55t^3 - 45t^4 + 34t^5 - 23t^6 + 14t^7 - 7t^8 + 3t^9 - t^{10}$$

By using our formulas we find the following:

$$H(\frac{3}{2}) + H(\frac{54}{25}) + H(\frac{532}{87}) + \frac{3-1}{2} = 2 + \frac{3}{2} + \frac{11}{2} + 1 = 10$$

and

$$L(\frac{3}{2}) + L(\frac{54}{25}) + L(\frac{532}{87}) + \frac{3-1}{2} = -2 - \frac{13}{2} - \frac{3}{2} + 1 = -9$$

So for K our formulas accurately give the maxexp(V(K)) and the minexp(V(K)).

Now that we have Lemma 4.2.1 we will use it to prove the following lemma that deals with Montesinos knots constructed from tangles that end in negative twists.

Lemma 4.2.2. *Let K be a Montesinos link given by* $K = N(A_1 + A_2 + \cdots + A_s)$ *and each* A_i *is a rational tangle* $A_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} >$ *such that each* $a_{i,j}$ *is even and not zero. Further suppose that each* $a_{i,n_i} < 0$ *and K has the orientation as shown in Figure 4.3. Then the greatest exponent of V(K) is given by:*

$$maxexp(V(K)) = \sum_{i=1}^{s} H(A_i) - \frac{s-1}{2}$$

and the least exponent of V(L) is given by:

$$minexp(V(K)) = \sum_{i=1}^{s} L(A_i) - \frac{s-1}{2}$$

Proof. Let K be defined as above in Lemma 4.2.2. Also let $A'_i =< -a_{i,1}, -a_{i,2}, \ldots, -a_{i,n_i} >$ for each i = 1, 2 . . . , s and let K' = $N(A'_1, A'_2, \ldots, A'_s)$. By Lemma 3.2.9 we find that

$$H(A_i) = -L(A'_i) \qquad and \qquad L(A_i) = -H(A'_i). \tag{4.10}$$

Now we can note that $-a_{i,n_i} > 0$ for i = 1, 2 . . . , s, so by Lemma 4.2.1 we know that

$$maxexp(V(K')) = \sum_{i=1}^{s} H(A'_i) + \frac{s-1}{2}$$

and

$$minexp(V(K')) = \sum_{i=1}^{s} L(A'_i) + \frac{s-1}{2}.$$

Next we can note that K and K' are mirror images and thus from Theorem 2.2.3 we have $V(K)(t) = V(K')(t^{-1})$. By combining this with Equation 4.10 we have the following:

$$maxexp(V(K)) = -minexp(V(K')) = -(\sum_{i=1}^{s} L(A_i') + \frac{s-1}{2})$$
$$= \sum_{i=1}^{s} H(A_i) - \frac{s-1}{2}$$

and

$$minexp(V(K)) = -maxexp(V(K')) = -(\sum_{i=1}^{s} H(A_i') + \frac{s-1}{2})$$
$$= \sum_{i=1}^{s} L(A_i) - \frac{s-1}{2}.$$

□

The next lemma will concern Montesinos links constructed from tangles that end in vertical twists. We can use the same notation as we have throughout and make the last horizontal twist zero to achieve this. For this lemma we will use the fact that $N < a_1, a_2, \dots, a_n, 0 >= N < a_1, a_2, \dots, a_{n-1} >$ and $D < a_1, a_2, \dots, a_n, 0 >=$ $N < -a_1, -a_2, \dots, -a_n >$. An example is illustrated in Figure 4.4.

