2005

Continued Radicals

Jamie Johnson
Western Kentucky University

CONTINUED RADICALS

A Thesis
Presented to
The Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
Of the Requirements for the Degree
Master of Science

By
Jamie Johnson
May, 2005

CONTINUED RADICALS

Date Recommended _____12/10/04_____

Dr. Tom Richmond, Chair

Dr. John Spraker

Dr. Daniel C. Biles

Elmer Gray, Dean, Graduate Studies and Research, May 4, 2005

Contents

CONTINUED RADICALS

Name: Jamie Johnson Date: May, 2005 Pages: 35

Directed by Dr. Tom Richmond
Department of Mathematics Western Kentucky University

Abstract

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers and $f_j(x) = \sqrt{a_j + x}$, then $f_1 \circ f_2 \circ \cdots \circ f_n(0)$ is a nested radical with terms $a_1, \ldots, a_n$. If it exists, the limit as $n \to \infty$ of such an expression is a continued radical. We consider the set of real numbers $S(M)$ representable as an infinite nested radical whose terms $a_1, a_2, \ldots$ are all from a finite set M. We give conditions on the set M for $S(M)$ to be (a) an interval, and (b) homeomorphic to the Cantor set.

CHAPTER 1

Foundations and Motivations

An iterated function system is a composition of several functions that are contractions. Continued fractions, series, and infinite products are all classic examples of iterated function systems, and have been fruitfully investigated by many mathematicians in the past. Another type of iterated function system, which appears to have received significantly less investigation, is the continued radical of form $\sqrt{a_1 + \sqrt{a_2 + \cdots}}$ with $a_i \geq 1$ for every i.

Two of the most well known results regarding continued radicals are that the continued radical

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

equals the golden ratio, φ, and that the continued radical

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

equals 2 [**5**].

Another theorem regarding continued radicals states that any real number can be represented as a continued radical of the form

$$a_1 + \sqrt{a_2 + \sqrt{a_3 + \sqrt{a_4 + \cdots}}}$$

where $a_1 \in \mathbb{Z}$ and a_i is 0, 1, or 2, for $i > 1$ [**9**].

Using this theorem as a basis, we construct an algorithm for representing real numbers with continued radicals that are less restrictive than those predicted by this theorem.

We will consider nested radicals of form

$$\sqrt{a_1+\sqrt{a_2+\sqrt{a_3+\cdots+\sqrt{a_n}}}}$$

which we will denote by $S_n = \sqrt{a_1, a_2, a_3, \ldots, a_n}$. The infinite nested radical

$$\sqrt{a_1+\sqrt{a_2+\sqrt{a_3+\cdots}}}\,, \qquad \text{denoted by} \quad \sqrt{a_1, a_2, a_3, \ldots}$$

is understood to be

$$\lim_{n\to\infty} \sqrt{a_1, a_2, a_3, \ldots, a_n} = \lim_{n\to\infty} S_n,$$

and is called a continued radical.

It is not sufficient to algebraically verify that $\sqrt{1,1,1,\ldots} = \varphi$; it is also necessary to verify that the sequence of partial roots,

$$\sqrt{1}, \sqrt{1,1}, \sqrt{1,1,1}, \ldots$$

is convergent; in this case, convergence is easily verifiable.

Thus, when considering arbitrary continued roots, the convergence behavior of their partial roots must be considered. Laugwitz [**5**] and Sizer [**9**] have determined necessary and sufficient conditions for this type of convergence. These conditions are provided in the following theorem.

Theorem 1.1. [**5**] *Suppose $a_n \geq 0$ for every $n \geq 1$, and define the sequence $(S_n)_{n=1}^{\infty}$ by*

$$S_1 = \sqrt{a_1}$$

$$S_2 = \sqrt{a_1, a_2}$$

$$\vdots$$

$$S_n = \sqrt{a_1, a_2, \ldots, a_n}$$

$$\vdots$$

The sequence $(S_n)_{n=1}^{\infty}$ converges if and only if there exists a real number b such that

$$2^{-n}\log a_n \leq b \text{ for every } n,$$

or, equivalently, if and only if there exists a $b > 0$ such that

$$a\_n \leq b^{2^n} \text{ for every } n,$$

or if and only if

$$\{ \sqrt[2^n]{a_n} : n \in \mathbb{N}\} \text{ is bounded.}$$

Proof. Let $a_n \geq 0$ for every $n \geq 1$. Choose $b > 0$ and define $q_n = b^{2^n}$ for every $n \geq 1$. Now, the corresponding sequence $(Q_n)_{n=1}^{\infty}$, satisfying

$$Q_1 = \sqrt{q_1}$$

$$Q_2 = \sqrt{q_1, q_2}$$

$$\vdots$$

$$Q_n = \sqrt{q_1, q_2, \ldots, q_n}$$

$$\vdots$$

can be rewritten in factored form as

$$Q_1 = \sqrt{b^2} = b\sqrt{1}$$

$$Q_2 = \sqrt{b^2 + \sqrt{b^4}} = b\sqrt{1 + \sqrt{1}}$$

$$\vdots$$

$$Q_n = \sqrt{b^2 + \sqrt{b^4 + \ldots \sqrt{b^{2n}}}} = b\sqrt{1 + \sqrt{1 + \ldots \sqrt{1}}}$$

$$\vdots$$

Thus, the sequence converges to $b\frac{1+\sqrt{5}}{2}$.

Together, Propositions 2.1 and 2.2 imply that if $0 \leq a_n \leq q_n$, and $(Q_n)_{n=1}^{\infty}$ is convergent then $(S_n)_{n=1}^{\infty}$ (as previously defined) is also convergent. Thus, we can show that $(S_n)_{n=1}^{\infty}$ converges if, for some $b > 0$,

$$a_n \leq b^{2^n} = q_n \text{ for every } n,$$

or equivalently

$$2^{-n} \log a_n \leq b \;\text{ for every } n.$$

Conversely, if the sequence $(S_n)_{n=1}^{\infty}$ satisfying

$$S_1 = \sqrt{a_1} = a_1^{(2^{-1})}$$

$$S_2 = \sqrt{a_1, a_2} > \sqrt[4]{a_2} = a_2^{(2^{-2})}$$

$$\vdots$$

$$S_n = \sqrt{a_1, a_2, \ldots, a_n} > \sqrt[2^n]{a_n} = a_n^{(2^{-n})}$$

$$\vdots$$

is convergent, then there exists a b that is the upper bound of $(S_n)_{n=1}^{\infty}$. It follows that $a_n \leq b^{2^n}$; formulated differently, we have

$$2^{-n} \log a_n \leq b \text{ for every } n.$$

$\square$

It is noteworthy that this theorem is a specific case of a more general convergence theorem in which the square root is replaced with an exponent, α, where $0 < \alpha < 1$.

