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Generalized Probabilistic Bowling Distributions

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GENERALIZED PROBABILISTIC BOWLING DISTRIBUTIONS

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GENERALIZED PROBABILISTIC BOWLING DISTRIBUTIONS

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GENERALIZED PROBABILISTIC BOWLING DISTRIBUTIONS

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Have you ever wondered if you are better than the average bowler? If so, there are a variety of ways to compute the average score of a bowling game, including methods that account for a bowler's skill level. In this thesis, we discuss several different ways to generate bowling scores randomly. For each distribution, we give results for the expected value and standard deviation of each frame's score, the expected value of the game's final score, and the correlation coefficient between the score of the first and second roll of a single frame. Furthermore, we shall generalize the results in each distribution for an N-frame game on M pins.

Additionally, we shall generalize the number of possible games when bowling N frames on M pins. Then, we shall derive the frequency distribution of each frame's scores and the arithmetic mean for frames $i = 1,2,\ldots,N$ on M pins. Finally, to summarize the variety of distributions, we shall make tables that display the results obtained from each distribution used to model a particular bowler's score. We evaluate the special case when bowling 10 frames on 10 pins, which represents a standard bowling game.

Chapter 1: Introduction

Have you ever wondered if you are better than the average bowler? If so, there are a variety of ways to compute the average score of a bowling game, including methods that account for different random distributions of scores and a bowler's skill level. In this thesis, we shall analyze five different ways of randomly generating scores in a frame of bowling. For each of these distributions, we shall derive the expected value and standard deviation of the score of each frame, the expected value of the game's final score, and the correlation between the values of the two rolls in each frame.

For a standard 10-frame bowling game, Cooper and Kennedy [1] have derived the average score of all possible games when the final scores are weighted according to their frequencies. In this case, the average is the usual arithmetic mean. They found that there are approximately 5.7 billion billion possible ways to bowl a game and that the average score is about 80. In [2], Cooper and Kennedy used a generating function to determine the distribution of the frequencies of final scores of all possible bowling games.

Throughout the work of Cooper and Kennedy, they did not apply a probability distribution to individual frames or rolls. Since they only evaluate the distribution of a standard bowling game in its entirety, we will derive the distribution of each frame's score in Chapter 2. We shall also generalize the number of possible games when bowling N frames on M pins, which agree with the results from [1] when $N = 10$ frames and $M = 10$ pins. Moreover, we shall derive the arithmetic mean for frames $i = 1,2,\ldots,N$ on M pins, as well as derive the frequency distribution of each frame's scores.

In Chapter 3, we shall develop a new probability distribution for each pair of rolls in a frame, as well as for individual rolls in a frame on M pins, where each pair is equally likely. The last frame must be dealt with separately because the probabilities vary when a spare or strike is rolled. We shall apply Neal's argument to find the average score of an N-frame game on M pins. Next, we shall derive the distribution of each frame's score not only to verify the result but also to compute the variance and standard deviation of a single frame. Then we shall find the correlation between the score of the first and second roll in a single frame of bowling.

In [3], Neal applied a discrete uniform distribution to individual rolls for a standard bowling game on 10 pins for which pins are knocked down randomly. He found that the average score of a 10 frame random bowing game on 10 pins is approximately 91.4127. He also developed a method for accounting for spares and strikes. This method, referred to as Neal's argument, will be applied to other random distributions. In [4], we developed a new method to analyze Neal's random bowling distribution. The indicator function is applied to derive the average score for a ten-frame game, which allowed us to compute the variance and standard deviation of a single frame.

In Chapter 4, we shall generalize Neal's result in [3], and the results in [4], to an N- frame game on M pins. Additionally, we shall derive the distribution of each frame's score to verify the results for the mean and standard deviation. Then we shall find the correlation between the score of the first and second roll in a single frame of random bowling.

In [6], Neal developed a new discrete probability distribution $X_n \sim Fib(n)$ on the integer values $\{1,\ldots,n\}$. In Chapter 5, we shall adjust the distribution to the values $\{0,\ldots,M\}$ and use a $Fib(M+1)-1$ distribution when bowling N frames on M pins. Unlike Neal's uniform distribution in [3], the Fibonacci distribution allows us to model the results of a skilled bowler. We shall apply Neal's argument to find the average score of a skilled bowler for an N- frame game on M pins. Next, we shall derive the distribution of each frame's score to verify the result and to compute the variance and standard deviation of a single frame. Then, we shall find the correlation between the score of the first and second roll in a single frame of bowling.

In Chapter 6, we shall derive the average score of a particular non-skilled bowler when bowling N frames on M pins using an $M+1-Fib(M+1)$ Fibonacci distribution. We shall apply Neal's argument to find the average score of a non-skilled bowler for an N- frame game on M pins. Next, we shall derive the distribution of each frame's score to verify the result and to compute the variance and standard deviation of a single frame. Then, we shall find the correlation between the score of the first and second roll in a single frame of bowling.

In [5], Brown and Neal developed another bowling distribution for which pins are knocked down according to binomial distributions. They found that for a 10-frame game on 10 pins, with the probability of any pin being knocked down being 0.5, the average score is approximately 77.84. They also derived the correlation between the first and second roll in a single frame of bowling.

In Chapter 7, we shall apply the results from [5] to model a $100p\%$ bowler. Applying the distribution method, we shall derive the variance and standard deviation for an entire frame in the particular case when the probability of knocking down any pin at any time is p. This scenario refers to a $100p\%$ bowler, who knocks down a single pin $100p\%$ of the time. Additionally, we are able to verify the results obtained from [5] for the average score and the correlation coefficient when bowling N frames on M pins.

Scoring in a Standard Bowling Game

The score of a standard bowling game can range anywhere from 0 to 300 depending on the combinations of rolls in each frame. The following table displays an example of the scoring for the first nine frames of a standard bowling game:

Frame, i	1	2	3	4	5	6	7	8	9
Roll, (X_{i1}, X_{i2})	(8,1)	(2,3)	(10,0)	(10,0)	(10,0)	(3,7)	(2,4)	(7,3)	(3,4)
Score, f_i	9	5	30	23	20	12	6	13	7
Total	9	14	44	67	87	99	105	118	125

We see that the score f_i in the $i\,th$ frame is initially the sum of the two rolls. However, the score increases when a spare or strike is rolled. If a spare is rolled, then the next roll is added to f_i. If a strike is rolled, then the next two rolls are added to f_i. For example, we can see that a strike was rolled in the third frame. Additionally, strikes were rolled in the next two frames. Thus, the score of the third frame is 30, which is the highest score attainable in a single frame with 10 pins. Moreover, if the score is 30 for all 10 frames, then the final score results in 300.

Interestingly, if a spare or strike is rolled in the last frame then one or two additional rolls are given. Such additions require the need to analyze each possible case separately. The first case occurs when a non-spare or non-strike is rolled in the last frame, called an *open* frame. As a result, the score in the last frame is simply the sum of the two rolls. However, if a spare is rolled in the last frame, then one additional roll is given, and the result is added to f_{10}. Lastly, if a strike is rolled in the last frame, two additional rolls are given and the results are added to f_{10}. Specifically, when a strike is rolled in the last frame, the additional rolls could be either a strike or non-strike. Thus, the following table gives an example of each case in the last frame:

Case:	Open	Spare	Strike, then Non-strike	Strike, then Strike
Frame	10	10	10	10
Rolls	(8,1)	(2,8) + 6	(10,0) + (6,2)	(10,0) + (10,0) + 8
Score	9	16	18	28
Total	134	141	143	153

Similar to scoring a standard bowling game, we can attain the score of a generalized game when bowling *N* frames on *M* pins. A standard bowling game with 10 pins has a maximum score of 30 for a single frame and a maximum final score of 300. Similarly, a generalized game with *M* pins has a maximum score of $3M$ for a single frame and a maximum final score of $3MN$.

Mathematical Background

We now provide some background necessary for our work. In Chapter 4, we shall write a frame's score using an indicator function. The indicator function of a subset *A* of a set *X* is a function $1_A : X \to \{0,1\}$ defined as

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

In order to determine the mean of a frame, we will need some information on conditional averages.

Suppose $W = \sum_{i=1}^{n} Y_i \times 1_{A_i}$, where A_i is a partition. Then $E[W] = \sum_{i=1}^{n} E[Y_i \mid A_i] \times P(A_i)$.

If Y_i is independent of A_i, then $E[Y_i \mid A_i] = E[Y_i]$.

Otherwise, $E[Y_i \mid A_i] = \sum_{k} k \times P(Y_i = k | A_i) = \sum_{k} k \times \frac{P(Y_i = k \cap A_i)}{P(A_i)}$. In this case, the term $E[Y_i \mid A_i] \times P(A_i)$ in $E[W]$ becomes just $\sum_{k} k \times P(Y_i = k \cap A_i)$.

A special case of applying conditional expectation for scores 0 to M on a roll is as follows:

$$E[X] = \sum_{k=0}^{M} k \times P(X = k).$$

Then

$$E[X \mid X < M] = \frac{1}{P(X < M)} \sum_{k=0}^{M-1} k \times P(X = k),$$

and

$$E[X \mid X = M] = M.$$

These two conditional averages can be weighted as follows:

$$E[X] = E[X \mid X < M] \times P(X < M) + E[X \mid X = M] \times P(X = M).$$

Thus,

$$\begin{aligned} E[X \mid X < M] \times P(X < M) &= E[X] - E[X \mid X = M] \times P(X = M) \\ &= E[X] - M \times P(X = M). \end{aligned}$$

The mean can be applied to find the variance of a frame's score, which is defined as the following:

$$Var(X) = E[X^2] - (E[X])^2$$

The variance is then applied to calculate the standard deviation, which is given by

$$\sigma_X = \sqrt{Var(X)}.$$

The Fibonacci numbers are a sequence of numbers where the first number of the sequence is 1, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers in the sequence itself. In mathematical terms, it is defined by the following recurrence relation:

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 3. \end{cases}$$

In probability theory and statistics, the binomial distribution is the discrete probability distribution of the number of successes in a sequence of n independent success/failure experiments, each of which yields success with probability p.

In general, if the random variable X follows the binomial distribution with parameters n and p, we write $X \sim B(n, p)$. The probability of getting exactly k successes in n trials is given by the probability mass function:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} ,$$

for $k = 0,1,2,\ldots n$. If $X \sim B(n, p)$, then the expected value of X is $E[X] = np$.

Chapter 2: Arithmetic Mean for Frames of Bowling

In this chapter, we count the number of possible games when bowling N frames on M pins and derive the arithmetic mean of each frame.

2.1 Total Possibilities for a Bowling Game

In a bowling game, there are many possibilities for the score of each frame, which results in various final scores. Interestingly, there is a way to compute the total number of possible bowling games for N frames played on M pins.

Every frame can be represented as a set of rolls (X_1, X_2), where X_1 and X_2 are integers with $0 \le X_1 \le M$ and $0 \le X_2 \le M - X_1$. The list of all possible pairs is shown below:

$$\begin{array}{l}
(0,0)\ \ (0,1)\ \ (0,2)\ \ \ldots\ \ (0,M-1)\ \ (0,M) \\
(1,0)\ \ (1,1)\ \ \ldots\ \ (1,M-2)\ \ (1,M-1) \\
(2,0)\ \ \ldots\ \ (2,M-3)\ \ (2,M-2) \\
\vdots \\
(M-1,0)\ \ (M-1,1) \\
(M,0)\ .
\end{array}$$

Since there are $M+1$ rows in the list with the number of pairs ranging from $M+1$ to 1, the total number of possible pairs for a single frame, denoted by L, is given by

$$L = \sum_{n=1}^{M+1} n = \frac{(M+1)(M+2)}{2}$$

for frames $i = 1,2,\ldots,N-1$.

We must handle the $N\,th$ frame separately because the number of possibilities increases when a spare or strike is rolled in the last frame. So if a non-spare or non-strike is rolled in the last frame, called an *open* frame, then the following pairs represent all of the possibilities:

$$\begin{array}{l}(0,0)\ \ (0,1)\ \ (0,2)\ \ \ldots\ \ (0,M-2)\ \ (0,M-1)\\ (1,0)\ \ (1,1)\ \ (1,2)\ \ \ldots\ \ (1,M-2)\\ (2,0)\ \ (2,1)\ \ \ldots\ \ (2,M-3)\\ \vdots\\ (M-1,0)\ .\end{array}$$

Since the first row has M pairs and each row decreases by one, the total number of possible pairs in an open frame is

$$\sum_{n=1}^{M} n = \frac{M(M+1)}{2}\ .$$

If there is a spare in the last frame, the first roll X_1 can range from 0 to $M-1$, and the second roll must be $M-X_1$, which gives M possibilities. Since a spare was rolled, an additional ball is given that can range from 0 to M, which gives $M+1$ possibilities. So, there are $M(M+1)$ possibilities if a spare is rolled in the last frame.

If there is a strike in the last frame, then there are two cases. The first case is when a strike is followed by a non-strike. The strike gives one possibility. For a non-strike to be rolled next, every pair can be rolled except $(M,0)$, which gives $L-1$ possibilities. The second case is when a strike is rolled, followed by another strike. Since we are looking at the last frame, after both strikes are rolled an additional ball is given. This roll can range from 0 to M, which gives $M+1$ possibilities.

Thus, accounting for an open frame, a spare, and the two cases of strikes, the total number of possibilities for the last frame is

$$\begin{aligned}&\frac{M(M+1)}{2}+M(M+1)+\left(\frac{(M+1)(M+2)}{2}-1\right)+(M+1)\\&=\frac{M^2+M}{2}+\frac{2M^2+2M}{2}+\frac{M^2+3M+2-2}{2}+\frac{2M+2}{2}\\&=\frac{4M^2+8M+2}{2}\\&=2M^2+4M+1\ .\end{aligned}$$

Since there are $N-1$ frames, each having L possibilities, and the last frame having $2M^2+4M+1$ possibilities, the total number of possible games in an N-frame bowling game with M pins is

$$G(M,N)=\left(\frac{(M+1)(M+2)}{2}\right)^{N-1}\times(2M^2+4M+1)\ .$$

In the special case with $M=10$ pins and $N=10$ frames, we have

$$\begin{aligned}G(10{,}10)&=\left(\frac{(10+1)(10+2)}{2}\right)^{10-1}\times\left(2(10)^2+4(10)+1\right)\\&=66^9\cdot 241\\&=5{,}726{,}805{,}883{,}325{,}784{,}576\,.\end{aligned}$$

Thus, there are $66^9\cdot 241$ possible games in a standard bowling game, which is about 6 billion billion, or 6 quintillion games.

Also, in the case with $M=2$ pins and $N=3$ frames, the possible pairs for each frame are

$$(0,\,0)\ \ (0,\,1)\ \ (0,\,2)\ \ (1,\,0)\ \ (1,\,1)\ \ (2,\,0)\ .$$

So, the total number of possible games for 3 frames with 2 pins is

$$G(2{,}3)=\left(\frac{(2+1)(2+2)}{2}\right)^{3-1}\times\left(2(2)^2+4(2)+1\right)=6^2\cdot 17=612\,.$$

2.2 Total Possibilities of Scores in the First $N-1$ Frames

Similarly, we can find the number of ways to obtain the possible scores 0, 1, . . . , $3M$ in a single frame. In order to find the total possibilities of a score in a single frame, we can combine certain scores that follow a similar pattern. We first consider all frames other than the last frame, and let (x,y) be a pair of rolls, where $0\le x,y\le M$.

To obtain a score of k, where $0 \le k < M$, a frame must have a combination (x, y) that sums to k, which gives $k+1$ possibilities shown in the table below:

Score	Set of Possibilities	Number of Possibilities
k	$\{(0,k),(1,k-1),\ldots,(k,0)\}$	$k+1$

Thus, the total number of possibilities for scores from 0 to $M-1$, which occur in an open frame when the first and second roll sum to less than M, is

$$\sum_{k=0}^{M-1}(k+1) \ .$$

To obtain a score of $M+k$, where $0 \le k < M$, there are two cases. In the first case, a spare can be rolled, which is a combination (x, y) that sums to M where $x < M$, which give M possibilities. Then a pair (k, y) is rolled in the next frame, which gives $M+1-k$ possibilities for y. So, the first case has $M(M+1-k)$ possibilities. In the second case, a strike can be rolled, followed by a combination that sums to k in the next frame, which gives $k+1$ possibilities. The possibilities for a score $M+k$, where $0 \le k < M$ are shown in the table below.

Score	Set of Possibilities	Number of Possibilities
$M+k$	$\{(0,\ M),\ldots,(M-1,\ 1)\}\times\{(k,0),\ldots,(k,M-k)\}$ $\cup\{(M,0)\}\times\{(0,k),\ldots,(k,0)\}$	$M(M+1-k)+(k+1)$

Thus, the total number of possibilities of scores from M to $2M-1$ is

$$\sum_{k=0}^{M-1}\left(M(M+1-k)+k+1\right) \ .$$

To obtain a score of $2M$, there are three cases. In the first case, a spare can be rolled, followed by a strike in the next frame, which gives M possibilities. The second case occurs when a strike is rolled, followed by a spare in the next frame. The initial frame has one possibility, and the next frame includes any combination of (x, y) that add up to M, where $x < M$, giving M possibilities. So, there are M possibilities in the second case. Lastly, the third case occurs when two strikes are rolled, followed by a pair $(0, y)$. The two strikes each have one possibility and the next pair has $M+1$ possibilities. So, the possibilities for the score $2M$ are shown in the table below.

Score	Set of Possibilities	Number of Possibilities
$2M$	$\{(0, M),\ldots, (M-1, 1)\}\times\{(M,0)\}$ $\cup\{(M,0)\}\times\{(0, M),\ldots, (M-1,1)\}$ $\cup\{(M,0)\}\times\{(M,0)\}\times\{(0,0),\ldots,(0, M)\}$	$M+M+(M+1)$

Thus, the number of ways to score $2M$ is $M+M+(M+1)$.

Finally, to obtain a score of $2M+k$, where $1 \le k \le M$, two strikes must be rolled, followed by a pair (k, y). So, there are $M+1-k$ possibilities as shown in the table below.

Score	Set of Possibilities	Number of Possibilities
$2M+k$	$\{(M,0)\}\times\{(M,0)\}\times\{(k,0),\ldots,(k, M-k)\}$	$M+1-k$

Thus, the total number of possibilities of scores from $2M+1$ to $3M$ is

$$\sum_{k=0}^{M-1}(M+1-k) \ .$$

Now we can combine parts from the score $2M$ with scores $M+k$ and scores $2M+k$. We can see that the first part $M+M$ follows the exact pattern as the scores from M to $2M-1$, and the second part $M+1$ follows a similar pattern as the scores from $2M+1$ to $3M$. So we can rewrite the first two parts of the score $2M$ as $M(1)+1(M+1)-1$, and combine these possibilities with the scores from M to $2M-1$, which results in the following summation:

$$\sum_{k=0}^{M}\big(M(M+1-k)+(k+1)\big)-1 \ .$$

Finally, the last part of the score $2M$ can be combined with the scores from $2M+1$ to $3M$, which results in the following summation:

$$\sum_{k=0}^{M}(M+1-k) \ .$$

Thus, the total number of ways $A\big(f(i)\big)$ to obtain scores for frames $i=1,2,\ldots,N-1$ is given by

$$\begin{aligned}
A\big(f(i)\big) &= \sum_{k=0}^{M-1}(k+1)+\sum_{k=0}^{M}\big[M(M+1-k)+(k+1)\big]-1+\sum_{k=0}^{M}(M+1-k) \\
&= \sum_{k=1}^{M}k+\sum_{k=0}^{M}M(M+1-k)+\sum_{k=0}^{M}(k+1)-1+\sum_{k=0}^{M}(M+1-k) \\
&= \sum_{k=1}^{M}k+(M+1)\sum_{k=0}^{M}(M+1-k)+\sum_{k=0}^{M}(k+1)-1 \\
&= \sum_{k=1}^{M}k+(M+1)\sum_{k=1}^{M+1}k+\sum_{k=1}^{M+1}k-1 \\
&= \sum_{k=1}^{M}k+(M+2)\sum_{k=1}^{M+1}k-1 \\
&= \frac{M(M+1)}{2}+(M+2)\frac{(M+1)(M+2)}{2}-1 \\
&= \frac{M^2+M}{2}+\frac{M^3+5M^2+8M+4}{2}-\frac{2}{2} \\
&= \frac{M^2+M+M^3+5M^2+8M+4-2}{2} \\
&= \frac{M^3+6M^2+9M+2}{2}.
\end{aligned}$$

2.3 Summation of all Possible Scores in the First $N-1$ Frames

In order to compute the arithmetic mean of a single frame, we first need to sum all the possible scores from 0 to $3M$ while accounting for their frequencies. Each of the scores from 0 to $M-1$ can occur $k+1$ ways. Thus, the sum of these scores is

$$\sum_{k=0}^{M-1} k(k+1) \ .$$

Looking at scores from M to $2M-1$, along with the first two parts of the score $2M$, we can multiply each score by the number of ways it can occur. Thus, the sum of these scores is

$$\sum_{k=0}^{M-1} (M+k)\big[M(M+1-k)+(k+1)\big]-2M \ .$$

Finally, scores from $2M+1$ to $3M$, along with the last part of score $2M$, each have $M+1-k$ possibilities. Thus, the sum of these scores is

$$\sum_{k=0}^{M} (2M+k)(M+1-k) \ .$$

Thus, the sum of all scores in a single frame, $B\big(f(i)\big)$, for frames $i=1,2,\ldots,N-1$ is

$$\begin{aligned}
B\big(f(i)\big) &= \sum_{k=0}^{M-1} k(k+1) + \sum_{k=0}^{M} (M+k)\big[M(M+1-k)+(k+1)\big]-2M + \sum_{k=0}^{M} (2M+k)(M+1-k) \\
&= \frac{M(M^2-1)}{3} + \frac{M(M+1)(4M^2+13M+10)}{6} - 2M + \frac{7M(M+1)(M+2)}{6} \\
&= \frac{2M(M^2-1)}{6} + \frac{M(M+1)(4M^2+13M+10)}{6} - \frac{12M}{6} + \frac{7M(M+1)(M+2)}{6} \\
&= \frac{2M(M^2-1) + M(M+1)(4M^2+13M+10) - 12M + 7M(M+1)(M+2)}{6} \\
&= \frac{2M^4+13M^3+22M^2+5M}{6} \ .
\end{aligned}$$

2.4 Arithmetic Mean of the First $N-1$ Frames

In order to compute the arithmetic mean of a single frame, $\overline{X}_{f(i)}$, for frames $i = 1,2,\ldots,N-1$ with M pins, we must take the sum of all scores, $B(f(i))$, and divide it by the total number of ways to accrue scores, $A(f(i))$, which is shown to be

$$\begin{aligned}
\overline{X}_{f(i)} &= \frac{\frac{2M^4+13M^3+22M^2+5M}{6}}{\frac{M^3+6M^2+9M+2}{2}} \\
&= \frac{2M^4+13M^3+22M^2+5M}{6} \div \frac{M^3+6M^2+9M+2}{2} \\
&= \frac{2M^4+13M^3+22M^2+5M}{6} \cdot \frac{2}{M^3+6M^2+9M+2} \\
&= \frac{2(2M^4+13M^3+22M^2+5M)}{3(M^3+6M^2+9M+2)} \\
&= \frac{4M^4+26M^3+44M^2+10M}{3M^3+18M^2+27M+6} \quad .
\end{aligned}$$

For the special case with $M = 10$ pins, the number of ways to obtain the scores from 0 to 30 for frames $i = 1,2,\ldots,N-1$ is listed below:

Score	0	1	2	3	4	5	6	7	8	9	10
# of Ways	1	2	3	4	5	6	7	8	9	10	111

Score	11	12	13	14	15	16	17	18	19	20
# of Ways	102	93	84	75	66	57	48	39	30	31

Score	21	22	23	24	25	26	27	28	29	30
# of Ways	10	9	8	7	6	5	4	3	2	1

Thus, the arithmetic mean, $\overline{X}_{f(i)}$, for frames $i = 1,2,\ldots,9$ with $M = 10$ pins is

$$\begin{aligned}
\overline{X}_{f(i)} &= \frac{4(10)^4 + 26(10)^3 + 44(10)^2 + 10(10)}{3(10)^3 + 18(10)^2 + 27(10) + 6} \\
&= \frac{40000 + 26000 + 4400 + 100}{3000 + 1800 + 270 + 6} \\
&= \frac{70500}{5076} \\
&= \frac{125}{9} \\
&\approx 13.8889 \ .
\end{aligned}$$

2.5 Total Possibilities of Scores in the Last Frame

Interestingly, the $N th$ frame has a different arithmetic mean because the possibilities of scores $n \geq M$ are reduced in the last frame. Thus, we must consider the $N th$ frame separately.

In order to find the total possibilities of all scores in the $N th$ frame, we again look at each score from $0,1,\ldots,3M$ and combine certain scores that follow a similar pattern. We again let (x, y) be a pair of rolls, where $0 \leq x, y \leq M$. To obtain a score of k, where $k < M$, frame N must have a combination (x, y) that sums to k. So there are $k+1$ possibilities.

To obtain a score of $M + k$, where $0 \leq k < M$, there are two cases. In the first case, a spare can be rolled in the last frame, which is a combination (x, y) that sums to M where $x < M$, which gives M possibilities. Then, k must be scored on the additional roll, which only has one possibility. Therefore, the first case has M possibilities. In the second case, a strike can be rolled in frame N, followed by a combination that sums to k in the next pair, which gives $k+1$ possibilities. Thus, the total number of ways to score $M + k$ is $M(1) + 1(k+1) = M + k + 1$.

To obtain a score of $2M$, there are three cases. In the first case, a spare can be rolled in frame N , followed by a strike on the additional roll, which gives M possibilities. The second case occurs when a strike is rolled in frame N, followed by a spare in the next two additional rolls. Frame N has one possibility, and the next two rolls can include any combination of (x, y) that sum to M, where $x < M$, giving M possibilities. Therefore, there are M possibilities in the second case. Lastly, the third case occurs when two strikes are rolled, followed by 0 on the last additional roll, which each have one possibility. Thus, the number of ways to score $2M$ in the last frame is $M(1) + 1(M) + 1(1)(1) = 2M + 1$.

Finally, to obtain a score of $2M + k$, where $1 \leq k \leq M$, two strikes must be rolled, followed by k on the additional roll, which only has one possibility.

As shown previously in Section 2.1, the total number of possibilities of scores in the last frame is $2M^2 + 4M + 1 = A(f(N))$.

For the special case with $M = 10$ pins, the number of ways to obtain the scores from 0 to 30 for the last frame is listed below:

Score	0	1	2	3	4	5	6	7	8	9	10
# of Ways	1	2	3	4	5	6	7	8	9	10	11

Score	11	12	13	14	15	16	17	18	19	20
# of Ways	12	13	14	15	16	17	18	19	20	21

Score	21	22	23	24	25	26	27	28	29	30
# of Ways	1	1	1	1	1	1	1	1	1	1

2.6 Summation of all Possible Scores in the Last Frame

In order to compute the arithmetic mean of the last frame, we again need to sum all the possible scores from 0 to $3M$. The scores from 0 to $2M$ each have $k + 1$ possibilities; thus the sum of these scores is

$$\sum_{k=0}^{2M} k(k+1) \ .$$

The scores from $2M+1$ to $3M$ each have one possibility. So, the sum of these scores is

$$\sum_{k=1}^{M}(2M+k) \ .$$

Thus, the sum of all scores in the last frame is

$$\begin{aligned}
B'\big(f(N)\big) &= \sum_{k=0}^{2M} k(k+1) + \sum_{k=1}^{M}(2M+k) \\
&= \frac{4M(M+1)(2M+1)}{3} + \frac{M(5M+1)}{2} \\
&= \frac{8M(M+1)(2M+1)}{6} + \frac{3M(5M+1)}{6} \\
&= \frac{8M(M+1)(2M+1) + 3M(5M+1)}{6} \\
&= \frac{16M^3 + 39M^2 + 11M}{6} \ .
\end{aligned}$$

In order to compute the arithmetic mean of the last frame with M pins, we must take the sum of all scores and divide it by the total number of ways to accrue scores in the last frame which is given by

$$\begin{aligned}
\overline{X}_{f(N)} &= \frac{\dfrac{16M^3 + 39M^2 + 11M}{6}}{2M^2 + 4M + 1} \\
&= \frac{16M^3 + 39M^2 + 11M}{12M^2 + 24M + 6} \ .
\end{aligned}$$

In the special case of $N = 10$ frames with $M = 10$ pins, we have

$$\begin{aligned}
\overline{X}_{f(10)} &= \frac{16(10)^3 + 39(10)^2 + 11(10)}{12(10)^2 + 24(10) + 6} \\
&= \frac{3335}{241} \\
&\approx 13.8382 \ .
\end{aligned}$$

2.7 Conclusion of the Arithmetic Mean

Unlike the properties of the expected value, the arithmetic mean of a ten-frame game cannot simply be computed by summing all the individual arithmetic means because the arithmetic mean is not a linear operator. That is,

$$\frac{1}{n}\sum_{i=1}^{n}a_i + \frac{1}{m}\sum_{i=1}^{m}b_i \neq \frac{a_1+\cdots+a_n+b_1+\cdots+b_m}{n+m}.$$

In order to calculate the arithmetic mean for a ten-frame game with ten pins, we would need to compute the possibilities of each score, ranging from 0 to 300, which is a tedious task. However, the arithmetic mean of a single frame, $\overline{X}_{f(i)}$, for frames $i = 1,2,\ldots,N-1$, can be computed, which is approximately 13.8889. In addition, the arithmetic mean of the $N\,th$ frame can be computed, which is approximately 13.8382.

Interestingly, Cooper and Kennedy constructed a generating function in [3] which determines the distribution of the scores of all possible bowling games. The derivation states that the most common score out of all the possibilities is **77**. This is the *mode* of the score distribution, which has a 3% occurrence. They found that the arithmetic mean of the distribution is about 79.7.

The arithmetic mean generates an interesting result, but does not give the most accurate information compared to other methods discussed in the upcoming chapters. We have seen that the arithmetic mean assumes that each possibility is equally likely to occur, whereas, the mean computed with random distributions looks at the probability of outcomes on the first roll and then the second roll. Overall, the arithmetic mean generates a higher average score of a bowling game compared to the results using random distributions. Regardless of which method generates a higher average, we are far more interested in feasibility. Since the second roll is dependent on the first roll, it is sensible to look at the probability of each roll separately. Thus, we are more interested in the results using random distributions.

Chapter 3: The Hohn Distribution

In this chapter, we apply two methods to derive the average score when bowling N frames on M pins using the Hohn distribution. First, we apply Neal's method to derive the mean of a single frame, and then we verify these results by applying the distribution method. In addition, we apply the distribution method to compute the variance and standard deviation for a single frame. Lastly, we will find the correlation between the score of the first and second rolls in a single frame of bowling.

3.1 The Rules of the Hohn Distribution

In general, there are N frames, and two rolls per frame (X_{i1}, X_{i2}). The sets of rolls can be represented as: (X_{11}, X_{12}), (X_{21}, X_{22}) , . . . , (X_{N1}, X_{N2}), where each pair is chosen at random. In section 2.1, we derived the total possibilities of pairs in a single frame to be $L = (M+1)(M+2)/2$. To create the Hohn distribution, we assume that each pair for a frame is equally likely to be chosen from the entire set of L possible pairs.

The standard scoring rules of bowling are applied to the Hohn distribution. If $X_{i1} = M$, then X_{i2} must equal 0. Also, the score f_i in the $i\,th$ frame is initially the sum of these two rolls. However, the score of a single frame will increase with the occurrence of spares or strikes.

If a spare is rolled in a single frame, then the next roll $X_{(i+1)1}$ is added to f_i. For frames i = 1, 2, . . . , $N-1$, this roll $X_{(i+1)1}$ has a distribution equivalent to the first roll in one of the random pairs listed above. On the other hand, in the $N\,th$ frame, an additional ball must be rolled. We can use a discrete uniform distribution for the additional ball by assuming that each of the values 0, 1, . . . , M are equally likely. Interestingly, this roll $X_{(N+1)1}$ has a distribution different to the first roll in a single frame.