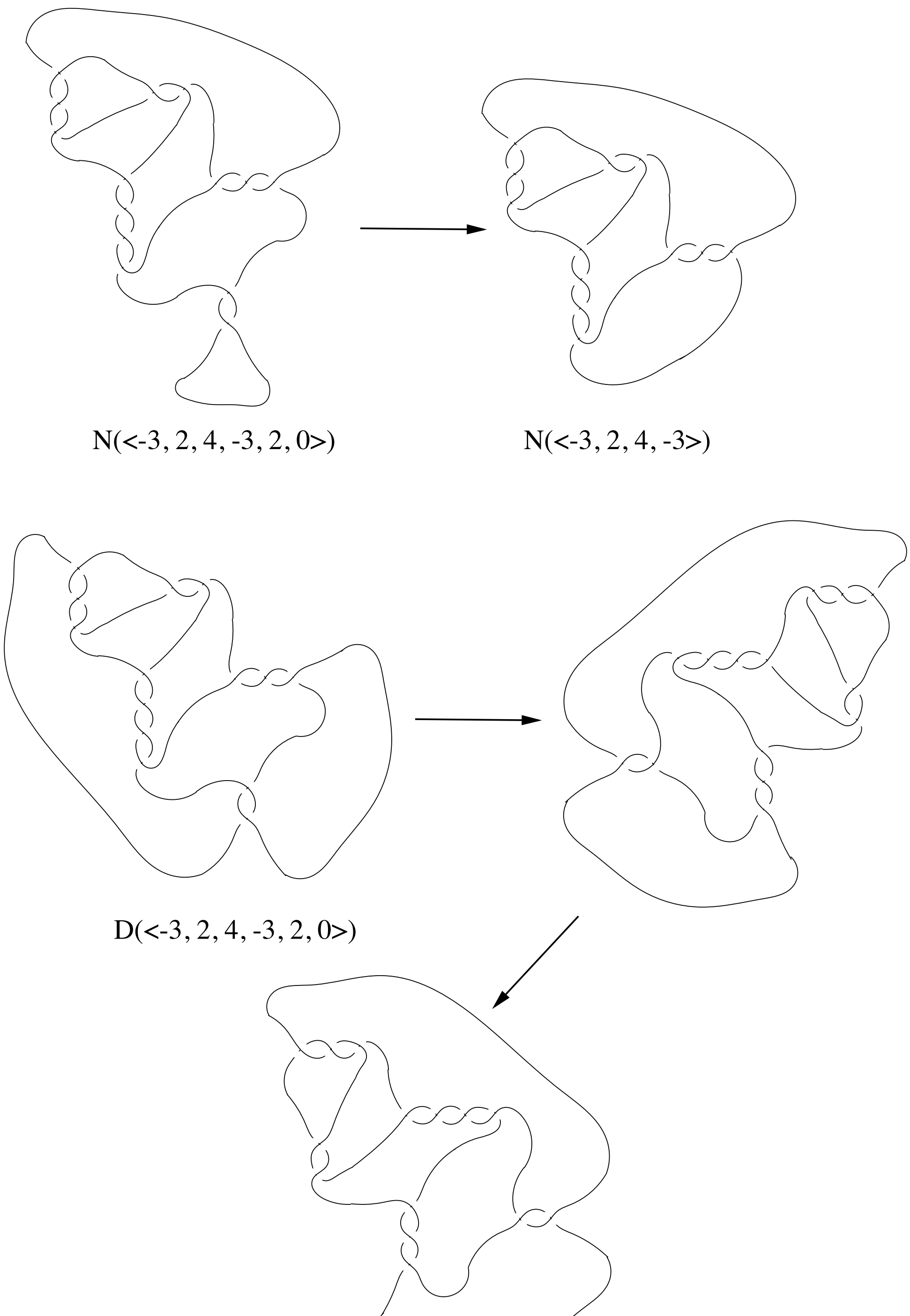

FIGURE 4.4. A figure to illustrate the relation between rational tangles ending in vertical twists and those ending in horizontal twists

LEMMA 4.2.3. *Let* $K = N(A_1 + A_2 + \cdots + A_s)$ *where* $s \geq 2$ *and each* A_i *is a tangle constructed as* $A_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}, 0 >$ *and each* $a_{i,j}$ *is even and nonzero. Furthermore suppose that each* a_{i,n_i} *is positive and* K *has the orientation as shown in Figure 4.3. Then*

$$maxexp(V(K)) = \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{1}{2}$$

and

$$minexp(V(K)) = \sum_{i=1}^{s} -H(\hat{A}_i) - \frac{1}{2}$$

Where $\hat{A}_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} >$

PROOF. To prove this lemma we will use induction on the value of s, similar to the proof for Lemma 4.2.1. Before we get to the induction portion of the proof we will first need to prove some preliminary results for rational tangles. Let A_i be a rational tangle as described in Lemma 4.2.3. Then from the facts above we know $N(A_i) = N(< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i-1} >)$ and $D(A_i) = N(< -a_{i,1}, -a_{i,2}, \ldots, -a_{i,n_i} >)$. Also let $\hat{A}_i$ be defined as in Lemma 4.2.3, then we can note that $D(A_i)$ and $N(\hat{A}_i)$ are mirror images. By Theorem 3.2.8 we know that

$$maxexp(V(N(\hat{A}_i))) = H(\hat{A}_i) \qquad and \qquad minexp(V(N(\hat{A}_i))) = L(\hat{A}_i).$$