THEOREM 1.2. [**5**] *Let α satisfy $0 < \alpha < 1$. Suppose $a_n \geq 0$ for every $n \geq 1$, and the sequence $(S_n)_{n=1}^{\infty}$ satisfies*

$$S_1 = {a_1}^{\alpha}$$

$$S_2 = (a_1 + a_2^{\alpha})^{\alpha}$$

$$\vdots$$

$$S_n = (a_1 + (a_2 + (\cdots + a_n^{\alpha})^{\alpha} \cdots)^{\alpha})^{\alpha}$$

$$\vdots$$

Then the sequence $(S_n)_{n=1}^{\infty}$ converges if and only if there exists a $c \in \mathbb{R}$ such that

$$\alpha^n \log a_n \leq c \text{ for every } n.$$

CHAPTER 2

Introduction

Continued radicals of the form we are considering are studied in [**5**], [**9**], and [**2**], and briefly in [**8**] and [**3**]. Variations of such continued radicals have been called continued roots or nested radicals. Ramanujan considered several continued radicals (see [**2**]) and showed that

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}.$$

Continued radicals can be related to solutions of certain polynomial or radical equations (see [**2**] and [**1**]). Laugwitz [**5**] studies chain operations or iterated function systems of form $\lim_{n\to\infty}(f_1 \circ f_2 \circ \cdots \circ f_n)(x)$. Observe that infinite series, infinite products, continued fractions, and continued radicals may all be so represented. Much work has been done on convergence criteria for continued radicals (see [**1**], [**9**], and [**5**]). Our emphasis is on the forms of the sets $S(M)$ of real numbers which are representable as an infinite continued radical whose terms $a_1, a_2, \ldots$ are all from a finite set M. The analogous problem for continued fractions has been considered in [**7**] (see also [**4**]).

Perhaps the most familiar continued radical is $\sqrt{1, 1, 1, \ldots}$, whose value is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$. The popular verification of this relies on the self-similarity of the continued radical: If $S = \sqrt{1, 1, 1, \ldots}$, then $S^2 = 1 + S$, and S must be the positive root of this quadratic equation. This argument has overlooked the serious issue of convergence. With similar careless regard for convergence, one might incorrectly conclude that

$$T = \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1 - \cdots}}}}$$

is the positive solution of $T^2 = 1 - T$ and thus $T = \frac{-1+\sqrt{5}}{2} \approx .61803$. However, the partial expressions

$$\sqrt{1},\ \sqrt{1-\sqrt{1}},\ \sqrt{1-\sqrt{1-\sqrt{1}}},\ \sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1}}}},\ \ldots$$

alternate $1, 0, 1, 0, \ldots$, and thus do not converge.

To guarantee that each partial expression S_n of a continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ is defined, each *term* a_i must be nonnegative. We will restrict our attention to continued radicals $\sqrt{a_1, a_2, a_3, \ldots}$ whose terms a_i are whole numbers, though we note that the fundamental result below holds for any nonnegative terms.

Proposition 2.1.

(a) *If $a_i \geq b_i \geq 0$ for all $i \in \mathbb{N}$, then $\sqrt{a_1, a_2, \ldots, a_k} \geq \sqrt{b_1, b_2, \ldots, b_k}$ for all $k \in \mathbb{N}$.*

(b) *If $a_i \geq 0$ for $i \in \mathbb{N}$, then the sequence $\sqrt{a_1}, \sqrt{a_1, a_2}, \sqrt{a_1, a_2, a_3}, \ldots$ of partial expressions of the infinite continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ is an increasing sequence.*

Part (a) follows from a direct calculation, using the fact that $f(x) = \sqrt{x}$ is an increasing function. Part (b) follows from part (a).

Observe that the converse of Part (a) does not hold. If $(a_i)_{i=1}^{\infty} = (8, 4, 0, 0, 0, \ldots)$ and $(b_i)_{i=1}^{\infty} = (6, 9, 0, 0, 0, \ldots)$, then $\sqrt{a_1, a_2, \ldots, a_k} \geq \sqrt{b_1, b_2, \ldots, b_k}$ for all $k \in \mathbb{N}$, but $a_2 \not\geq b_2$.

The result below follows from Theorem 1.1. To illustrate common techniques for continued radicals, we give a direct proof here.

Proposition 2.2. *If the sequence $(a_i)_{i=1}^{\infty}$ of nonnegative numbers is bounded above, then $\sqrt{a_1, a_2, a_3, \ldots}$ converges.*

Proof. Suppose $(a_i)_{i=1}^{\infty}$ is bounded above by $M \geq 2$. We will show that the increasing sequence $S_1 = \sqrt{a_1}$, $S_2 = \sqrt{a_1, a_2}$, $S_3 = \sqrt{a_1, a_2, a_3}$, $\ldots$ of partial expressions is bounded above by M^2. By Proposition 2.1 (a),

$$S_k = \sqrt{a_1, a_2, \ldots, a_k} \leq \sqrt{M, M, \ldots, M} = q_k$$

where the number of Ms in the latter continued radical is k. Now $S_1 < q_1 = \sqrt{M} < M^2$. Suppose $q_k < M^2$. Then $q_{k+1} = \sqrt{M + q_k} < \sqrt{M + M^2} < \sqrt{2M^2} = M\sqrt{2} < M^2$. Thus, $(q_k)_{k=1}^{\infty}$ and therefore $(S_k)_{k=1}^{\infty}$ is bounded above by M^2, as needed. $\square$

The converse of the proposition fails; Theorem 1.1 implies that $\sqrt{1, 2, 3, 4, \ldots}$ converges even though $(1, 2, 3, 4, \ldots)$ is not bounded.

Proposition 2.2 shows that for any nonnegative number n, $\sqrt{n, n, n, \ldots}$ converges. We will denote the value of $\sqrt{n, n, n, \ldots}$ by φ_n. Now $\varphi_n^2 = n + \varphi_n$ and if $n \in \mathbb{N}$, the quadratic formula shows that

$$\sqrt{n, n, n, \ldots} = \varphi_n = \frac{1 + \sqrt{4n+1}}{2}\,.$$

It is easy to verify that φ_n is a nonzero integer if and only if n is twice a triangular number. Specifically, $\varphi_n = k \in \mathbb{N}$ if and only if $n = k(k-1) = 2T_{k-1}$ for some integer $k \geq 2$. Recall that a triangular number is an integer of form $T_m = 1 + 2 + \cdots + m = \frac{m(m+1)}{2} = \binom{m+1}{2}$ for $m \in \mathbb{N}$.

We will consider continued radicals whose terms all come from a finite set $M = \{m_1, m_2, \ldots, m_p\}$ of integral values. We will determine which such sets M permit us to represent all real numbers from an interval and which such sets M leave gaps in the set of numbers representable. We will consider uniqueness of representation. First we consider sets M of nonnegative terms in Chapter 3. Allowing zero as a term complicates matters, and this case is considered in Chapter 4. In Chapter 5 we consider the interesting pattern of gaps in the numbers representable by continued radicals whose terms all come from a two-element set $\{m_1, m_2\} \subseteq \mathbb{N}$.

The results of the following chapters, with the noted exception of Theorem 4.4, were developed independently, have not been found in the literature, and appear to be original.

CHAPTER 3

Continued radicals with nonzero terms

Let us consider continued radicals $\sqrt{a_1, a_2, a_3, \ldots}$ whose terms a_i come from a set $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \cdots < m_p$. We will be interested in sets M of "term values" which allow all points of a nondegenerate interval to be represented. To insure that no gaps occur in the set of values representable as a continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ with terms from M, it is necessary that the largest value representable with $a_1 = m_i$ equal or exceed the smallest value representable with $a_1 = m_{i+1}$ (for $i = 1, \ldots, p-1$). Obviously, if this condition is not met, then the values that fall between the largest value representable with $a_1 = m_i$ and the smallest value representable with $a_1 = m_{i+1}$ that cannot be represented as such.