If a strike is rolled in a single frame, then the next two rolls are added to f_i. These two rolls are actually another pair chosen at random from the entire set with all pairs again equally likely. However, if the pair $(M, 0)$ is chosen, then an additional roll is added to f_i. This roll has the same distribution as in the case of a spare. Combining the rules, we obtain:

Theorem 3.1.1. The score in the $i\,th$ frame is given by

$$\begin{aligned} f_i = {} & X_{i1} + X_{i2} + X_{(i+1)1} \times 1_{\{(X_{i1}, X_{i2}) = M \cap X_{i1} < M\}} \\ & + (M + X_{(i+2)1}) \times 1_{\{X_{i1} = M\}} \times 1_{\{X_{(i+1)1} = M\}} \\ & + 1_{\{X_{i1} = M\}} \times (X_{(i+1)1} + X_{(i+1)2}) \times 1_{\{X_{(i+1)1} < M\}} \end{aligned}$$

for $i = 1, 2, \ldots, N$, and the final score is $\sum_{i=1}^{N} f_i$.

We can use this form of f_i to compute its expected value. $X_{(i+1)1}$ is independent of having a spare in the previous frame, and $M + X_{(i+2)1}$ is independent of having two previous strikes in a row. Thus

$$\begin{aligned} E[f_i] = {} & E[X_{i1} + X_{i2}] + E[X_{(i+1)1}] \times P\left(X_{i1} + X_{i2} = M \cap X_{i1} < M\right) \\ & + E[M + X_{(i+2)1}] \times P\left(X_{i1} = M\right) \times P\left(X_{(i+1)1} = M\right) \\ & + P\left(X_{i1} = M\right) \times E[X_{(i+1)1} + X_{(i+2)1} \,|\, X_{(i+1)1} < M] \times P\left(X_{(i+1)1} < M\right) \end{aligned}$$

We shall compute these four terms separately in the next sections.

3.2 Mean of a Single Frame Pair

Because a pair is chosen at random, each frame has L possibilities. We can define the probability of any single pair in frames $1, 2, \ldots, N$ to be the following:

$$P\left((X_{i1}, X_{i2})\right) = \frac{1}{L} = \frac{2}{(M+1)(M+2)} \ .$$

Because in our list of pairs, a sum of k can occur $k+1$ ways, for $0 \le k \le M$, the total sum of all pairs is $\sum_{k=0}^{M} k(k+1)$. Because each pair is equally likely with probability $1/L$, we have for frames $i = 1, 2, \ldots, N$

$$
\begin{aligned}
E[X_{i1} + X_{i2}] &= \frac{2}{(M+1)(M+2)} \sum_{k=1}^{M} k(k+1) \\
&= \frac{2}{(M+1)(M+2)} \sum_{k=1}^{M} (k^2 + k) \\
&= \frac{2}{(M+1)(M+2)} \left(\sum_{k=1}^{M} k^2 + \sum_{k=1}^{M} k \right) \\
&= \frac{2}{(M+1)(M+2)} \left(\frac{M(M+1)(2M+1)}{6} + \frac{M(M+1)}{2} \right) \\
&= \frac{1}{(M+2)} \left(\frac{M(2M+1)}{3} + \frac{3M}{3} \right) \\
&= \frac{1}{(M+2)} \left(\frac{2M(M+2)}{3} \right) \\
&= \frac{2M}{3} \ .
\end{aligned}
$$

Thus, without taking into account strikes and spares, the average score per frame is $2M/3$. Now, we can use the Neal argument from [5] to account for the average addition to each frame in the case of spares and strikes.

3.3 Accounting for Spares in the First $N-1$ Frames

Interestingly, the results for the $N th$ frame differ from the other frames when accounting for spares and strikes. For this reason, we must look at the last frame separately. First we will compute the mean for all other frames. If a spare is rolled in a frame, then the next roll is added to the score. Since the next roll is on a new set of pins, the average of the first value of a new pair is

$$
\begin{aligned}
E[X_{(i+1)1}] &= \frac{2}{(M+1)(M+2)}\left(\sum_{k=0}^{M} k(M+1-k)\right) \\
&= \frac{2}{(M+1)(M+2)}\left(\sum_{k=1}^{M} k(M+1-k)\right) \\
&= \frac{2}{(M+1)(M+2)}\left(M\sum_{k=1}^{M} k + \sum_{k=1}^{M} k - \sum_{k=1}^{M} k^2\right) \\
&= \frac{2}{(M+1)(M+2)}\left(\frac{M(M)(M+1)}{2} + \frac{M(M+1)}{2} - \frac{M(M+1)(2M+1)}{6}\right) \\
&= \frac{1}{(M+2)}\left(M^2 + M - \frac{M(2M+1)}{3}\right) \\
&= \frac{1}{(M+2)}\left(\frac{3M^2}{3} + \frac{3M}{3} - \frac{M(2M+1)}{3}\right) \\
&= \frac{1}{(M+2)}\left(\frac{M(M+2)}{3}\right) \\
&= \frac{M}{3} \ .
\end{aligned}
$$

However, the probability of a spare must be found in order to determine the average amount added to each frame's score. A spare occurs if $X_{i1} + X_{i2} = M$ but $X_{i1} < M$. The following pairs are considered spares:

$$
(0, M), (1, M-1), (2, M-2), \ldots, (M-1, 1) \ .
$$

Since a spare occurs M times out of the total possibilities, the probability of a spare is simply M/L. Thus, the average addition for spares in frames i = 1, 2, . . . , $N-1$, is given by

$$
\begin{aligned}
& E[X_{(i+1)1}] \times P\left(X_{i1} + X_{i2} = M \cap X_{i1} < M\right) \\
&= \frac{M}{3} \times \frac{M}{L} \\
&= \frac{M^2}{3L} \ .
\end{aligned}
$$

3.4 Accounting for Strikes in the First $N-1$ Frames

If a strike is rolled in frame i = 1, 2, . . . , $N-1$, then the next two rolls are added to the frame's score. There are two cases to consider: either the next roll after the strike is also a strike, or the next roll is not a strike. In the first case, the average addition to the original frame is M from the second strike, plus the average score of $M/3$ from the first roll on a new pair of pins. This first case occurs only in the event of two strikes in a row which has probability $1/L \times 1/L = 1/L^2$.

Thus the average addition to a frame in this case is

$$
\begin{aligned}
&(M + E[X_{(i+2)1}]) \times P(X_{i1} = M) \times P(X_{(i+1)1} = M) \\
&= \left(M + \frac{M}{3}\right) \times \frac{1}{L} \times \frac{1}{L} \\
&= \frac{4M}{3L^2}.
\end{aligned}
$$

If a strike is rolled, with probability $1/L$, and the next roll is not another strike, then the average addition to the frame's score is given by the weighted average of the next two rolls given that the first of these rolls is not M. Equivalently, it is the unconditional average $2M/3$ of the next two rolls minus the conditional average of the next two rolls given that the first roll *is* M. Thus, the average addition to a frame in this situation is

$$
\begin{aligned}
&P\left(X_{i1} = M\right) \times E[X_{(i+1)1} + X_{(i+1)2} \big| X_{(i+1)1} < M] \times P(X_{(i+1)1} < M) \\
&= \frac{1}{L} \times \left(E[X_{(i+1)1} + X_{(i+1)2}] - E[X_{(i+1)1} + X_{(i+1)2} \big| X_{(i+1)1} = M \times P(X_{(i+1)1} = M)\right) \\
&= \frac{1}{L} \times \left(\frac{2M}{3} - M \times \frac{1}{L}\right) \\
&= \left(\frac{2M}{3L} - \frac{M}{L^2}\right) \\
&= \frac{2ML - 3M}{3L^2} .
\end{aligned}
$$

Finally, the average score for frames $i = 1,2,\ldots, N-1$ is

$$E[f_i] = \frac{2M}{3} + \frac{M^2}{3L} + \frac{4M}{3L^2} + \frac{2ML-3M}{3L^2}$$
$$= \frac{2ML^2}{3L^2} + \frac{M^2L}{3L^2} + \frac{4M}{3L^2} + \frac{2ML-3M}{3L^2}$$
$$= \frac{2ML^2 + M^2L + M + 2ML}{3L^2} \ .$$

Substituting $L = (M+1)(M+2)/2$ and simplifying, we obtain

$$E[f_i] = \frac{2M\left((M+1)(M+2)^3 + 2\right)}{3(M+1)^2(M+2)^2} \ .$$

For the special case with $M = 10$ pins, we have

$$E[f_i] = \frac{2(10)\left((11)(12)^3 + 2\right)}{3(11)^2(12)^2} = \frac{47525}{6534} \approx 7.27349 \ .$$

3.5 Mean of the Nth Frame

Now we must look at the last frame separately to derive the mean of an N-frame game. If the last frame is a spare, then another roll is chosen at random from 0 to M. If the last frame is a strike $(M,0)$, then another pair is chosen at random from the entire set of pairs. If this second choice is also a strike, then one roll is chosen at random from 0 to M. So, frame N has the same average for the first and second roll, but varies slightly when accounting for spares and strikes. Therefore, without taking into account strikes and spares, the mean for frame N is $2M/3$.

If a spare is rolled in the Nth frame, then another ball is rolled and added to the score. Because pins are knocked down at random, the values of 0 through M are all equally likely to occur with probability $1/(M+1)$. Thus, the average score of the next roll is

$$E[X_{(N+1)1}] = \frac{1}{M+1}\sum_{k=0}^{M} k$$
$$= \frac{1}{M+1} \times \left(\frac{M(M+1)}{2}\right)$$
$$= \frac{M}{2} \ .$$

Since the probability of a spare in the $N\,th$ frame remains M/L, the average addition for spares in the $N\,th$ frame is given by

$$E[X_{(N+1)1}] \times P(X_{N1} + X_{N2} = M \cap X_{N1} < M)$$
$$= \frac{M}{2} \times \frac{M}{L} = \frac{M^2}{2L} \ .$$

If a strike is rolled in the $N\,th$ frame, then the next two rolls are added to the frame's score. Similar to frames $1,2,\ldots N-1$, there are two cases in this situation: either the next roll after the strike is also a strike, or the next roll is not a strike. In the first case, the average addition to the original frame remains M from the second strike, but the average score from the next roll on a new set of pins slightly differs because only one extra ball is rolled, so the average score added is $M/2$. This first case occurs only in the event of two strikes in a row. Thus, the probability of the first case remains $1/L^2$. Therefore, the average addition to the $N\,th$ frame in this case is

$$(M + E[X_{(N+2)1}]) \times P(X_{N1} = M) \times P(X_{(N+1)1} = M)$$
$$= \left(M + \frac{M}{2}\right) \times \frac{1}{L} \times \frac{1}{L} = \frac{3M}{2L^2} \ .$$

If a strike is rolled, and the next roll is not another strike, then the average addition to the frame's score remains the same as in frames $i = 1, 2, \ldots, N-1$. Thus, the average addition to the $N\,th$ frame in this situation is still $(2ML - 3M)/3L^2$.

Thus, the average score for the $N th$ frame is

$$
\begin{aligned}
E[f_N] &= \frac{2M}{3} + \frac{M^2}{2L} + \frac{3M}{2L^2} + \frac{2ML - 3M}{3L^2} \\
&= \frac{4ML^2}{6L^2} + \frac{3M^2L}{6L^2} + \frac{9M}{6L^2} + \frac{4ML - 6M}{6L^2} \\
&= \frac{4ML^2 + 3M^2L + 3M + 4ML}{6L^2} \ .
\end{aligned}
$$

Substituting $L = (M+1)(M+2)/2$ and simplifying, we obtain

$$
E[f_N] = \frac{M(M+1)(M+2)\big(2(M+1)(M+2) + 3M + 4\big) + 6M}{3(M+1)^2(M+2)^2} \ .
$$

For the special case with $M = 10$ pins, we have

$$
E[f_N] = \frac{10(11)(12)\big(2(11)(12) + 30 + 4\big) + 60}{3(11)^2(12)^2} = \frac{32785}{4356} \approx 7.5264 \ .
$$

3.6 Conclusion of the Mean

The average score for an N-frame bowling game can be computed by applying the linearity property of the mean. That is, the expected value of the sum of two or more random variables is the sum of each individual expected value:

$$
E[f_1 + f_2 + \ldots + f_{N-1} + f_N] = \sum_{i=1}^{N} E[f_i] \ .
$$

Applying this result, we obtain the following theorem:

Theorem 3.6.1. The average score for an N-frame game of bowling on M pins with the Hohn distribution is given by

$$\sum_{i=1}^{N} E[f_i] = (N-1)\left(\frac{2M((M+1)(M+2)^3+2)}{3(M+1)^2(M+2)^2}\right) + \frac{M(M+1)(M+2)(2(M+1)(M+2)+3M+4)+6M}{3(M+1)^2(M+2)^2},$$

where f_i represents the score of each frame.

Thus, the average score for a ten-frame game with $M = 10$ pins is

$$\sum_{i=1}^{10} E[f_i] = (10-1)\left(\frac{2(10)((11)(12)^3+2)}{3(11)^2(12)^2}\right) + \frac{10(11)(12)(2(11)(12)+30+4)+60}{3(11)^2(12)^2}$$
$$= 9\left(\frac{47525}{6534}\right) + \frac{32785}{4356} = \frac{317935}{4356} \approx 72.98783 .$$

3.7 Distribution Method for a Single Frame

We can verify the results from above using the distribution method. Unlike the Neal argument, this method will allow us to compute the variance of an N-frame game, as well as the standard deviation. The probabilities for scores $k \geq M$ vary in the Nth frame, so we must look at the Nth frame separately.

First, we can find the distribution for frames $i = 1,2,\ldots,N-1$, with M pins. Thus, we must find the probability of obtaining the scores $0,1,\ldots,3M$. Since there are L possibilities for the first roll, the following table shows the probability for scores $0,1,\ldots,3M$, where $0 \leq k \leq M-1$:

Score	Set of Possibilities	Probability
k	$\{(0,k),(1,k-1),\dots,(k,0)\}$	$\frac{k+1}{L}$
$M+k$	$\{(0,M),\dots,(M-1,1)\}\times\{(k,0),\dots,(k,M-k)\}$ $\cup\{(M,0)\}\times\{(0,k),\dots,(k,0)\}$	$\frac{M}{L}\times\frac{M+1-k}{L}$ $+\frac{1}{L}\times\frac{k+1}{L}$
$2M$	$\{(0,M),\dots,(M-1,1)\}\times\{(M,0)\}$ $\cup\{(M,0)\}\times\{(0,M),\dots,(M-1,1)\}$ $\cup\{(M,0)\}\times\{(M,0)\}\times\{(0,0),\dots,(0,M)\}$	$\frac{M}{L}\times\frac{1}{L}+\frac{1}{L}\times\frac{M}{L}$ $+\frac{1}{L}\times\frac{1}{L}\times\frac{M+1}{L}$
$2M+1+k$	$\{(M,0)\}\times\{(M,0)\}\times\{(k+1,0),\dots,(k+1,M-(k+1))\}$	$\frac{1}{L}\times\frac{1}{L}\times\frac{M-k}{L}$

We can verify that all of the possible scenarios are given and make up a partition by summing all of the probabilities, and to do so we can combine similarities into common summations.

Thus, the sum of probabilities for the scores from 0 to $M-1$ can be combined to the following summation:

$$\sum_{k=0}^{M-1}\left(\frac{k+1}{L}\right)=\frac{1}{L}\sum_{k=0}^{M-1}(k+1)\ .$$

Also, the sum of probabilities for the scores from M to $2M-1$ can be combined to the following:

$$\sum_{k=0}^{M-1}\left(\frac{M}{L}\cdot\frac{M+1-k}{L}+\frac{1}{L}\cdot\frac{k+1}{L}\right)=\frac{1}{L^2}\sum_{k=0}^{M-1}\left(M\left(M+1-k\right)+k+1\right)\ .$$

The score $2M$ is unique in a sense because the probability varies when two strikes are rolled in a row. So, instead of combining the probability in a summation, we will write out the probability for the score $2M$ as

$$\frac{M}{L}\times\frac{1}{L}+\frac{1}{L}\times\frac{M}{L}+\frac{1}{L}\times\frac{1}{L}\times\frac{M+1}{L}\ .$$

Finally, the sum of probabilities for the scores from $2M+1$ to $3M$ can be combined to the following summation:

$$\frac{1}{L^3}\sum_{k=1}^{M}(M+1-k)\,.$$

Now we can combine parts from the score $2M$ with scores $M+k$ and scores $2M+1+k$. We can see that the first part follows the exact pattern as scores $M+k$, and the second part follows a similar pattern to the scores $M+k$. So, we can rewrite the first two parts of the probability for the score $2M$ as

$$\frac{M}{L}\times\frac{1}{L}+\frac{1}{L}\frac{M+1}{L}-\frac{1}{L^2},$$

and combine these values with the probabilities for the scores from M to $2M-1$, which results in the following summation:

$$\frac{1}{L^2}\sum_{k=0}^{M}\left(M(M+1-k)+k+1\right)-\frac{1}{L^2}\ .$$

Finally, the last part of the score $2M$ can be combined with the probabilities for the scores from $2M+1$ to $3M$, which results in the following summation:

$$\frac{1}{L^3}\sum_{k=0}^{M}(M+1-k)\,.$$

Therefore, the sum of probabilities for all the scores $0,1,\ldots,3M$ is

$$\sum_{k=0}^{3M}P(f_i=k)=\frac{1}{L}\sum_{k=0}^{M-1}(k+1)\ +\frac{1}{L^2}\sum_{k=0}^{M}\left(M(M+1-k)+k+1\right)-\frac{1}{L^2}+\frac{1}{L^3}\sum_{k=0}^{M}(M+1-k)\,.$$

Using *Mathematica* to simplify the sum; we obtain

$$\sum_{k=0}^{3M}P(f_i=k)=\frac{1}{L}\times\frac{M(M+1)}{2}\ +\frac{1}{L^2}\times\frac{(M+1)^2(M+2)}{2}-\frac{1}{L^2}+\frac{1}{L^3}\times\frac{(M+1)(M+2)}{2}\ .$$

Substituting $L=(M+1)(M+2)/2$ and simplifying, we obtain

$$\begin{aligned}
\sum_{k=0}^{3M} P(f_i=k) &= \frac{1}{L}\times\frac{M(M+1)}{2} + \frac{M+1}{L^2}\times\frac{(M+1)(M+2)}{2} - \frac{1}{L^2} + \frac{1}{L^3}\times\frac{(M+1)(M+2)}{2} \\
&= \frac{1}{L}\times\frac{M(M+1)}{2} + \frac{M+1}{L} - \frac{1}{L^2} + \frac{1}{L^2} \\
&= \frac{M(M+1)+2(M+1)}{2L} \\
&= \frac{(M+1)(M+2)}{2L} \\
&= \frac{(M+1)(M+2)}{2}\times\frac{1}{L} \\
&= L\times\frac{1}{L} \\
&= 1.
\end{aligned}$$

In order to compute the mean of a single frame, $E[f_i]$, for frames $i=1,2,\ldots,N-1$, with M pins, we must multiply each score by its probability. Thus, the weighted sum for scores from 0 to $M-1$ can be combined to the following summation:

$$\frac{1}{L}\sum_{k=0}^{M-1} k(k+1) = \frac{1}{L}\sum_{k=0}^{M-1}(k^2+k) .$$

Also, the weighted sum for scores from M to $2M-1$, along with the first two parts for the score $2M$ can be combined to the following:

$$\frac{1}{L^2}\sum_{k=0}^{M}(M+k)\big(M(M+1-k)+k+1\big) - \frac{2M}{L^2} .$$

Finally, the weighted sum for scores from $2M+1$ to $3M$, along with the last part for the score $2M$, can be combined to the following summation:

$$\frac{1}{L^3}\sum_{k=0}^{M}(2M+k)(M+1-k) .$$

Therefore, the mean of a single frame with M pins for frames $i=1,2,\ldots,N-1$ is the following:

$$E[f_i] = \frac{1}{L}\sum_{k=0}^{M-1}(k^2+k) + \frac{1}{L^2}\sum_{k=0}^{M}(M+k)\big(M(M+1-k)+k+1\big) - \frac{2M}{L^2}$$
$$+\frac{1}{L^3}\sum_{k=0}^{M}(2M+k)(M+1-k)\ .$$

Using *Mathematica* to simplify the sum, we obtain

$$E[f_i] = \frac{1}{L}\times\frac{M(M^2-1)}{3} + \frac{1}{L^2}\times\frac{M(M+1)(M+2)(4M+5)}{6} - \frac{2M}{L^2}$$
$$+\frac{1}{L^3}\times\frac{7M(M+1)(M+2)}{6}\ .$$

Since $L = (M+1)(M+2)/2$, we can simplify the sum to be

$$\begin{aligned}
E[f_i] &= \frac{M^3-M}{3L} + \frac{1}{L^2}\times\frac{(M+1)(M+2)}{2}\times\frac{M(4M+5)}{3} - \frac{2M}{L^2} \\
&\quad + \frac{1}{L^3}\times\frac{(M+1)(M+2)}{2}\times\frac{7M}{3} \\
&= \frac{M^3-M}{3L} + \frac{1}{L^2}\times L\times\frac{M(4M+5)}{3} - \frac{2M}{L^2} + \frac{1}{L^3}\times L\times\frac{7M}{3} \\
&= \frac{M^3-M}{3L} + \frac{4M^2+5M}{3L} - \frac{6M}{3L^2} + \frac{7M}{3L^2} \\
&= \frac{M^3+4M^2+4M}{3L} + \frac{M}{3L^2} = \frac{L(M^3+4M^2+4M)+M}{3L^2} \\
&= \frac{LM(M+2)^2+M}{3L^2} = \frac{2M\big((M+1)(M+2)^3+2\big)}{3(M+1)^2(M+2)^2}\ .
\end{aligned}$$

This result agrees with the result previously obtained in Section 3.4.

As we can see, the Hohn distribution method for deriving the average score of a single frame, $E[f_i]$, where $i = 1,2,\ldots,N-1$, is similar to the derivation of the arithmetic mean. However, the Hohn distribution looks at the probability of scores based upon each pair (X_{i1}, X_{i2}) of a frame being equally likely out of L possibilities. On the other hand, the arithmetic mean assumes that each outcome is equally likely with regard to the total possibilities, which is denoted as A possibilities. Recall from Section 2.2, $A = (M^3 + 6M^2 + 9M + 2)/2$.

We can compare the probabilities of each score in a single frame for both methods, shown in the table below, where $0 \le k \le M - 1$.

Score	Probability with Arithmetic Mean	Probability with Hohn Distribution
k	$\frac{k+1}{A}$	$\frac{k+1}{L}$
$M+k$	$\frac{M(M+1-k)+k+1}{A}$	$\frac{M(M+1-k)+k+1}{L^2}$
$2M$	$\frac{3M+1}{A}$	$\frac{2M}{L^2} + \frac{M+1}{L^3}$
$2M+1+k$	$\frac{M-k}{A}$	$\frac{M-k}{L^3}$

We can see that the probability for scores from 0 to $M-1$ is much higher in the Hohn distribution while scores from M to $3M$ are much lower. For example, in the special case where $M = 10$, the probability of the score 20 in the Hohn distribution is 1/216, which is approximately 0.46%. On the other hand, using the arithmetic mean distribution, the probability is 31/846, which is approximately 3.66%. Therefore, the result for the average score of a single frame is much lower using the Hohn distribution compared to the arithmetic mean distribution. Since the arithmetic mean assumes that each possibility is equally likely, we are more interested in the results using the Hohn distribution. Thus, we will continue to derive the variance and standard deviation of a single frame using the Hohn distribution.

3.8 Variance of a Single Frame

In order to compute the variance, $Var(f_i)$, of a single frame for frames $i = 1,2,\ldots, N-1$, we first need to compute $E[{f_i}^2]$, which can be found by squaring each score, then multiplying it by the probability of the score and summing the results. The probabilities remain the same, so the computation is similar to the mean of a single frame. Thus, $E[{f_i}^2]$ can be written as the following:

$$E[{f_i}^2] = \frac{1}{L}\sum_{k=0}^{M-1}(k^3 + k^2) + \frac{1}{L^2}\sum_{k=0}^{M}(M+k)^2\left(M(M+1-k)+k+1\right) - \frac{(2M)^2}{L^2}$$
$$+ \frac{1}{L^3}\sum_{k=0}^{M}(2M+k)^2(M+1-k)\ .$$

Using *Mathematica* to simplify the sum; we obtain

$$E[{f_i}^2] = \frac{1}{L}\times\frac{M(M^2-1)(3M-2)}{12} + \frac{1}{L^2}\times\frac{M(M+1)(M+2)(11M^2+18M+1)}{12}$$
$$-\frac{4M^2}{L^2} + \frac{1}{L^3}\times\frac{M(M+1)(M+2)(33M+1)}{12}.$$

Since $L = (M+1)(M+2)/2$, we can simplify the sum to be

$$\begin{aligned}
E[{f_i}^2] &= \frac{1}{L}\times\frac{M(M^2-1)(3M-2)}{12} + \frac{1}{L^2}\times\frac{(M+1)(M+2)}{2}\times\frac{M(11M^2+18M+1)}{6} - \frac{4M^2}{L^2} \\
&\quad + \frac{1}{L^3}\times\frac{(M+1)(M+2)}{2}\times\frac{M(33M+1)}{6} \\
&= \frac{1}{L}\times\frac{M(M^2-1)(3M-2)}{12} + \frac{1}{L}\times\frac{M(11M^2+18M+1)}{6} - \frac{24M^2}{6L^2} + \frac{M(33M+1)}{6L^2} \\
&= \frac{(3M^4-2M^3-3M^2+2M)+(22M^3+36M^2+2M)}{12L} + \frac{9M^2+M}{6L^2} \\
&= \frac{L(3M^4+20M^3+33M^2+4M)+18M^2+2M}{12L^2} \\
&= \frac{(M+1)(M+2)(3M^4+20M^3+33M^2+4M)+36M^2+4M}{6(M+1)^2(M+2)^2}\ .
\end{aligned}$$

Thus, the variance of the score for frames $i = 1,2,\ldots,N-1$ is

$$Var(f_i) = E[{f_i}^2] - \left(E[f_i]\right)^2$$

For the special case with $M = 10$ pins, we have for frames $i = 1,2,\ldots,N-1$

$$E[f_i^{\,2}] = \frac{(11)(12)\left(3(10)^4 + 20(10)^3 + 33(10)^2 + 40\right) + 36(10)^2 + 40}{6(11)^2(12)^2}$$
$$= \frac{880565}{13068} \approx 67.3833\ .$$

Thus, the variance is

$$Var(f_i) = \frac{880565}{13068} - \left(\frac{47525}{6534}\right)^2 = \frac{309090115}{21346578} \approx 14.4796\,.$$

The computation of the variance can be applied to calculate the standard deviation, which is very applicable to the random bowling game. The result describes the measure of the spread of each frame's score. The standard deviation for frames $i = 1,2,\ldots,N-1$ with $M = 10$ pins is then given by

$$\sigma(f_i) = \sqrt{Var(f_i)} = \sqrt{\left(\frac{880565}{13068}\right) - \left(\frac{47525}{6534}\right)^2} \approx 3.8052\ .$$

3.9 Distribution Method for the Nth Frame

On the last frame, there is variation in the probabilities of scores M and higher. Thus, we need to look at the distribution of frame N separately, which is shown in the table below, where $0 \le k \le M-1$.

Score	Set of Possibilities	Probability
k	$\{(0,k),(1,k-1),\ldots,(k,0)\}$	$\frac{k+1}{L}$
$M+k$	$\{(0,M),\ldots,(M-1,1)\}\times\{k\}$ $\cup\{(M,0)\}\times\{(0,k),\ldots,(k,0)\}$	$\frac{M}{L}\times\frac{1}{M+1}$ $+\frac{1}{L}\times\frac{k+1}{L}$
$2M$	$\{(0,M),\ldots,(M-1,1)\}\times\{M\}$ $\cup\{(M,0)\}\times\{(0,M),\ldots,(M-1,1)\}$ $\cup\{(M,0)\}\times\{(M,0)\}\times\{0\}$	$\frac{M}{L}\times\frac{1}{M+1}+\frac{1}{L}\times\frac{M}{L}$ $+\frac{1}{L}\times\frac{1}{L}\times\frac{1}{M+1}$
$2M+1+k$	$\{(M,0)\}\times\{(M,0)\}\times\{k+1\}$	$\frac{1}{L}\times\frac{1}{L}\times\frac{1}{M+1}$

We again can verify that all of the possible scenarios are given and make up a partition by summing all of the probabilities. The probabilities for scores from 0 to $M-1$ can be combined to the following summation:

$$\sum_{k=0}^{M-1}\left(\frac{k+1}{L}\right)=\frac{1}{L}\sum_{k=0}^{M-1}(k+1)\ .$$

Also, the probabilities for scores from M to $2M-1$ can be combined to the following summation:

$$\sum_{k=0}^{M-1}\left(\frac{M}{L}\times\frac{1}{M+1}+\frac{1}{L}\times\frac{k+1}{L}\right)=\frac{M}{L(M+1)}\sum_{k=0}^{M-1}1+\frac{1}{L^2}\sum_{k=0}^{M-1}(k+1)\ .$$

The score $2M$ is unique in a sense because the probability varies when two strikes are rolled in a row. Instead of combining the probability in a summation, we will write out the probability for the score $2M$ as

$$\frac{M}{L}\times\frac{1}{(M+1)}+\frac{1}{L}\times\frac{M}{L}+\frac{1}{L}\times\frac{1}{L}\times\frac{1}{(M+1)}=\frac{M}{L(M+1)}+\frac{M}{L^2}+\frac{1}{L^2(M+1)}\ .$$

Finally, the probabilities for the scores from $2M+1$ to $3M$ can be combined to the following summation:

$$\sum_{k=0}^{M-1}\left(\frac{1}{L}\cdot\frac{1}{L}\cdot\frac{1}{(M+1)}\right)=\frac{1}{L^2(M+1)}\sum_{k=0}^{M-1}1\,.$$

Now we can combine parts from the score $2M$ with scores $M+k$ and scores $2M+1+k$. We can see that the first part follows the exact pattern as scores $M+k$, and the second part follows a similar pattern to the scores $M+k$. We can rewrite the first two parts of the probability for the score $2M$ as

$$\frac{M}{L(M+1)}+\frac{M+1}{L^2}-\frac{1}{L^2}$$

and combine these values with the probabilities for the scores from M to $2M-1$, which results in the following summation:

$$\frac{M}{L(M+1)}\sum_{k=0}^{M}1+\frac{1}{L^2}\sum_{k=0}^{M}(k+1)-\frac{1}{L^2}\ .$$

Finally, the last part of the score $2M$ can be combined with the probabilities for the scores from $2M+1$ to $3M$, which results in the following summation:

$$\frac{1}{L^2(M+1)}\sum_{k=0}^{M}1\,.$$

Therefore, the sum of probabilities for all the scores $0,1,\ldots,3M$ in the $N\,th$ frame is

$$\sum_{k=0}^{3M}P(f_N=k)=\frac{1}{L}\sum_{k=0}^{M-1}(k+1)+\frac{M}{L(M+1)}\sum_{k=0}^{M}1+\frac{1}{L^2}\sum_{k=0}^{M}(k+1)-\frac{1}{L^2}+\frac{1}{L^2(M+1)}\sum_{k=0}^{M}1\,.$$

Using *Mathematica* to simplify the sum; we obtain

$$\sum_{k=0}^{3M} P(f_N = k) = \frac{1}{L} \times \frac{M(M+1)}{2} + \frac{M}{L(M+1)} \times (M+1) + \frac{1}{L^2} \times \frac{(M+1)(M+2)}{2} - \frac{1}{L^2}$$
$$+ \frac{1}{L^2(M+1)} \times (M+1) \ .$$

Substituting $L = (M+1)(M+2)/2$ and simplifying, we obtain

$$\begin{aligned}\sum_{k=0}^{3M} P(f_N = k) &= \frac{M(M+1)}{2L} + \frac{M}{L} + \frac{1}{L} - \frac{1}{L^2} + \frac{1}{L^2} \\ &= \frac{M(M+1) + 2(M+1)}{2L} \\ &= \frac{(M+1)(M+2)}{2L} \\ &= \frac{(M+1)(M+2)}{2} \times \frac{1}{L} \\ &= L \times \frac{1}{L} \\ &= 1 \ .\end{aligned}$$

In order to compute the mean of the last frame, $E[f_N]$, with M pins, we can apply the same method used in Section 3.7. Therefore, the mean of the last frame with M pins is the following:

$$E[f_N] = \frac{1}{L} \sum_{k=0}^{M-1} (k^2 + k) + \frac{M}{L(M+1)} \sum_{k=0}^{M} (M+k) + \frac{1}{L^2} \sum_{k=0}^{M} (M+k)(k+1) - \frac{2M}{L^2}$$
$$+ \frac{1}{L^2(M+1)} \sum_{k=0}^{M} (2M+k) \ .$$

Using *Mathematica* to simplify the sum; we obtain

$$E[f_N] = \frac{1}{L} \times \frac{M(M^2-1)}{3} + \frac{M}{L(M+1)} \times \frac{3M(M+1)}{2} + \frac{1}{L^2} \times \frac{5M(M+1)(M+2)}{6} - \frac{2M}{L^2}$$
$$+ \frac{1}{L^2(M+1)} \times \frac{5M(M+1)}{2} \ .$$

Substituting $L=(M+1)(M+2)/2$ and simplifying, we obtain

$$\begin{aligned}
E[f_N] &= \frac{M^3-M}{3L}+\frac{3M^2}{2L}+\frac{1}{L^2}\times\frac{(M+1)(M+2)}{2}\times\frac{5M}{3}-\frac{2M}{L^2}+\frac{5M}{2L^2} \\
&= \frac{M^3-M}{3L}+\frac{3M^2}{2L}+\frac{5M}{3L}-\frac{4M}{2L^2}+\frac{5M}{2L^2} \\
&= \frac{2LM^3-2LM}{6L^2}+\frac{9LM^2}{6L^2}+\frac{10LM}{6L^2}-\frac{12M}{6L^2}+\frac{15M}{6L^2} \\
&= \frac{L(2M^3+9M^2+8M)+3M}{6L^2} \\
&= \frac{M(M+1)(M+2)\big(2(M+1)(M+2)+3M+4\big)+6M}{3(M+1)^2(M+2)^2} \quad .
\end{aligned}$$

This result agrees with the result previously obtained in Section 3.5.