So now we can use Theorem 2.2.3 to find that

$$maxexp(V(D(A_i))) = -L(\hat{A}_i) \qquad and \qquad minexp(V(D(A_i))) = -H(\hat{A}_i). \quad (4.11)$$

Now we can use Definition 3.2.6 to find the following:

$$
\begin{aligned}
maxexp(V(N(A_i))) = H(N(< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i-1} >))\\
=&\frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})+1) + \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})-1)\\
&-\frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i-1}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1)\\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2)\\
=&\frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})+1) + \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})-1)\\
&+\frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j})-1)(-sign(a_{i,j-1})sign(a_{i,j})+1)\\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2)
\end{aligned}
$$

and

$$
\begin{aligned}
minexp(V(N(A_i))) = L(N(< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i-1} >))\\
=&-\frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})-1) - \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1)\\
&-\frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i-1}sign(a_{i,j})-1)(-sign(a_{i,j-1})sign(a_{i,j})+1)\\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2)\\
=&-\frac{1}{2}\sum_{j\equiv n_i \ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})-1) - \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1)\\
&+\frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1)\\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2)
\end{aligned}
$$

Next we can find the following for $a_{i,n_i-1} > 0$:

$$\begin{aligned}
&\frac{1}{2}a_{i,n_i}(sign(a_{i,n_i}) - 1) = 0\\
&\frac{1}{2}a_{i,n_i}(sign(a_{i,n_i}) + 1) = a_{i,n_i}\\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i - 1)\ mod2) = \begin{cases} -\frac{1}{2} & \text{if } n_i \text{ is even} \\ 0 & \text{if } n_i \text{ is odd} \end{cases}\\
&\frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) = \begin{cases} 0 & \text{if } n_i \text{ is even} \\ \frac{1}{2} & \text{if } n_i \text{ is odd} \end{cases}\\
&\frac{1}{4}((-1)^{n_i}sign(a_{i,n_i}) - 1)(-sign(a_{i,n_i-1})sign(a_{i,n_i}) + 1) = 0\\
&\frac{1}{4}((-1)^{n_i}sign(a_{i,n_i}) + 1)(-sign(a_{i,n_i-1})sign(a_{i,n_i}) + 1) = 0
\end{aligned}$$

So then we can find

$$\begin{aligned}
&maxexp(V(N(A_i))) + a_{i,n_i} + \frac{1}{2}\\
=&\frac{1}{2}\sum_{j\equiv n_i\ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j}) + 1) + \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j}) - 1)\\
&+\frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j}) - 1)(-sign(a_{i,j-1})sign(a_{i,j}) + 1)\\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i - 1)\ mod2) + \frac{1}{2}sign(a_{i,n_i-1})((n_i - 1)\ mod2)\\
&+\frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) + \frac{1}{2}a_{i,n_i}(sign(a_{i,n_i}) + 1)\\
=&\frac{1}{2}\sum_{j\equiv n_i\ mod2}^{n_i} a_{i,j}(sign(a_{i,j}) + 1) + \frac{1}{2}\sum_{j\equiv(n_i+1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j}) - 1)\\
&+\frac{1}{4}\sum_{j=2}^{n_i}((-1)^{n_i}sign(a_{i,j}) - 1)(-sign(a_{i,j-1})sign(a_{i,j}) + 1)\\
&+\frac{1}{2}sign(a_{i,n_i})(n_i\ mod2)\\
=&-L(\hat{A}_i)
\end{aligned} \tag{4.12}$$