Thus, it is necessary that

$$\sqrt{m_i, m_p, m_p, m_p, \ldots} \geq \sqrt{m_{i+1}, m_1, m_1, m_1, \ldots} \qquad \text{for every } i \in \{1, \ldots, p-1\},$$

or equivalently,

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \text{for every } i \in \{1, \ldots, p-1\}.$$

In fact, this condition will be necessary and sufficient, as we will see in Theorem 3.2. First, we need a lemma.

Lemma 3.1. *Suppose $(a_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ are sequences of real numbers with $a_n \geq 1$ for every $n \in \mathbb{N}$ and $r_n \geq 2$ for every $n \in \mathbb{N}$, and for $k \in \mathbb{N}$, define*

$$\begin{aligned} h_k(x) &= \sqrt[r_1]{a_1 + \sqrt[r_2]{a_2 + \sqrt[r_3]{a_3 + \cdots + \sqrt[r_k]{a_k + x}}}} \\ &= \left(a_1 + \left(a_2 + \left(a_3 + \cdots + (a_k + x)^{\frac{1}{r_k}} \cdots\right)^{\frac{1}{r_3}}\right)^{\frac{1}{r_2}}\right)^{\frac{1}{r_1}}. \end{aligned}$$

Then the derivative of h_k over $[0,\infty)$ is positive and bounded above by 2^{-k}. That is, $0 < h_k'(c) \leq 2^{-k}$ for all $c \geq 0$.

Proof. Applying the chain rule for differentiation and recalling that $a_n \geq 1$ for all $n \in \mathbb{N}$, we find that

$$h_k'(c) = \frac{1}{r_1}(w_1)^{\frac{1}{r_1}-1} \cdot \frac{1}{r_2}(w_2)^{\frac{1}{r_2}-1} \cdot \frac{1}{r_3}(w_3)^{\frac{1}{r_3}-1} \cdots \frac{1}{r_k}(w_k)^{\frac{1}{r_k}-1}$$

where $w_n \geq 1$ for $n = 1, \ldots, k$. Observing that $r_n \geq 2$ for $n = 1, \ldots, k$, we see that each $\frac{1}{r_n} - 1$ is negative, so with $p_n = -(\frac{1}{r_n} - 1) > 0$, we have

$$0 < h_k'(c) = \frac{1}{r_1 r_2 r_3 \cdots r_k (w_1)^{p_1}(w_2)^{p_2} \cdots (w_k)^{p_k}} \leq \frac{1}{r_1 r_2 r_3 \cdots r_k} \leq \frac{1}{2^k} = 2^{-k}.$$

□

Theorem 3.2. *Suppose $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \cdots < m_p$ and*

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \text{for every } i \in \{1, \ldots, p-1\}.$$

Then the set, $S(M)$, of numbers representable as a continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ with terms $a_i \in M$ is the interval $[\varphi_{m_1}, \varphi_{m_p}]$.

Proof. We present an algorithm to construct a representation $\sqrt{a_1, a_2, a_3, \ldots}$ ($a_i \in M$ for every $i \in \mathbb{N}$) for any given $b \in [\varphi_{m_1}, \varphi_{m_p}]$. First, note that, clearly, $S(M) \subseteq [\varphi_{m_1}, \varphi_{m_p}]$. Conversely, suppose $b \in [\varphi_{m_1}, \varphi_{m_p}]$ is given. Set $a_1 = m_p$ if $b \geq \sqrt{m_p + \varphi_{m_1}}$ and otherwise take $a_1 = m_i$ where m_i is the largest element of M for which $b \in [\sqrt{m_i + \varphi_{m_1}}, \sqrt{m_{i+1} + \varphi_{m_1}})$. Having found $a_1, a_2, \ldots, a_{n-1}$, we take a_n to be the largest element m_i of M for which

$$\sqrt{a_1, \ldots, a_{n-1}, m_i, m_1, m_1, m_1, \ldots} = \sqrt{a_1, \ldots, a_{n-1}, m_i + \varphi_{m_1}} \leq b.$$

The sequence of partial expressions $(S_n)_{n=1}^{\infty}$ determined by the terms $(a_1, a_2, a_3, \ldots)$ is bounded above by b, and therefore must converge. We will show that it converges to b by considering the auxiliary sequence $(b_n)_{n=1}^{\infty}$ defined by

$$\begin{aligned} b_{2n} &= \sqrt{a_1, \ldots, a_n, m_p, m_p, m_p, \ldots} = \sqrt{a_1, \ldots, a_n + \varphi_{m_p}} \quad \text{for every } n \in \mathbb{N} \\ b_{2n-1} &= \sqrt{a_1, \ldots, a_n, 0, 0, 0, \ldots} = \sqrt{a_1, \ldots, a_n} = S_n \quad \text{for every } n \in \mathbb{N}. \end{aligned}$$

Clearly $S_n = b_{2n-1} \leq b$ for all $n \in \mathbb{N}$. We will now show by induction that $b \leq b_{2n}$ for all $n \in \mathbb{N}$. For $n = 1$, suppose $a_1 = m_i < m_p$. Then

$$b_2 = \sqrt{a_1 + \varphi_{m_p}} = \sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} > b$$

by the assignment of $a_1 = m_i$. If $a_1 = m_p$, then $b_2 = \sqrt{m_p + \varphi_{m_p}} = \varphi_{m_p} \geq b \in [\varphi_{m_1}, \varphi_{m_p}]$. Now suppose we have shown that $b_{2n-2} \geq b$. Consider $b_{2n} = \sqrt{a_1, \ldots, a_{n-1}, a_n + \varphi_{m_p}}$. If $a_n = m_p$, then

$$b_{2n} = \sqrt{a_1, \ldots, a_{n-1}, m_p + \varphi_{m_p}} = \sqrt{a_1, \ldots, a_{n-1} + \varphi_{m_p}} \geq b$$

by the induction hypothesis. If $a_n = m_i < m_p$, then

$$\begin{aligned} b_{2n} &= \sqrt{a_1, \ldots, a_{n-1}, m_i + \varphi_{m_p}} \\ &= \sqrt{a_1, \ldots, a_{n-1} + \sqrt{m_i + \varphi_{m_p}}} \\ &\geq \sqrt{a_1, \ldots, a_{n-1} + \sqrt{m_{i+1} + \varphi_{m_1}}} \\ &\geq b \end{aligned}$$

where the second-to-last inequality follows from the induction hypothesis, and the last inequality follows from the choice of $a_n = m_i$ to be the largest m_k for which $\sqrt{a_1, \ldots, a_{n-1}, m_k + \varphi_{m_1}} \leq b$. This completes the proof that $b_{2n-1} \leq b \leq b_{2n}$ for every $n \in \mathbb{N}$.