Similarly, the average score in the $N\,th$ frame of the Hohn distribution is much lower than the results found using the arithmetic mean distribution. The Hohn distribution looks at the probability of scores based upon each pair (X_{i1}, X_{i2}) being equally likely out of L possibilities. On the other hand, the arithmetic mean gives the probability of each score out of the total possibilities, which was previously computed in Section 2.1 as $A'(f(N)) = 2M^2+4M+1$.

Looking at the table below, we can compare the probabilities of each score in the $N\,th$ frame for both methods, where $0 \le k \le M-1$.

Score	Probability with Arithmetic Mean	Probability with Hohn Distribution
k	$\frac{k+1}{A'}$	$\frac{k+1}{L}$
$M+k$	$\frac{M+k+1}{A'}$	$\frac{M}{L(M+1)}+\frac{k+1}{L^2}$
$2M$	$\frac{2M+1}{A'}$	$\frac{M}{L(M+1)}+\frac{M}{L^2}+\frac{1}{L^2(M+1)}$
$2M+1+k$	$\frac{1}{A'}$	$\frac{1}{L^2(M+1)}$

We can see that the probability in the Nth frame for scores from 0 to $M-1$ is much higher in the Hohn distribution, while the probability for scores from M to $3M$ is much lower. For example, in the special case where $M=10$, the probability of the score 20 in the Hohn distribution is 257/15972, which is approximately 1.61%. On the other hand, using the arithmetic mean distribution, the probability is 21/241, which is approximately 8.71%. Therefore, the results for the average score in the Nth frame is much lower using the Hohn distribution compared to the arithmetic mean distribution. Again, we are more interested in the results using the Hohn distribution, so we will use it to continue to derive the variance and standard deviation of the Nth frame.

In order to compute the variance of the last frame, $Var(f_N)$, we first need to compute $E[f_N^2]$, which is given by

$$E[f_N^2] = \frac{1}{L}\sum_{k=0}^{M-1}(k^3+k^2) + \frac{M}{L(M+1)}\sum_{k=0}^{M}(M+k)^2 + \frac{1}{L^2}\sum_{k=0}^{M}(M+k)^2(k+1) - \frac{(2M)^2}{L^2} + \frac{1}{L^2(M+1)}\sum_{k=0}^{M}(2M+k)^2 .$$

Using *Mathematica* to simplify the sum; we obtain

$$E[f_N^2] = \frac{1}{L}\times\frac{M(M^2-1)(3M-2)}{12} + \frac{M}{L(M+1)}\times\frac{M(M+1)(14M+1)}{6} + \frac{1}{L^2}\times\frac{M(M+1)(M+2)(17M+1)}{12} - \frac{4M^2}{L^2} + \frac{1}{L^2(M+1)}\times\frac{M(M+1)(38M+1)}{6} .$$

Since $L=(M+1)(M+2)/2$, we can further simplify the sum to be

$$E[f_N^2] = \frac{3M^4 - 2M^3 - 3M^2 + 2M}{12L} + \frac{2M^2(14M+1)}{12L}$$

$$+ \frac{1}{L^2} \times \frac{(M+1)(M+2)}{2} \times \frac{2M(17M+1)}{12} - \frac{24M^2}{6L^2} + \frac{M(38M+1)}{6L^2}$$

$$= \frac{3M^4 - 2M^3 - 3M^2 + 2M}{12L} + \frac{28M^3 + 2M^2}{12L} + \frac{34M^2 + 2M}{12L} - \frac{24M^2}{6L^2}$$

$$+ \frac{38M^2 + M}{6L^2}$$

$$= \frac{L(3M^4 + 26M^3 + 33M^2 + 4M)}{12L^2} + \frac{14M^2 + M}{6L^2}$$

$$= \frac{L(3M^4 + 26M^3 + 33M^2 + 4M) + 28M^2 + 2M}{12L^2}$$

$$= \frac{(M+1)(M+2)\left(3M^4 + 26M^3 + 33M^2 + 4M\right) + 56M^2 + 4M}{6(M+1)^2(M+2)^2}.$$

Thus, the variance of the score of the $N\,th$ frame is

$$Var(f_N) = E[f_N^2] - \left(E[f_N]\right)^2$$

For the special case with $M = 10$ pins, we have

$$E[f_N^2] = \frac{(11)(12)\left(3(10)^4 + 26(10)^3 + 33(10)^2 + 40\right) + 56(10)^2 + 40}{6(11)^2(12)^2}$$

$$= \frac{326605}{4356} \approx 74.9782 \ .$$

Thus, the variance of the last frame is

$$Var(f_N) = \frac{326605}{4356} - \left(\frac{32785}{4356}\right)^2 = \frac{347835155}{18974736} \approx 18.3315,$$

and the standard deviation for the last frame is given by

$$\sigma(f_N) = \sqrt{Var(f_N)} = \sqrt{\frac{326605}{4356} - \left(\frac{32785}{4356}\right)^2} \approx 4.2815 \ .$$

Unlike the mean, the variance of a ten-frame game cannot simply be computed by summing each individual variance because the frames are not independent of one another. That is, f_i depends on f_{i+1} in the cases of strikes and spares. However, the variance of a single frame can be computed, which in the case of $M = 10$ pins is approximately 14.4796 in the first $N-1$ frames and 18.3315 in the Nth frame. In addition, the variance can be used to calculate the standard deviation, which is approximately 3.8052 in the first $N-1$ frames and 4.2815 in the Nth frame.

3.10 Correlation Coefficient Using the Hohn Distribution

We now shall find the correlation between the score of the first and second roll in a single frame of the Hohn distribution.

Definition 3.10.1. The correlation coefficient ρ_{X_1,X_2} between two random variables X_1 and X_2 with expected values $E[X_1]$ and $E[X_2]$ and standard deviations σ_{X_1} and σ_{X_2} is defined as

$$\rho_{X_1,X_2} = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}} \ .$$

In Section 3.3, we previously computed the average of the first roll on a set of pins to be $M/3$. Because the values on the second roll occur equally as often as those on the first roll, the average of the second ball also equals $M/3$, which is shown below:

$$E[X_2] = \frac{2}{(M+1)(M+2)} \sum_{k=0}^{M} k(M+1-k) \ .$$

Using *Mathematica* to simplify the sum; we obtain

$$E[X_2] = \frac{2}{(M+1)(M+2)} \times \frac{M(M+1)(M+2)}{6} = \frac{M}{3} \ .$$

In order to find the standard deviation of the first and second roll, we need to compute $E[X_i^2]$. Since the probability of each value remains the same, the only difference in the computation is squaring each score before multiplying it by the probability. Therefore,

$$E[X_i^2] = \frac{2}{(M+1)(M+2)} \sum_{k=0}^{M} k^2 (M+1-k) \ .$$

Using *Mathematica* to simplify the sum; we obtain

$$E[X_i^2] = \frac{2}{(M+1)(M+2)} \times \frac{M(M+1)^2(M+2)}{12} = \frac{M(M+1)}{6} \ .$$

So, the variance of both the first and second roll can be computed as the following:

$$\begin{aligned} Var(X_i) &= E[X_i^2] - \left(E[X_i]\right)^2 \\ &= \frac{M(M+1)}{6} - \left(\frac{M}{3}\right)^2 \\ &= \frac{3M(M+1)}{18} - \frac{2M^2}{18} \\ &= \frac{M(M+3)}{18} \ . \end{aligned}$$

Thus, the standard deviation of both the first and second roll is

$$\sigma_{X_i} = \sqrt{E[X_i^2] - (E[X_i])^2} = \sqrt{\frac{M(M+3)}{18}} \ .$$

Lastly, we need to compute $E[X_1 X_2]$, which is the average score of the first roll multiplied by the second roll. The probability of each individual product is still $1/L$, so $E[X_1 X_2]$ is given by

$$E[X_1 X_2] = \frac{2}{(M+1)(M+2)} \left(\sum_{n=0}^{M} n \sum_{k=0}^{M-n} k \right) .$$

Using *Mathematica* to simplify the sum, we obtain

$$E[X_1 X_2] = \frac{2}{(M+1)(M+2)} \times \frac{M(M+1)(M-1)(M+2)}{24} = \frac{M(M-1)}{12} \ .$$

Therefore, the correlation coefficient between rolls in a frame is computed to be

$$\begin{aligned}
\rho_{X_1,X_2} &= \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}} \\
&= \frac{\frac{M(M-1)}{12} - \left(\frac{M}{3}\right)\left(\frac{M}{3}\right)}{\sqrt{\frac{M(M+3)}{18}}\sqrt{\frac{M(M+3)}{18}}} \\
&= \frac{\frac{3M^2-3M}{36} - \frac{4M^2}{36}}{\frac{M(M+3)}{18}} \\
&= \frac{-M(M+3)}{36} \times \frac{18}{M(M+3)} \\
&= -\frac{1}{2} \ .
\end{aligned}$$

We can see that the correlation between the first and second roll in a single frame is not dependent on the number of pins in a frame. Moreover, the second roll is dependent on the first roll, and its value decreases as the value of the first roll increases, which is why the correlation coefficient is negative. However, we cannot conclude that there is a strong correlation between these values in the Hohn distribution.

Chapter 4: Generalization of the Neal Distribution

In this chapter, we generalize the Neal distribution from [5] and the Hohn method in [4] to derive the average score when bowling N frames on M pins where pins are knocked down according to discrete uniform distributions. In addition, we apply the distribution method to verify these results and compute the variance and standard deviation for a single frame. Lastly, we will find the correlation between the score of the first and second roll in a single frame of this type of random bowling.

4.1 Generalized Rules for the Neal Distribution

In general, there are N frames and two rolls per frame (X_{i1}, X_{i2}), where X_{i1} is a randomly chosen integer from 0 to M and X_{i2} is a randomly chosen integer from 0 to $M - X_{i1}$ for frames $i = 1,2,\ldots,N$. In each case, each integer in the range is equally likely to be chosen. If $X_{i1} = M$, then X_{i2} must equal 0. The score, f_i, in the $i\,th$ frame is initially the sum of these two rolls. However, if $X_{i1} + X_{i2} = M$ and $X_{i1} < M$, which is a spare, then the next roll $X_{(i+1)1}$ is added to f_i. This roll $X_{(i+1)1}$ again is chosen randomly from 0 to M and is independent of any previous roll.

A strike occurs if $X_{i1} = M$. If so, then the next two rolls are added to f_i. Since X_{i2} will equal 0, the next roll is actually $X_{(i+1)1}$ and is independently performed on a new set of pins. If $X_{(i+1)1} = M$ also, then the following roll is $X_{(i+2)1}$, which is also on a new set of pins. However, if $X_{(i+1)1} < M$, then the second roll of the additional two rolls is $X_{(i+1)2}$ and is chosen randomly from 0 to $M - X_{(i+1)1}$.

4.2 The Average Sum of Two Rolls

Because pins are knocked down at random on the first roll of each frame, the values of 0 through M are all equally likely to occur with probability $1/(M+1)$. That is,

X_{i1} is a discrete uniform random variable with range $\{0,1,\ldots,M\}$, denoted by $X_{i1} \sim DU[0, M]$. Thus for frames $i = 1,2,\ldots,N,$ we have

$$E[X_{i1}] = \frac{1}{M+1}\left(\sum_{k=0}^{M} k\right) = \frac{1}{M+1}\left(\frac{M(M+1)}{2}\right) = \frac{M}{2} \ .$$

The range of the second roll is dependent on the first roll. For instance, if a 6 is rolled on the first ball, then the score on the second roll is chosen at random from the integers 0 through $M-6$. So,

$$X_{i2} \sim DU[0, M - X_{i1}] = \sum_{j=0}^{M} DU[0, j] \times 1_{\{X_{i1} = M-j\}} \ .$$

The average score of the second roll is then given by

$$\begin{aligned} E[X_{i2}] &= \sum_{j=0}^{M} E\big[DU[0, j]\big] \times P(X_{i1} = M - j) \\ &= \sum_{j=0}^{M} \frac{j}{2} \times \frac{1}{M+1} \\ &= \frac{1}{2(M+1)} \times \frac{M(M+1)}{2} \\ &= \frac{M}{4} \ . \end{aligned}$$

Thus, without taking into account strikes and spares, the average score per frame is $E[X_{i1}] + E[X_{i2}] = 3M/4$.

4.3 Accounting for Spares

If a spare is rolled in a frame, then the next roll is added to the score. Since the next roll is on a new set of pins, the average score is $M/2$. However, the probability of a spare must be found in order to determine the average amount added to each frame's score. A spare occurs if $X_{i1} + X_{i2} = M$, but $X_{i1} < M$. For example, if $X_{i1} = 3$, which occurs with probability $1/(M+1)$, then X_{i2} must equal $M-3$, which occurs with probability $1/(M-2)$ since the second roll is only on $M-3$ pins. In general, the average addition for spares is given by

$$E[X_{(i+1)1}] \times P(X_{i1} + X_{i2} = M \;\cap\; X_{i1} < M)$$
$$= \frac{M}{2} \times \sum_{y=0}^{M-1} P(X_{i1} = y) \times P(X_{i2} = M - y)$$
$$= \frac{M}{2} \times \sum_{y=0}^{M-1} \frac{1}{M+1} \times \frac{1}{M+1-y} .$$

Definition 4.3.1. The harmonic function is be defined by

$$Har(n) = \sum_{y=1}^{n} \frac{1}{y} .$$

Using the definition of harmonic function, we obtain the following identity:

$$\sum_{y=0}^{M-1} \frac{1}{M+1-y} = \sum_{y=2}^{M+1} \frac{1}{y} = \sum_{y=1}^{M+1} \frac{1}{y} - 1 = Har(M+1) - 1 .$$

Thus, the average addition for spares can be written as

$$\frac{M}{2} \times \frac{1}{M+1} \times \sum_{y=0}^{M-1} \frac{1}{M+1-y} = \frac{M}{2(M+1)} \times \left(Har(M+1) - 1\right) .$$

4.4 Accounting for Strikes

If a strike is rolled in a frame, then the next two rolls are added to the frame's score. There are two cases in this situation: either the next roll after the strike is also a strike, or the next roll is not a strike. In the first case, the average addition to the original frame is M from the second strike, plus the average score of $M/2$ from the next roll on a new set of pins. This first case occurs only in the event of two strikes in a row which has probability $1/(M+1)^2$. Thus the average addition to a frame in this case is

$$(M + E[X_{(i+2)1}]) \times P\left(X_{i1} = M\right) \times P(X_{(i+1)1} = M)$$
$$= \left(M + \frac{M}{2}\right) \times \frac{1}{M+1} \times \frac{1}{M+1} = \frac{3M}{2(M+1)^2}.$$

If a strike is rolled with probability $1/(M+1)$, and the next roll is not another strike, then the average addition to the frame's score is given by the average of the next two rolls given that the first of these rolls is not M. Equivalently, it is the unconditional average $3M/4$ of the next two rolls minus the conditional average of the next two rolls given that the first roll *is* M (and the second roll is automatically a 0). So the average addition to a frame in this situation is

$$P\left(X_{i1} = M\right) \times E[X_{(i+1)1} + X_{(i+1)2} \big| X_{(i+1)1} < M] \times P\left(X_{(i+1)1} < M\right)$$
$$= \frac{1}{M+1} \times (E[X_{(i+1)1} + X_{(i+1)2}] - E[X_{(i+1)1} + X_{(i+1)2} \big| X_{(i+1)1} = M] \times P(X_{(i+1)1} = M))$$
$$= \frac{1}{M+1} \times \left(\frac{3M}{4} - M \times \frac{1}{M+1}\right)$$
$$= \frac{1}{M+1} \times \left(\frac{3M(M+1)}{4(M+1)} - \frac{4M}{4(M+1)}\right) = \frac{3M^2 - M}{4(M+1)^2}.$$

4.5 The Average Score

Finally, the average score per frame in random bowling with the generalized Neal distribution is

$$E[f_i] = \frac{M}{2} + \frac{M}{4} + \frac{M}{2(M+1)} \times \left(Har(M+1) - 1\right) + \frac{3M}{2(M+1)^2} + \frac{3M^2 - M}{4(M+1)^2}$$
$$= \frac{3M(M+1)^2}{4(M+1)^2} + \frac{M}{2(M+1)} \times Har(M+1) - \frac{2M(M+1)}{4(M+1)^2} + \frac{6M}{4(M+1)^2} + \frac{3M^2 - M}{4(M+1)^2}$$
$$= \frac{3M(M^2 + 2M + 1) - 2M(M+1) + 6M + 3M^2 - M}{4(M+1)^2} + \frac{M}{2(M+1)} \times Har(M+1)$$
$$= \frac{3M^3 + 7M^2 + 6M}{4(M+1)^2} + \frac{M}{2(M+1)} \times Har(M+1).$$

For the special case with $M = 10$ pins, we have

$$E[f_i] = \frac{3(10)^3 + 7(10)^2 + 6(10)}{4(10+1)^2} + \frac{10}{2(10+1)} \sum_{y=1}^{11} \frac{1}{y} = \frac{557471}{60984} \approx 9.14127,$$

and the final score for a ten-frame game is approximately 91.4127.

4.6 Generalization of the Hohn Extension

In [4], Hohn developed a new argument to derive the average score of a ten-frame bowling game having the Neal distribution. This method used a combination of probability and indicator functions which also allowed us to derive the variance and standard deviation of a single frame. We now shall generalize this Hohn extension to a game with N frames and M pins. By generalizing the results, we can make comparisons to the results of other distributions used to derive the average score of a bowling game.

To derive the average score, we shall use an indicator function to represent the frame's score. There are four possible disjoint events that could occur in each frame. The probability of each event will be calculated and combined with conditional averages to compute the average score per frame for the Neal distribution. Then the variance and standard deviation are calculated using the same indicator function argument.

4.7 Computing the Probability of Each Event

A frame has four possible scenarios, denoted by A_1, A_2, A_3, and A_4. A_1 is the event that the first and second rolls add up to be less than M. A_2 is the event that the first two rolls add up to M, but the first was less than M, which is known as a spare. A_3 is the event that the first roll is M, and the following roll is also M, which is two strikes in a row. Finally, A_4 is the event that the first roll is M, but the following roll is less than M. The four possible events are shown below in set notation:

$$A_1 = \{X_1 + X_2 \leq M - 1\} \qquad A_2 = \{X_1 + X_2 = M \cap X_1 < M\}$$
$$A_3 = \{X_1 = M \cap X_3 = M\} \qquad A_4 = \{X_1 = M \cap X_3 < M\}$$

where X_1 is the first roll, X_2 is the second roll if X_1 is not a strike, X_3 is used if X_1 is a strike or if $X_1 + X_2$ is a spare, and X_4 is used if X_1 and X_3 are both strikes. The events are disjoint, and the four events comprise of all cases for a single frame.

To compute $P(A_1)$, we consider each possibility for the first roll, which can range from 0 to $M-1$ along with each possibility for the second roll subject to what was previously rolled. There is a pattern in the computation, which can be simplified into the following summation:

$$\begin{aligned} P(A_1) &= \sum_{k=0}^{M-1} P(X_1 = k \,\cap\, 0 \le X_2 \le M-1-k) \\ &= \sum_{k=0}^{M-1} \frac{1}{M+1} \times \sum_{y=0}^{k} \frac{1}{M+1-y} = \frac{1}{M+1} \sum_{k=0}^{M-1} \sum_{y=0}^{k} \frac{1}{M+1-y} \ . \end{aligned}$$

But the sum of probabilities of all possible outcomes for rolls in a frame is

$$\frac{1}{M+1} \sum_{k=0}^{M} \left(\sum_{y=0}^{k} \frac{1}{M+1-y} \right) = \sum_{k=0}^{M} P(X_{i1} + X_{i2} = k) = 1 \ ,$$

so the probability of A_1 can be written as

$$\begin{aligned} P(A_1) &= \frac{1}{M+1} \sum_{k=0}^{M} \left(\sum_{y=0}^{k} \frac{1}{M+1-y} \right) - \frac{1}{M+1} \sum_{y=0}^{M} \frac{1}{M+1-y} \\ &= 1 - \frac{1}{M+1} \sum_{y=0}^{M} \frac{1}{M+1-y} = 1 - \frac{Har(M+1)}{M+1}. \end{aligned}$$

Similarly, the probability of a spare is given by

$$\begin{aligned} P(A_2) &= \sum_{k=0}^{M-1} P(X_1 = k \cap X_2 = M-k) \\ &= \sum_{k=0}^{M-1} \frac{1}{M+1} \times \frac{1}{M+1-k} = \frac{1}{M+1} \sum_{k=0}^{M-1} \frac{1}{M+1-k} \\ &= \frac{Har(M+1)-1}{M+1}. \end{aligned}$$

The probability of two strikes in a row is given by

$$P(A_3) = P(X_1 = M \cap X_3 = M) = \frac{1}{M+1} \times \frac{1}{M+1} = \frac{1}{(M+1)^2} \ ,$$

and the probability of a strike followed by a non-strike is

$$P(A_4) = P(X_1 = M \cap 0 \le X_3 \le M-1) = \frac{1}{M+1} \times \frac{M}{M+1} = \frac{M}{(M+1)^2} \, .$$

We now have

$$\begin{aligned} P(A_1) + P(A_2) + P(A_3) + P(A_4) &= 1 - \frac{Har(M+1)}{M+1} + \frac{Har(M+1)-1}{M+1} + \frac{1}{(M+1)^2} + \frac{M}{(M+1)^2} \\ &= 1 - \frac{1}{M+1} + \frac{M+1}{(M+1)^2} \\ &= 1 \, . \end{aligned}$$

The probabilities sum to one, which verifies that all of the possible scenarios are given and make up a partition.

4.8 Computing the Mean of a Random Bowling Game Frame

Using indicator functions, a frame's score can be written as

$$f_i = (X_1 + X_2) \times 1_{A_1} + (M + X_3) \times 1_{A_2} + (2M + X_4) \times 1_{A_3} + (M + X_3 + X_4) \times 1_{A_4}$$

where 1_{A_i} is equal to one if that event occurs, and equals zero otherwise. For example, if the frame resulted in an $M-2$ on the first roll, a 2 on the second roll, and an $M-6$ on the third roll, then event A_2 has occurred. So 1_{A_2} is equal to one, and all other indicators are equal to zero. The frame's score would result in the following:

$$\begin{aligned} f_i &= (X_1 + X_2) \times 0 + (M + X_3) \times 1 + (2M + X_4) \times 0 + (M + X_3 + X_4) \times 0 \\ &= (M + X_3) = (M + M - 6) = 2M - 6 \, . \end{aligned}$$

Because the four events are disjoint and comprise a partition, the mean of each frame is the sum of the weighted conditional averages given by

$$\begin{aligned} E[f_i] &= E[(X_1 + X_2)|A_1] \times P(A_1) + E[(M + X_3)|A_2] \times P(A_2) \\ &+ E[(2M + X_4)|A_3] \times P(A_3) + E[(M + X_3 + X_4)|A_4] \times P(A_4). \end{aligned}$$

The first weighted conditional average is given by

$$\begin{aligned} E[X_1 + X_2 \mid A_1] \times P(A_1) &= \left(\sum_{k=0}^{M-1} k \times P(X_1 + X_2 = k \mid A_1) \right) \times P(A_1) \\ &= \left(\sum_{k=0}^{M-1} k \times \frac{P(X_1 + X_2 = k \ \cap \ A_1)}{P(A_1)} \right) \times P(A_1) \\ &= \sum_{k=0}^{M-1} k \times P(X_1 + X_2 = k) \\ &= \sum_{k=0}^{M-1} k \sum_{y=0}^{k} P(X_1 = y \cap X_2 = k - y) \\ &= \sum_{k=0}^{M-1} k \sum_{y=0}^{k} \frac{1}{M+1} \times \frac{1}{M+1-y} \\ &= \frac{1}{M+1} \sum_{k=0}^{M-1} k \sum_{y=0}^{k} \frac{1}{M+1-y} . \end{aligned}$$

However from Section 4.2, the average score per frame without taking into account strikes and spares was found to be

$$\frac{1}{M+1} \sum_{k=0}^{M} k \left(\sum_{y=0}^{k} \frac{1}{M+1-y} \right) = \frac{3M}{4};$$

thus, the first weighted conditional average can be simplified to

$$E[X_1 + X_2 \mid A_1] \times P(A_1) = \frac{1}{M+1} \sum_{k=0}^{M} \left(k \sum_{y=0}^{k} \frac{1}{M+1-y} \right) - \frac{M}{M+1} \sum_{y=0}^{M} \frac{1}{M+1-y}$$
$$= \frac{3M}{4} - \frac{M}{M+1} \sum_{y=0}^{M} \frac{1}{M+1-y}$$
$$= \frac{3M}{4} - \frac{M}{M+1} \times Har(M+1) \ .$$

Next, we consider the term dealing with event A_2. The principles of conditional averages are used to simplify this part because $M + X_3$ is independent of A_2. Also, using the properties of the mean and the results in Section 4.2 for the average score of a single roll, which is $M/2$, the weighted average results in the following:

$$\begin{aligned} E[(M + X_3) \mid A_2] \times P(A_2) &= E[M + X_3] \times P(A_2) \\ &= \left(E[M] + E[X_3]\right) \times P(A_2) \\ &= \left(M + \frac{M}{2}\right) \times P(A_2) \\ &= \frac{3M}{2} \times \frac{Har(M+1) - 1}{M+1}. \end{aligned}$$

Similarly, because $2M + X_4$ is independent of A_3, we have

$$\begin{aligned} E[(2M + X_4) \mid A_3] \times P(A_3) &= E[2M + X_4] \times P(A_3) \\ &= \left(E[2M] + E[X_4]\right) \times P(A_3) \\ &= \left(2M + \frac{M}{2}\right) \times \frac{1}{(M+1)^2} \\ &= \frac{5M}{2(M+1)^2}. \end{aligned}$$

For the term dealing with event A_4, we note that $X_3 + X_4$ is dependent on A_4, but M is independent of A_4. Thus, the last weighted conditional average can be written as

$$E[(M + X_3 + X_4) \mid A_4] \times P(A_4) = E[M] \times P(A_4) + E[(X_3 + X_4) \mid A_4] \times P(A_4) \ .$$

Using the principles of conditional averages and the independence of rolls X_1 and X_3 the second term may be written as

$$
\begin{aligned}
&E[(X_3+X_4)\,|\,A_4]\times P(A_4) \\
&= E[(X_3+X_4)\,|\,X_1 = M \cap X_3 < M]\times P(X_1 = M \cap X_3 < M) \\
&= \sum_k k\times P(X_3+X_4 = k\ |\ X_1 = M \cap X_3 < M\ )\times P(X_1 = M \cap X_3 < M) \\
&= \sum_k k\times \frac{P(X_3+X_4 = k \cap X_1 = M \cap X_3 < M\ )}{P(X_1 = M \cap X_3 < M)}\times P(X_1 = M)\times P(X_3 < M) \\
&= \sum_k k\times \frac{P(X_3+X_4 = k \cap X_3 < M\ )\times P(X_1 = M)}{P(X_1 = M)\times P(X_3 < M)}\times P(X_1 = M)\times P(X_3 < M) \\
&= \sum_{k=0}^{M} k\times P(X_3+X_4 = k \cap X_3 < M)\times P(X_1 = M)\ .
\end{aligned}
$$

For further simplification, this last expression can be broken into two summations. The first sum, where X_3+X_4 ranges from 0 to $M-1$, is similar to the probability of event A_1. The second sum, where X_3+X_4 equals M, is similar to the probability of event A_2. The sums are shown to be

$$
\begin{aligned}
&E[(X_3+X_4)|A_4]\times P(A_4) \\
&= \left(\sum_{k=0}^{M-1} k\times P(X_3+X_4 = k) + M\sum_{k=0}^{M-1} P(X_3 = k \cap X_4 = M-k)\right)\times P(X_1 = M) \\
&= \left(E[X_1+X_2|A_1]\times P(A_1) + M\times P(A_2)\right)\times \frac{1}{M+1} \\
&= \frac{1}{M+1}\times\left[\left(\frac{3M}{4} - \frac{M}{M+1}\times Har(M+1)\right) + M\times\frac{Har(M+1)-1}{M+1}\right] \\
&= \frac{3M}{4(M+1)} - \frac{M}{(M+1)^2}\times Har(M+1) + \frac{M}{(M+1)^2}\times Har(M+1) - \frac{M}{(M+1)^2} \\
&= \frac{3M(M+1)}{4(M+1)^2} - \frac{4M}{4(M+1)^2} \\
&= \frac{M(3M-1)}{4(M+1)^2}.
\end{aligned}
$$

Thus, the weighted conditional average involving event A_4 yields

$$\begin{aligned} E[(M+X_3+X_4)\,|\,A_4]\times P(A_4) &= E[M]\times P(A_4)+E[(X_3+X_4)\,|\,A_4]\times P(A_4)\ . \\ &= M\times\frac{M}{(M+1)^2}+\frac{M(3M-1)}{4(M+1)^2} \\ &= \frac{M(7M-1)}{4(M+1)^2}. \end{aligned}$$

Therefore, by combining all four weighted conditional averages, $E[F]$ can be written as

$$\begin{aligned} E[f_i] &= \frac{3M}{4}-\frac{M}{M+1}\times Har(M+1)+\frac{3M}{2}\times\frac{Har(M+1)-1}{M+1}+\frac{5M}{2(M+1)^2}+\frac{M(7M-1)}{4(M+1)^2} \\ &= \frac{3M(M+1)^2}{4(M+1)^2}+\left(\frac{3M}{2(M+1)}-\frac{M}{M+1}\right)\times Har(M+1)-\frac{6M(M+1)}{4(M+1)^2}+\frac{10M}{4(M+1)^2} \\ &\quad+\frac{M(7M-1)}{4(M+1)^2} \\ &= \frac{3M(M+1)^2-6M(M+1)+10M+M(7M-1)}{4(M+1)^2}+\frac{M}{2(M+1)}\times Har(M+1) \\ &= \frac{3M^3+6M^2+3M-6M^2-6M+10M+7M^2-M}{4(M+1)^2}+\frac{M}{2(M+1)}\times Har(M+1) \\ &= \frac{3M^3+7M^2+6M}{4(M+1)^2}+\frac{M}{2(M+1)}\times Har(M+1). \end{aligned}$$

This result agrees with the result previously obtained in Section 4.5.