and

$$
\begin{aligned}
&minexp(V(N(A_i))) + \frac{1}{2} \\
&= -\frac{1}{2}\sum_{j\equiv n_i\ mod2}^{n_i-2} a_{i,j}(sign(a_{i,j})-1) - \frac{1}{2}\sum_{j\equiv(n_i-1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1) \\
&\quad + \frac{1}{4}\sum_{j=2}^{n_i-1}((-1)^{n_i}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1) \\
&\quad - \frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2) + \frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2) \\
&\quad + \frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) + \frac{1}{2}a_{i,n_i}(sign(a_{i,n_i})-1) \\
&= -\frac{1}{2}\sum_{j\equiv n_i\ mod2}^{n_i} a_{i,j}(sign(a_{i,j})-1) - \frac{1}{2}\sum_{j\equiv(n_i+1)\ mod2}^{n_i-1} a_{i,j}(sign(a_{i,j})+1) \\
&\quad + \frac{1}{4}\sum_{j=2}^{n_i}((-1)^{n_i}sign(a_{i,j})+1)(-sign(a_{i,j-1})sign(a_{i,j})+1) \\
&\quad + \frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) \\
&= -H(\hat{A}_i)
\end{aligned} \tag{4.13}
$$

Now that we have this result for $a_{i,n_i-1} > 0$ let us consider $a_{i,n_i-1} < 0$.

$$
\begin{aligned}
&\frac{1}{2}a_{i,n_i}(sign(a_{i,n_i})-1) = 0 \\
&\frac{1}{2}a_{i,n_i}(sign(a_{i,n_i})+1) = a_{i,n_i} \\
&-\frac{1}{2}sign(a_{i,n_i-1})((n_i-1)\ mod2) = \begin{cases} \frac{1}{2} & \text{if } n_i \text{ is even} \\ 0 & \text{if } n_i \text{ is odd} \end{cases} \\
&\frac{1}{2}sign(a_{i,n_i})(n_i\ mod2) = \begin{cases} 0 & \text{if } n_i \text{ is even} \\ \frac{1}{2} & \text{if } n_i \text{ is odd} \end{cases} \\
&\frac{1}{4}((-1)^{n_i}sign(a_{i,n_i})-1)(-sign(a_{i,n_i-1})sign(a_{i,n_i})+1) = \begin{cases} 0 & \text{if } n_i \text{ is even} \\ -1 & \text{if } n_i \text{ is odd} \end{cases} \\
&\frac{1}{4}((-1)^{n_i}sign(a_{i,n_i})+1)(-sign(a_{i,n_i-1})sign(a_{i,n_i})+1) = \begin{cases} 1 & \text{if } n_i \text{ is even} \\ 0 & \text{if } n_i \text{ is odd} \end{cases}
\end{aligned}
$$

Thus by a similar argument as the one shown above we find that

$$maxexp(V(N(A_i))) + a_{i,n_i} - \frac{1}{2} = -L(\hat{A}_i)$$
$$minexp(V(N(A_i))) + \frac{1}{2} = -H(\hat{A}_i)$$

Therefore for any tangle A_i we have the following:

$$maxexp(V(N(A_i))) \leq -L(\hat{A}_i) - \frac{3}{2} \tag{4.14}$$
$$minexp(V(N(A_i))) = -H(\hat{A}_i) - \frac{1}{2} \tag{4.15}$$

Now that we have 4.11, 4.14 and 4.15 we can start to prove the lemma by induction. If s = 2 then K = $N(A_1 + A_2)$ where A_1 and A_2 are a tangles as described in Lemma 4.2.3. Then from 4.14 and 4.15 we have

$$maxexp(V(N(A_1))) \leq -L(\hat{A}_1) - \frac{3}{2}$$
$$minexp(V(N(A_1))) = -H(\hat{A}_1) - \frac{1}{2}$$
$$maxexp(V(N(A_2))) \leq -L(\hat{A}_2) - \frac{3}{2}$$
$$minexp(V(N(A_2))) = -H(\hat{A}_2) - \frac{1}{2}$$

and from 4.11 we have

$$maxexp(V(D(A_1))) = -L(\hat{A}_1)$$
$$minexp(V(D(A_1))) = -H(\hat{A}_1)$$
$$maxexp(V(D(A_2))) = -L(\hat{A}_2)$$
$$minexp(V(D(A_2))) = -H(\hat{A}_2)$$