For a fixed $k \in \mathbb{N}$, let h_k be defined as in Lemma 3.1, with $r_n = 2$ for every $n \in \mathbb{N}$. Now given $i > j > 2k$, we have $b_i = h_k(x_1)$ for some $x_1 \in [0, \varphi_{m_p}]$ and $b_j = h_k(x_0)$ for some $x_0 \in [0, \varphi_{m_p}]$. The mean value theorem applies to h_k over the interval with endpoints x_0 and x_1, so

$$|b_i - b_j| = |h_k(x_1) - h_k(x_0)| = h_k'(c)|x_1 - x_0|$$

for some c between x_0 and x_1. Now $|x_1 - x_0| \leq \varphi_{m_p}$ and, by Lemma 3.1, $h'_k(c) \leq 2^{-k}$, so we have $|b_i - b_j| \leq \varphi_{m_p} 2^{-k}$ for any $i, j > k$. Given any $\epsilon > 0$, we may find $k \in \mathbb{N}$ such that $\varphi^2_{m_p} 2^{-k} < \epsilon$, and it follows that $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence and must converge. Now the two subsequences $(b_{2n-1})_{n=1}^{\infty}$ and $(b_{2n})_{n=1}^{\infty}$ of the convergent sequence $(b_n)_{n=1}^{\infty}$ must have the same limit L, and the inequality $S_n = b_{2n-1} \leq b \leq b_{2n}$ for every $n \in \mathbb{N}$ shows that $L \leq b \leq L$, so $L = b$. In particular, $(b_{2n-1})_{n=1}^{\infty} = (S_n)_{n=1}^{\infty}$ must converge to b. $\square$

The argument of the last paragraph above can be used to show that if $M = \{m_1, \ldots, m_p\}$ where $0 < m_1 < \ldots < m_p$ and $a_n \in M$ for $n = 1, \ldots, k, \ldots, j$, then the continued radical $\sqrt{a_1, \ldots, a_k}$ and $\sqrt{a_1, \ldots, a_j}$ differ by no more than $2^{-k}\varphi_{m_p}$.

In choosing sets of values $M = \{m_1, \ldots, m_p\}$ to serve as terms of continued radical representations of the elements of an interval $[\varphi_{m_1}, \varphi_{m_p}]$, we might want the most efficient selection of terms. Suppose there exists $i \in \{1, 2, \ldots,\ p-1\}$ such that $\sqrt{m_i + \varphi_{m_p}} > \sqrt{m_{i+1} + \varphi_{m_1}}$. Then the algorithm of Theorem 3.2 applied to $b = \sqrt{m_i + \varphi_{m_p}} = \sqrt{m_i, m_p, m_p, m_p, \ldots}$ greedily chooses the initial term as large as possible and produces a representation of form $b = \sqrt{m_{i+1}, a_2, a_3, \ldots}$. Thus, we do not have uniqueness of representation if $\sqrt{m_i + \varphi_{m_p}} > \sqrt{m_{i+1} + \varphi_{m_1}}$ for some $i \in \{1, \ldots, p-1\}$. Thus, in choosing the values $\{m_1, \ldots, m_p\}$ most efficiently, the inequalities

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \text{for every } i \in \{1, \ldots, p-1\}$$

to insure that every value in $[\varphi_{m_1}, \varphi_{m_p}]$ is representable should be equalities

$$\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}} \quad \text{for every } i \in \{1, \ldots, p-1\}$$

to prevent unnecessary duplication of representation. These equalities will not eliminate all duplication of representation, but will limit it to situations involving overlapping endpoints of the intervals $(I_i)_{i=1}^{p}$ where $I_i = [\sqrt{m_i + \varphi_{m_1}}, \sqrt{m_i + \varphi_{m_p}}]$ contains all the points representable as $\sqrt{m_i, a_2, a_3, \ldots}$. If there exists $b \in [\varphi_{m_1}, \varphi_{m_p}]$ having distinct representations $b = \sqrt{a_1, a_2, \ldots} = \sqrt{b_1, b_2, \ldots}$ with $a_1 \neq b_1$, then $a_1 = m_i$ implies $b \in I_i$ and $b_1 = m_j$ implies $b \in I_j$. Since $i \neq j$ and $I_i \cap I_j \neq \emptyset$, we must have (assuming, without loss of generality, that $i < j$) $b = \sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$, the shared endpoint of adjacent intervals I_i and I_j. It follows that the only duplication of representations must be of form $\sqrt{c_1, \ldots, c_z, a_1, a_2, \ldots} = \sqrt{c_1, \ldots, c_z, b_1, b_2, \ldots}$, where $\sqrt{a_1, a_2, \ldots} = \sqrt{b_1, b_2, \ldots}$ are as above. That is, the only possible duplication of representation must be of form

$$\begin{aligned}\sqrt{c_1, \ldots, c_z, m_i + \varphi_{m_p}} &= \sqrt{c_1, \ldots, c_z, m_{i+1} + \varphi_{m_1}} \\ \sqrt{c_1, \ldots, c_z, m_i, m_p, m_p, m_p, \ldots} &= \sqrt{c_1, \ldots, c_z, m_{i+1}, m_1, m_1, m_1, \ldots},\end{aligned}$$

where repeating the largest value m_p is equal to raising the preceding term from m_i to m_{i+1} and repeating the smallest value m_1. (Compare to the decimal equation $1.3\overline{999} = 1.4\overline{000}$.)

To investigate when the equality $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$, required for minimal duplication of representation, may occur, we start with a lemma.

Lemma 3.3. *Suppose z and w are distinct natural numbers. Then $\sqrt{z} - \sqrt{w} \in \mathbb{Z}$ if and only if $\sqrt{z}$ and $\sqrt{w}$ are natural numbers.*

Proof. Suppose $z, w \in \mathbb{N}$ and $\sqrt{z} - \sqrt{w} = r \in \mathbb{Z} \setminus \{0\}$. Then $r^2 = z - 2\sqrt{zw} + w \in \mathbb{N} \Rightarrow \sqrt{zw} \in \mathbb{Q} \Rightarrow \sqrt{zw} \in \mathbb{N}$. Now $r\sqrt{z} = (\sqrt{z} - \sqrt{w})\sqrt{z} = z - \sqrt{zw} = s \in \mathbb{Z}$ (recalling that $\sqrt{zw} \in \mathbb{N}$). Dividing $r\sqrt{z} = s$ by $r \neq 0$ shows $\sqrt{z} \in \mathbb{Q}$ and thus $\sqrt{z} \in \mathbb{N}$. Similarly, $r\sqrt{w} \in \mathbb{Z}$ implies $\sqrt{w} \in \mathbb{N}$. The converse is immediate. $\square$

Now suppose $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$. Squaring both sides of this equation leads to $\varphi_{m_p} - \varphi_{m_1} = m_{i+1} - m_i \in \mathbb{N}$. Since $\varphi_x = \frac{1+\sqrt{4x+1}}{2}$, we find that $\varphi_{m_p} - \varphi_{m_1} \in \mathbb{N}$ if and only if $\sqrt{4m_p + 1} - \sqrt{4m_1 + 1}$ is an even integer. By Lemma 3.3, $\sqrt{4m_p + 1}$ and $\sqrt{4m_1 + 1}$ must both be integers, and as square roots of odd numbers, they must both be odd integers (so their difference is even). Adding 1 to each and dividing by 2 shows that φ_{m_p} and φ_{m_1} are integers.

Thus, to avoid unnecessary duplication of representation, we must have $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$, or $m_{i+1} = m_i + \varphi_{m_p} - \varphi_{m_1}$, and φ_{m_1} and φ_{m_p} must both be integers, say $n+1$ and $j+1$. Recall that this occurs if and only if $m_1 = n(n+1)$ and $m_p = j(j+1)$ for some values $n, j \in \mathbb{N}$ with $j > n$, and it follows that

$$M = \{n(n+1), n(n+1) + 1(j-n), n(n+1) + 2(j-n), \ldots, j(j+1)\}$$

has $j + n + 2$ equally spaced terms.

We summarize our results below.