4.9 Frame Variance for the Neal Distribution

Using our setup, we can obtain the variance of a frame's score which was not previously found by Neal. In order to compute the variance, we first need to find the square of the frame's score . Recall that the frame's score can be represented as the following:

$$f_i = (X_1+X_2)\times 1_{A_1}+(M+X_3)\times 1_{A_2}+(2M+X_4)\times 1_{A_3}+(M+X_3+X_4)\times 1_{A_4}.$$

We note that the four events, A_i, are disjoint, meaning only one event can occur in a given frame. Therefore, many terms are able to cancel out using the background on disjoint sets and the indicator function. Since the events are disjoint, anytime a product includes two different indicator functions, the result is zero because if one indicator was equal to one, the other indicator is equal to zero. Also, if a product includes two identical indicators the result remains one, so there is no need to square the indicator. Therefore, ${f_i}^2$ can be written as follows:

$$\begin{aligned}{f_i}^2 &= (X_1 + X_2)^2 \times 1_{A_1} + (M + X_3)^2 \times 1_{A_2} + (2M + X_4)^2 \times 1_{A_3} \\ &\quad + (M + X_3 + X_4)^2 \times 1_{A_4}.\end{aligned}$$

Next, the mean of ${f_i}^2$ must be calculated. Similar to $E[f_i]$, $E[{f_i}^2]$ can be written as the following:

$$\begin{aligned}E[{f_i}^2] &= E[(X_1 + X_2)^2 \big| A_1] \times P(A_1) + E[(M + X_3)^2 \big| A_2] \times P(A_2) \\ &\quad + E[(2M + X_4)^2 \big| A_3] \times P(A_3) + E[(M + X_3 + X_4)^2 \big| A_4] \times P(A_4).\end{aligned}$$

In order to calculate $E[{f_i}^2]$, we separately consider the four terms and apply the same properties that were used to compute $E[f_i]$. The first term yields the following:

$$\begin{aligned}E[(X_1 + X_2)^2 | A_1] \times P(A_1) &= \sum_{k=0}^{M-1} k^2 \times P(X_1 + X_2 = k) \\ &= \sum_{k=0}^{M-1} k^2 \sum_{y=0}^{k} \frac{1}{M+1} \times \frac{1}{M+1-y} \\ &= \frac{1}{M+1} \sum_{k=0}^{M-1} \left(k^2 \sum_{y=0}^{k} \frac{1}{M+1-y} \right).\end{aligned}$$

However, using *Mathematica*, we obtain the following identity:

$$\frac{1}{M+1}\sum_{k=0}^{M} k^2\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)=\frac{M(22M+5)}{36} \ .$$

Thus, the first weighted conditional average can be simplified to

$$\begin{aligned}
E[(X_1+X_2)^2\,|\,A_1]\times P(A_1) &= \frac{1}{M+1}\sum_{k=0}^{M}\left(k^2\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{M^2}{M+1}\sum_{y=0}^{M}\frac{1}{M+1-y} \\
&= \frac{M(22M+5)}{36}-\frac{M^2}{M+1}\sum_{y=0}^{M}\frac{1}{M+1-y} \\
&= \frac{M(22M+5)}{36}-\frac{M^2}{M+1}\times Har(M+1) \ .
\end{aligned}$$

The second weighted conditional average yields

$$\begin{aligned}
E[(M+X_3)^2|A_2]\times P(A_2) &= E[(M+X_3)^2]\times P(A_2) \\
&= E[M^2+2MX_3+X_3^2]\times P(A_2) \\
&= (M^2+2ME[X_3]+E[X_3^2])\times P(A_2) \\
&= \left(M^2+2M\times\frac{M}{2}+\sum_{k=0}^{M}k^2\times P(X_3=k)\right)\times P(A_2) \\
&= \left(2M^2+\frac{1}{M+1}\sum_{k=0}^{M}k^2\right)\times P(A_2) \\
&= \left(2M^2+\frac{1}{M+1}\times\frac{M(M+1)(2M+1)}{6}\right)\times P(A_2) \\
&= \left(\frac{12M^2}{6}+\frac{2M^2+M}{6}\right)\times\frac{Har(M+1)-1}{M+1} \\
&= \frac{M(14M+1)}{6}\times\frac{Har(M+1)-1}{M+1}.
\end{aligned}$$

The third weighted conditional average yields

$$
\begin{aligned}
E[(2M+X_4)^2 \mid A_3] \times P(A_3) &= E[\,(2M+X_4)^2] \times P(A_3) \\
&= E[4M^2 + 4MX_4 + X_4{}^2] \times P(A_3) \\
&= \left(4M^2 + 4M \times \frac{M}{2} + \sum_{k=0}^{M} k^2 \times P(X_4 = k) \right) \times P(A_3) \\
&= \left(6M^2 + \frac{1}{M+1} \times \frac{M(M+1)(2M+1)}{6} \right) \times \frac{1}{(M+1)^2} \\
&= \left(\frac{36M^2}{6} + \frac{2M^2+M}{6} \right) \times \frac{1}{(M+1)^2} \\
&= \frac{M(38M+1)}{6(M+1)^2} \; .
\end{aligned}
$$

Finally, in Section 4.8, we derived $E[(X_3+X_4) \mid A_4] \times P(A_4)$ to be $M(3M-1)/(4(M+1)^2)$, and similarly we can derive $E[(X_3+X_4)^2 \mid A_4] \times P(A_4)$ as the following:

$$
\begin{aligned}
E[(X_3+X_4)^2 \mid A_4] \times P(A_4) &= (E[(X_1+X_2)^2 \mid A_1] \times P(A_1) + M^2 \times P(A_2)) \times \frac{1}{M+1} \\
&= \frac{M(22M+5)}{36(M+1)} - \frac{M^2}{(M+1)^2} \times Har(M+1) + M^2 \times \frac{Har(M+1)-1}{(M+1)^2} \\
&= \frac{M(22M+5)}{36(M+1)} - \frac{M^2}{(M+1)^2} \, .
\end{aligned}
$$

Therefore, the fourth weighted conditional average yields

$$
\begin{aligned}
&E[(M+X_3+X_4)^2 \,|\, A_4]\times P(A_4)\\
&= E[M^2+2MX_3+2MX_4+X_3{}^2+2X_3X_4+X_4{}^2 \,|\, A_4]\times P(A_4)\\
&= E[(M^2+2MX_3+2MX_4)+(X_3{}^2+2X_3X_4+X_4{}^2) \,|\, A_4]\times P(A_4)\\
&= E[M^2+2M(X_3+X_4)+(X_3+X_4)^2 \,|\, A_4]\times P(A_4)\\
&= M^2\times P(A_4)+2M\times E[(X_3+X_4) \,|\, A_4]\times P(A_4)+E[(X_3+X_4)^2 \,|\, A_4]\times P(A_4)\\
&= M^2\times\frac{M}{(M+1)^2}+2M\times\frac{M(3M-1)}{4(M+1)^2}+\frac{M(22M+5)}{36(M+1)}-\frac{M^2}{(M+1)^2}\\
&= \frac{36M^3}{36(M+1)^2}+\frac{54M^3-18M^2}{36(M+1)^2}+\frac{22M^3+27M^2+5M}{36(M+1)^2}-\frac{36M^2}{36(M+1)^2}\\
&= \frac{112M^3-27M^2+5M}{36(M+1)^2}.
\end{aligned}
$$

Combining all four terms results in the following:

$$
\begin{aligned}
E[f_i{}^2] &= \frac{M(22M+5)}{36}-\frac{M^2}{M+1}\times Har(M+1)+\frac{M(14M+1)}{6}\times\frac{Har(M+1)-1}{M+1}\\
&\quad+\frac{M(38M+1)}{6(M+1)^2}+\frac{112M^3-27M^2+5M}{36(M+1)^2}\\
&= \frac{M(22M+5)(M+1)^2}{36(M+1)^2}+\left(\frac{M(14M+1)}{6(M+1)}-\frac{M^2}{M+1}\right)\times Har(M+1)\\
&\quad-\frac{6M(14M+1)(M+1)}{36(M+1)^2}+\frac{6M(38M+1)}{36(M+1)^2}+\frac{112M^3-27M^2+5M}{36(M+1)^2}\\
&= \frac{22M^4+77M^3+143M^2+10M}{36(M+1)^2}+\frac{M(8M+1)}{6(M+1)}\times Har(M+1).
\end{aligned}
$$

Therefore, the variance of the score of each frame with the Neal distribution is equal to

$$
\begin{aligned}
Var(f_i) &= E[f_i{}^2] - (E[f_i])^2 \\
&= \left(\frac{22M^4 + 77M^3 + 143M^2 + 10M}{36(M+1)^2} + \frac{M(8M+1)}{6(M+1)} \times Har(M+1) \right) \\
&\quad - \left(\frac{3M^3 + 7M^2 + 6M}{4(M+1)^2} + \frac{M}{2(M+1)} \times Har(M+1) \right)^2,
\end{aligned}
$$

and the standard deviation for each frame is then given by

$$
\begin{aligned}
\sigma_{f_i} &= \sqrt{Var(f_i)} \\
&= \sqrt{\begin{array}{l} \dfrac{22M^4 + 77M^3 + 143M^2 + 10M}{36(M+1)^2} + \dfrac{M(8M+1)}{6(M+1)} \times Har(M+1) \\ - \left(\dfrac{3M^3 + 7M^2 + 6M}{4(M+1)^2} + \dfrac{M}{2(M+1)} \times Har(M+1) \right)^2 \end{array}}.
\end{aligned}
$$

For the special case when $M = 10$ pins, we have

$$
E[f_i{}^2] = \frac{22(10)^4 + 77(10)^3 + 143(10)^2 + 10(10)}{36(11)^2} + \frac{10(8(10)+1)}{6(11)} \sum_{k=1}^{11} \frac{1}{k} = \frac{735533}{6776},
$$

$$
Var(f_i) = \left(\frac{735533}{6776} \right) - \left(\frac{557471}{60984} \right)^2 \approx 24.987,
$$

and

$$
\sigma_{f_i} = \sqrt{\left(\frac{735533}{6776} \right) - \left(\frac{557471}{60984} \right)^2} \approx 4.9987.
$$

4.10 Conclusion of Mean and Variance for the Neal Distribution

The average score for an N-frame bowling game can be computed by applying the linearity property of the mean. Therefore, we obtain the following theorem:

Theorem 4.10.1. The average score for an N-frame game of random bowling with the Neal distribution on M pins is given by

$$\sum_{i=1}^{N} E[f_i] = N \times \left[\frac{3M^3 + 7M^2 + 6M}{4(M+1)^2} + \frac{M}{2(M+1)} \times Har(M+1) \right],$$

where f_i represents the score of each frame.

So, the average score for a ten-frame game with $M = 10$ pins is

$$\sum_{i=1}^{10} E[f_i] = 10 \times \frac{557471}{60984} \approx 91.4127\,.$$

Therefore, the average score of a random bowler is approximately 91.4. Unlike the mean, the variance of a ten-frame game cannot simply be computed by summing each individual variance because the frames are not independent of one another. That is, f_i depends on f_{i+1} in the cases of strikes and spares. However, the variance of a single frame can be computed, which is approximately 24.987. In addition, the variance can be applied to calculate the standard deviation, which is approximately 4.9987.

4.11 Distribution Method for the Neal Distribution

We can verify the results for the mean and variance using the distribution method. To find the distribution of a frame with M pins, we must find the probability for scores $0,1,\ldots,3M$. On the first roll of each frame, the values of 0 through M are all equally likely to occur with probability $1/(M+1)$. Thus, the following table shows the probability for scores $n = 0,1,\ldots,3M$ in a single frame, where $0 \le k \le M-1$:

Score	Set of Possibilities	Probability
k	$\{(0,k),(1,k-1),\ldots,(k,0)\}$	$\frac{1}{M+1}\sum_{y=0}^{k}\frac{1}{M+1-y}$
$M+k$	$\{(0,M),\ldots,(M-1,1)\}\times\{k\}$ $\cup\{(M,0)\}\times\{(0,k),\ldots,(k,0)\}$	$\frac{1}{(M+1)^2}\sum_{y=0}^{M-1}\frac{1}{M+1-y}$ $+\frac{1}{(M+1)^2}\sum_{y=0}^{k}\frac{1}{M+1-y}$
$2M$	$\{(0,M),\ldots,(M-1,1)\}\times\{M\}$ $\cup\{(M,0)\}\times\{(0,M),\ldots,(M-1,1)\}$ $\cup\{(M,0)\}\times\{(M,0)\}\times\{0\}$	$\frac{1}{(M+1)^2}\sum_{y=0}^{M-1}\frac{1}{M+1-y}$ $+\frac{1}{(M+1)^2}\sum_{y=0}^{M-1}\frac{1}{M+1-y}$ $+\frac{1}{(M+1)^3}$
$2M+1+k$	$\{(M,0)\}\times\{(M,0)\}\times\{k+1\}$	$\frac{1}{(M+1)^3}$

We can verify that all of the possible scenarios are given and make up a partition by summing all of the probabilities, and we can combine similarities into common summations. Thus, the probabilities of scores from 0 to $M-1$ can be combined to the following summation, which is equivalent to $P(A_1)$:

$$\sum_{k=0}^{M-1}\left(\frac{1}{M+1}\sum_{y=0}^{k}\frac{1}{M+1-y}\right)=1-\frac{Har(M+1)}{M+1} \quad .$$

Also, the probabilities of scores from M to $2M-1$ can be combined to the following:

$$\frac{1}{(M+1)^2}\sum_{k=0}^{M-1}\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\sum_{y=0}^{k}\frac{1}{M+1-y}\right).$$

The score $2M$ is unique, so we will begin by writing its probability as

$$\frac{1}{(M+1)^2}\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\frac{1}{(M+1)^2}\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\frac{1}{(M+1)^3}\ .$$

Since the first part follows the exact pattern as scores $M+k$, and the second part follows a similar pattern, we can rewrite the first two parts of the probability of score $2M$ to be

$$\left(\frac{1}{(M+1)^2}\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\frac{1}{(M+1)^2}\sum_{y=0}^{M}\frac{1}{M+1-y}\right)-\frac{1}{(M+1)^2}$$

and combine these parts with the probabilities of scores from M to $2M-1$, which results in the following summation:

$$\begin{aligned}
&\frac{1}{(M+1)^2}\sum_{k=0}^{M}\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{1}{(M+1)^2}\\
&=\frac{1}{(M+1)^2}\sum_{k=0}^{M}\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}\right)+\frac{1}{(M+1)^2}\sum_{k=0}^{M}\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{1}{(M+1)^2}\\
&=\frac{1}{(M+1)^2}\times\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}\right)\times\sum_{k=0}^{M}1+\frac{1}{M+1}\times\left(\frac{1}{M+1}\sum_{k=0}^{M}\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{1}{(M+1)^2}\\
&=\frac{1}{(M+1)^2}\times\big(Har(M+1)-1\big)\times(M+1)+\frac{1}{M+1}\times 1-\frac{1}{(M+1)^2}\\
&=\frac{Har(M+1)-1}{M+1}+\frac{M}{(M+1)^2}\,.
\end{aligned}$$

Finally, the last part of the score $2M$ can be combined with the probabilities of scores from $2M+1$ to $3M$, which results in the following summation:

$$\frac{1}{(M+1)^3}\sum_{k=0}^{M}1=\frac{1}{(M+1)^3}\times(M+1)=\frac{1}{(M+1)^2}\ .$$

Therefore, the sum of probabilities for all the scores $n=0,1,\ldots,3M$ is

$$\sum_{n=0}^{3M} P(F_i = n) = 1 - \frac{Har(M+1)}{M+1} + \frac{Har(M+1)-1}{M+1} + \frac{M}{(M+1)^2} + \frac{1}{(M+1)^2}$$

$$= 1 - \frac{1}{M+1} + \frac{M+1}{(M+1)^2}$$

$$= 1\ .$$

The probabilities sum to one, which verifies that all of the possible scores are given and make up a partition.

In order to derive the mean, we can take the sum of each score multiplied by the probability of that score occurring. We can combine the mean of the scores from 0 to $M-1$ into the following summation, which is equivalent to $E[X_1 + X_2 \mid A_1] \times P(A_1)$:

$$\sum_{k=0}^{M-1} k\left(\frac{1}{M+1} y \sum_{i=0}^{k} \frac{1}{M+1-y}\right) = \frac{3M}{4} - \frac{M}{M+1} \times Har(M+1)\,.$$

Also, scores from M to $2M-1$, along with the first two parts of score $2M$ can be combined to the following:

$$\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{2M}{(M+1)^2}$$

$$=\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}\right)+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{2M}{(M+1)^2}$$

$$=\frac{1}{(M+1)^2}\times\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}\right)\times\sum_{k=0}^{M}(M+k)+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{2M}{(M+1)^2}$$

$$=\frac{1}{(M+1)^2}\times\left(Har(M+1)-1\right)\times\frac{3M(M+1)}{2}+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{2M}{(M+1)^2}$$

$$=\frac{3M}{2(M+1)}\times\left(Har(M+1)-1\right)+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{2M}{(M+1)^2}$$

$$=\frac{3M}{2(M+1)}\times\left(Har(M+1)-1\right)+\frac{M}{M+1}\times\left(\frac{1}{M+1}\sum_{k=0}^{M}\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)\right)$$

$$+\frac{1}{M+1}\times\left(\frac{1}{M+1}\sum_{k=0}^{M}k\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)\right)-\frac{2M}{(M+1)^2}$$

$$=\frac{3M}{2(M+1)}\times Har(M+1)-\frac{3M}{2(M+1)}+\frac{M}{M+1}\times 1+\frac{1}{M+1}\times\frac{3M}{4}-\frac{2M}{(M+1)^2}$$

$$=\frac{3M}{2(M+1)}\times Har(M+1)-\frac{6M(M+1)}{4(M+1)^2}+\frac{4M(M+1)}{4(M+1)^2}+\frac{3M(M+1)}{4(M+1)^2}-\frac{8M}{4(M+1)^2}$$

$$=\frac{3M}{2(M+1)}\times Har(M+1)+\frac{M(M-7)}{4(M+1)^2}.$$

Finally, the weighted sum for scores from $2M+1$ to $3M$, along with the last part for the score $2M$ can be combined to form the following summation:

$$\frac{1}{(M+1)^3}\sum_{k=0}^{M}\left(2M+k\right)=\frac{1}{(M+1)^3}\times\frac{5M\left(M+1\right)}{2}=\frac{5M}{2\left(M+1\right)^2}.$$

Therefore, the average score in a frame can be computed as

$$E[f_i] = \frac{3M}{4} - \frac{M}{M+1} \times Har(M+1) + \frac{3M}{2(M+1)} \times Har(M+1) + \frac{M(13M+5)}{4(M+1)^2} + \frac{5M}{2(M+1)^2}$$
$$= \frac{3M(M+1)^2}{4(M+1)^2} + \left(\frac{3M}{2(M+1)} - \frac{M}{M+1}\right) \times Har(M+1) + \frac{M(M-7)}{4(M+1)^2} + \frac{10M}{4(M+1)^2}$$
$$= \frac{3M(M^2+2M+1) + M^2 - 7M + 10M}{4(M+1)^2} + \left(\frac{3M}{2(M+1)} - \frac{M}{M+1}\right) \times Har(M+1)$$
$$= \frac{3M^3 + 7M^2 + 6M}{4(M+1)^2} + \frac{M}{2(M+1)} \times Har(M+1).$$

This result agrees with the result previously obtained in Section 4.8.

4.12 Verification of Frame Variance

We can also verify the variance derived in the Hohn extension. First, we need to compute $E[f_i^2]$, which can be found by squaring each score, multiplying it by the probability of the score, and then summing these products. The probability remains the same, so the computation is similar to the mean of a single frame.

We can combine the mean of the squared scores from 0 to $M-1$ as the following summation, which is equivalent to $E[(X_1 + X_2)^2 \mid A_1] \times P(A_1)$:

$$\frac{1}{M+1} \sum_{k=0}^{M-1} \left(k^2 \sum_{y=0}^{k} \frac{1}{M+1-y} \right) = \frac{M(22M+5)}{36} - \frac{M^2}{M+1} \times Har(M+1).$$

Also, the mean of the squared scores from M to $2M-1$, along with parts of $2M$ can be combined into the following summation, which is simplified throughout using *Mathematica*:

$$\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)^2\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}+\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{(2M)^2}{(M+1)^2}$$

$$=\frac{1}{(M+1)^2}\times\left(\sum_{y=0}^{M-1}\frac{1}{M+1-y}\right)\times\sum_{k=0}^{M}(M+k)^2$$

$$+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)^2\left(\sum_{y=0}^{n}\frac{1}{M+1-y}\right)-\frac{(2M)^2}{(M+1)^2}$$

$$=\frac{1}{(M+1)^2}\times\left(Har(M+1)-1\right)\times\frac{M(M+1)(14M+1)}{6}$$

$$+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)^2\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{4M^2}{(M+1)^2}$$

$$=\frac{M(14M+1)}{6(M+1)}\times\left(Har(M+1)-1\right)+\frac{1}{(M+1)^2}\sum_{k=0}^{M}(M+k)^2\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{4M^2}{(M+1)^2}$$

$$=\frac{M(14M+1)}{6(M+1)}\times\left(Har(M+1)-1\right)+\frac{M^2}{(M+1)^2}\sum_{k=0}^{M}\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)$$

$$+\frac{2M}{(M+1)^2}\sum_{k=0}^{M}k\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)+\frac{1}{(M+1)^2}\sum_{k=0}^{M}k^2\left(\sum_{y=0}^{k}\frac{1}{M+1-y}\right)-\frac{4M^2}{(M+1)^2}$$

$$=\frac{M(14M+1)}{6(M+1)}\times Har(M+1)-\frac{M(14M+1)}{6(M+1)}+\frac{M^2}{(M+1)}\times 1+\frac{2M}{(M+1)}\times\frac{3M}{4}$$

$$+\frac{1}{(M+1)}\times\frac{M(22M+5)}{36}-\frac{4M^2}{(M+1)^2}$$

$$=\frac{M(14M+1)}{6(M+1)}\times Har(M+1)-\frac{6M(14M+1)}{36(M+1)}+\frac{90M^2}{36(M+1)}+\frac{M(22M+5)}{36(M+1)}-\frac{144M^2}{36(M+1)^2}$$

$$=\frac{M(14M+1)}{6(M+1)}\times Har(M+1)+\frac{M(28M-1)(M+1)}{36(M+1)^2}-\frac{144M^2}{36(M+1)^2}$$

$$=\frac{M(14M+1)}{6(M+1)}\times Har(M+1)+\frac{28M^3-117M^2-M}{36(M+1)^2}.$$

Finally, the weighted sum for squared scores from $2M+1$ to $3M$, along with the last part of $2M$, can be combined to form the following summation:

$$\frac{1}{(M+1)^3}\sum_{k=0}^{M}(2M+k)^2 = \frac{1}{(M+1)^3}\times\frac{M(M+1)(38M+1)}{6} = \frac{M(38M+1)}{6(M+1)^2} \ .$$

Thus, $E[f_i{}^2]$ can be written as the following:

$$\begin{aligned}
E[f_i{}^2] &= \frac{M(22M+5)}{36} - \frac{M^2}{M+1}\times Har(M+1) + \frac{M(14M+1)}{6(M+1)}\times Har(M+1) \\
&+ \frac{28M^3 - 117M^2 - M}{36(M+1)^2} + \frac{M(38M+1)}{6(M+1)^2} \\
&= \frac{M(22M+5)(M+1)^2}{36(M+1)^2} + \left(\frac{M(14M+1)}{6(M+1)} - \frac{M^2}{M+1}\right)\times Har(M+1) \\
&+ \frac{28M^3 - 177M^2 - M}{36(M+1)^2} + \frac{6M(38M+1)}{36(M+1)^2} \\
&= \frac{22M^4 + 49M^3 + 32M^2 + 5M + 28M^3 - 117M^2 - M + 228M^2 + 6M}{36(M+1)^2} \\
&+ \left(\frac{M(14M+1)}{6(M+1)} - \frac{6M^2}{6(M+1)}\right)\times Har(M+1) \\
&= \frac{22M^4 + 77M^3 + 143M^2 + 10M}{36(M+1)^2} + \frac{M(8M+1)}{6(M+1)}\times Har(M+1)\,.
\end{aligned}$$

This result agrees with the result previously obtained in Section 4.9, so the variance and standard deviation will also agree with the Hohn Extension.

4.13 Correlation Coefficient in Random Bowling

We can find the correlation between the score of the first and second roll in a single frame of the Neal distribution, defined as

$$\rho_{X_1,X_2} = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}} \ .$$

In Section 4.2, we computed the average of the first roll on a set of pins to be $M/2$. We now compute the average square of the first roll as follows:

$$E[X_1{}^2] = \frac{1}{M+1}\sum_{k=0}^{M} k^2 = \frac{1}{M+1}\left(\frac{M(M+1)(2M+1)}{6}\right) = \frac{M(2M+1)}{6} .$$

Applying this result, we obtain

Theorem 4.13.1. For $X_1 \sim DU[0,M]$, the variance of X_1 is

$$Var(X_1) = E[X_1^2] - \left(E[X_1]\right)^2 = \frac{M(2M+1)}{6} - \left(\frac{M}{2}\right)^2 = \frac{M(M+2)}{12},$$

and the standard deviation of X_1 is

$$\sigma_{X_1} = \sqrt{Var(X_1)} = \sqrt{\frac{M(M+2)}{12}} .$$

In Section 4.2, we also computed the average of the second roll on a set of pins to be $M/4$, and we can note that $X_2 \sim \sum_{j=0}^{M} DU[0,j] \times 1_{\{X_1 = M-j\}}$. Thus, we can compute the average square of the second roll as follows:

$$\begin{aligned}
E[X_2^2] &= \sum_{j=0}^{M} E[X^2 \,|\, X \sim DU[0,j]] \times P\left(X_1 = M-j\right) \\
&= \sum_{j=0}^{M} \frac{j(2j+1)}{6} \times \frac{1}{M+1} = \frac{1}{6(M+1)}\left(2\sum_{j=0}^{M} j^2 + \sum_{j=0}^{M} j\right) \\
&= \frac{1}{6(M+1)}\left(2 \times \frac{M(M+1)(2M+1)}{6} + \frac{M(M+1)}{2}\right) \\
&= \frac{1}{6}\left(\frac{2M(2M+1)}{6} + \frac{3M}{6}\right) = \frac{M(4M+5)}{36} .
\end{aligned}$$

Now we can compute the variance of X_2 to be

$$Var(X_2) = E[X_2^2] - \left(E[X_2]\right)^2 = \frac{M(4M+5)}{36} - \left(\frac{M}{4}\right)^2 = \frac{M(7M+20)}{144},$$

and the standard deviation of X_2 is

$$\sigma_{X_2} = \sqrt{E[X_2^2] - \left(E[X_2]\right)^2} = \sqrt{\frac{M(7M+20)}{144}} .$$

We can verify the results computed above for $E[X_2]$ and $E[X_2^2]$ by using the distribution of the second roll of the Neal distribution, which is shown to be

$$\begin{aligned} E[X_2] &= \sum j \times P(X_2 = j) \\ &= \sum_{j=0}^{M} j \left(\frac{1}{M+1} \sum_{y=0}^{M-j} \frac{1}{M+1-y} \right) \\ &= \frac{1}{M+1} \sum_{j=0}^{M} j \left(\sum_{y=0}^{M-j} \frac{1}{M+1-y} \right). \end{aligned}$$

For the special case with $M = 10$ pins, we have

$$E[X_2] = \frac{1}{11} \sum_{j=0}^{10} j \left(\sum_{y=0}^{10-j} \frac{1}{11-y} \right) = \frac{1}{11} \times \frac{55}{2} = 2.5 .$$

This result is equivalent to the results derived in Section 4.2, which can be generalized to $M/4$. We can also find $E[X_2^2]$ in a similar manner as follows:

$$\begin{aligned} E[X_2^2] &= \sum j^2 \times P(X_2 = j) \\ &= \sum_{j=0}^{M} j^2 \left(\frac{1}{M+1} \sum_{y=0}^{M-j} \frac{1}{M+1-y} \right) \\ &= \frac{1}{M+1} \sum_{j=0}^{M} j^2 \left(\sum_{y=0}^{M-j} \frac{1}{M+1-y} \right). \end{aligned}$$

In the special case where $M = 10$ pins, we have

$$E[X_2^2] = \frac{1}{11}\sum_{j=0}^{10} j^2 \left(\sum_{i=0}^{10-j} \frac{1}{11-i} \right) = \frac{275}{22} = 12.5 \ .$$

This is equivalent to the results for $E[X_2^2]$ derived previously in this section, which can be generalized to $M\left(4M+5\right)/36$.

Lastly, we need to compute $E[X_1 X_2]$, which is shown to be

$$\begin{aligned}
E[X_1 X_2] &= \sum k \times y \times P\big((X_1, X_2) = (k, y)\big) \\
&= \sum_{k=0}^{M} \sum_{y=0}^{M-k} \left(k \times y \times \frac{1}{M+1} \times \frac{1}{M-k+1} \right) \\
&= \frac{1}{M+1} \sum_{k=0}^{M} \frac{k}{M-k+1} \sum_{y=0}^{M-k} y \\
&= \frac{1}{M+1} \sum_{k=0}^{M} \frac{k}{M-k+1} \left(\frac{(M-k)(M-k+1)}{2} \right) \\
&= \frac{1}{2(M+1)} \sum_{k=0}^{M} (kM - k^2) \\
&= \frac{1}{2(M+1)} \left(M \sum_{k=0}^{M} k - \sum_{k=0}^{M} k^2 \right) \\
&= \frac{1}{2(M+1)} \left(M \frac{M(M+1)}{2} - \frac{M(M+1)(2M+1)}{6} \right) \\
&= \frac{1}{2} \left(\frac{3M^2}{6} - \frac{2M^2 + M}{6} \right) \\
&= \frac{M(M-1)}{12} \ .
\end{aligned}$$

Therefore, the correlation coefficient between the first and second roll of a random bowler with the Neal distribution is computed to be

$$
\begin{aligned}
\rho_{X_1,X_2} &= \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}} \\
&= \frac{\dfrac{M(M-1)}{12} - \left(\dfrac{M}{2}\right)\left(\dfrac{M}{4}\right)}{\sqrt{\dfrac{M(M+2)}{12}} \times \sqrt{\dfrac{M(7M+20)}{144}}} \\
&= \left(\frac{M^2 - M}{12} - \frac{M^2}{8}\right) \div \left[\sqrt{\frac{M^2(M+2)(7M+20)}{1728}}\right] \\
&= \left(\frac{2M^2 - 2M}{24} - \frac{3M^2}{24}\right) \times \left[\sqrt{\frac{1728}{M^2(M+2)(7M+20)}}\right] \\
&= \left(\frac{-M(M+2)}{24}\right) \times \left[\frac{24\sqrt{3}}{M\sqrt{(M+2)(7M+20)}}\right] \\
&= -\frac{\sqrt{3}(M+2)}{\sqrt{(M+2)(7M+20)}} \\
&= -\sqrt{\frac{3(M+2)}{7M+20}} \;.
\end{aligned}
$$

For the special case with $M = 10$ pins, we have

$$
\rho_{X_1,X_2} = -\sqrt{\frac{3(10+2)}{70+20}} = -\frac{\sqrt{10}}{5} \approx -0.6324 \;.
$$

We can see that the correlation between the first and second roll of a single frame is dependent on the number of pins. For example with $M = 5$ pins, we have $\rho_{X_1,X_2} \approx -0.6179$. Moreover, the closer the coefficient is to either –1 or 1, the stronger the correlation between variables. Therefore, we can observe that there is a stronger linear relationship between these values using the Neal distribution compared to using the Hohn distribution. Also, the correlation coefficient is negative because the value of the second roll decreases as the value of the first roll increases.

Chapter 5: The Fibonacci Distribution for a Skilled Bowler

In this chapter, we apply two methods to derive the average score of a particular skilled bowler when bowling N frames on M pins using the Fibonacci distribution. First, we apply Neal's method to derive the mean of a single frame, and then we verify these results by applying the distribution method. In addition, we apply the distribution method to compute the variance and standard deviation for a single frame. Lastly, we will find the correlation between the score of the first and second roll in a single frame of bowling.