Then we will use Corollary 3.1.5 to find the maxexp and minexp of V(K).

$$maxexp(V(K)) \leq max \begin{cases} maxexp(V(N(A_1))V(N(A_2))) - \frac{1}{2} \leq -L(\hat{A}_1) - L(\hat{A}_2) - \frac{5}{2} \\ maxexp(V(D(A_1))V(D(A_2))) - \frac{1}{2} = -L(\hat{A}_1) - L(\hat{A}_2) - \frac{1}{2} \\ maxexp(V(N(A_1))V(D(A_2))) - 1 \leq -L(\hat{A}_1) - L(\hat{A}_2) - \frac{5}{2} \\ maxexp(V(D(A_1))V(N(A_2))) - 1 \leq -L(\hat{A}_1) - L(\hat{A}_2) - \frac{5}{2} \end{cases}$$
$$= -L(\hat{A}_1) - L(\hat{A}_2) - \frac{1}{2}$$

and

$$minexp(V(K)) \geq min \begin{cases} minexp(V(N(A_1))V(N(A_2))) + \frac{1}{2} = -H(\hat{A}_1) - H(\hat{A}_2) - \frac{1}{2} \\ minexp(V(D(A_1))V(D(A_2))) + \frac{1}{2} = -H(\hat{A}_1) - H(\hat{A}_2) + \frac{1}{2} \\ minexp(V(N(A_1))V(D(A_2))) + 1 = -H(\hat{A}_1) - H(\hat{A}_2) + \frac{1}{2} \\ minexp(V(D(A_1))V(N(A_2))) + 1 = -H(\hat{A}_1) - H(\hat{A}_2) + \frac{1}{2} \end{cases}$$
$$= -H(\hat{A}_1) - H(\hat{A}_2) - \frac{1}{2}$$

We must have equality with both of these because of the crossing number argument used before. Thus for s = 2 the Lemma is true. Now assume that the lemma is true for s = k - 1, we need to prove it is true for s = k. Consider K = $N(A_1 + A_2 + \cdots + A_k)$. Let A = $(A_1 + A_2 + \cdots + A_{k-1})$. Using the inductive hypothesis as well as 4.14 and 4.15 we find

$$maxexp(V(N(A))) = \sum_{i=1}^{k-1} -L(\hat{A}_i) - \frac{1}{2}$$
$$minexp(V(N(A))) = \sum_{i=1}^{k-1} -H(\hat{A}_i) - \frac{1}{2}$$
$$maxexp(V(N(A_k))) \leq -L(\hat{A}_k) - \frac{3}{2}$$
$$minexp(V(N(A_k))) = -H(\hat{A}_k) - \frac{1}{2}$$

Next we can note that $D(A) = (D(A_1)\#D(A_2)\#\ldots\#D(A_{k-1}))$ and thus by Theorem 2.2.5 we know V(D(A)) $= (V(D(A_1))V(D(A_2))\ldots V(D(A_{k-1})))$. Then by using 4.11 we find

$$\begin{aligned} maxexp(V(D(A))) &= \sum_{i=1}^{k-1} -L(\hat{A}_i) \\ minexp(V(D(A))) &= \sum_{i=1}^{k-1} -H(\hat{A}_i) \\ maxexp(V(D(A_k))) &= -L(\hat{A}_k) \\ minexp(V(D(A_k))) &= -H(\hat{A}_k) \end{aligned}$$

So again we can use Corollary 3.1.5 to find the maxexp and minexp of V(K).