THEOREM 3.4. *Suppose* $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N}$ *where* $0 < m_1 < m_2 < \cdots < m_p$ *and*

$$\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}} \quad \text{for every } i \in \{1, \ldots, p-1\}.$$

Then φ_1 *and* φ_p *are natural numbers,* $m_{i+1} = m_i + \varphi_{m_p} - \varphi_{m_1}$ *for each* $i \in \{1, \ldots, p-1\}$*, each number in the interval* $[\varphi_{m_1}, \varphi_{m_p}]$ *is representable as a continued radical* $\sqrt{a_1, a_2, a_3, \ldots}$ *with terms* $a_i \in M$*, and the representation is unique except for expressions of form*

$$\sqrt{c_1, \ldots, c_z, m_i, m_p, m_p, m_p, \ldots} = \sqrt{c_1, \ldots, c_z, m_{i+1}, m_1, m_1, m_1, \ldots}.$$

EXAMPLE 3.5. *Let* $M = \{2, 4, 6, 8, 10, 12\}$*. Now* $m_1 = 2 = 1(2) = 2T_1$*,* $m_p = 12 = 3(4) = 2T_3$*,* $\varphi_{m_1} = \varphi_2 = 2$*, and* $\varphi_{m_p} = \varphi_{12} = 4$*. Observe that* $\varphi_{m_p} - \varphi_{m_1} = 4 - 2 = 2$*, and our set* $M = \{m_1, \ldots, m_p\}$ *does satisfy* $m_{i+1} = m_i + \varphi_{m_p} - \varphi_{m_1}$ *(i.e.,* $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$*) for all* $i = 1, \ldots, p-1$*. Thus, every real number* $b \in [2, 4] = [\varphi_{m_1}, \varphi_{m_p}]$ *has a representation* $\sqrt{a_1, a_2, \ldots}$ *where* $a_i \in \{2, 4, 6, 8, 10, 12\}$ *for every* $i \in \mathbb{N}$*. Duplication of representations occur, for example, in*

$$\begin{aligned} \sqrt{8, 4, 12, 12, 12, \ldots} &= \sqrt{8, 4 + \varphi_{12}} = \sqrt{8, 4 + 4} \\ &= \sqrt{8, 6 + 2} = \sqrt{8, 6 + \varphi_2} = \sqrt{8, 6, 2, 2, 2, \ldots}. \end{aligned}$$

To find the representation of $\pi \in [2,4]$*, note that*

$$\sqrt{8} \approx 2.82842712 \leq \pi < \sqrt{10} \approx 3.16227766,$$

so our algorithm assigns $a_1 = 6$*, the largest value of* $m_i \in M$ *for which* $\sqrt{m_i + \varphi_2} = \sqrt{m_i + 2} \leq \pi$*. Having found* a_1*, we note that*

$$\sqrt{6, 12 + \varphi_2} = \sqrt{6, 14} \approx 3.12116282 \leq \pi,$$

so we have $a_2 = 12$*. Next,*

$$\begin{aligned} \sqrt{6, 12, 6 + \varphi_2} &= \sqrt{6, 12, 8} \approx 3.13859358 \\ &\leq \pi < \sqrt{6, 12, 10} = \sqrt{6, 12, 8 + \varphi_2} \approx 3.14545337, \end{aligned}$$

so $a_3 = 6$ *and* $\sqrt{a_1, a_2, a_3} = \sqrt{6, 12, 6}$*. Next,*

$$\begin{aligned} \sqrt{6, 12, 6, 6 + \varphi_2} &= \sqrt{6, 12, 6, 8} \approx 3.14153977 \\ &\leq \pi < \sqrt{6, 12, 6, 10} \approx 3.14268322, \end{aligned}$$

so $a_4 = 6$ *and* $\sqrt{a_1, a_2, a_3, a_4} = \sqrt{6, 12, 6, 6}$*. Continuing, we find that*

$$\pi = \sqrt{6, 12, 6, 6, 2, 2, 10, 4, 4, 2, 8, 10, 12, 6, \ldots}.$$

CHAPTER 4

Representation allowing 0 as a term

Allowing zero as a term in our continued radicals introduces some minor complications. We will assume our terms a_n all come from a set $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N} \cup \{0\}$ where $0 = m_1 < m_2 < \cdots < m_p$. To prevent gaps in the set,$S(M)$, of numbers representable with these terms, the largest value of form $\sqrt{m_i, a_2, a_3, \ldots}$ must equal or exceed the smallest value of form $\sqrt{m_{i+1}, b_2, b_3, \ldots}$. That is, we must have

$$\begin{aligned} \sqrt{m_i, m_p, m_p, m_p, \ldots} &\geq \sqrt{m_{i+1}, 0, 0, 0, \ldots} \\ \sqrt{m_i + \varphi_{m_p}} &\geq \sqrt{m_{i+1}}. \end{aligned}$$

However, note that besides the single value $\sqrt{m_{i+1}} = \sqrt{m_{i+1}, 0, 0, 0, \ldots}$, every other value representable as $\sqrt{m_{i+1}, b_2, b_3, \ldots}$ must be strictly greater than

$$\sqrt{m_{i+1}, 1} = \sqrt{m_{i+1} + 1} = \lim_{k \to \infty} \sqrt{m_{i+1}, 0, 0, \ldots, 0, 0, b_k, 0, 0, \ldots} \quad \text{where } b_k \neq 0.$$

We saw in the previous section that if all term values $m_i \in M$ are positive, then the numbers representable as $\sqrt{m_i, a_2, \ldots}$ where $a_n \in M$ for every $n \in \mathbb{N}$ formed a closed interval $I_i = [\sqrt{m_i + \varphi_{m_1}}, \sqrt{m_i + \varphi_{m_p}}]$ (assuming the m_i's were chosen so that $I_i \cap I_{i+1} \neq \emptyset$ for $i = 1, \ldots, p-1$). We now see that if $m_1 = 0$, then the numbers representable as $\sqrt{m_i, a_2, \ldots}$ where $a_n \in M$ for every $n \in \mathbb{N}$ will be a subset of $\{\sqrt{m_i}\} \cup (\sqrt{m_i + 1}, \sqrt{m_i + \varphi_{m_p}}] = J_i$. To prevent any gaps in $S(M)$, it is necessary that $\bigcup\{J_i : i = 1, 2, \ldots, p\}$ forms a solid interval. Consequently, it is necessary that

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + 1} \quad \text{for every } i = 1, 2, \ldots, p-1.$$

Squaring both sides of this equation gives the necessary condition that

$$m_{i+1} \leq m_i + (\varphi_{m_p} - 1) \quad \text{for every } i = 1, 2, \ldots, p-1.$$

One would again expect that choosing the values of $m_1, \ldots, m_p$ so that the above inequalities are equalities will result in the most efficient representation of the largest possible interval using the smallest number of terms. For equality to hold, we must have that $\varphi_{m_p} \in \mathbb{N}$, and thus $m_p = (q+1)q$ for some $q \in \mathbb{N}$. Then since $\varphi_{(q+1)q} = q+1$, the equation $m_{i+1} = m_i + (\varphi_{m_p} - 1)$ becomes $m_{i+1} = m_i + q$, and thus $M = \{0, q, 2q, 3q, \ldots, (q+1)q\}$. The theorem below confirms these expectations.