5.1 The Rules of the Fibonacci Distribution for a Skilled Bowler

We shall use the scoring rules of a standard bowling game for which there are ten frames and two rolls per frame (X_{i1}, X_{i2}), where X_{i1} is an integer ranging from 0 to 10. However, the probabilities for the first roll shall be obtained from the following Fibonacci distribution:

Score	0	1	2	3	4	5	6	7	8	9	10
Probability	$\frac{1}{232}$	$\frac{1}{232}$	$\frac{2}{232}$	$\frac{3}{232}$	$\frac{5}{232}$	$\frac{8}{232}$	$\frac{13}{232}$	$\frac{21}{232}$	$\frac{34}{232}$	$\frac{55}{232}$	$\frac{89}{232}$

An analysis of this distribution, denoted $X_n \sim Fib(n)$ when applied to the integers $\{1, . . . , n\}$, is given in [6]. In particular, the sum of the first n Fibonacci numbers is given as $F_{n+2} - 1$. However the table above shows a $Fib(11) - 1$ distribution which is applied to the integers $\{0, . . . , 10\}$. This distribution models a particular skilled bowler who tends to score high on the first roll.

In general, we shall use N frames and two rolls per frame $\left(X_{i1}, X_{i2}\right)$, where X_{i1} is an integer ranging from 0 to M with probabilities obtained from the following Fibonacci distribution:

Score	0	1	...	$M-1$	M
Probability	$\frac{F_1}{F_{M+3}-1}$	$\frac{F_2}{F_{M+3}-1}$	...	$\frac{F_M}{F_{M+3}-1}$	$\frac{F_{M+1}}{F_{M+3}-1}$

So, $X_{i1} \sim Fib(M+1)-1$ for frames $i = 1, \ldots, N$. We note that $F_{M+3}-1$ gives the sum of the first $M+1$ Fibonacci numbers used to determine the weights of the distribution on the integers $\{0, \ldots, M\}$.

Next, X_{i2} is an integer ranging from 0 to $M - X_{i1}$. The second roll X_{i2} is dependent upon the first. For instance, if $M = 10$ and $X_{i1} = 6$, then the second roll X_{i2} has the following distribution:

Score	0	1	2	3	4
Probability	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{5}{12}$

In general, $X_{i2} \sim Fib(M+1-X_{i1})-1$, for frames $i = 1, \ldots, N$:

Score	0	1	...	$M - X_{i1}$
Probability	$\frac{F_1}{F_{M+3-X_{i1}}-1}$	$\frac{F_2}{F_{M+3-X_{i1}}-1}$	...	$\frac{F_{M+1-X_{i1}}}{F_{M+3-X_{i1}}-1}$

5.2 The Average Sum of Two Rolls

If pins are knocked down by our skilled bowler, then on the first roll of each frame the values of k from 0 through M are likely to occur with probability $F_{k+1}/(F_{M+3}-1)$. Thus for frames $i = 1,2,\ldots,N$

$$E[X_{i1}] = \sum_{k=0}^{M} k \times \frac{F_{k+1}}{F_{M+3}-1} = \frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k(F_{k+1}).$$

For the special case with $M = 10$ pins, using *Mathematica* to simplify the sum, we have

$$E[X_{i1}] = \frac{1}{F_{13}-1}\sum_{k=0}^{10} k(F_{k+1}) = \frac{489}{58} \approx 8.4310.$$

We can derive the expected value of X_{i2} by finding the distribution of the second roll. For X_{i2} to equal k, the first roll can vary from 0 to $M-k$. The set of possibilities is then $\{(0,k),(1,k),\ldots,(M-k,k)\}$, and then

$$\begin{aligned} P(X_{i2}=k) &= \sum_{y=0}^{M-k} P(X_{i1}=y)\times P(X_{i2}=k \mid X_{i1}=y) \\ &= \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{M+3}-1}\times\frac{F_{k+1}}{F_{M-y+3}-1} \\ &= \frac{F_{k+1}}{F_{M+3}-1}\sum_{y=0}^{M-k}\frac{F_{y+1}}{F_{M+3-y}-1}. \end{aligned}$$

Therefore, the average score of the second roll is given by

$$\begin{aligned} E[X_{i2}] &= \sum_{k=0}^{M} k\times P\left(X_{i2}=k\right) \\ &= \sum_{k=0}^{M} k\left[\frac{F_{k+1}}{F_{M+3}-1}\sum_{y=0}^{M-k}\frac{F_{y+1}}{\mathrm{F}_{M+3\text{-y}}-1}\right] \\ &= \frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k\left(F_{k+1}\right)\sum_{y=0}^{M-k}\frac{F_{y+1}}{F_{M+3-y}-1}. \end{aligned}$$

For the special case with $M=10$ pins, using *Mathematica* to simplify the sum, we have

$$E\left[X_{i2}\right] = \frac{1}{F_{13}-1}\sum_{k=0}^{10} k\left(F_{k+1}\right)\sum_{y=0}^{10-k}\frac{F_{y+1}}{F_{13-y}-1} = \frac{14179157}{13321440} \approx 1.06434.$$

So without taking into account strikes and spares, the average score per frame is

$$E[X_{i1}]+E[X_{i2}]=\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k(F_{k+1})\right]+\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k(F_{k+1})\sum_{y=0}^{M-k}\frac{F_{y+1}}{F_{M+3\text{-}y}-1}$$
$$=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k(F_{k+1})\left[1+\sum_{y=0}^{M-k}\frac{F_{y+1}}{F_{M+3\text{-}y}-1}\right].$$

For the special case with $M=10$ pins, we have

$$E[X_{i1}]+E[X_{i2}]=\frac{126492677}{13321440}\approx 9.49542.$$

Proposition 5.2.2. The following summations are equivalent to the average score per frame without taking into account strikes and spares.

$$E[X_{i1}]+E[X_{i2}]=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k(F_{k+1})\left[1+\sum_{y=0}^{M-k}\frac{F_{y+1}}{F_{M+3\text{-}y}-1}\right]$$
$$=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right].$$

5.3 Accounting for Spares

If a spare is rolled in a frame, then the next roll is added to the score. Since the next roll is on a new set of pins, the average score is

$$E\left[X_{(i+1)1}\right]=\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k(F_{k+1})\right].$$

In order to determine the average amount added to each frame's score, the probability of a spare must be found. A spare occurs if $X_{i1}+X_{i2}=M$ and $X_{i1}<M$. For example, if $X_{i1}=3$, which occurs with probability $F_4/(F_{M+3}-1)$, then X_{i2} must equal $M-3$, which occurs with probability $F_{M-2}/(F_M-1)$ since the second roll is only on $M-3$ pins. In general, the average addition for spares is given by

$$
\begin{aligned}
&E\left[X_{(i+1)1}\right] \times P\left(X_{i1} + X_{i2} = M \cap X_{i1} < M\right) \\
&= \frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k\left(F_{k+1}\right)\right] \times \sum_{y=0}^{M-1} P\left(X_{i1} = y\right) \times P\left(X_{i2} = M - y\right) \\
&= \frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k\left(F_{k+1}\right)\right] \times \sum_{y=0}^{M-1}\left(\frac{F_{y+1}}{F_{M+3} - 1} \times \frac{F_{M+1-y}}{F_{M+3-y} - 1}\right) \\
&= \frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k\left(F_{k+1}\right)\right] \times \left(\frac{1}{F_{M+3} - 1} \sum_{y=0}^{M-1} \frac{F_{y+1} \times F_{M+1-y}}{F_{M+3-y} - 1}\right).
\end{aligned}
$$

5.4 Accounting for Strikes

If a strike is rolled in a frame, then the next two rolls are added to the frame's score. There are two cases in this situation: either the next roll after the strike is also a strike, or the next roll is not a strike. In the first case, the average addition to the original frame is M from the second strike, plus the average score from the next roll on a new set of pins, which is equivalent to $E[X_{i1}]$. This first case occurs only in the event of two strikes in a row which each have probability $F_{M+1}/(F_{M+3} - 1)$. Thus the average addition to a frame in this case is

$$
\begin{aligned}
&\left(M + E[X_{(i+2)1}]\right) \times P(X_{i1} = M) \times P(X_{(i+1)1} = M) \\
&= \left(M + \frac{1}{F_{M+3} - 1}\left(\sum_{k=0}^{M} k\, F_{k+1}\right)\right) \times \left(\frac{F_{M+1}}{F_{M+3} - 1}\right)^2 .
\end{aligned}
$$

If a strike is rolled with probability $F_{M+1}/(F_{M+3} - 1)$, and the next roll is not another strike, then the average addition to the frame's score is given by the average of the next two rolls given that the first of these rolls is not M. Equivalently, it is the unconditional average of the next two rolls minus the weighted conditional average of the next two rolls given that the first roll *is* M (and the second roll is automatically a 0). The average addition to a frame in this situation is

$$P(X_{i1} = M) \times E[X_{(i+1)1} + X_{(i+1)2} \big| X_{(i+1)1} < M] \times P(X_{(i+1)1} < M)$$
$$= \frac{F_{M+1}}{F_{M+3} - 1} \times \left(E[X_{(i+1)1} + X_{(i+1)2}] - E[X_{(i+1)1} + X_{(i+1)2} \big| X_{(i+1)1} = M] \times P(X_{(i+1)1} = M \right)$$
$$= \frac{F_{M+1}}{F_{M+3} - 1} \times \left(\frac{1}{F_{M+3} - 1} \sum_{k=0}^{M} k \left[\sum_{y=0}^{k} \frac{F_{y+1} \times F_{k+1-y}}{F_{M+3-y} - 1} \right] - M \times \frac{F_{M+1}}{F_{M+3} - 1} \right).$$

5.5 The Average Score

Combining the average score of two rolls with the average additions for spares and strikes, the average score per frame for our skilled bowler is

$$E[f_i] = \frac{1}{F_{M+3} - 1} \sum_{k=0}^{M} k \left[\sum_{y=0}^{k} \frac{F_{y+1} \times F_{k+1-y}}{F_{M+3-y} - 1} \right]$$
$$+ \frac{1}{F_{M+3} - 1} \left[\sum_{k=0}^{M} k(F_{k+1}) \right] \times \left(\frac{1}{F_{M+3} - 1} \sum_{y=0}^{M-1} \frac{F_{y+1} \times F_{M+1-y}}{F_{M+3-y} - 1} \right)$$
$$+ \left(M + \frac{1}{F_{M+3} - 1} \left[\sum_{k=0}^{M} k(F_{k+1}) \right] \right) \times \left(\frac{F_{M+1}}{F_{M+3} - 1} \right)^2$$
$$+ \frac{F_{M+1}}{F_{M+3} - 1} \times \left(\frac{1}{F_{M+3} - 1} \sum_{k=0}^{M} k \left[\sum_{y=0}^{k} \frac{F_{y+1} \times F_{k+1-y}}{F_{M+3-y} - 1} \right] - M \times \frac{F_{M+1}}{F_{M+3} - 1} \right)$$
$$= \left(1 + \frac{F_{M+1}}{F_{M+3} - 1} \right) \times \frac{1}{F_{M+3} - 1} \sum_{k=0}^{M} k \left[\sum_{y=0}^{k} \frac{F_{y+1} \times F_{k+1-y}}{F_{M+3-y} - 1} \right]$$
$$+ \frac{1}{F_{M+3} - 1} \left[\sum_{k=0}^{M} k(F_{k+1}) \right] \times \left(\frac{1}{F_{M+3} - 1} \sum_{y=0}^{M-1} \frac{F_{y+1} \times F_{M+1-y}}{F_{M+3-y} - 1} + \left(\frac{F_{M+1}}{F_{M+3} - 1} \right)^2 \right).$$

For the special case with $M = 10$ pins, we have

$$
\begin{aligned}
E[f_i] &= \left(1+\frac{F_{11}}{F_{13}-1}\right)\times\frac{1}{F_{13}-1}\sum_{k=0}^{10}k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{13-y}-1}\right] \\
&\quad+\frac{1}{F_{13}-1}\left[\sum_{k=0}^{10}k\left(F_{k+1}\right)\right]\times\left(\frac{1}{F_{13}-1}\sum_{y=0}^{9}\frac{F_{y+1}\times F_{11-y}}{F_{13-y}-1}+\left(\frac{F_{11}}{F_{13}-1}\right)^2\right) \\
&= \left(1+\frac{89}{232}\right)\times\frac{126492677}{13321440}+\frac{489}{58}\times\left(\frac{33239363}{115452480}+\left(\frac{89}{232}\right)^2\right) \\
&= \frac{225076163731}{13392487680} \\
&\approx 16.80615,
\end{aligned}
$$

and the final score for a ten-frame game is approximately 168.0615.

5.6 Distribution Method for the Skilled Bowler

We can verify the results for the mean using the distribution method. To find the distribution of a frame with M pins, we must find the probability for scores $0,1,\ldots,3M$. As pins are knocked down on the first roll of each frame, the values of 0 through M are given increasing probabilities based on the Fibonacci sequence. Thus, the following table shows the probability for scores $0,1,\ldots,3M$ in a single frame, where $0 \le k \le M-1$:

Score	Set of Possibilities	Probability
k	$\{(0,k),(1,k-1),\ldots,(k,0)\}$	$\frac{1}{F_{M+3}-1}\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}$
$M+k$	$\{(0,M),\ldots,(M-1,1)\}\times\{k\}$ $\cup\{(M,0)\}\times\{(0,k),\ldots,(k,0)\}$	$\frac{F_{k+1}}{(F_{M+3}-1)^2}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}$ $+\frac{F_{M+1}}{(F_{M+3}-1)^2}\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}$
$2M$	$\{(0,M),\ldots,(M-1,1)\}\times\{M\}$ $\cup\{(M,0)\}\times\{(0,M),\ldots,(M-1,1)\}$ $\cup\{(M,0)\}\times\{(M,0)\}\times\{0\}$	$\frac{F_{M+1}}{(F_{M+3}-1)^2}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}$ $+\frac{F_{M+1}}{(F_{M+3}-1)^2}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}$ $+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}$
$2M+1+k$	$\{(M,0)\}\times\{(M,0)\}\times\{k+1\}$	$\frac{(F_{M+1})^2\times F_{k+2}}{(F_{M+3}-1)^3}$

We can verify that all of the possible scenarios are given and make up a partition by summing all of the probabilities, and we can combine similarities into common summations. Thus, the probabilities of scores from 0 to $M-1$ can be combined to the following summation:

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right].$$

Because the sum of the first two rolls must range from 0 to M with probability 1, we have:

Proposition 5.6.1. The sum of probabilities of all possible outcomes for rolls in a frame is

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]=\sum_{k=0}^{M}P(X_{i1}+X_{i2}=k)=1.$$

Thus, the probabilities of scores from 0 to $M-1$ can be written as the following summation:

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]-\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}$$
$$=1-\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}.$$

Also, the probabilities of scores from M to $2M-1$ can be combined to the following:

$$\frac{1}{(F_{M+3}-1)^2}\left(\sum_{k=0}^{M-1}F_{k+1}\left[\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\right]+F_{M+1}\sum_{k=0}^{M-1}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]\right).$$

The score $2M$ is unique, so we will begin by writing its probability as $\frac{1}{(F_{M+3}-1)^2}\left(F_{M+1}\sum_{k=0}^{M-1}\frac{F_{k+1}\times F_{M+1-k}}{F_{M+3-k}-1}+F_{M+1}\sum_{k=0}^{M-1}\frac{F_{k+1}\times F_{M+1-k}}{F_{M+3-k}-1}\right)+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}$.

Since the first part follows the exact pattern as scores from M to $2M-1$, and the second part follows a similar pattern, we can rewrite the first two parts of the probability of the score $2M$ to be

$$\frac{F_{M+1}}{(F_{M+3}-1)^2}\sum_{k=0}^{M-1}\frac{F_{k+1}\times F_{M+1-k}}{F_{M+3-k}-1}+\frac{F_{M+1}}{(F_{M+3}-1)^2}\sum_{k=0}^{M}\frac{F_{k+1}\times F_{M+1-k}}{(F_{M+3-k}-1)}-\frac{(F_{M+1})^2}{(F_{M+3}-1)^2},$$

and combine these parts with the probabilities of scores from M to $2M-1$, which results in the following summation:

$$\frac{1}{(F_{M+3}-1)^2}\left(\sum_{k=0}^{M}F_{k+1}\left[\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\right]+F_{M+1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]\right)-\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}.$$

Proposition 5.6.2. The sum of probabilities of all possible outcomes for the first roll in a frame is

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} F_{k+1} = \sum_{k=0}^{M} P(X_{i1}+X_{i2}=k)=1.$$

Using *Proposition 5.6.1* and *Proposition 5.6.2*, the probabilities of scores from M to $2M-1$ and the first two parts of the score $2M$ can be written as the following summation:

$$\begin{aligned}
&\frac{1}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{k+1}\right)\\
&+\frac{F_{M+1}}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]\right)-\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}\\
&=\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}+\frac{F_{M+1}}{F_{M+3}-1}-\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}\\
&=\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}-\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}.
\end{aligned}$$

Finally, the last part of the score $2M$ can be combined with the probabilities of the scores from $2M+1$ to $3M$, and using *Proposition 5.6.2*, we have the following summation:

$$\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}F_{k+1}=\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{k+1}\right)=\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}.$$

Therefore, the sum of probabilities for all the scores $n=0,1,\ldots,3M$ is

$$\begin{aligned}
\sum_{n=0}^{3M}P(f_i=n)&=\left(1-\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\right)\\
&\quad+\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}-\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}\right)+\left(\frac{(F_{M+1})^2}{(F_{M+3}-1)^2}\right)\\
&=1.
\end{aligned}$$

The probabilities sum to one, which verifies that all of the possible scores are given and make up a partition.

In order to compute the mean of a single frame with M pins, we must multiply each score by its probability and sum the resulting products. Thus, the weighted sum for scores from 0 to $M-1$ can be combined to the following summation:

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1} k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right].$$

Using the results from Section 5.2 for $E[X_{i1}]+E[X_{i2}]$, we can simplify the summation to be

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}$$
$$=E[X_{i1}]+E[X_{i2}]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}$$
$$=E[X_{i1}]+E[X_{i2}]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}-\frac{M(F_{M+1})}{F_{M+3}-1}.$$

Also, the weighted sum for scores from M to $2M-1$, along with the first two parts for the score $2M$, can be combined to the following:

$$\frac{1}{(F_{M+3}-1)^2}\times\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times\sum_{k=0}^{M}(M+k)F_{k+1}$$
$$+\frac{F_{M+1}}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]-\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2}.$$

Using *Proposition 5.6.1*, *Proposition 5.6.2,* and the results from Section 5.2, we can simplify the summation to be

$$\frac{M}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{k+1}\right)$$

$$+\frac{1}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}kF_{k+1}\right)$$

$$+M\times\frac{F_{M+1}}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]\right)$$

$$+\frac{F_{M+1}}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]\right)-\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2}$$

$$=\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}+\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times E[X_{i1}]$$

$$+\frac{M(F_{M+1})}{F_{M+3}-1}+\frac{F_{M+1}}{F_{M+3}-1}\times\left(E[X_{i1}]+E[X_{i2}]\right)-\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2}.$$

Finally, the weighted sum for scores from $2M+1$ to $3M$, along with the last part for the score $2M$ can be combined to the following summation:

$$\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(2M+k)F_{k+1}$$

$$=\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}F_{k+1}+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{n=0}^{M}k(F_{k+1})$$

$$=\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{k+1}\right)+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\times E[X_{i1}]$$

$$=\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2}+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\times E[X_{i1}].$$

Therefore, the mean score of a single frame with M pins is the following:

$$E[f_i] = E[X_{i1}] + E[X_{i2}] - \frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1} - \frac{M(F_{M+1})}{F_{M+3}-1}$$
$$+\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1} + \frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times E[X_{i1}]$$
$$+\frac{M(F_{M+1})}{F_{M+3}-1} + \frac{F_{M+1}}{F_{M+3}-1}\times\left(E[X_{i1}]+E[X_{i2}]\right) - \frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2}$$
$$+\frac{2M(F_{M+1})^2}{(F_{M+3}-1)^2} + \frac{\left(F_{M+1}\right)^2}{(F_{M+3}-1)^3}\times E[X_{i1}]$$
$$=\left(1+\frac{F_{M+1}}{F_{M+3}-1}\right)\times\left(E[X_{i1}]+E[X_{i2}]\right) + \frac{1}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1}\times E[X_{i1}]$$
$$+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\times E[X_{i1}].$$

Substituting the summations for $E[X_{i1}]$ and $E[X_{i1}]+E[X_{i2}]$ back into the equation, we obtain

$$E[f_i] = \left(1+\frac{F_{M+1}}{F_{M+3}-1}\right)\times\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]$$
$$+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{k+1}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1} + \left(\frac{F_{M+1}}{F_{M+3}-1}\right)^2\right).$$

This result agrees with the result previously obtained in Section 5.5.

5.7 Frame Variance for the Skilled Bowler

We can also derive the variance of the score of a single frame for our skilled bowler. First, we need to compute $E[{f_i}^2]$, which can be found by squaring each score, then multiplying it by the probability of the score and summing the results. The probabilities remain the same, so the computation is similar to the mean of a single frame.

We can combine the weighted sum of the squared scores from 0 to $M-1$ to be

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1}k^2\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right].$$

Also, the mean of the squared scores from M to $2M-1$, along with parts of $2M$ can be combined into the following summation:

$$\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2\left[\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}\times F_{k+1}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}\times F_{M+1}}{F_{M+3-y}-1}\right]$$
$$-\frac{(2M)^2(F_{M+1})^2}{(F_{M+3}-1)^2}.$$

Finally, the weighted sum for squared scores from $2M+1$ to $3M$, along with the last part of $2M$, can be combined into the following summation:

$$\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(2M+k)^2F_{k+1}.$$

Using *Proposition 5.6.2* and the results from Section 5.2, we can simplify the summation to be

$$\frac{(2M)^2(F_{M+1})^2}{(F_{M+3}-1)^2}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}(F_{k+1})\right)+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\times\sum_{k=0}^{M}(4Mk+k^2)(F_{k+1})$$
$$=\frac{(2M)^2(F_{M+1})^2}{(F_{M+3}-1)^2}+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(4Mk+k^2)(F_{k+1}).$$

Thus, $E[f_i^2]$ can be written as the following:

$$E[f_i^{\,2}] = \frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1} k^2 \left[\sum_{y=0}^{k} \frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]$$
$$+\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2\left[\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}\times F_{k+1}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}\times F_{M+1}}{F_{M+3-y}-1}\right]$$
$$-\frac{(2M)^2(F_{M+1})^2}{(F_{M+3}-1)^2}+\frac{(2M)^2(F_{M+1})^2}{(F_{M+3}-1)^2}+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(4Mk+k^2)(F_{k+1})$$
$$=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1} k^2 \left[\sum_{y=0}^{k} \frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right]+\frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(4Mk+k^2)(F_{k+1})$$
$$+\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2\left[\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}\times F_{k+1}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}\times F_{M+1}}{F_{M+3-y}-1}\right].$$

For the special case with $M = 10$ pins, we have

$$E[f_i^{\,2}] = \frac{1}{F_{13}-1}\sum_{k=0}^{9} k^2 \left[\sum_{y=0}^{k} \frac{F_{y+1}\times F_{k+1-y}}{F_{13-y}-1}\right]+\frac{(F_{11})^2}{(F_{13}-1)^3}\sum_{k=0}^{10}(40k+k^2)(F_{k+1})$$
$$+\frac{1}{(F_{13}-1)^2}\sum_{k=0}^{10}(10+k)^2\left[\sum_{y=0}^{9}\frac{F_{y+1}\times F_{11-y}\times F_{k+1}}{F_{13-y}-1}+\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}\times F_{11}}{F_{13-y}-1}\right]$$
$$=\frac{374883839}{15743520}+\frac{(89)^2}{(232)^3}\times 95584+\frac{1}{(232)^2}\times\left[\frac{25461352058}{4785}+89\times\frac{65959088699}{746460}\right]$$
$$=\frac{1018082288639}{3090574080}$$
$$\approx 329.415\,.$$

Therefore the variance of each frame is equal to

$$Var(f_i) = E[f_i^{\,2}] - (E[f_i])^2$$

$$= \left[\begin{array}{l} \dfrac{1}{F_{M+3}-1}\displaystyle\sum_{k=0}^{M-1} k^2 \left[\sum_{y=0}^{k} \frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right] + \frac{(F_{M+1})^2}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(4Mk+k^2)(F_{k+1}) \\ + \dfrac{1}{(F_{M+3}-1)^2}\displaystyle\sum_{k=0}^{M}(M+k)^2 \left[\begin{array}{l} \displaystyle\sum_{y=0}^{M-1} \frac{F_{y+1}\times F_{M+1-y}\times F_{k+1}}{F_{M+3-y}-1} \\ + \displaystyle\sum_{y=0}^{k} \frac{F_{y+1}\times F_{k+1-y}\times F_{M+1}}{F_{M+3-y}-1} \end{array}\right] \end{array}\right]$$

$$- \left[\begin{array}{l} \left(1+\dfrac{F_{M+1}}{F_{M+3}-1}\right)\times \dfrac{1}{F_{M+3}-1}\displaystyle\sum_{k=0}^{M} k\left[\sum_{y=0}^{k}\frac{F_{y+1}\times F_{k+1-y}}{F_{M+3-y}-1}\right] \\ + \dfrac{1}{F_{M+3}-1}\left[\displaystyle\sum_{k=0}^{M} k\left(F_{k+1}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{y+1}\times F_{M+1-y}}{F_{M+3-y}-1} + \left(\frac{F_{M+1}}{F_{M+3}-1}\right)^2\right) \end{array}\right]^2.$$

For the special case with $M = 10$ pins, we have

$$Var(f_i) = E[f_i^{\,2}] - (E[f_i])^2 = \left(\frac{1018082288639}{3090574080}\right) - \left(\frac{225076163731}{13392487680}\right)^2 \approx 46.9686,$$

and the standard deviation is

$$\sigma_{f_i} = \sqrt{Var(f_i)} = \sqrt{\left(\frac{1018082288639}{3090574080}\right) - \left(\frac{225076163731}{13392487680}\right)^2} \approx 6.8534.$$

5.8 Conclusion of Mean and Variance for the Skilled Bowler

The average score for an N-frame bowling game can be computed by applying the linearity property of the mean. Therefore, we obtain

Theorem 5.8.1. The average score for an N-frame game of bowling on M pins with the skilled Fibonacci distribution is given by

$$\sum_{i=1}^{N} E[f_i] = N \times \left[\begin{array}{l} \left(1 + \frac{F_{M+1}}{F_{M+3} - 1}\right) \times \frac{1}{F_{M+3} - 1} \sum_{k=0}^{M} k \left[\sum_{y=0}^{k} \frac{F_{y+1} \times F_{k+1-y}}{F_{M+3-y} - 1} \right] \\ + \frac{1}{F_{M+3} - 1} \left[\sum_{k=0}^{M} k(F_{k+1}) \right] \times \left(\frac{1}{F_{M+3} - 1} \sum_{y=0}^{M-1} \frac{F_{y+1} \times F_{M+1-y}}{F_{M+3-y} - 1} + \left(\frac{F_{M+1}}{F_{M+3} - 1} \right)^2 \right) \end{array} \right],$$

where f_i represents the score of each frame.

Thus, the average score for a ten-frame game with $M = 10$ pins is

$$\sum_{i=1}^{10} E[f_i] = 10 \times \frac{225076163731}{13392487680} \approx 168.0615.$$

Therefore, the average score of our skilled bowler is approximately 168.0615. Unlike the mean, the variance of a ten-frame game cannot simply be computed by summing each individual variance because the frames are not independent of one another. That is, f_i depends on f_{i+1} in the cases of strikes and spares. However, the variance of a single frame can be computed, which is approximately 46.9686. In addition, the variance can be used to calculate the standard deviation, which is approximately 6.8534.

5.9 Correlation Coefficient for the Skilled Bowler

We now shall find the correlation between the score of the first and second roll in a single frame, which is given by

$$\rho_{X_1, X_2} = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X_1} \sigma_{X_2}}.$$

In Section 5.2, we previously computed the average score of the first roll on a set of pins to be

$$E[X_1] = \frac{1}{F_{M+3} - 1} \left[\sum_{k=0}^{M} k F_{k+1} \right].$$

In order to find the standard deviation for the score of the first roll, we need to compute $E[X_1^2]$. Since the probability of each score remains the same, the only difference in the computation is squaring each score before multiplying it by the probability. Therefore,

$$E[X_1^2] = \frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M} k^2 F_{k+1}\right].$$

For the special case with $M = 10$ pins, we have

$$E[X_1^2] = \frac{1}{F_{13}-1}\left[\sum_{k=0}^{10} k^2 F_{k+1}\right] = \frac{2168}{29}.$$

Thus, we can compute the standard deviation of X_1 to be

$$\begin{aligned}\sigma_{X_1} &= \sqrt{E[X_1^2] - (E[X_1])^2} \\ &= \sqrt{\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M} k^2 F_{k+1}\right] - \left(\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M} kF_{k+1}\right]\right)^2}.\end{aligned}$$

In the special case with $M = 10$ pins, we have

$$\sigma_{X_1} = \sqrt{\frac{2168}{29} - \left(\frac{489}{58}\right)^2} = \sqrt{\frac{12367}{3364}} \approx 1.9174.$$

Also in Section 5.2, we previously computed the average of the second roll on a set of pins to be

$$E[X_2] = \frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M} k(F_{k+1}) \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{M+3\text{-}y}-1}\right].$$

In order to find the standard deviation for the score of the second roll, we need to compute $E[X_2^2]$, which is given by

$$\begin{aligned}
E[X_2{}^2] &= \sum k^2 \times P(X_2 = k) \\
&= \sum_{k=0}^{M} k^2 \left[\frac{F_{k+1}}{F_{M+3}-1} \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{M+3-y}-1} \right] \\
&= \frac{1}{F_{M+3}-1} \sum_{k=0}^{M} k^2 \left(F_{k+1}\right) \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{M+3-y}-1}.
\end{aligned}$$

For the special case with $M = 10$ pins, we have

$$E[X_2{}^2] = \frac{1}{F_{13}-1} \sum_{k=0}^{10} k^2 \left(F_{k+1}\right) \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{13\text{-y}}-1} = \frac{677818099}{173178720} \approx 3.91398.$$

Thus, the standard deviation of X_2 is

$$\begin{aligned}
\sigma_{X_2} &= \sqrt{E[X_2{}^2] - \left(E[X_2]\right)^2} \\
&= \sqrt{\frac{1}{F_{M+3}-1} \sum_{k=0}^{M} k^2 \left(F_{k+1}\right) \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{M+3-y}-1} - \left(\frac{1}{F_{M+3}-1} \sum_{k=0}^{M} k\left(F_{k+1}\right) \sum_{y=0}^{M-k} \frac{F_{y+1}}{F_{M+3-y}-1} \right)^2}.
\end{aligned}$$

For the special case with $M = 10$ pins, we have

$$\sigma_{X_2} = \sqrt{\frac{677818099}{173178720} - \left(\frac{14179157}{13321440}\right)^2} = \sqrt{\frac{6415882724744123}{2306989927756800}} \approx 1.6677.$$

Lastly, we need to compute $E\left[X_1 X_2\right]$. Using background information on the mean, we can multiply the product of the first and second roll by the probability of each possible combination. Thus, the expected value of $X_1 X_2$ is found to be

$$
\begin{aligned}
E[X_1X_2] &= \sum k\times y\times P((X_1,X_2)=(k,y)) \\
&= \sum_{k=0}^{M}\sum_{y=0}^{M-k}\left[k\times y\times\frac{F_{k+1}}{F_{M+3}-1}\times\frac{F_{y+1}}{F_{M+3-k}-1}\right] \\
&= \frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\frac{F_{k+1}}{F_{M+3-k}-1}\sum_{y=0}^{M-k}yF_{y+1}\,.
\end{aligned}
$$

For the special case with $M=10$ pins, we have

$$
E[X_1X_2]=\frac{1}{F_{13}-1}\sum_{k=0}^{10}k\frac{F_{k+1}}{F_{13-k}-1}\sum_{y=0}^{10-k}yF_{y+1}=\frac{31369}{5104}\,.
$$

Therefore, the correlation coefficient between the first and second roll of our skilled Fibonacci bowler is computed to be

$$
\rho_{X_1,X_2}=\frac{E[X_1X_2]-E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}}
$$

$$
=\frac{\left[\begin{array}{l}\dfrac{1}{F_{M+3}-1}\displaystyle\sum_{n=0}^{M}n\frac{F_{n+1}}{F_{M+3-n}-1}\sum_{k=0}^{M-n}kF_{k+1} \\ -\left(\dfrac{1}{F_{M+3}-1}\left[\displaystyle\sum_{n=0}^{M}nF_{n+1}\right]\right)\left(\dfrac{1}{F_{M+3}-1}\displaystyle\sum_{n=0}^{M}nF_{n+1}\sum_{k=0}^{M-n}\frac{F_{1+k}}{F_{M+3-k}-1}\right)\end{array}\right]}{\left[\begin{array}{l}\sqrt{\left(\dfrac{1}{F_{M+3}-1}\left[\displaystyle\sum_{n=0}^{M}n^2F_{n+1}\right]-\left(\dfrac{1}{F_{M+3}-1}\left[\displaystyle\sum_{n=0}^{M}nF_{n+1}\right]\right)^2\right)} \\ \times\sqrt{\left(\begin{array}{l}\dfrac{1}{F_{M+3}-1}\displaystyle\sum_{n=0}^{M}n^2F_{n+1}\sum_{k=0}^{M-n}\frac{F_{1+k}}{F_{M+3-k}-1} \\ -\left(\dfrac{1}{F_{M+3}-1}\displaystyle\sum_{n=0}^{M}nF_{n+1}\sum_{k=0}^{M-n}\frac{F_{1+k}}{F_{M+3-k}-1}\right)^2\end{array}\right)}\end{array}\right]}\,.
$$

For the special case with $M=10$ pins, we have

$$\begin{aligned}
\rho_{X_1,X_2} &= \frac{E[X_1X_2]-E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}} \\
&= \frac{\dfrac{31369}{5104}-\left(\dfrac{489}{58}\right)\left(\dfrac{14179157}{13321440}\right)}{\sqrt{\left(\dfrac{12367}{3364}\right)}\times\sqrt{\left(\dfrac{6415882724744123}{2306989927756800}\right)}} \\
&= \frac{-62063138508785879517}{79345221656910569141} \\
&\approx -0.8844\,.
\end{aligned}$$

We can see that the correlation between the first and second roll in a single frame is dependent on the number of pins. For example with $M = 5$ pins, we have $\rho_{X_1,X_2} \approx -0.8309$. Also, the correlation coefficient is negative because the value of the second roll decreases as the value of the first roll increases. Moreover, there is a strong linear relationship between the first and second roll using the Fibonacci distribution for our skilled bowler. Additionally, the linear relationship is much stronger than that of the Neal distribution and the Hohn distribution.