$$maxexp(V(K)) \leq max \begin{cases} maxexp(V(N(A))V(N(A_k))) - \frac{1}{2} \leq \sum_{i=1}^{k} -L(\hat{A}_i) - \frac{5}{2} \\ maxexp(V(D(A))V(D(A_k))) - \frac{1}{2} = \sum_{i=1}^{k} -L(\hat{A}_i) - \frac{1}{2} \\ maxexp(V(N(A))V(D(A_k))) - 1 = \sum_{i=1}^{k} -L(\hat{A}_i) - \frac{3}{2} \\ maxexp(V(D(A))V(N(A_k))) - 1 \leq \sum_{i=1}^{k} -L(\hat{A}_i) - \frac{5}{2} \end{cases}$$

$$= \sum_{i=1}^{k} -L(\hat{A}_i) - \frac{1}{2}$$

and

$$minexp(V(K)) \geq min \begin{cases} minexp(V(N(A))V(N(A_k))) + \frac{1}{2} = \sum_{i=1}^{k} -H(\hat{A}_i) - \frac{1}{2} \\ minexp(V(D(A))V(D(A_k))) + \frac{1}{2} = \sum_{i=1}^{k} -H(\hat{A}_i) + \frac{1}{2} \\ minexp(V(N(A))V(D(A_k))) + 1 = \sum_{i=1}^{k} -H(\hat{A}_i) + \frac{1}{2} \\ minexp(V(D(A))V(N(A_k))) + 1 = \sum_{i=1}^{k} -H(\hat{A}_i) + \frac{1}{2} \end{cases}$$

$$= \sum_{i=1}^{k} -H(\hat{A}_i) - \frac{1}{2}$$

Again we have equality with both of these because of the crossing number argument. Thus our lemma is true. $\square$

LEMMA 4.2.4. *Let* $K = N(A_1 + A_2 + \cdots + A_s)$ *where* $s \geq 2$ *and each* A_i *is a tangle constructed as* $A_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}, 0 >$ *and each* $a_{i,j}$ *is even and nonzero. Furthermore suppose that each* a_{i,n_i} *is negative and* K *has the orientation as shown in Figure 4.3. Then*

$$maxexp(V(K)) = \sum_{i=1}^{s} -L(\hat{A}_i) + \frac{1}{2}$$

and

$$minexp(V(K)) = \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{1}{2}$$

Where $\hat{A}_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} >$.

The proof to this lemma is similar to the proof of Lemma 4.2.2 so we leave it to the curious reader. Now that we have the previous lemmas we will use them to prove the next theorem which will give a general formula for the maxexp and minexp of V(K) where K is a Montesinos knot constructed from rational tangles that end in vertical twists. It is important to discuss a Montesinos link of this form because a nonalternating diagram of a Montesinos link that contains a tangle ending in a horizontal twist cannot be minimal [**7**]. The Lemmas 4.2.1, 4.2.2, 4.2.3, and 4.2.4 all deal with alternating links. The next theorem is for one type of non-alternating Montesinos links.

THEOREM 4.2.5. *Let* $K = N(A_1 + A_2 + \cdots + A_s)$ *where* $s \geq 4$ *and each* A_i *is a tangle constructed as* $A_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}, 0 >$ *and each* $a_{i,j}$ *is even and nonzero. Furthermore suppose u of these rational tangles end in* $a_{i,n_i} > 0$ *and v = s - u of these rational tangles end in* $a_{i,n_i} < 0$ *and* $1 < u < s - 1$. *Also let K have the orientation* *as shown in Figure 4.3. Then*

$$maxexp(V(K)) = \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{1}{2}$$

and

$$minexp(V(K)) = \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{1}{2}$$

where $\hat{A}_i =< a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} >$.

Proof. Let K be a link as described above. Then by Lemma 4.1.3 we can assume that for A_i where $i = 1, 2, \ldots, u$ that $a_{i,n_i} > 0$ and for A_i where $i = u+1, u+2, \ldots, s$ that $a_{i,n_i} < 0$ without changing V(K). Let A = $(A_1 + A_2 + \cdots + A_u)$ and B = $(A_{u+1} + A_{u+2} + \cdots + A_s)$. Then from Lemma 4.2.3 and Lemma 4.2.4 we find the following:

$$\begin{aligned}
maxexp(V(N(A))) &= \sum_{i=1}^{u} -L(\hat{A}_i) - \frac{1}{2}\\
maxexp(V(N(B))) &= \sum_{i=u+1}^{s} -L(\hat{A}_i) + \frac{1}{2}\\
minexp(V(N(A))) &= \sum_{i=1}^{u} -H(\hat{A}_i) - \frac{1}{2}\\
minexp(V(N(B))) &= \sum_{i=u+1}^{s} -H(\hat{A}_i) + \frac{1}{2}
\end{aligned}$$