THEOREM 4.1. *Suppose $q \in \mathbb{N}$. Any $b \in (1, q+1]$ can be represented as an infinite continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ where $a_i \in \{0, q, 2q, 3q, \ldots, (q+1)q\} = M$ for every $i \in \mathbb{N}$. Furthermore, if any $b \in (1, q+1]$ can be represented as an infinite continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ where $a_i \in J$ for every $i \in \mathbb{N}$ and $|J| \leq |M|$, then $J = M$. That is, M is the unique set of $q+2$ nonnegative integer terms allowing every $b \in (1, q+1]$ to be represented, and there is no set of $q+1$ or fewer nonnegative integer terms which allow every $b \in (1, q+1]$ to be represented.*

Proof: Given $q \in \mathbb{N}$ and $b \in (1, q+1]$, we present an algorithm to generate a sequence $(a_i)_{i=1}^{\infty}$ such that $b = \sqrt{a_1, a_2, a_3, \ldots}$ where each $a_i \in M = \{m_i : i = 1, \ldots, q+2\}$ and $m_i = (i-1)q$. The algorithm is greedy, taking each a_i as large as possible subject to the restriction that $\sqrt{a_1, \ldots, a_k} \leq b$ for every $k \in \mathbb{N}$. Set $a_1 = m_i$ if $b = \sqrt{m_i}$ and otherwise take a_1 to be the largest m_i for which $\sqrt{m_i + 1} < b$ (or $\sqrt{m_i + 1} \leq b$ in case $q = 1$). Having found $a_1, a_2, \ldots, a_{n-1}$, take $a_n = m_i$ if $\sqrt{a_1, \ldots, a_{n-1}, m_i} = b$ and otherwise take a_n to be the largest m_i for which $\sqrt{a_1, \ldots, a_{n-1}, m_i + 1} < b$ (or $\sqrt{a_1, \ldots, a_{n-1}, m_i + 1} \leq b$ in case $q = 1$).

The sequence $(S_n)_{n=1}^{\infty}$ of partial expressions determined by the terms $(a_1, a_2, a_3, \ldots)$ generated by the algorithm is bounded above by b and therefore converges. Now the algorithm assigns $a_i = 0$ for all $i > n$ if and only if $b = S_n = \sqrt{a_1, \ldots, a_n}$ for some $n \in \mathbb{N}$, and in this case $(S_n)_{n=1}^{\infty}$ is eventually constantly b and thus clearly converges to b. Thus, we will assume that $(a_n)_{n=1}^{\infty}$ is not eventually constantly 0. Letting $(b_n)_{n=1}^{\infty}$ be the auxiliary sequence as defined in Theorem 3.2, we have, as before, $b_{2n-1} \leq b \leq b_{2n}$ for all $n \in \mathbb{N}$. Let $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ be the subsequence of nonzero terms of $(a_n)_{n=1}^{\infty}$. Considering the effect of the zeros in the sequence $(a_n)_{n=1}^{\infty}$, for $i, j \geq 2n_k$, we have $b_i = h_k(x_0)$ and $b_i = h_k(x_1)$ for some $x_0, x_1 \in [0, (q+1)]$ where

$$h_k(x) = \sqrt[r_1]{a_{n_1} + \sqrt[r_2]{a_{n_2} + \sqrt[r_3]{a_{n_3} + \cdots + \sqrt[r_k]{a_{n_k} + x}}}},$$

where $r_1 = 2^{n_1}$, and $r_i = 2^{(n_i - n_{i-1})}$ for $i = 2, \ldots, k$. By the mean value theorem, there exists c between x_0 and x_1 with

$$|b_i - b_j| = |h_k(x_1) - h_k(x_0)| = |x_1 - x_0| h_k'(c) \leq (q+1)2^{-k},$$

where the last inequality follows from Lemma 3.1. As before, $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence and must converge to b, and consequently, the subsequence $(b_{2n-1})_{n=1}^{\infty} = (S_n)_{n=1}^{\infty}$ converges to b.

Now the elements of M were chosen so that the necessary inequalities $m_{i+1} \leq m_i + (\varphi_{m_p} - 1)$ for every $i = 1, 2, \ldots, p-1$ were actually equalities. Consequently, if the continued radicals whose terms come from $J = \{0, n_1, \ldots, n_s\}$ where $|J| \leq |M|$ also cover the interval $(1, q+1]$, then $\varphi_{n_s} \geq q+1$ so $n_s \geq (q+1)q$. Now if $J \neq M$, it follows that at least one pair of consecutive entries of J differ by more than the uniform distance j between consecutive entries of M, contrary to the fact that the entries of M were already chosen as widely spaced as possible. $\square$

We now turn our attention to the question of uniqueness of representation. Uniqueness of representation fails if the set M of possible values for the terms contains

$0, n$ and φ_n for some integer $n \in \mathbb{N}$, for then

$$\sqrt{\varphi_n, 0, 0, 0, \ldots} = \sqrt{0, n, n, n, \ldots}.$$

In case $M = \{0, q, 2q, 3q, \ldots, (q+1)q\} = \{m_1, \ldots, m_{q+2}\}$, we will see that the converse of the implication is also true. Let us first assume $q \geq 2$.

Suppose $\sqrt{c_1, c_2, \ldots, c_z, a_1, a_2, a_3, \ldots}$ and $\sqrt{c_1, c_2, \ldots, c_z, b_1, b_2, b_3, \ldots}$ are distinct representations of $b \in (1, q+1]$ where $a_1 \neq b_1$. Squaring both sides of the equation

$$\sqrt{c_1, c_2, \ldots, c_z, a_1, a_2, a_3, \ldots} = \sqrt{c_1, c_2, \ldots, c_z, b_1, b_2, b_3, \ldots}$$

and subtracting c_i repeatedly as i ranges from 1 to z gives

$$\sqrt{a_1, a_2, a_3, \ldots} = \sqrt{b_1, b_2, b_3, \ldots} \text{ where } a_1 \neq b_2.$$

Now from the algorithm of Theorem 4.1, the numbers representable as $\sqrt{m_i, a_2, \ldots}$ where $a_n \in M_q$ for every $n \in \mathbb{N}$ are the elements of $J_i = \{\sqrt{m_i}\} \cup I_i$ where $I_i = (\sqrt{m_i + 1}, \sqrt{m_i + \varphi_{m_p}}]$. The intervals I_i are mutually disjoint, so any duplication of representation can only occur for a number of form $\sqrt{m_i} \in I_{i-1}$. Now if we have distinct representations of a number

$$\sqrt{m_i} = \sqrt{m_{i-1}, b_2, b_3, b_4, \ldots},$$

squaring both sides and recalling that $m_i - m_{i-1} = q$ gives

$$q = \sqrt{b_2, b_3, b_4, \ldots} \quad (b_n \in \{0, q, 2q, \ldots, (q+1)q\} \quad \text{for every } n \geq 2).$$

Now we show that the only such representations of q are

$$\begin{aligned} q &= \sqrt{q^2, 0, 0, 0, \ldots} \\ &= \sqrt{(q-1)q, (q-1)q, \ldots, (q-1)q, q^2, 0, 0, 0, \ldots} \\ &= \sqrt{(q-1)q, (q-1)q, (q-1)q, \ldots} \end{aligned}$$