Chapter 6:

The Fibonacci Distribution for a Non-Skilled Bowler

In this chapter, we apply two methods to derive the average score of a particular non-skilled bowler when bowling N frames on M pins using the Fibonacci distribution. First, we apply Neal's method to derive the mean of a single frame, and then we verify these results by applying the distribution method. In addition, we apply the distribution method to compute the variance and standard deviation for a single frame. Lastly, we will find the correlation between the score of the first and second rolls in a single frame of bowling.

6.1 The Rules of the Fibonacci Distribution for a Non-Skilled Bowler

We shall use the scoring rules of a standard bowling game for which there are ten frames and two rolls per frame, (X_{i1}, X_{i2}), where X_{i1} is an integer ranging from 0 to 10. The probabilities for the first roll shall be obtained from the following Fibonacci distribution:

Score	0	1	2	3	4	5	6	7	8	9	10
Probability	$\frac{89}{232}$	$\frac{55}{232}$	$\frac{34}{232}$	$\frac{21}{232}$	$\frac{13}{232}$	$\frac{8}{232}$	$\frac{5}{232}$	$\frac{3}{232}$	$\frac{2}{232}$	$\frac{1}{232}$	$\frac{1}{232}$

An analysis of this distribution, denoted $Y_n \sim n+1-X_n$ when applied to the integers $\{1, \ldots, n\}$, is given in [6]. In particular, the sum of the first n Fibonacci numbers is given as $F_{n+2}-1$. The table above shows a $11-Fib(11)$ distribution which is applied to the integers $\{0, \ldots, 10\}$. This distribution models a particular non-skilled bowler who tends to score low on the first roll.

To generalize this, suppose there are N frames and two rolls per frame, (X_{i1}, X_{i2}), where X_{i1} is an integer ranging from 0 to M with probabilities obtained from the following Fibonacci distribution:

Score	0	1	…	$M-1$	M
Probability	$\frac{F_{M+1}}{F_{M+3}-1}$	$\frac{F_M}{F_{M+3}-1}$	…	$\frac{F_2}{F_{M+3}-1}$	$\frac{F_1}{F_{M+3}-1}$

Thus, $X_{i1} \sim M+1-Fib(M+1)$ for frames $i=1,\ldots,N$. We note that $F_{M+3}-1$ gives the sum of the first $M+1$ Fibonacci numbers used to determine the weights of the distribution on the integers $\{0, \ldots, M\}$.

Next, X_{i2} is an integer ranging from 0 to $M-X_{i1}$. The second roll X_{i2} is conditional upon the first. For instance, if $M=10$ and $X_{i1}=6$, then the second roll X_{i2} has the following distribution:

Score	0	1	2	3	4
Probability	$\frac{5}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

In general, $X_{i2} \sim M+1-Fib(M+1-X_{i1})$ for frames $i = 1, \ldots, N$:

Score	0	1	…	$M-X_{i1}$
Probability	$\frac{F_{M+1-X_{i1}}}{F_{M+3-X_{i1}}-1}$	$\frac{F_{M-X_{i1}}}{F_{M+3-X_{i1}}-1}$	…	$\frac{F_1}{F_{M+3-X_{i1}}-1}$

6.2 The Average Sum of Two Rolls

If pins are knocked down by our non-skilled bowler, then on the first roll of each frame the values of 0 through M are likely to occur with probability $F_{M+1-k}/(F_{M+3}-1)$. Thus for frames $i=1,2,\ldots,N$

$$E[X_{i1}] = \sum_{k=0}^{M} k \times \frac{F_{M+1-k}}{F_{M+3}-1} = \frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k(F_{M+1-k}).$$

For the special case with $M=10$ pins, using *Mathematica* to simplify the sum, we have

$$E[X_{i1}] = \frac{1}{F_{13}-1}\sum_{k=0}^{10} k(F_{11-k}) = \frac{91}{58} \approx 1.569\,.$$

We can derive the expected value of X_{i2} by finding the distribution of the second roll. For X_{i2} to equal k, then the first roll can vary from 0 to $M-k$. The set of possibilities is $\{(0,k),(1,k),\ldots,(M-k,k)\}$, and

$$\begin{aligned} P(X_{i2}=k) &= \sum_{y=0}^{M-k} P(X_{i1}=y)\times P(X_{i2}=k \mid X_{i1}=y) \\ &= \sum_{y=0}^{M-k} \frac{F_{M+1-y}}{F_{M+3}-1}\times\frac{F_{M+1-y-k}}{F_{M-y+3}-1} \\ &= \frac{1}{F_{M+3}-1}\sum_{y=0}^{M-k}\frac{F_{M+1-y}\times F_{M+1-y-k}}{F_{M+3-y}-1}\,. \end{aligned}$$

Therefore, the average score of the second roll is given by

$$\begin{aligned} E[X_{i2}] &= \sum_{k=0}^{M} k\times P(X_{i2}=k) \\ &= \sum_{k=0}^{M} k\left[\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-k}\frac{F_{M+1-y}\times F_{M+1-y-k}}{F_{M+3-y}-1}\right] \\ &= \frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k\sum_{y=0}^{M-k}\frac{F_{M+1-y}\times F_{M+1-y-k}}{F_{M+3-y}-1}\,. \end{aligned}$$

For the special case with $M=10$ pins, using *Mathematica* to simplify the sum, we have

$$E[X_{i2}] = \frac{1}{F_{13}-1}\sum_{k=0}^{10} k\sum_{y=0}^{10-k}\frac{F_{11-y}\times F_{11-y-k}}{F_{13-y}-1} = \frac{19873123}{13321440} \approx 1.49181\,.$$

Thus, without taking into account strikes and spares, the average score per frame is

$$E[X_{i1}]+E[X_{i2}]=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k(F_{M+1-k})+\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\sum_{y=0}^{M-k}\frac{F_{M+1-y}\times F_{M+1-y-k}}{F_{M+3-y}-1}$$

$$=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left[F_{M+1-k}+\sum_{y=0}^{M-k}\frac{F_{M+1-y}\times F_{M+1-y-k}}{F_{M+3-y}-1}\right].$$

For the special case with $M=10$ pins, we have

$$E[X_1]+E[X_2]=\frac{91}{58}+\frac{19873123}{13321440}=\frac{40774003}{13321440}\approx 3.06078$$

Proposition 6.2.2. The following summations are equivalent to the average score per frame without taking into account strikes and spares.

$$E[X_{i1}]+E[X_{i2}]=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left[F_{M+1-k}+\sum_{y=0}^{M-k}\frac{F_{M+1-y}\times F_{M+1-y-k}}{F_{M+3-y}-1}\right]$$

$$=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k(F_{M+1-k})\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right].$$

6.3 Accounting for Spares

If a spare is rolled in a frame, then the score on the next roll is added to the score. Since the next roll is on a new set of pins, the average score is

$$E[X_{(i+1)1}]=\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k(F_{M+1-k})\right].$$

However, we must multiply by the probability of a spare in order to determine the average amount added to each frame's score. In general, the average addition for spares is given by

$$E[X_{(i+1)1}]\times P\left(X_{i1}+X_{i2}=M\cap X_{i1}<M\right)$$
$$=\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\times\sum_{y=0}^{M-1}P\left(X_{i1}=y\right)\times P\left(X_{i2}=M-y\right)$$
$$=\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\times\sum_{y=0}^{M-1}\left(\frac{F_{M+1-y}}{F_{M+3}-1}\times\frac{1}{F_{M+3-y}-1}\right)$$
$$=\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right).$$

6.4 Accounting for Strikes

If a strike is rolled in a frame, then the next two rolls are added to the frame's score. There are two cases in this situation: either the next roll after the strike is also a strike, or the next roll is not a strike. In the first case, the average addition to the original frame is M from the second strike, plus the average score from the next roll on a new set of pins, which is equivalent to $E\left[X_{i1}\right]$. This first case occurs only in the event of two strikes in a row which each have probability $1/\left(F_{M+3}-1\right)$. Thus the average addition to a frame in this case is

$$\left(M+E[X_{(i+2)1}]\right)\times P(X_{i1}=M)\times P(X_{(i+1)1}=M)$$
$$=\left(M+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\right)\times\left(\frac{1}{F_{M+3}-1}\right)^2.$$

However, if a strike is rolled with probability $1/\left(F_{M+3}-1\right)$, and the next roll is not another strike, then the average addition to the frame's score is given by the average of the next two rolls, given that the first of these rolls is not M. Equivalently, it is the unconditional average of the next two rolls minus the weighted conditional average of the next two rolls given that the first roll *is* M. Thus, the average addition to a frame in this situation is

$$P(X_{i1}=M)\times E[X_{(i+1)1}+X_{(i+1)2}\big|X_{(i+1)1}<M]\times P(X_{(i+1)1}<M)$$
$$=\frac{1}{F_{M+3}-1}\times\Big(E\big[X_{(i+1)1}+X_{(i+1)2}\big]-E\big[X_{(i+1)1}+X_{(i+1)2}\big|X_{(i+1)1}=M\big]\times P(X_{(i+1)1}=M)\Big)$$
$$=\frac{1}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-M\times\frac{1}{F_{M+3}-1}\right).$$

6.5 The Average Score

Combining the average roll of two balls with the average additions for spares and strikes, the average score per frame for our non-skilled bowler is

$$\begin{aligned}
E\left[f_i\right]=&\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\\
&+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right)\\
&+\left(M+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\right)\times\left(\frac{1}{F_{M+3}-1}\right)^2\\
&+\frac{1}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-M\times\frac{1}{F_{M+3}-1}\right)\\
=&\left(1+\frac{1}{F_{M+3}-1}\right)\times\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\\
&+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\left(\frac{1}{F_{M+3}-1}\right)^2\right).
\end{aligned}$$

For the special case with $M=10$ pins, we have

$$E[f_i] = \left(1 + \frac{1}{F_{13} - 1}\right) \times \frac{1}{F_{13} - 1} \sum_{k=0}^{10} k(F_{11-k}) \left[\sum_{y=0}^{k} \frac{F_{11-y}}{F_{13-y} - 1} \right]$$

$$+ \frac{1}{F_{13} - 1} \left[\sum_{k=0}^{10} k(F_{11-k}) \right] \times \left(\frac{1}{F_{13} - 1} \sum_{y=0}^{9} \frac{F_{11-y}}{F_{13-y} - 1} + \left(\frac{1}{F_{13} - 1} \right)^2 \right)$$

$$= \left(1 + \frac{1}{232}\right) \times \frac{40774003}{13321440} + \frac{91}{58} \times \left(\frac{43710701}{2424502080} + \left(\frac{1}{232} \right)^2 \right)$$

$$= \frac{3195951397}{1030191360}$$

$$\approx 3.10229.$$

and the average final score for a ten-frame game is approximately 31.023 .

6.6 Distribution Method for the Non-Skilled Bowler

We can verify the results for the mean using the distribution method. To find the distribution of the score on a frame with M pins, we must find the probability for scores $0,1,\ldots,3M$. As pins are knocked down on the first roll of each frame, the values of 0 through M are given decreasing probabilities based on the Fibonacci sequence. Thus, the following table shows the probability for scores $0,1,\ldots,3M$ in a single frame, where $0 \le k \le M - 1$:

Score	Set of Possibilities	Probability
k	$\{(0,k),(1,k-1),\ldots,(k,0)\}$	$\dfrac{F_{M+1-k}}{F_{M+3}-1}\displaystyle\sum_{y=0}^{k}\dfrac{F_{M+1-y}}{F_{M+3-y}-1}$
$M+k$	$\{(0,M),\ldots,(M-1,1)\}\times\{k\}$ $\cup\{(M,0)\}\times\{(0,k),\ldots,(k,0)\}$	$\dfrac{F_{M+1-k}}{(F_{M+3}-1)^2}\displaystyle\sum_{y=0}^{M-1}\dfrac{F_{M+1-y}}{F_{M+3-y}-1}$ $+\dfrac{F_{M+1-k}}{(F_{M+3}-1)^2}\displaystyle\sum_{y=0}^{k}\dfrac{F_{M+1-y}}{F_{M+3-y}-1}$
$2M$	$\{(0,M),\ldots,(M-1,1)\}\times\{M\}$ $\cup\{(M,0)\}\times\{(0,M),\ldots,(M-1,1)\}$ $\cup\{(M,0)\}\times\{(M,0)\}\times\{0\}$	$\dfrac{1}{(F_{M+3}-1)^2}\displaystyle\sum_{y=0}^{M-1}\dfrac{F_{M+1-y}}{F_{M+3-y}-1}$ $+\dfrac{1}{(F_{M+3}-1)^2}\displaystyle\sum_{y=0}^{M-1}\dfrac{F_{M+1-y}}{F_{M+3-y}-1}$ $+\dfrac{F_{M+1}}{(F_{M+3}-1)^3}$
$2M+1+k$	$\{(M,0)\}\times\{(M,0)\}\times\{k+1\}$	$\dfrac{F_{M-k}}{(F_{M+3}-1)^3}$

We can verify that all of the possible scenarios are given and make up a partition by summing all of the probabilities, and we can combine similarities into common summations. Thus, the probabilities of scores from 0 to $M-1$ can be combined to the following summation:

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1}\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right].$$

Because the sum of the first two rolls must range from 0 to M with probability 1, we have:

Proposition 6.6.1. The sum of probabilities of all possible scores for rolls in a frame is

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]=\sum_{k=0}^{M}P(X_{i1}+X_{i2}=k)=1.$$

Thus, the probabilities of scores from 0 to $M-1$ can be written as the following summation:

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}$$
$$=1-\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}.$$

Also, the probabilities of scores from M to $2M-1$ can be combined to the following:

$$\frac{1}{\left(F_{M+3}-1\right)^2}\sum_{k=0}^{M-1}\left(F_{M+1-k}\right)\left[\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right].$$

The score $2M$ is unique, so we will begin by writing its probability as

$$\frac{1}{\left(F_{M+3}-1\right)^2}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\frac{1}{\left(F_{M+3}-1\right)^2}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\frac{F_{M+1}}{\left(F_{M+3}-1\right)^3}.$$

Since the first part follows the exact pattern as scores from M to $2M-1$, and the second part follows a similar pattern, we can rewrite the first two parts of the probability of the score $2M$ to be

$$\frac{F_{M+1}}{\left(F_{M+3}-1\right)^2}\sum_{k=0}^{M-1}\frac{F_{k+1}\times F_{M+1-k}}{F_{M+3-k}-1}+\frac{F_{M+1}}{\left(F_{M+3}-1\right)^2}\sum_{k=0}^{M}\frac{F_{k+1}\times F_{M+1-k}}{\left(F_{M+3-k}-1\right)}-\frac{\left(F_{M+1}\right)^2}{\left(F_{M+3}-1\right)^2},$$

and combine these parts with the probabilities of scores from M to $2M-1$, which results in the following summation:

$$\frac{1}{\left(F_{M+3}-1\right)^2}\sum_{k=0}^{M}\left(F_{M+1-k}\right)\left[\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-\frac{1}{\left(F_{M+3}-1\right)^2}.$$

Proposition 6.6.2. The sum of probabilities of all possible scores for the first roll in a frame is

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} F_{M+1-k} = \sum_{k=0}^{M} P(X_{i1}+X_{i2}=k)=1.$$

Using *Proposition 6.6.1* and *Proposition 6.6.2*, the probabilities of scores from M to $2M-1$ and the first two parts of the score $2M$ can be written as the following summation:

$$\begin{aligned}
&\frac{1}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{M+1-k}\right)\\
&+\frac{1}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\right)-\frac{1}{(F_{M+3}-1)^2}\\
&=\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\frac{1}{F_{M+3}-1}-\frac{1}{(F_{M+3}-1)^2}\\
&=\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}-\frac{1}{(F_{M+3}-1)^2}.
\end{aligned}$$

Finally, the last part of the score $2M$ can be combined with the probabilities of the scores from $2M+1$ to $3M$, and using *Proposition 6.6.2*, we have the following summation:

$$\frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}F_{M+1-k}=\frac{1}{(F_{M+3}-1)^2}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{M+1-k}\right)=\frac{1}{(F_{M+3}-1)^2}.$$

Therefore, the sum of probabilities for all the scores $n=0,1,\ldots,3M$ is

$$\begin{aligned}
\sum_{n=0}^{3M}P(f_i=n)&=\left(1-\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right)\\
&\quad+\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}-\frac{1}{(\mathrm{F}_{M+3}-1)^2}\right)+\left(\frac{1}{(F_{M+3}-1)^2}\right)\\
&=1.
\end{aligned}$$

The probabilities sum to one, which verifies that all of the possible scores are given and make up a partition.

In order to compute the mean of a single frame with M pins, we must multiply each score by its probability and sum the resulting products. Thus, the weighted sum for scores from 0 to $M-1$ can be combined to the following summation:

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1} k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right].$$

Using the results from Section 6.2 for $E[X_{i1}]+E[X_{i2}]$, we can simplify the summation to

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}$$
$$=E[X_{i1}]+E[X_{i2}]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M}\frac{F_{M+1-y}}{F_{M+3-y}-1}$$
$$=E[X_{i1}]+E[X_{i2}]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}-\frac{M}{F_{M+3}-1}.$$

Also, the weighted sum for scores from M to $2M-1$, along with the first two parts for the score $2M$ can be combined to the following:

$$\frac{1}{\left(F_{M+3}-1\right)^2}\times\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times\sum_{k=0}^{M}(M+k)F_{M+1-k}$$
$$+\frac{1}{\left(F_{M+3}-1\right)^2}\sum_{k=0}^{M}(M+k)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-\frac{2M}{\left(F_{M+3}-1\right)^2}.$$

Using *Proposition 6.6.1*, *Proposition 6.6.2,* and the results from Section 6.2, we can simplify the summation to be

$$\frac{M}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{M+1-k}\right)$$
$$+\frac{1}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right)$$
$$+M\times\frac{1}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\right)$$
$$+\frac{1}{F_{M+3}-1}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\right)-\frac{2M}{\left(F_{M+3}-1\right)^2}$$
$$=\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times E[X_{i1}]$$
$$+\frac{M}{F_{M+3}-1}+\frac{1}{F_{M+3}-1}\times\left(E[X_{i1}]+E[X_{i2}]\right)-\frac{2M}{\left(F_{M+3}-1\right)^2}.$$

Finally, the weighted sum for scores from $2M+1$ to $3M$, along with the last part for the score $2M$ can be combined to the following summation:

$$\frac{1}{\left(F_{M+3}-1\right)^3}\sum_{k=0}^{M}\left(2M+k\right)F_{M+1-k}$$
$$=\frac{2M}{\left(F_{M+3}-1\right)^3}\sum_{k=0}^{M}F_{M+1-k}+\frac{1}{\left(F_{M+3}-1\right)^3}\sum_{n=0}^{M}k\left(F_{M+1-k}\right)$$
$$=\frac{2M}{\left(F_{M+3}-1\right)^2}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{M+1-k}\right)+\frac{1}{\left(F_{M+3}-1\right)^3}\times E[X_{i1}]$$
$$=\frac{2M}{\left(F_{M+3}-1\right)^2}+\frac{1}{\left(F_{M+3}-1\right)^3}\times E[X_{i1}].$$

Therefore, the mean of a single frame with M pins is the following:

$$E[f_i]=E[X_{i1}]+E[X_{i2}]-\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}-\frac{M}{F_{M+3}-1}$$
$$+\frac{M}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times E[X_{i1}]$$
$$+\frac{M}{F_{M+3}-1}+\frac{1}{F_{M+3}-1}\times\left(E[X_{i1}]+E[X_{i2}]\right)-\frac{2M}{\left(F_{M+3}-1\right)^2}$$
$$+\frac{2M}{\left(F_{M+3}-1\right)^2}+\frac{1}{\left(F_{M+3}-1\right)^3}\times E[X_{i1}]$$
$$=\left(1+\frac{1}{F_{M+3}-1}\right)\times\left(E[X_{i1}]+E[X_{i2}]\right)+\frac{1}{F_{M+3}-1}\times\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}\times E[X_{i1}]$$
$$+\frac{1}{\left(F_{M+3}-1\right)^3}\times E[X_{i1}].$$

Substituting the summations for $E[X_{i1}]$ and $E[X_{i1}]+E[X_{i2}]$ back into the equation, we obtain

$$E[f_i]=\left(1+\frac{1}{F_{M+3}-1}\right)\times\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]$$
$$+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M}k\left(F_{M+1-k}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\left(\frac{1}{F_{M+3}-1}\right)^2\right).$$

This result agrees with the result previously obtained in Section 6.5.

6.7 Frame Variance for the Non-Skilled Bowler

We can also derive the variance of a single frame for our non-skilled bowler. First, we need to compute $E[f_i^{\,2}]$, which can be found by squaring each score, then multiplying it by the probability of the score and summing the results. The probabilities remain the same, so the computation is similar to the mean of a single frame.

We can combine the weighted sum of the squared scores from 0 to $M-1$ to be

$$\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1}k^2\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right].$$

Also, the mean of the squared scores from M to $2M-1$, along with parts of $2M$ can be combined into the following summation:

$$\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2(F_{M+1-k})\left[\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]-\frac{(2M)^2}{(F_{M+3}-1)^2}.$$

Finally, the weighted sum for squared scores from $2M+1$ to $3M$, along with the last part of $2M$, can be combined into the following summation:

$$\frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(2M+k)^2 F_{M+1-k}.$$

Using *Proposition 6.6.2* and the results from Section 6.2, we can simplify the summation to be

$$\begin{aligned}
&\frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}(2M+k)^2 F_{M+1-k}\\
&\frac{(2M)^2}{(F_{M+3}-1)^2}\times\left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M}F_{M+1-k}\right)+\frac{1}{(F_{M+3}-1)^3}\times\sum_{k=0}^{M}\left(4Mk+k^2\right)F_{M+1-k}\\
&=\frac{(2M)^2}{(F_{M+3}-1)^2}+\frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}\left(4Mk+k^2\right)F_{M+1-k}.
\end{aligned}$$

Thus, $E[f_i^2]$ can be written as the following:

$$\begin{aligned}
E[f_i^2]&=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1}k^2(F_{M+1-k})\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\\
&+\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2(F_{M+1-k})\left[\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\\
&-\frac{(2M)^2}{(F_{M+3}-1)^2}+\frac{(2M)^2}{(F_{M+3}-1)^2}+\frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}\left(4Mk+k^2\right)F_{M+1-k}\\
&=\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1}k^2(F_{M+1-k})\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]+\frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}\left(4Mk+k^2\right)F_{M+1-k}\\
&+\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2(F_{M+1-k})\left[\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right].
\end{aligned}$$

Therefore the variance of each frame is equal to

$$Var(f_i) = E[f_i{}^2] - (E[f_i])^2$$

$$= \left[\begin{aligned} &\frac{1}{F_{M+3}-1}\sum_{k=0}^{M-1} k^2 (F_{M+1-k})\left[\sum_{y=0}^{k} \frac{F_{M+1-y}}{F_{M+3-y}-1}\right] + \frac{1}{(F_{M+3}-1)^3}\sum_{k=0}^{M}\left(4Mk+k^2\right)F_{M+1-k} \\ &+\frac{1}{(F_{M+3}-1)^2}\sum_{k=0}^{M}(M+k)^2(F_{M+1-k})\left[\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right]\end{aligned}\right]$$

$$-\left[\begin{aligned} &\left(1+\frac{1}{F_{M+3}-1}\right)\times\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k(F_{M+1-k})\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right] \\ &+\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M} k(F_{M+1-k})\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\left(\frac{1}{F_{M+3}-1}\right)^2\right)\end{aligned}\right]^2.$$

For the special case with $M = 10$ pins, we have

$$\begin{aligned} E[f_i{}^2] &= \frac{1}{F_{13}-1}\sum_{k=0}^{9} k^2(F_{11-k})\left[\sum_{y=0}^{k}\frac{F_{11-y}}{F_{13-y}-1}\right] + \frac{1}{(F_{13}-1)^3}\sum_{k=0}^{10}\left(40k+k^2\right)F_{11-k} \\ &\quad+\frac{1}{(F_{13}-1)^2}\sum_{k=0}^{10}(10+k)^2(F_{11-k})\left[\sum_{y=0}^{9}\frac{F_{11-y}}{F_{13-y}-1}+\sum_{y=0}^{k}\frac{F_{11-y}}{F_{13-y}-1}\right] \\ &= \frac{16382290943}{1212251040}+\frac{999}{780448}+\left[\frac{43579568897}{17577640080}+\frac{214524137453}{281242241280}\right] \\ &= \frac{120842275339}{7211339520} \\ &\approx 16.7573. \end{aligned}$$

Thus, the variance is

$$Var(f_i) = E[f_i{}^2] - (E[f_i])^2 = \left(\frac{120842275339}{7211339520}\right) - \left(\frac{3195951397}{1030191360}\right)^2 \approx 7.1331,$$

and the standard deviation is

$$\sigma_{f_i} = \sqrt{Var(f_i)} = \sqrt{\left(\frac{120842275339}{7211339520}\right) - \left(\frac{3195951397}{1030191360}\right)^2} \approx 2.6708.$$

6.8 Conclusion of Mean and Variance for the Non-Skilled Bowler

The average score for an N-frame bowling game can be computed by applying the linearity property of the mean. Therefore, we obtain

Theorem 6.8.1. The average score for an N-frame game of bowling on M pins with the Fibonacci Distribution for our non-skilled bowler is given by

$$\sum_{i=1}^{N} E[f_i] = N \times \left[\begin{array}{l} \left(1+\frac{1}{F_{M+3}-1}\right)\times\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k\left(F_{M+1-k}\right)\left[\sum_{y=0}^{k}\frac{F_{M+1-y}}{F_{M+3-y}-1}\right] \\ +\frac{1}{F_{M+3}-1}\left[\sum_{k=0}^{M} k\left(F_{M+1-k}\right)\right]\times\left(\frac{1}{F_{M+3}-1}\sum_{y=0}^{M-1}\frac{F_{M+1-y}}{F_{M+3-y}-1}+\left(\frac{1}{F_{M+3}-1}\right)^2\right) \end{array} \right],$$

where f_i represents the score of each frame.

Thus, the average score for a ten-frame game with $M = 10$ pins is

$$\sum_{i=1}^{10} E[f_i] = 10 \times \frac{3195951397}{1030191360} \approx 31.023.$$

Unlike the mean, the variance of a ten-frame game cannot simply be computed by summing each individual variance because the frames are not independent of one another. That is, f_i depends on f_{i+1} in the cases of strikes and spares. However, the variance of a single frame can be computed, which is approximately 7.1331. In addition, the variance can be applied to calculate the standard deviation, which is approximately 2.6708.

6.9 Correlation Coefficient for the Non-Skilled Bowler

We now shall find the correlation between the score of the first and second roll in a single frame, defined as

$$\rho_{X_1,X_2} = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}}.$$

In Section 6.2, we previously computed the average of the first roll on a set of pins to be

$$E[X_1] = \frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k\left(F_{M+1-k}\right)\right].$$

In order to find the standard deviation for the score of the first roll, we need to compute $E[X_1{}^2]$. Since the probability of each score remains the same, the only difference in the computation is squaring each score before multiplying it by the probability. Therefore,

$$E[X_1{}^2] = \frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k^2\left(F_{M+1-k}\right)\right].$$

For the special case with $M = 10$ pins, we have

$$E[X_1{}^2] = \frac{1}{F_{13} - 1}\left[\sum_{k=0}^{10} k^2\left(F_{11-k}\right)\right] = \frac{178}{29}.$$

We can compute the standard deviation of X_1 to be

$$\begin{aligned}\sigma_{X_1} &= \sqrt{E[X_1{}^2] - \left(E[X_1]\right)^2} \\ &= \sqrt{\frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k^2\left(F_{M+1-k}\right)\right] - \left(\frac{1}{F_{M+3} - 1}\left[\sum_{k=0}^{M} k\left(F_{M+1-k}\right)\right]\right)^2}.\end{aligned}$$

In the special case with $M = 10$ pins, we have

$$\sigma_{X_1} = \sqrt{\frac{178}{29} - \left(\frac{91}{58}\right)^2} = \sqrt{\frac{12367}{3364}} \approx 1.9174.$$

Interestingly, the variance and standard deviation of the first roll for our non-skilled bowler is equivalent to that of our skilled bowler. This result agrees with the result obtained in [6], which states that the values in the *Fib*(n) distribution occur more frequently near n, while the values in the $n+1-Fib(n)$ distribution occur more frequently near 1; however, the two distributions have the same spread.