Next we can note that $D(A) = (D(A_1)\#D(A_2)\# \ldots \#D(A_u))$ and thus by Theorem 2.2.5 we know V(D(A)) = $(V(D(A_1))V(D(A_2)) \ldots V(D(A_u)))$ and similarly V(D(B))

$= (V(D(A_{u+1}))V(D(A_{u+2}))\dots V(D(A_s)))$. Then by using 4.11 we find

$$maxexp(V(D(A))) = \sum_{i=1}^{u} -L(\hat{A}_i)$$
$$maxexp(V(D(B))) = \sum_{i=u+1}^{s} -L(\hat{A}_i)$$
$$minexp(V(D(A))) = \sum_{i=1}^{u} -H(\hat{A}_i)$$
$$minexp(V(D(B))) = \sum_{i=u+1}^{s} -H(\hat{A}_i)$$

Then by Corollary 3.1.5 we have the following:

$$maxexp(V(K)) \leq max \begin{cases} maxexp(V(N(A))V(N(B))) - \frac{1}{2} = \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{1}{2} \\ maxexp(V(D(A))V(D(B))) - \frac{1}{2} = \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{1}{2} \\ maxexp(V(N(A))V(D(B))) - 1 = \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{3}{2} \\ maxexp(V(D(A))V(N(B))) - 1 = \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{1}{2} \end{cases}$$
$$= \sum_{i=1}^{s} -L(\hat{A}_i) - \frac{1}{2} \tag{4.16}$$

and

$$minexp(V(K)) \geq min \begin{cases} minexp(V(N(A))V(N(B))) + \frac{1}{2} = \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{1}{2} \\ minexp(V(D(A))V(D(B))) + \frac{1}{2} = \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{1}{2} \\ minexp(V(N(A))V(D(B))) + 1 = \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{1}{2} \\ minexp(V(D(A))V(N(B))) + 1 = \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{3}{2} \end{cases}$$
$$= \sum_{i=1}^{s} -H(\hat{A}_i) + \frac{1}{2} \tag{4.17}$$

From [**7**] we know that a non alternating diagram of K where the all tangles A_i are alternating and end in vertical twists is a minimal diagram. Note that this implies $Cr(K) = \sum_{i=1}^{s} Cr(D(A_i))$. Moreover from [**10**] we also know that $Br(V(K)) =$

$Cr(K) - 1$. From our calculations we know that

$$\begin{aligned} Br(V(K)) &\leq \sum_{i=1}^{s}(H(\hat{A}_i) - L(\hat{A}_i)) - 1 \\ &= \sum_{i=1}^{s} Br(D(A_i)) - 1 = \sum_{i=1}^{s} Cr(D(A_i)) - 1 = Cr(K) - 1. \end{aligned}$$

From the previous line we have that equality must hold and therefore $Br(V(K)) = \sum_{i=1}^{s}(H(\hat{A}_i) - L(\hat{A}_i)) - 1$ and likewise we must have equality in Equation 4.16 and 4.17. $\square$

CHAPTER 5

Conclusions

The main purpose of this thesis was to find formulas for the maximum and minimum exponents of the Jones polynomial for certain classes of knots and links. To accomplish this goal we started by investigating the maximum and minimum exponent for knots and links constructed from special rational tangles namely those of the form $\frac{\beta}{\alpha}$ where β is odd and α is even or when β is even and α is odd. We then went on to use the formulas for finding the maximum and minimum exponent of the Jones polynomial of Montesinos knots. The Montesinos knots that we could draw some conclusions about were those constructed from rational tangles ending in horizontal twists and Montesinos knots ending in vertical twists. The next step in furthering this research would be to investigate what would happen if we have a Montesinos knot constructed by mixing tangles that end in vertical twists and horizontal twists.

Another open question is how do we deal with Montesinos knots that contain rational tangles of the form $\frac{\beta}{\alpha}$ where β is odd and α is odd. It is clear to us that if we did have an odd number of these tangles and no tangles of the form $\frac{\beta}{\alpha}$ where β is odd and α is even then the knot cannot have the orientation as shown in Figure 4.3 and thus we would have to approach this case with different methods from those used in this thesis.

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