Suppose $q = \sqrt{b_2, b_3, b_4, \ldots}$ where $b_n \in \{0, q, 2q, \ldots, (q+1)q\}$ for $n = 2, 3, 4, \ldots$ and $(b_i)_{i=2}^{\infty} \neq (q^2, 0, 0, 0, \ldots)$. Clearly $b_2 \neq (q+1)q = q^2 + q$, for then $\sqrt{b_2, b_3, b_4, \ldots} > q$

and $b_2 \neq q^2$ for then $(b_i)_{i=2}^{\infty} = (q^2, 0, 0, 0, \ldots)$. Furthermore, $b_2 \not\leq (q-2)q$, for if $b_2 \leq (q-2)q$, then

$$\begin{aligned}\sqrt{b_2, b_3, b_4, \ldots} &\leq \sqrt{(q-2)q, (q+1)q, (q+1)q, (q+1)q, \ldots} \\ &= \sqrt{q^2 - 2q + \varphi_{(q+1)q}} \\ &= \sqrt{q^2 - 2q + q + 1} \\ &< q \quad \text{since } q \geq 2.\end{aligned}$$

Thus, we must have $b_2 = (q-1)q$ and we now have

$$q = \sqrt{(q-1)q, b_3, b_4, \ldots}.$$

Squaring this equation yields

$$q = \sqrt{b_3, b_4, \ldots}.$$

Repeating the argument above, we find that $(b_3, b_4, b_5, \ldots) = (q^2, 0, 0, \ldots)$ or $b_3 = (q-1)q$. Iterating, we find that $(b_2, b_3, b_4, \ldots)$ is either constantly $(q-1)q$ or has a finite number of initial terms equal to $(q-1)q$ followed by $q^2, 0, 0, 0, \ldots$. This completes our claim about the possible representations of $q = \sqrt{b_2, b_3, b_4, \ldots}$.

Now it follows that the only possible duplicate representations of $\sqrt{a_1, a_2, \ldots} = \sqrt{b_1, b_2, \ldots}$ in which $a_1 \neq b_1$ are of form

$$\begin{aligned}\sqrt{m_i} &= \sqrt{m_{i-1}, q^2, 0, 0, 0, \ldots} \\ &= \sqrt{m_{i-1}, (q-1)q, (q-1)q, \ldots, (q-1)q, q^2, 0, 0, 0, \ldots} \\ &= \sqrt{m_{i-1}, (q-1)q, (q-1)q, (q-1)q, \ldots}.\end{aligned}$$

Furthermore, all possible duplicate representations are of form

$$\sqrt{c_1, \ldots, c_z, a_1, a_2, \ldots} = \sqrt{c_1, \ldots, c_z, b_1, b_2, \ldots}$$

where $\sqrt{a_1, a_2, a_3, \ldots}$ and $\sqrt{b_1, b_2, b_3, \ldots}$ are as above.

Example 4.2. *To illustrate this kind of duplication of representation, consider $q = 2$. Let the set of terms allowed be $M = \{0, 2, 4, 6\}$, then*

$$\begin{aligned} q = 2 &= \sqrt{4, 0, 0, 0, \ldots} \\ &= \sqrt{2, 2, 2, \ldots} \\ &= \sqrt{2, 2, \ldots, 2, 2 + \sqrt{2, 2, 2, \ldots}} \\ &= \sqrt{2, 2, \ldots, 2, 2 + 2} \\ &= \sqrt{2, 2, \ldots, 2, 4, 0, 0, 0, \ldots} \\ &= 2 \end{aligned}$$

We summarize our results, stated contrapositively, below.

Theorem 4.3. *A real number $b \in (\sqrt{q}, q + 1)$ has a unique representation as $\sqrt{a_1, a_2, a_3, \ldots}$ where $a_i \in M_q = \{0, q, 2q, \ldots, (q + 1)q\}$ with $q \geq 2$ if and only if it cannot be represented as a terminating continued radical $\sqrt{a_1, a_2, \ldots, a_z, 0, 0, 0, \ldots}$. A number $b \in (\sqrt{q}, q+1)$ has a terminating continued radical representation $\sqrt{a_1, a_2, \ldots, a_z, 0, 0, 0, \ldots}$ if and only if it has a continued radical representation ending in repeating $(q - 1)q$'s. Observe that*

$$\sqrt{q, 0, 0, 0, \ldots} \quad \text{and} \quad \sqrt{(q + 1)q, (q + 1)q, (q + 1)q, \ldots}$$

respectively are the unique representations of $\sqrt{q}$ and $\varphi_{(q+1)q} = q + 1$.

Proof. The first statement was proved above. The second statement follows from the equation

$$\sqrt{c_1, \ldots, c_z, m_i, 0, 0, \ldots} = \sqrt{c_1, \ldots, c_z, m_{i-1}, (q - 1)q, (q - 1)q, \ldots}$$

for $q \geq 2$ and $i \in \{2, \ldots, q + 2\}$. $\square$

To complete the uniqueness discussion, we now consider the special case $q = 1$ in which our terms are selected from $M_1 = \{0, 1, 2\}$. The arguments required for this case are similar to those given above, but the illustrative nature of a specific case may be helpful.

THEOREM 4.4. [**9**] *Any number* $b \in (1,2)$ *can be represented as a continued radical* $\sqrt{a_1, a_2, \ldots}$ *where* $a_i \in \{0,1,2\}$. *This representation is unique unless* b *has such a representation ending in repeating* $0s$. *A number* $b \in (1,2)$ *has such a representation ending in repeating* $0s$ *if and only if it has such a representation ending in repeating* $2s$.

Note that $\sqrt{1,0,0,0,0,\ldots} = 1$ *and* $\sqrt{2,2,2,2,2,\ldots} = 2$ *are the unique representations of* 1 *and* 2.

Proof: The existence of such a representation of $b \in (1,2)$ follows from Theorem 4.1.

Suppose $\sqrt{c_1, \ldots, c_z, a_1, a_2, a_3, \ldots}$ and $\sqrt{c_1, \ldots, c_z, b_1, b_2, b_3, \ldots}$ are distinct representations of $b \in (1,2)$ and $a_1 \neq b_1$. Now if $\sqrt{a_1, a_2, a_3, \ldots} = b' = \sqrt{b_1, b_2, b_3, \ldots}$, the observation that

$$\sqrt{0, x_2, x_3, \ldots} \in \{0\} \cup \in [1, \sqrt{2}]$$
$$\sqrt{1, x_2, x_3, \ldots} \in \{1\} \cup [\sqrt{2}, \sqrt{3}]$$
$$\sqrt{2, x_2, x_3, \ldots} \in \{\sqrt{2}\} \cup [\sqrt{3}, 2]$$

implies that either

$$b' = \sqrt{2} = \sqrt{2,0,0,0,\ldots} = \sqrt{1,1,0,0,0,\ldots} = \sqrt{0,2,2,2,\ldots}$$

or

$$b' = \sqrt{3} = \sqrt{2,1,0,0,0,\ldots} = \sqrt{1,2,2,2,\ldots}\ .$$

Inserting the initial terms $c_1, \ldots, c_z$, we find that any distinct representations of b by continued radicals end in repeating zeros or repeating twos.