Also in Section 6.2, we previously computed the average of the second roll on a set of pins to be

$$E[X_2] = \frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k \sum_{y=0}^{M-k} \frac{F_{M+1-y} \times F_{M+1-y-k}}{F_{M+3-y}-1}.$$

In order to find the standard deviation for the score of the second roll, we need to compute $E[X_2{}^2]$. We can find $E[X_2{}^2]$ by taking the square of each score ranging from $0,1,\ldots M$, multiplying it by the probability of the score, and summing the results, shown to be

$$\begin{aligned} E[X_2{}^2] &= \sum k^2 \times P(X_2 = k) \\ &= \frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k^2 \sum_{y=0}^{M-k} \frac{F_{M+1-y} \times F_{M+1-y-k}}{F_{M+3-y}-1}. \end{aligned}$$

For the special case with $M = 10$ pins, we have

$$E[X_2{}^2] = \frac{1}{F_{13}-1}\sum_{k=0}^{10} k^2 \sum_{y=0}^{10-k} \frac{F_{11-y} \times F_{11-y-k}}{F_{13-y}-1} = \frac{2221855511}{404083680} \approx 5.4985.$$

Thus, the standard deviation of X_2 is

$$\begin{aligned} \sigma_{X_2} &= \sqrt{E[X_2{}^2] - (E[X_2])^2} \\ &= \sqrt{\frac{1}{F_{13}-1}\sum_{k=0}^{10} k^2 \sum_{y=0}^{10-k} \frac{F_{11-y} \times F_{11-y-k}}{F_{13-y}-1} - \left(\frac{1}{F_{M+3}-1}\sum_{k=0}^{M} k \sum_{y=0}^{M-k} \frac{F_{M+1-y} \times F_{M+1-y-k}}{F_{M+3-y}-1}\right)^2}. \end{aligned}$$

For the special case with $M = 10$ pins, we have

$$\sigma_{X_2} = \sqrt{\frac{2221855511}{404083680} - \left(\frac{19873123}{13321440}\right)^2} = \sqrt{\frac{52855312018012781}{16148929494297600}} \approx 1.809.$$

Lastly, we need to compute $E[X_1X_2]$. Using background information on the mean, we can multiply the product of the first and second roll by the probability of each possible combination, and sum the products. Thus, the expected value of X_1X_2 is found to be

$$\begin{aligned}
E[X_1X_2] &= \sum k \times y \times P((X_1, X_2) = (k, y)) \\
&= \sum_{k=0}^{M} \sum_{y=0}^{M-k} \left[k \times y \times \frac{F_{M+1-k}}{F_{M+3}-1} \times \frac{F_{M+1-k-y}}{F_{M+3-k}-1} \right] \\
&= \frac{1}{F_{M+3}-1} \sum_{k=0}^{M} k \frac{F_{M+1-k}}{F_{M+3-k}-1} \sum_{y=0}^{M-k} y(F_{M+1-k-y}).
\end{aligned}$$

For the special case with $M = 10$ pins, we have

$$E[X_1X_2] = \frac{1}{F_{13}-1} \sum_{k=0}^{10} k \frac{F_{11-k}}{F_{13-k}-1} \sum_{y=0}^{10-k} y(F_{11-k-y}) = \frac{31477}{15312}.$$

Therefore, the correlation coefficient between the first and second roll of our non-skilled Fibonacci bowler is computed to be

$$\rho_{X_1,X_2} = \frac{E[X_1X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}}$$

$$= \frac{\left[\begin{array}{l} \dfrac{1}{F_{M+3}-1} \displaystyle\sum_{k=0}^{M} k \frac{F_{M+1-k}}{F_{M+3-k}-1} \sum_{y=0}^{M-k} y(F_{M+1-k-y}) \\ -\left(\dfrac{1}{F_{M+3}-1} \left[\displaystyle\sum_{k=0}^{M} k(F_{M+1-k}) \right] \right) \left(\dfrac{1}{F_{M+3}-1} \displaystyle\sum_{k=0}^{M} k \sum_{y=0}^{M-k} \frac{F_{M+1-y} \times F_{M+1-y-k}}{F_{M+3-y}-1} \right) \end{array} \right]}{\left[\begin{array}{l} \sqrt{\dfrac{1}{F_{M+3}-1} \left[\displaystyle\sum_{k=0}^{M} k^2 (F_{M+1-k}) \right] - \left(\dfrac{1}{F_{M+3}-1} \left[\displaystyle\sum_{k=0}^{M} k(F_{M+1-k}) \right] \right)^2} \\ \times \sqrt{\begin{array}{l} \dfrac{1}{F_{13}-1} \displaystyle\sum_{k=0}^{10} k^2 \sum_{y=0}^{10-k} \frac{F_{11-y} \times F_{11-y-k}}{F_{13-y}-1} \\ -\left(\dfrac{1}{F_{M+3}-1} \displaystyle\sum_{k=0}^{M} k \sum_{y=0}^{M-k} \frac{F_{M+1-y} \times F_{M+1-y-k}}{F_{M+3-y}-1} \right)^2 \end{array}} \end{array} \right]}.$$

For the special case with $M = 10$ pins, we have

$$\begin{aligned}
\rho_{X_1,X_2} &= \frac{E[X_1X_2] - E[X_1]E[X_2]}{\sigma_{X_1}\sigma_{X_2}} \\
&= \frac{\frac{31477}{15312} - \left(\frac{91}{58}\right)\left(\frac{19873123}{13321440}\right)}{\left(\sqrt{\frac{12367}{3364}}\right)\left(\sqrt{\frac{52855312018012781}{16148929494297600}}\right)} \\
&= \frac{-\frac{2043294134190922849}{596978008997990400}}{\sqrt{\frac{653661643726764062627}{543249988188171264 00}}} \\
&\approx -0.9867
\end{aligned}$$

We can see that the correlation between the first and second roll in a single frame is dependent on the number of pins. Also, the correlation coefficient is negative because the value of the second roll decreases as the first roll increases. Moreover, there is a very strong linear relationship between the first and second roll using the Fibonacci distribution for our non-skilled bowler. Additionally, the relationship is much stronger than that of the Neal distribution, the Hohn distribution, and the Skilled Fibonacci distribution.

Chapter 7: The Binomial Distribution for a 100*p*% Bowler

In this chapter, we apply the results from [5] to model a $100p\%$ bowler. Neal and Brown considered a scenario in which pins are knocked down according to binomial distributions. On the first ball of each frame, each pin has equal probability p_1 of being knocked down and pins fall independently of each other, creating the $b(M, p_1)$ distribution. On the second ball of each frame, each remaining pin has equal probability p_2 of being knocked down, and again pins fall independently of each other. Spares and strikes are accounted for according to the regular rules of bowling. Under those conditions, they applied a method to derive the average score for a bowling game. However, this method only allowed them to derive the variance of a single roll. Therefore, we shall derive the distribution of each frame's score that facilitates the means to derive the variance and standard deviation for an entire frame in the particular case when $p_1 = p_2$. This scenario refers to a $100p\%$ bowler, who knocks down a single pin $100p\%$ of the time. Additionally, we are able to verify the results obtained from [5] for the average score and the correlation coefficient when bowling N frames on M pins with $p_1 = p_2 = p$.

7.1 Distribution Method for a 100*p*% Bowler

We can verify Neal and Brown's results in [5] for the mean using the distribution method. To find the distribution of a frame with M pins, we must find the probability for scores $0,1,\ldots,3M$. As pins are knocked down on the first roll of each frame, the values of 0 through M are likely to occur with probability

$$P(X = k) = \binom{M}{k} p^k (1-p)^{M-k}.$$

We can verify that all of the possible scenarios are given and make up a partition by summing all of the probabilities, and we can combine similarities into common summations. Thus, the probabilities of scores from 0 to $M-1$ can be combined to the following summation:

$$\sum_{n=0}^{M-1}\sum_{k=0}^{n}\binom{M}{k}p^{k}(1-p)^{M-k}\times\binom{M-k}{n-k}p^{n-k}(1-p)^{(M-k)-(n-k)}$$
$$=\sum_{n=0}^{M-1}\sum_{k=0}^{n}\binom{M}{k}\binom{M-k}{n-k}p^{n}(1-p)^{2M-k-n}.$$

Because the sum of the first two rolls must range from 0 to M with probability 1, we have the following proposition:

Proposition 7.1.1. The sum of probabilities of all possible outcomes for rolls in a frame is

$$\sum_{n=0}^{M}\sum_{k=0}^{n}\binom{M}{k}\binom{M-k}{n-k}p^{n}(1-p)^{2M-k-n}=\sum_{k=0}^{M}P(X_{i1}+X_{i2}=k)=1.$$

Thus, the probabilities of scores from 0 to $M-1$ can be written as the following summation:

$$\sum_{n=0}^{M}\sum_{k=0}^{n}\binom{M}{k}\binom{M-k}{n-k}p^{n}(1-p)^{2M-k-n}-p^{M}\sum_{k=0}^{M}\binom{M}{k}\binom{M-k}{M-k}(1-p)^{2M-k-M}$$
$$=1-p^{M}\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}.$$

Also, the probabilities of scores from M to $2M-1$ can be combined to the following:

$$\sum_{n=0}^{M-1}\sum_{k=0}^{M-1}\binom{M}{k}p^{k}(1-p)^{M-k}\times\binom{M-k}{M-k}p^{M-k}(1-p)^{(M-k)-(M-k)}\times\binom{M}{n}p^{n}(1-p)^{M-n}$$
$$+\sum_{n=0}^{M-1}\binom{M}{M}p^{M}(1-p)^{M-M}\times\sum_{k=0}^{n}\binom{M}{k}p^{k}(1-p)^{M-k}\times\binom{M-k}{n-k}p^{n-k}(1-p)^{(M-k)-(n-k)}$$
$$=\sum_{n=0}^{M-1}\sum_{k=0}^{M-1}\binom{M}{k}\binom{M}{n}p^{M+n}(1-p)^{2M-k-n}+\sum_{n=0}^{M-1}\sum_{k=0}^{n}\binom{M}{k}\binom{M-k}{n-k}p^{M+n}(1-p)^{2M-k-n}.$$

The score $2M$ is unique, so we will begin by writing its probability as

$$\sum_{k=0}^{M-1}\binom{M}{k}p^k(1-p)^{M-k}\times\binom{M-k}{M-k}p^{M-k}(1-p)^{(M-k)-(M-k)}\times\binom{M}{M}p^M(1-p)^{M-M}$$
$$+\binom{M}{M}p^M(1-p)^{M-M}\times\sum_{k=0}^{M-1}\binom{M}{k}p^k(1-p)^{M-k}\times\binom{M-k}{M-k}p^{M-k}(1-p)^{(M-k)-(M-k)}$$
$$+p^M\times p^M\times(1-p)^M$$
$$=\sum_{k=0}^{M-1}\binom{M}{k}p^{2M}(1-p)^{M-k}+\sum_{k=0}^{M-1}\binom{M}{k}p^{2M}(1-p)^{M-k}+p^{2M}(1-p)^M.$$

Since the first part follows the exact pattern as scores from M to $2M-1$, and the second part follows a similar pattern, we can rewrite the first two parts of the probability of the score $2M$ to be

$$\sum_{k=0}^{M-1}\binom{M}{k}p^{2M}(1-p)^{M-k}+\sum_{k=0}^{M}\binom{M}{k}p^{2M}(1-p)^{M-k}-p^{2M}\binom{M}{M}(1-p)^{M-M}$$
$$=\sum_{k=0}^{M-1}\binom{M}{k}p^{2M}(1-p)^{M-k}+\sum_{k=0}^{M}\binom{M}{k}p^{2M}(1-p)^{M-k}-p^{2M}.$$

and combine these parts with the probabilities of scores from M to $2M-1$, which results in the following summation:

$$\sum_{n=0}^{M}\left[\sum_{k=0}^{M-1}\binom{M}{k}\binom{M}{n}p^{M+n}(1-p)^{2M-k-n}\right]$$
$$+\sum_{n=0}^{M}\left[\sum_{k=0}^{n}\binom{M}{k}\binom{M-k}{n-k}p^{M+n}(1-p)^{2M-k-n}\right]-p^{2M}.$$

Proposition 7.1.2. The sum of probabilities of all possible outcomes for the first roll in a frame is

$$\sum_{k=0}^{M}\binom{M}{k}p^k(1-p)^{M-k}=\sum_{k=0}^{M}P(X_{i1}+X_{i2}=k)=1.$$

Using *Proposition 7.1.1* and *Proposition 7.1.2*, the probabilities of scores from M to $2M-1$ and the first two parts of the score $2M$ can be written as the following summation:

$$
\begin{aligned}
& p^M \times \left[\sum_{k=0}^{M-1} \binom{M}{k} (1-p)^{M-k}\right] \times \left(\sum_{n=0}^{M} \binom{M}{n} p^n (1-p)^{M-n}\right) \\
& + p^M \times \left(\sum_{n=0}^{M}\sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^n (1-p)^{2M-k-n}\right) - p^{2M} \\
& = p^M \times \left[\sum_{k=0}^{M-1} \binom{M}{k} (1-p)^{M-k}\right] + p^M - p^{2M} \\
& = p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k} - p^{2M}.
\end{aligned}
$$

Finally, the last part of the score $2M$ can be combined with the probabilities of the scores from $2M+1$ to $3M$, and using *Proposition 7.6.2*, we have the following summation:

$$\binom{M}{M} p^M (1-p)^{M-M} \times \binom{M}{M} p^M (1-p)^{M-M} \times \sum_{k=0}^{M} \binom{M}{k} p^k (1-p)^{M-k} = p^{2M}.$$

Therefore, the sum of probabilities for all the scores $n = 0,1,\ldots,3M$ is

$$\sum_{n=0}^{3M} P(f_i = n) = \left(1 - p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k}\right) + \left(p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k} - p^{2M}\right) + \left(p^{2M}\right) = 1.$$

The probabilities sum to one, which verifies that all of the possible scores are given and make up a partition.

In order to compute the mean of a single frame with M pins, we must multiply each score by its probability, and sum the resulting products. Thus, the weighted sum for scores from 0 to $M-1$ can be combined to the following summation:

$$\sum_{n=0}^{M-1} n \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^n (1-p)^{2M-k-n}.$$

Proposition 7.1.3. The following summation represents the average score per frame without taking into account strikes and spares, so it can be simplified to

$$\sum_{n=0}^{M} n \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^{n}(1-p)^{2M-k-n} = Mp(2-p).$$

Thus, the weighted sum for scores from 0 to $M-1$ can be written as

$$\begin{aligned}
&\sum_{n=0}^{M} n \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^{n}(1-p)^{2M-k-n} - M \sum_{k=0}^{M} \binom{M}{k}\binom{M-k}{M-k} p^{M}(1-p)^{2M-k-M} \\
&= Mp(2-p) - Mp^{M} \sum_{k=0}^{M} \binom{M}{k}(1-p)^{M-k}.
\end{aligned}$$

Also, the weighted sum for scores from M to $2M-1$, along with the first two parts for the score $2M$ can be combined to the following:

$$\begin{aligned}
&p^{M}\left[\sum_{k=0}^{M-1} \binom{M}{k}(1-p)^{M-k}\right] \times \sum_{n=0}^{M} (M+n) \binom{M}{n} p^{n}(1-p)^{M-n} \\
&+ p^{M} \sum_{n=0}^{M} (M+n) \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^{n}(1-p)^{2M-k-n} - 2Mp^{2M}.
\end{aligned}$$

Using *Proposition 7.1.3*, we can simplify the summation to be

$$\begin{aligned}
&p^{M} \times \left[\sum_{k=0}^{M-1} \binom{M}{k}(1-p)^{M-k}\right] \times \left(M \times \sum_{n=0}^{M} \binom{M}{n} p^{n}(1-p)^{M-n} + \sum_{n=0}^{M} n \binom{M}{n} p^{n}(1-p)^{M-n} \right) \\
&+ Mp^{M} \times \left(\sum_{n=0}^{M} \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^{n}(1-p)^{2M-k-n} \right) \\
&+ p^{M} \times \left(\sum_{n=0}^{M} n \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^{n}(1-p)^{2M-k-n} \right) - 2Mp^{2M} \\
&= p^{M} \times \left[\sum_{k=0}^{M-1} \binom{M}{k}(1-p)^{M-k}\right] \times (M + Mp) + Mp^{M} + p^{M}(Mp(2-p)) - 2Mp^{2M} \\
&= Mp^{M} \times \left[\sum_{k=0}^{M} \binom{M}{k}(1-p)^{M-k}\right] + Mp^{M+1}\left[\sum_{k=0}^{M} \binom{M}{k}(1-p)^{M-k}\right] + M\left(p^{M+1} - p^{M+2} - 2p^{2M}\right).
\end{aligned}$$

Finally, the weighted sum for scores from $2M+1$ to $3M$, along with the last part for the score $2M$ can be combined to the following summation:

$$
\begin{aligned}
&p^{2M}\sum_{k=0}^{M}(2M+k)\binom{M}{k}p^k(1-p)^{M-k} \\
&=2Mp^{2M}\times\left(\sum_{k=0}^{M}\binom{M}{k}p^k(1-p)^{M-k}\right)+p^{2M}\times\left(\sum_{k=0}^{M}k\binom{M}{k}p^k(1-p)^{M-k}\right) \\
&=2Mp^{2M}+p^{2M}\times Mp \\
&=Mp^{2M}(2+p).
\end{aligned}
$$

Therefore, the mean of a single frame with M pins is the following:

$$
\begin{aligned}
E[f_i]&=Mp(2-p)-Mp^M\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k} \\
&\quad+Mp^M\left[\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right]+Mp^{M+1}\left[\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right] \\
&\quad-M\left(p^{M+2}+p^{M+1}-2p^{2M}\right)+Mp^{2M}(2+p) \\
&=2Mp-Mp^2+Mp^{M+1}\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}-Mp^{M+2}+Mp^{M+1}+Mp^{2M+1} \\
&=Mp\left(2-p+p^M\left[\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right]-p^{M+1}+p^M+p^{2M}\right) \\
&=Mp\left(2-p+p^M\left[\sum_{k=0}^{M-1}\binom{M}{k}(1-p)^{M-k}+1\right]-p^{M+1}+p^M+p^{2M}\right) \\
&=Mp\left(2-p+p^M\left[p^M-p+2+\sum_{k=0}^{M-1}\binom{M}{k}(1-p)^{M-k}\right]\right)
\end{aligned}
$$

This result agrees with the result previously obtained in [5] for the particular case when $p_1=p_2=p$.

7.2 Frame Variance for a 100p% Bowler

We can also derive the variance of a single frame for our $100p\%$ bowler. First, we need to compute $E[f_i^{\,2}]$, which can be found by squaring each score, then multiplying it by the probability of the score and summing the results. The probabilities remain the same, so the computation is similar to the mean of a single frame.

We can combine the weighted sum of the squared scores from 0 to $M-1$ to be

$$\sum_{n=0}^{M-1} n^2 \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^n (1-p)^{2M-k-n}.$$

Proposition 7.2.1. The following summation can be simplified as

$$\sum_{n=0}^{M} n^2 \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^n (1-p)^{2M-k-n} = M(M-1)p^2(2-p)^2 + Mp(2-p).$$

Thus, the mean of the weighted sum of the squared scores from 0 to $M-1$ can be written as

$$\sum_{n=0}^{M} n^2 \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^n (1-p)^{2M-k-n} - M^2 \sum_{k=0}^{M} \binom{M}{k}\binom{M-k}{M-k} p^M (1-p)^{2M-k-M}$$
$$= M(M-1)p^2(2-p)^2 + Mp(2-p) - M^2 p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k}.$$

Also, the mean of the squared scores from M to $2M-1$, along with parts of $2M$ can be combined into the following summation:

$$p^M \left[\sum_{k=0}^{M-1} \binom{M}{k} (1-p)^{M-k} \right] \times \sum_{n=0}^{M} (M+n)^2 \binom{M}{n} p^n (1-p)^{M-n}$$
$$+ p^M \sum_{n=0}^{M} (M+n)^2 \sum_{k=0}^{n} \binom{M}{k}\binom{M-k}{n-k} p^n (1-p)^{2M-k-n} - (2M)^2 p^{2M}.$$

Proposition 7.2.2. The following summation can be simplified as

$$\sum_{k=0}^{M} k^2 \binom{M}{k} p^k (1-p)^{M-k} = M(M-1)p^2 + Mp\,.$$

Using the propositions in Section 7.1 and 7.2, we can simplify the summation to be

$$
\begin{aligned}
& p^M \left[\sum_{k=0}^{M-1} \binom{M}{k} (1-p)^{M-k} \right] \times \begin{pmatrix} M^2 \times \left(\sum_{n=0}^{M} \binom{M}{n} p^n (1-p)^{M-n} \right) \\ + 2M \times \left(\sum_{n=0}^{M} n \binom{M}{n} p^n (1-p)^{M-n} \right) \\ + \left(\sum_{n=0}^{M} n^2 \binom{M}{n} p^n (1-p)^{M-n} \right) \end{pmatrix} \\
& + p^M \begin{pmatrix} M^2 \left(\sum_{n=0}^{M} \sum_{k=0}^{n} \binom{M}{k} \binom{M-k}{n-k} p^n (1-p)^{2M-k-n} \right) \\ + 2M \left(\sum_{n=0}^{M} n \sum_{k=0}^{n} \binom{M}{k} \binom{M-k}{n-k} p^n (1-p)^{2M-k-n} \right) \\ + \left(\sum_{n=0}^{M} n^2 \sum_{k=0}^{n} \binom{M}{k} \binom{M-k}{n-k} p^n (1-p)^{2M-k-n} \right) \end{pmatrix} - (2M)^2 p^{2M} \\
& = p^M \left[\sum_{k=0}^{M-1} \binom{M}{k} (1-p)^{M-k} \right] \times \left(M^2 \times 1 + 2M \times Mp + M(M-1)p^2 + Mp \right) \\
& + p^M \left(M^2 \times 1 + 2M \times Mp(2-p) + M(M-1)p^2 (2-p)^2 + Mp(2-p) \right) - (2M)^2 p^{2M} \\
& = \left(M^2 + 2M^2 p + M(M-1)p^2 + Mp \right) \left[p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k} \right] \\
& - p^M \left(M^2 + 2M^2 p + M(M-1)p^2 + Mp \right) \\
& + p^M \left(M^2 + 2M^2 p(2-p) + M(M-1)p^2 (2-p)^2 + Mp(2-p) \right) - 4M^2 p^{2M} \\
& = \left[M^2 p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k} \right] + \left(2M^2 p + M(M-1)p^2 + Mp \right) \left[p^M \sum_{k=0}^{M} \binom{M}{k} (1-p)^{M-k} \right] \\
& + p^M (1-p) \left(2M^2 p + M(M-1)p^2 (3-p) + Mp \right) - 4M^2 p^{2M}\,.
\end{aligned}
$$

Finally, the weighted sum for squared scores from $2M+1$ to $3M$, along with the last part of $2M$, can be combined into the following summation:

$$p^{2M}\sum_{k=0}^{M}(2M+k)^2\binom{M}{k}p^k(1-p)^{M-k}.$$

Using *Proposition 7.1.2* and *Proposition 7.2.2*, we can simplify the summation to be

$$\begin{aligned}
&4M^2p^{2M}\times\left(\sum_{k=0}^{M}\binom{M}{k}p^k(1-p)^{M-k}\right)+4Mp^{2M}\times\left(\sum_{k=0}^{M}k\binom{M}{k}p^k(1-p)^{M-k}\right)\\
&+p^{2M}\times\left(\sum_{k=0}^{M}k^2\binom{M}{k}p^k(1-p)^{M-k}\right)\\
&=4M^2p^{2M}+4Mp^{2M}\times Mp+p^{2M}\times\left(M(M-1)p^2+Mp\right)\\
&=4M^2p^{2M}+p^{2M}\left(4M^2p+M(M-1)p^2+Mp\right).
\end{aligned}$$

Thus, $E[f_i^2]$ can be written as the following:

$$\begin{aligned}
E[f_i^2]&=M(M-1)p^2(2-p)^2+Mp(2-p)-M^2p^M\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\\
&+\left[M^2p^M\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right]\\
&+\left(2M^2p+M(M-1)p^2+Mp\right)\left[p^M\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right]\\
&+p^M(1-p)\left(2M^2p+M(M-1)p^2(3-p)+Mp\right)-4M^2p^{2M}\\
&+4M^2p^{2M}+p^{2M}\left(4M^2p+M(M-1)p^2+Mp\right)\\
&=M(M-1)p^2(2-p)^2+Mp(2-p)\\
&+\left(2M^2p+M(M-1)p^2+Mp\right)\left[p^M\sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right]\\
&+p^M(1-p)\left(2M^2p+M(M-1)p^2(3-p)+Mp\right)\\
&+p^{2M}\left(4M^2p+M(M-1)p^2+Mp\right)
\end{aligned}$$

$$= Mp\begin{pmatrix} (M-1)p(2-p)^2 + (2-p) + (2M + (M-1)p + 1)\left[p^M \sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right] \\ + p^M(1-p)(2M + (M-1)p + 1) + p^M(1-p)(M-1)p(2-p) \\ + p^{2M}(2M + (M-1)p + 1) + 2Mp^{2M} \end{pmatrix}$$

$$= Mp\begin{pmatrix} (2-p)\left[(M-1)p(1-p) + (M-1)p + 1 + p^M(1-p)(M-1)p\right] + 2Mp^{2M} \\ + p^M(2M + (M-1)p + 1)\left[p^M + (1-p) + \sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right] \end{pmatrix}$$

$$= Mp\begin{pmatrix} (2-p)\left[(M-1)p(1-p)(1+p^M) + (M-1)p + 1\right] + 2Mp^{2M} \\ + p^M(2M + (M-1)p + 1)\left[p^M + (1-p) + \sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right] \end{pmatrix}$$

Therefore the variance of each frame is equal to

$$\begin{aligned} Var(f_i) &= E[f_i{}^2] - (E[f_i])^2 \\ &= Mp\begin{pmatrix} (2-p)\left[(M-1)p(1-p)(1+p^M) + (M-1)p + 1\right] + 2Mp^{2M} \\ + p^M(2M + (M-1)p + 1)\left[p^M + (1-p) + \sum_{k=0}^{M}\binom{M}{k}(1-p)^{M-k}\right] \end{pmatrix} \\ &\quad - \left[Mp\left(2 - p + p^M\left[p^M - p + 2 + \sum_{k=0}^{M-1}\binom{M}{k}(1-p)^{M-k}\right]\right)\right]^2. \end{aligned}$$

For the special case with $M = 10$ pins and $p = 0.9$, the average is

$$\begin{aligned} E[f_i] &= Mp\left(2 - p + p^M\left[p^M - p + 2 + \sum_{k=0}^{M-1}\binom{M}{k}(1-p)^{M-k}\right]\right) \\ &= \frac{1944743916248436400218}{10^{20}} \\ &\approx 19.4474, \end{aligned}$$

and the square of the average is

$$E[{f_i}^2] = 10\left(\frac{9}{10}\right)\left(\begin{aligned}&\left(2-\left(\frac{9}{10}\right)\right)\left[9\left(\frac{9}{10}\right)\left(\frac{1}{10}\right)\left(1+\left(\frac{9}{10}\right)^{10}\right)+9\left(\frac{9}{10}\right)+1\right]+20\left(\frac{9}{10}\right)^{20}\\&+\left(\frac{9}{10}\right)^{10}\left(20+9\left(\frac{9}{10}\right)+1\right)\left[\left(\frac{9}{10}\right)^{10}+\left(\frac{1}{10}\right)+\sum_{k=0}^{M}\binom{10}{k}\left(\frac{1}{10}\right)^{10-k}\right]\end{aligned}\right)$$

$$= 9\left(\begin{aligned}&\left(\frac{11}{10}\right)\left[\left(\frac{81}{10^2}\right)\left(\frac{13486784401}{10^{10}}\right)+\left(\frac{91}{10}\right)\right]+20\left(\frac{9}{10}\right)^{20}\\&+\left(\frac{9}{10}\right)^{10}\left(\frac{291}{10}\right)\left[\left(\frac{9}{10}\right)^{10}+\left(\frac{1}{10}\right)+\frac{25937424601}{10^{10}}\right]\end{aligned}\right)$$

$$= 9\left(\left(\frac{11}{10}\right)\left[\frac{10192429536481}{10^{12}}\right]+20\left(\frac{9}{10}\right)^{20}+\left(\frac{9}{10}\right)^{10}\left(\frac{291}{10}\right)\left[\frac{30424209002}{10^{10}}\right]\right)$$

$$= 9\left(\frac{112116724901291}{10^{13}}+\frac{243153309181138576020}{10^{20}}+\frac{3087005329203277694038 2}{10^{21}}\right)$$

$$= \frac{400619329865759364305238}{10^{21}}$$

$$\approx 400.6193.$$

Thus, the variance is

$$Var(f_i) = \left(\frac{400619329865759364305238}{10^{21}}\right) - \left(\frac{1944743916248436400218}{10^{20}}\right)^2 \approx 22.416\ ,$$

and the standard deviation is

$$\begin{aligned}\sigma_{f_i} &= \sqrt{Var(f_i)}\\&= \sqrt{\left(\frac{400619329865759364305238}{10^{21}}\right) - \left(\frac{1944743916248436400218}{10^{20}}\right)^2}\\&\approx 4.7348.\end{aligned}$$

7.3 Conclusion of Mean and Variance for a 100p% Bowler

The average score for an N-frame bowling game can be computed by applying the linearity property of the mean.

Theorem 7.3.1. The average score for an N-frame game of bowling on M pins with the Binomial distribution for a $100p\%$ bowler is given by

$$\sum_{i=1}^{N} E[f_i] = N \times \left[Mp\left(2 - p + p^M \left[p^M - p + 2 + \sum_{k=0}^{M-1} \binom{M}{k} (1-p)^{M-k} \right]\right)\right],$$

where f_i represents the score of each frame.

Thus, the average score for a ten-frame game with $M = 10$ pins and $p = 0.9$, our 90% bowler, is

$$\sum_{i=1}^{10} E[f_i] = 10 \times \frac{1944743916248436400218}{10^{20}} \approx 194.474 .$$

Unlike the mean, the variance of a ten-frame game cannot simply be computed by summing each individual variance because the frames are not independent of one another. That is, f_i depends on f_{i+1} in the cases of strikes and spares. However, the variance of a single frame can be computed, which is approximately 22.416. In addition, the variance can be used to calculate the standard deviation, which is approximately 4.735.

7.4 Correlation Coefficient for a 100*p*% Bowler

We now shall find the correlation between the score of the first and second roll in a single frame for the particular case when $p_1 = p_2 = p$. Using the results from [5], we obtain

$$\begin{aligned} \rho_{X_1, X_2} &= \frac{-\sqrt{p_1 p_2}}{\sqrt{(1 - p_2 + p_1 p_2)}} \\ &= \frac{-p}{\sqrt{(p^2 - p + 1)}} . \end{aligned}$$

For the special case with $M = 10$ pins and $p = 0.9$, we have

$$\rho_{X_1,X_2} = \frac{-\left(\frac{9}{10}\right)}{\sqrt{\left(\left(\frac{9}{10}\right)^2 - \left(\frac{9}{10}\right) + 1\right)}} = -\sqrt{\frac{81}{91}} \approx -0.9435\,.$$

We can see that the correlation coefficient is not dependent on the number of pins. Also, the correlation coefficient is negative because the value of the second roll decreases as the first roll increases. Moreover, there is a very strong relationship between the first and second roll using the binomial distribution for a 90% bowler.

Chapter 8: Synopsis

In this chapter, we make tables that display the results obtained from each distribution used to model a particular bowler's score. We evaluate the special case when bowling 10 frames on 10 pins, which represents a standard bowling game.

8.1 The Hohn Distribution

The table below shows the distribution of the first and second roll for a single frame, ranging from 0 to 10, followed by the approximated values of the expected value of the score of each roll and the correlation between the two values.

<table>
<tr><td colspan="11">Distribution of the First Roll: X_{i1}</td></tr>
<tr><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>$\frac{11}{66}$</td><td>$\frac{10}{66}$</td><td>$\frac{9}{66}$</td><td>$\frac{8}{66}$</td><td>$\frac{7}{66}$</td><td>$\frac{6}{66}$</td><td>$\frac{5}{66}$</td><td>$\frac{4}{66}$</td><td>$\frac{3}{66}$</td><td>$\frac{2}{66}$</td><td>$\frac{1}{66}$</td></tr>
<tr><td colspan="11">Distribution of the Second Roll: X_{i2}</td></tr>
<tr><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>$\frac{11}{66}$</td><td>$\frac{10}{66}$</td><td>$\frac{9}{66}$</td><td>$\frac{8}{66}$</td><td>$\frac{7}{66}$</td><td>$\frac{6}{66}$</td><td>$\frac{5}{66}$</td><td>$\frac{4}{66}$</td><td>$\frac{3}{66}$</td><td>$\frac{2}{66}$</td><td>$\frac{1}{66}$</td></tr>
<tr><td colspan="4">$E[X_{i1}]$</td><td colspan="3">$E[X_{i2}]$</td><td colspan="4">$\rho_{X_{i1},X_{i2}}$</td></tr>
<tr><td colspan="4">3.33</td><td colspan="3">3.33</td><td colspan="4">-0.5</td></tr>
</table>

The following table shows the approximated values of the mean, variance, and standard deviation for frames $i = 1,2,\ldots,9$.

Mean	Variance	Standard Deviation
$E[f_i]$	$Var(f_i)$	σ_{f_i}
7.27349	14.4796	3.8052

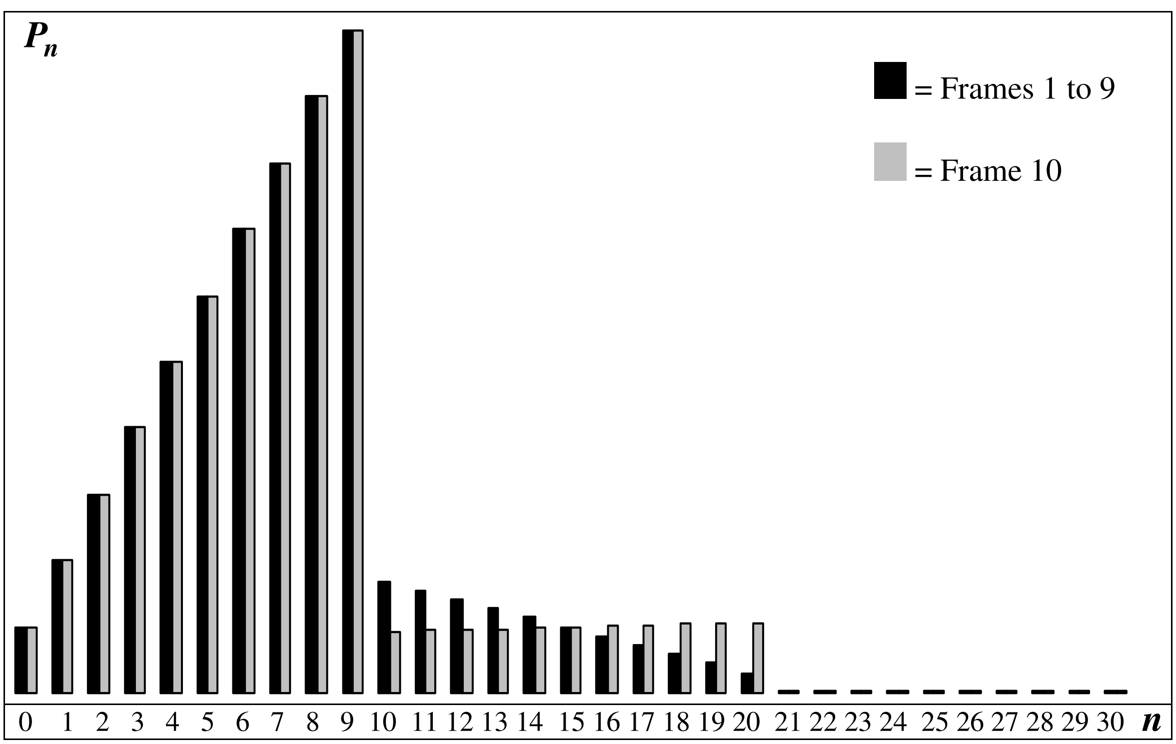
The table below shows the approximated values of the mean, variance, and standard deviation for the last frame, followed by the final score for a ten-frame game.

Mean	Variance	Standard Deviation
$E[f_{10}]$	$Var(f_{10})$	$\sigma_{f_{10}}$
7.5264	18.3315	4.2815
Final Mean Score for a Standard Bowling Game: **72.9878**		

Verification using *Mathematica*

The results for the distribution of scores ranging from 0 to 30 for a single frame can be verified using *Mathematica*. A program was written using piecewise functions to display a bar graph representing the probability distribution for the scores in a single frame. Since the last frame has a unique distribution, we must analyze the last frame separately. Thus, there is a bar graph shown below representing frames $i = 1,2,\ldots,9$ displayed in black, and the last frame displayed in grey.

Hohn Distribution



8.2 The Neal Distribution

The table below shows the distribution of the score of the first and second rolls for a single frame, ranging from 0 to 10, followed by the expected value of each roll and the approximated correlation between the two values.

	Distribution of the 1st Roll $X_{i1} \sim DU[0,10]$		Distribution of the 2nd Roll $X_{i2} \sim DU[0,10-X_{i1}]$	
Score	Exact	Approximation	Exact	Approximation
0	$\frac{1}{11}$	0.091	$\frac{83711}{304920}$	0.275
1	$\frac{1}{11}$	0.091	$\frac{55991}{304920}$	0.184
2	$\frac{1}{11}$	0.091	$\frac{42131}{304920}$	0.138
3	$\frac{1}{11}$	0.091	$\frac{32891}{304920}$	0.108
4	$\frac{1}{11}$	0.091	$\frac{25961}{304920}$	0.085
5	$\frac{1}{11}$	0.091	$\frac{20417}{304920}$	0.067
6	$\frac{1}{11}$	0.091	$\frac{15797}{304920}$	0.052
7	$\frac{1}{11}$	0.091	$\frac{1691}{43560}$	0.039
8	$\frac{1}{11}$	0.091	$\frac{299}{10890}$	0.027
9	$\frac{1}{11}$	0.091	$\frac{21}{1210}$	0.017
10	$\frac{1}{11}$	0.091	$\frac{1}{121}$	0.008
Mean:	$E[X_{i1}]=5$		$E[X_{i2}]=2.5$	
$\rho_{X_{i1},X_{i2}} \approx -0.6324$				

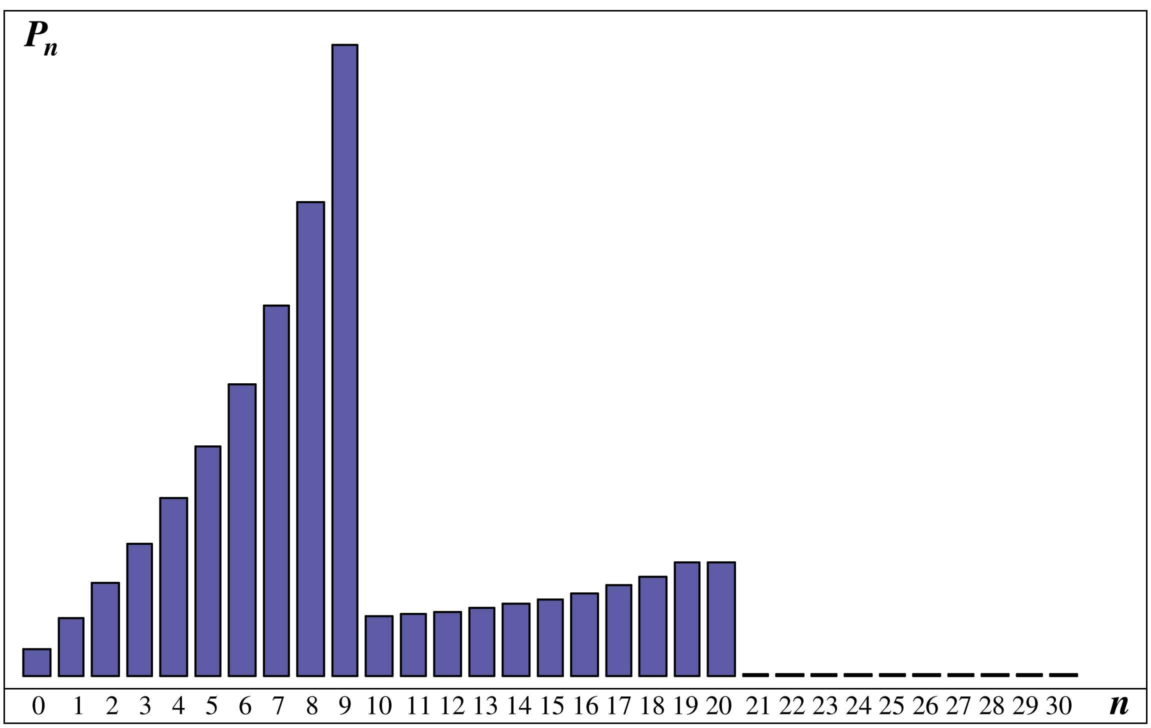
The table below shows the approximated values of the mean, variance, and standard deviation for the scores for frames $i = 1,2,\ldots,10$, followed by the final mean score of a random bowler for a ten-frame game.

Mean	Variance	Standard Deviation
$E[f_i]$	$Var(f_i)$	σ_{f_i}
9.1413	24.987	4.9987
Final Mean Score for a Standard Bowling Game: **91.4127**		

Verification using *Mathematica*

The results for the distribution of scores ranging from 0 to 30 for a single frame can be verified using *Mathematica*. A program was written using piecewise functions to display a bar graph representing the probability distribution for the scores in a single frame, which is shown below.

Neal Probability Distribution



8.3 Fibonacci Distribution for the Skilled Bowler

The table below shows the distribution of the score of the first and second roll for a single frame, ranging from 0 to 10, followed by the expected value of each roll and the approximated correlation between the two values.

	Distribution of the 1st Roll $X_{i1} \sim Fib(11)-1$		Distribution of the 2nd Roll $X_{i2} \sim Fib(11-X_{i1})-1$	
Score	Exact	Approximation	Exact	Approximation
0	$\frac{1}{232}$	0.0043	$\frac{193668203}{346357440}$	0.5592
1	$\frac{1}{232}$	0.0043	$\frac{60798323}{346357440}$	0.1755
2	$\frac{2}{232}$	0.0086	$\frac{39486046}{346357440}$	0.1140
3	$\frac{3}{232}$	0.0129	$\frac{21159609}{346357440}$	0.0611
4	$\frac{5}{232}$	0.0216	$\frac{12872215}{346357440}$	0.0372
5	$\frac{8}{232}$	0.0345	$\frac{7656904}{346357440}$	0.0221
6	$\frac{13}{232}$	0.0560	$\frac{4679285}{346357440}$	0.0135
7	$\frac{21}{232}$	0.0905	$\frac{2808645}{346357440}$	0.0081
8	$\frac{34}{232}$	0.1466	$\frac{1727370}{346357440}$	0.005
9	$\frac{55}{232}$	0.2371	$\frac{928125}{346357440}$	0.0026
10	$\frac{89}{232}$	0.3836	$\frac{572715}{346357440}$	0.0017
Mean:	$E[X_{i1}]=\frac{489}{58}\approx 8.431$		$E[X_{i2}]=\frac{14179157}{13321440}\approx 1.064$	
$\rho_{X_{i1},X_{i2}} \approx -0.8844$				

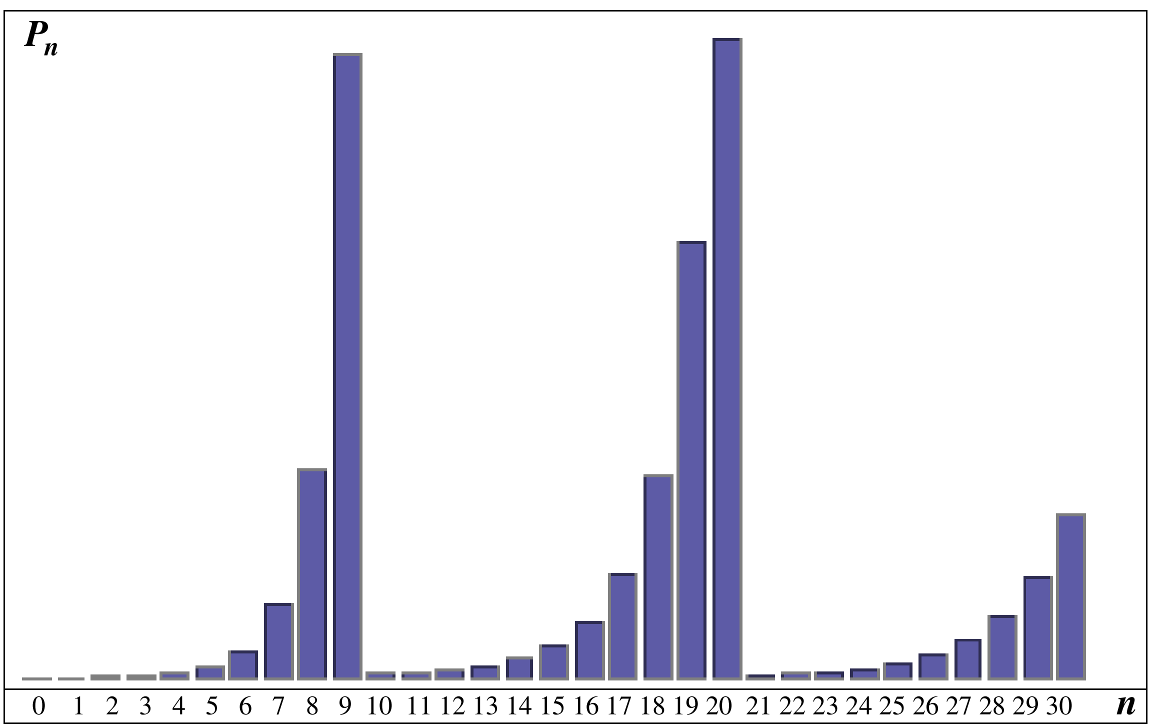
The table below shows the approximated values of the mean, variance, and standard deviation of scores for frames $i = 1,2,\ldots,10$, followed by the final mean score of our skilled bowler for a ten-frame game.

Mean	Variance	Standard Deviation
$E[f_i]$	$Var(f_i)$	σ_{f_i}
16.80615	46.9686	6.8534
Final Mean Score for a Standard Bowling Game**: 168.0615**		

Verification using *Mathematica*

The results for the distribution of scores ranging from 0 to 30 for a single frame can be verified using *Mathematica*. A program was written using piecewise functions to display a bar graph representing the probability distribution for the scores in a single frame, which is shown below.

Fibonacci Probability Distribution for a Skilled Bowler



8.4 Fibonacci Distribution for the Non-Skilled Bowler

The table below shows the distribution of the scores of the first and second roll for a single frame, ranging from 0 to 10, followed by the expected value of each roll and the approximated correlation between the two values.

	Distribution of the 1st Roll $X_{i1} \sim 11 - Fib(11)$		Distribution of the 2nd Roll $X_{i2} \sim 11 - Fib(11 - X_{i1})$	
Score	Exact	Approximation	Exact	Approximation
0	$\frac{89}{232}$	0.3836	$\frac{948788943}{2424502080}$	0.3913
1	$\frac{55}{232}$	0.2371	$\frac{581085335}{2424502080}$	0.2397
2	$\frac{34}{232}$	0.1466	$\frac{357253168}{2424502080}$	0.1474
3	$\frac{21}{232}$	0.0905	$\frac{218606947}{2424502080}$	0.0902
4	$\frac{13}{232}$	0.0560	$\frac{133421001}{2424502080}$	0.0550
5	$\frac{8}{232}$	0.0345	$\frac{80707186}{2424502080}$	0.0333
6	$\frac{5}{232}$	0.0216	$\frac{48359465}{2424502080}$	0.0199
7	$\frac{3}{232}$	0.0129	$\frac{28167545}{2424502080}$	0.0116
8	$\frac{2}{232}$	0.0086	$\frac{16075080}{2424502080}$	0.0066
9	$\frac{1}{232}$	0.0043	$\frac{8028405}{2424502080}$	0.0033
10	$\frac{1}{232}$	0.0043	$\frac{4009005}{2424502080}$	0.0017
Mean:	$E[X_{i1}] = \frac{91}{58} \approx 1.569$		$E[X_{i2}] = \frac{19873123}{13321440} \approx 1.49181$	
$\rho_{X_{i1}, X_{i2}} \approx -0.9867$				

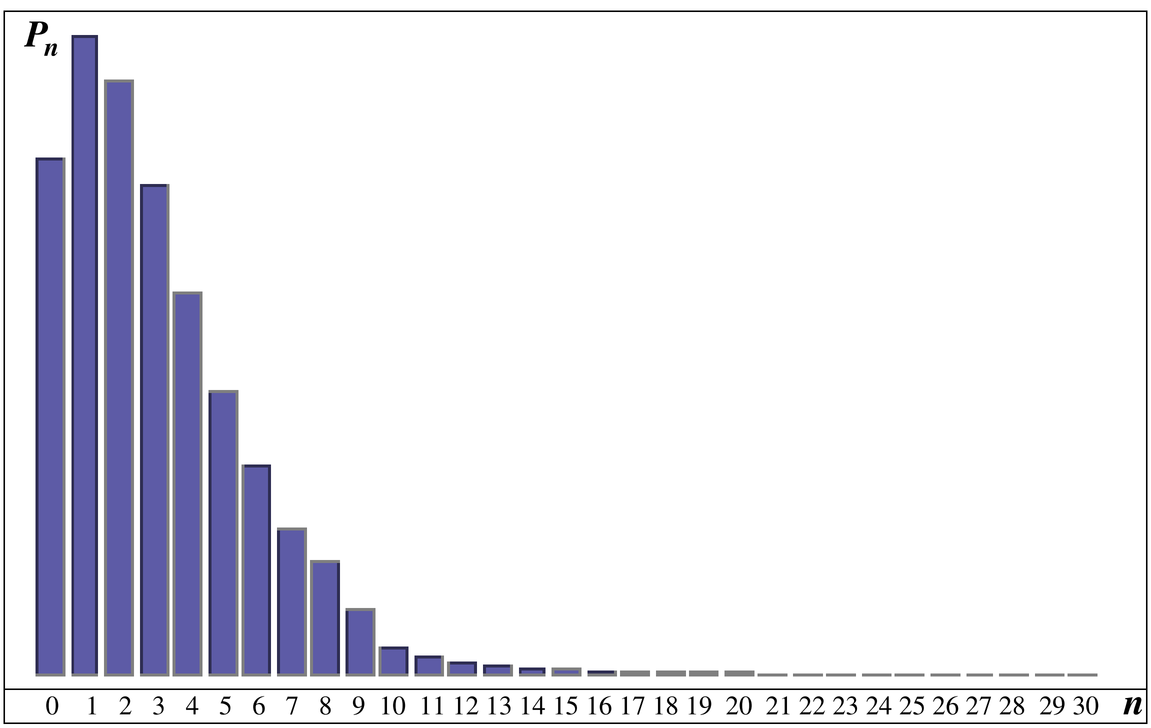
The table below shows the approximated values of the mean, variance, and standard deviation of scores for frames $i = 1,2,\ldots,10$, followed by the final mean score of our non-skilled bowler for a ten-frame game.

Mean	Variance	Standard Deviation
$E[f_i]$	$Var(f_i)$	σ_{f_i}
3.10229	7.1331	2.6708
Final Mean Score for a Standard Bowling Game: **31.023**		

Verification using *Mathematica*

The results for the distribution of scores ranging from 0 to 30 for a single frame can be verified using *Mathematica*. A program was written using piecewise functions to display a bar graph representing the probability distribution for the scores in a single frame, which is shown below.

Fibonacci Probability Distribution for a Non-Skilled Bowler



8.5 Binomial Distribution for a 90% Bowler

We can evaluate the results for a 90% bowler by substituting $p = 0.9$ into the equations. Thus, the table below shows the approximated distribution of the scores of the first and second rolls for a single frame, ranging from 0 to 10, followed by the expected value of each roll and the approximated correlation between the two values.

Score	Distribution of the First Roll $X_{i1} \sim B(10, 0.9)$	Distribution of the Second Roll $X_{i2} \sim B(10 - X_{i1}, 0.9)$
0	10×10^{-10}	0.389
1	9×10^{-9}	0.385
2	3.645×10^{-7}	0.171
3	8.748×10^{-6}	0.045
4	1.378×10^{-4}	0.008
5	0.001	9.286×10^{-4}
6	0.011	7.653×10^{-5}
7	0.057	4.325×10^{-6}
8	0.194	1.604×10^{-7}
9	0.387	3.526×10^{-9}
10	0.349	3.487×10^{-11}
Mean:	$E[X_{i1}] = 9$	$E[X_{i2}] = 0.9$
	$\rho_{X_{i1}, X_{i2}} \approx -0.9435$	

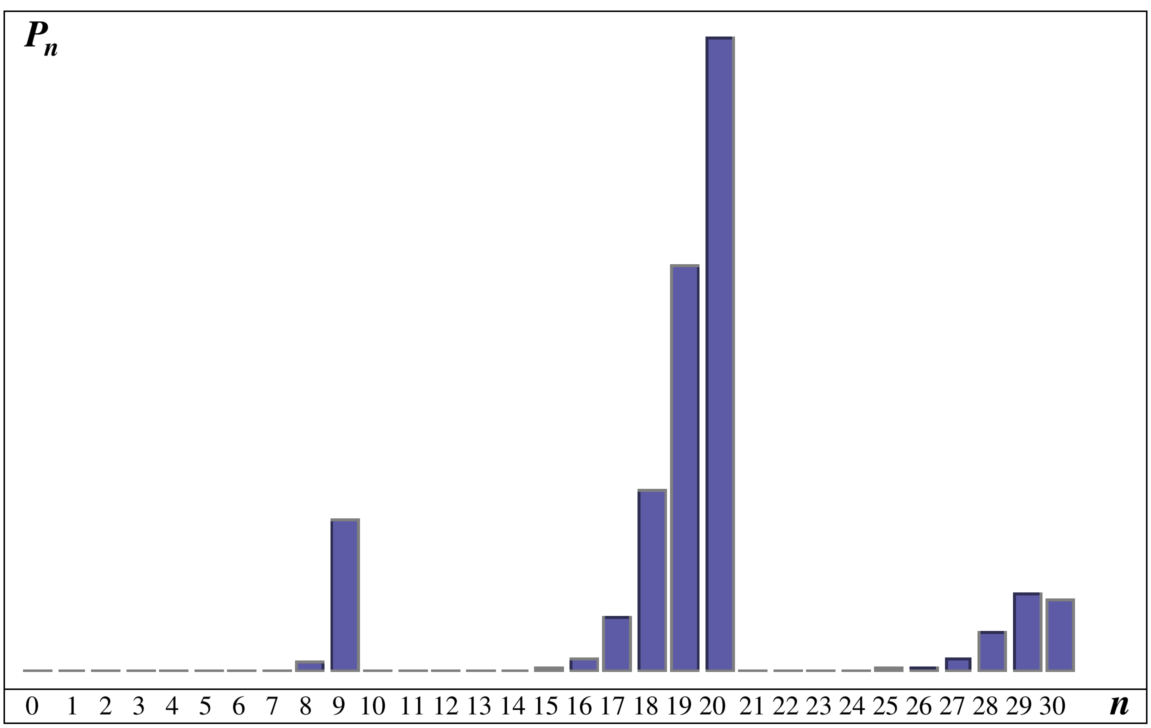
The table below shows the approximated values of the mean, variance, and standard deviation of scores for frames $i = 1,2,\ldots,10$, followed by the final mean score of a 90% bowler for a ten-frame game.

Mean	Variance	Standard Deviation
$E[f_i]$	$Var(f_i)$	σ_{f_i}
19.4474	22.416	4.7348
Final Mean Score for a Standard Bowling Game: **194.474**		

Verification using *Mathematica*

The results for the distribution of scores ranging from 0 to 30 for a single frame can be verified using *Mathematica*. A program was written using piecewise functions to display a bar graph representing the probability distribution for the scores in a single frame, which is shown below.

Binomial Distribution for a 90% Bowler



8.6 Distribution of Scores for a Single Frame

The table below shows the distribution of the scores ranging from 0 to 30 for a single frame for each probability distribution, followed by the expected value of each frame and the expected value of the game's final score.

Dist. Method	Hohn	Hohn	Neal	Fibonacci Skilled	Fibonacci Non-Skilled	Binomial 90% Bowler
Frames / Scores	$i=1,2,\ldots,9$	$i=10$	$i=1,2,\ldots,10$	$i=1,2,\ldots,10$	$i=1,2,\ldots,10$	$i=1,2,\ldots,10$
0	$\frac{1}{66}$	$\frac{1}{66}$	$\frac{1}{121}$	$\frac{1}{53824}$	$\frac{7921}{53824}$	$1.\times10^{-20}$
1	$\frac{2}{66}$	$\frac{2}{66}$	$\frac{21}{1210}$	$\frac{375}{7696832}$	$\frac{127435}{699712}$	9.9×10^{-18}
2	$\frac{3}{66}$	$\frac{3}{66}$	$\frac{299}{10890}$	$\frac{159}{962104}$	$\frac{651185}{3848416}$	4.411×10^{-15}
3	$\frac{4}{66}$	$\frac{4}{66}$	$\frac{1691}{43560}$	$\frac{31411}{69271488}$	$\frac{3226027}{23090496}$	1.164×10^{-12}
4	$\frac{5}{66}$	$\frac{5}{66}$	$\frac{15797}{304920}$	$\frac{8009}{6297408}$	$\frac{578485}{5328576}$	2.018×10^{-10}
5	$\frac{6}{66}$	$\frac{6}{66}$	$\frac{20417}{304920}$	$\frac{597359}{173178720}$	$\frac{3489593}{43294680}$	2.3965×10^{-8}
6	$\frac{7}{66}$	$\frac{7}{66}$	$\frac{25961}{304920}$	$\frac{1084181}{115452480}$	$\frac{4111643}{69271488}$	1.9771×10^{-6}
7	$\frac{8}{66}$	$\frac{8}{66}$	$\frac{32891}{304920}$	$\frac{686617}{26642880}$	$\frac{33260261}{808167360}$	0.00011185
8	$\frac{9}{66}$	$\frac{9}{66}$	$\frac{42131}{304920}$	$\frac{777137}{10823670}$	$\frac{38485481}{1212251040}$	0.00415235
9	$\frac{10}{66}$	$\frac{10}{66}$	$\frac{55991}{304920}$	$\frac{14969941}{69271488}$	$\frac{43710701}{2424502080}$	0.0913517
10	$\frac{111}{4356}$	$\frac{61}{4356}$	$\frac{58511}{3354120}$	$\frac{8357567}{6696243840}$	$\frac{2123526917}{281242241280}$	5.557×10^{-11}

11	$\frac{102}{4356}$	$\frac{62}{4356}$	$\frac{61283}{3354120}$	$\frac{8434997}{6696243840}$	$\frac{25869553}{5113495296}$	5.0013×10^{-9}
12	$\frac{93}{4356}$	$\frac{63}{4356}$	$\frac{64363}{3354120}$	$\frac{34088423}{13392487680}$	$\frac{10775059}{3195934560}$	2.0255×10^{-7}
13	$\frac{84}{4356}$	$\frac{64}{4356}$	$\frac{16957}{838530}$	$\frac{156566081}{40177463040}$	$\frac{1360019}{608749440}$	4.8613×10^{-6}
14	$\frac{75}{4356}$	$\frac{65}{4356}$	$\frac{17947}{838530}$	$\frac{26889725}{4017746304}$	$\frac{15989419}{10817009280}$	0.00007657
15	$\frac{66}{4356}$	$\frac{66}{4356}$	$\frac{9551}{419265}$	$\frac{452037307}{40177463040}$	$\frac{17034463}{17577640080}$	0.00082692
16	$\frac{57}{4356}$	$\frac{67}{4356}$	$\frac{10244}{419265}$	$\frac{44050319}{2232081280}$	$\frac{36246101}{56248448256}$	0.00620249
17	$\frac{48}{4356}$	$\frac{68}{4356}$	$\frac{44441}{1677060}$	$\frac{1444247869}{40177463040}$	$\frac{38485481}{93747413760}$	0.031934
18	$\frac{39}{4356}$	$\frac{69}{4356}$	$\frac{49061}{1677060}$	$\frac{254713691}{3652496640}$	$\frac{41098091}{140621120640}$	0.109093
19	$\frac{30}{4356}$	$\frac{70}{4356}$	$\frac{55991}{1677060}$	$\frac{151826483}{1004436576}$	$\frac{43710701}{281242241280}$	0.247143
20	$\frac{1331}{287496}$	$\frac{4621}{287496}$	$\frac{617161}{18447660}$	$\frac{5933597159}{26784975360}$	$\frac{91430407}{562484482560}$	0.387524
21	$\frac{10}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{7921}{12487168}$	$\frac{55}{12487168}$	1.0942×10^{-9}
22	$\frac{9}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{7921}{6243584}$	$\frac{17}{6243584}$	4.4315×10^{-8}
23	$\frac{8}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{23763}{12487168}$	$\frac{21}{12487168}$	1.0636×10^{-6}
24	$\frac{7}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{39605}{12487168}$	$\frac{13}{12487168}$	0.0000168
25	$\frac{6}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{7921}{1560896}$	$\frac{1}{1560896}$	0.00018091
26	$\frac{5}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{102973}{12487168}$	$\frac{5}{12487168}$	0.00135683

27	$\frac{4}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{166341}{12487168}$	$\frac{3}{12487168}$	0.00697797
28	$\frac{3}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{134657}{6243584}$	$\frac{1}{6243584}$	0.0235506
29	$\frac{2}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{435655}{12487168}$	$\frac{1}{12487168}$	0.0471013
30	$\frac{1}{287496}$	$\frac{6}{287496}$	$\frac{1}{1331}$	$\frac{704969}{12487168}$	$\frac{1}{12487168}$	0.0423912
$E[f_i]$	$\frac{47525}{6534}$	$\frac{32785}{4356}$	$\frac{557471}{60984}$	$\frac{225076163731}{13392487680}$	$\frac{3195951397}{1030191360}$	19.4474
$\sum_{i=1}^{10} E[f_i]$	**72.9878**		**91.4127**	**168.0615**	**31.023**	**194.474**

8.7 Approximated Distribution of Scores for a Single Frame

The table below shows the approximated probability distribution of the scores ranging from 0 to 30 for a single frame in each method.

Dist. Method	Hohn	Hohn	Neal	Fibonacci Skilled	Fibonacci Non-Skilled	Binomial 90% Bowler
Frames / Scores	$i = 1,2,\ldots,9$	$i = 10$	$i = 1,2,\ldots,10$	$i = 1,2,\ldots,10$	$i = 1,2,\ldots,10$	$i = 1,2,\ldots,10$
0	0.0151515	0.0151515	0.008264	0.0000186	0.147165	$1.\times10^{-20}$
1	0.030303	0.030303	0.0173554	0.0000487	0.182125	9.9×10^{-18}
2	0.045455	0.0454545	0.0274564	0.0001653	0.169209	4.411×10^{-15}
3	0.060606	0.0606061	0.03882	0.000453	0.139712	1.164×10^{-12}
4	0.075758	0.0757576	0.051807	0.0012718	0.108563	2.018×10^{-10}
5	0.090909	0.0909091	0.0669585	0.0034494	0.080601	2.3965×10^{-8}
6	0.106061	0.106061	0.0851404	0.0093907	0.0593555	1.9771×10^{-6}
7	0.121212	0.121212	0.107868	0.0257711	0.0411552	0.00011185
8	0.136364	0.136364	0.138171	0.0717998	0.0317471	0.00415235
9	0.151515	0.151515	0.183625	0.216105	0.0180287	0.0913517

10	0.025482	0.0140037	0.0174445	0.0012481	0.00755053	5.557×10^{-11}
11	0.023416	0.0142332	0.018271	0.0012597	0.00505907	5.0013×10^{-9}
12	0.02135	0.0144628	0.0191892	0.0025453	0.00337149	2.0255×10^{-7}
13	0.019284	0.0146924	0.0202223	0.0038969	0.00223412	4.8613×10^{-6}
14	0.017218	0.0149219	0.0214029	0.0066927	0.00147817	0.00007657
15	0.015152	0.0151515	0.0227803	0.011251	0.0009691	0.00082692
16	0.013085	0.0153811	0.0244332	0.0197351	0.00064439	0.00620249
17	0.011019	0.0156107	0.0264994	0.0359467	0.00041052	0.031934
18	0.00895	0.0158402	0.0292542	0.0697369	0.00029226	0.109093
19	0.006887	0.0160698	0.0333864	0.151156	0.00015542	0.247143
20	0.00463	0.0160907	0.0334547	0.221527	0.00016255	0.387524
21	0.0000348	0.000021	0.0007513	0.0006343	4.4045×10^{-6}	1.0942×10^{-9}
22	0.0000313	0.000021	0.0007513	0.0012687	2.7228×10^{-6}	4.4315×10^{-8}
23	0.0000278	0.000021	0.0007513	0 .001903	1.6817×10^{-6}	1.0636×10^{-6}
24	0.0000243	0.000021	0.0007513	0.0031717	1.0411×10^{-6}	0.0000168
25	0.0000209	0.000021	0.0007513	0.0050747	6.4066×10^{-7}	0.00018091
26	0.0000174	0.000021	0.0007513	0.0082463	4.0041×10^{-7}	0.00135683
27	0.0000139	0.000021	0.0007513	0.013321	2.4025×10^{-7}	0.00697797
28	0.0000104	0.000021	0.0007513	0.0215673	1.6016×10^{-7}	0.0235506
29	6.96×10^{-6}	0.000021	0.0007513	0.0348882	8.0082×10^{-8}	0.0471013
30	3.48×10^{-6}	0.000021	0.0007513	0.0564555	8.0082×10^{-8}	0.0423912
$E[f_i]$	7.27349	7.5264	9.14127	16.80615	3.1023	19.4474
$\sum_{i=1}^{10} E[f_i]$	**72.9878**		**91.4127**	**168.0615**	**31.023**	**194.474**

These numerical results represent the special case when bowling 10 frames on 10 pins, which correlate to a standard bowling game. Interestingly, we can see that the 90% bowler yields the highest mean score, while the Fibonacci distribution for a non-skilled bowler yields a significantly lower mean score compared to the other methods.

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