Suppose $b \in (1,2)$ and $b = \sqrt{a_1, \dots, a_n, 2, 2, 2, \dots}$, where $a_n \neq 2$. (Since $b \neq 2$, $n \geq 1$.) Now either $a_n = 0$ and

$$b = \sqrt{a_1, \dots, a_{n-1}, 0, 2, 2, 2, \dots} = \sqrt{a_1, \dots, a_{n-1}, 2, 0, 0, 0, \dots},$$

or $a_n = 1$ and

$$b = \sqrt{a_1, \dots, a_{n-1}, 1, 2, 2, 2, \dots} = \sqrt{a_1, \dots, a_{n-1}, 2, 1, 0, 0, \dots}.$$

Similarly, suppose $b \in (1,2)$ and $b = \sqrt{a_1, \dots, a_n, 0, 0, 0, \dots}$, where $a_n \neq 0$. If $a_n = 2 = \sqrt{2, 2, 2, \dots}$, then $b = \sqrt{a_1, \dots, a_{n-1}, 0, 2, 2, 2, \dots}$ gives the desired representation. If $a_n = 1$, then $b = \sqrt{a_1, \dots, a_{n-1}, 1}$ and without loss of generality, we may assume $a_{n-1} \neq 0$. If $a_{n-1} = 1$, then $\sqrt{a_{n-1}, a_n} = \sqrt{1, 1} = \sqrt{2} = \sqrt{0, 2, 2, 2, \dots}$, so $b = \sqrt{a_1, \dots, a_{n-2}, 0, 2, 2, 2, \dots}$ is the desired representation. If $a_{n-1} = 2$, then $\sqrt{a_{n-1}, a_n} = \sqrt{2, 1} = \sqrt{3} = \sqrt{1, 2, 2, 2, \dots}$, so $b = \sqrt{a_1, \dots, a_{n-2}, 1, 2, 2, 2, \dots}$ is the desired representation. $\square$

Finally, we observe that taking the terms of $\sqrt{a_1, a_2, a_3, \dots}$ to be elements of $\{0, q, 2q, 3q, \dots, (q+1)q\}$ where $q \in \mathbb{N}$, we can represent every element of $(1, q+1]$, and since this interval has length $q \geq 1$, it follows that every real number can be represented as $a_0 + \sqrt{a_1, a_2, a_3, \dots}$ where $a_0 \in \mathbb{Z}$ and $a_i \in M$ for every $i \in \mathbb{N}$.

CHAPTER 5

Continued radicals whose terms assume only two values

We note that one cannot represent all points of an interval using continued radicals whose terms come from a set of two values $M = \{m_1, m_2\} \subseteq \mathbb{N} \cup \{0\}$. If both values are nonnegative, then the requirement from Theorem 3.2 that $\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}}$ implies that

$$\sqrt{m_1 + \frac{1 + \sqrt{4m_2 + 1}}{2}} \geq \sqrt{m_2 + \frac{1 + \sqrt{4m_1 + 1}}{2}},$$

and it follows that $\sqrt{4m_2 + 1} - 2m_2 \geq \sqrt{4m_1 + 1} - 2m_1$. This last inequality must fail since $m_2 > m_1$ but $f(x) = \sqrt{4x + 1} - 2x$ is strictly decreasing on $[0, \infty)$. Similarly, if $M = \{0, m_2\}$ where $m_2 \in \mathbb{N}$, the requirement $\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + 1}$ to insure no gaps becomes

$$\sqrt{0 + \frac{1 + \sqrt{4m_2 + 1}}{2}} \geq \sqrt{m_2 + 1},$$

which is easily seen to have no positive solutions.

Furthermore, the set D of real numbers representable using terms from a two element set $M = \{m_1, m_2\} \subseteq \mathbb{N}$ will have a familiar pattern of gaps.

Theorem 5.1. *If m_1 and m_2 are natural numbers with $m_1 < m_2$, then the set $D = \{\sqrt{a_1, a_2, \ldots} : a_i \in \{m_1, m_2\}$ for every $i \in \mathbb{N}\} = S(\{m_1, m_2\})$ is homeomorphic to the Cantor ternary set $\mathcal{C}$.*

Proof. The Cantor set $\mathcal{C}$ is the set of real numbers in $[0, 1]$ which have ternary representations of form $0.c_1c_2c_3\ldots$ where each digit c_i is either 0 or 2. Each element of $\mathcal{C}$ has a unique such representation (though some have other representations using 1's.) Note that the arguments of the previous chapters insure that each element of D is determined by a unique sequence of terms in M. Letting $g(0) = m_1$ and $g(2) = m_2$, the function $h : \mathcal{C} \to D$ which maps $0.c_1c_2c_3\ldots$ to $\sqrt{g(c_1), g(c_2), g(c_3), \ldots}$ is a bijection. The continuity of h and h^{-1} follows from the fact that these functions both preserve limits: Suppose $(\overline{c_i})_{i=1}^{\infty}$ is a sequence in $\mathcal{C}$ converging to $\overline{c_0} \in \mathcal{C}$ where for $i \in \mathbb{N} \cup \{0\}$, $\overline{c_i}$ has ternary representation $0.c_{i,1}c_{i,2}c_{i,3}\ldots$ with $c_{i,n} \in \{0, 2\}$ for each $n \in \mathbb{N}$. Now the convergence of $\overline{c_i}$ to $\overline{c_0}$ implies that the sequences $(c_{i,1}, c_{i,2}, c_{i,3}, \ldots)$ of digits of $\overline{c_i}$ must "converge" to the sequence of digits of $\overline{c_0}$ in the sense that the sequence $(k_i)_{i=1}^{\infty}$ converges to ∞ where $k_i \in \mathbb{N} \cup \{\infty\}$ is the smallest value of k for which $c_{i,k} \neq c_{0,k}$. It follows that the corresponding sequence $(g(c_{i,1}), g(c_{i,2}), g(c_{i,3}), \ldots)$ of terms of $h(\overline{c_i})$ must "converge" to the sequence of terms of $h(\overline{c_0})$ in the same sense, so h preserves limits. A similar argument shows that h^{-1} also preserves limits. $\square$

The theorem above only used the assumption that the integers m_1, m_2 be nonzero to insure uniqueness of representation of continued radicals of form $\sqrt{a_1, a_2, \ldots}$. If $M = \{0, m_2\}$ where $m_2 \in \mathbb{N}$, and b has two representations as continued radicals having terms from M, removing any initial identical terms gives a number having two representations which differ in the first term. But $\sqrt{0, a_2, a_3, \ldots} \in \{0\} \cup (1, \sqrt{\varphi_{m_2}}\,]$ and $\sqrt{m_2, b_2, b_3, \ldots} \in \{\sqrt{m_2}\} \cup (\sqrt{m_2 + 1}, \varphi_{m_2}]$, so considering the remarks of the first paragraph of this chapter, the only possible duplication of representation could occur for $\sqrt{m_2}$ if it is less that or equal to $\sqrt{\varphi_{m_2}}$. This would lead to $-1 \leq \sqrt{4m_2 + 1} - 2m_2 = f(m_2)$ where $f(x)$ is as defined before Theorem 5.1. Since $f(x)$ is strictly decreasing and $f(2) = -1$, it follows that the only sets $M = \{m_1, m_2\}$ for which the previous proof fails to show $S(M) = D$ is homeomorphic to the Cantor ternary set are $M = \{0, 1\}$ and $M = \{0, 2\}$. With $M = \{0, 1\}$, for example, we have $\sqrt{a_1, \ldots, a_n, 1, 0, 0, 0, \ldots} = \sqrt{a_1, \ldots, a_n, 0, 1, 0, 0, \ldots}$, and thus h^{-1}, as defined in the proof of Theorem 5.1, is not well-defined at these points.

CHAPTER 6

Further Results

6.1. Periodic Representations

It was mentioned earlier that continued radical representations bear some structural similarity to decimal representations. If the algorithm for constructing a continued radical representation of a given number is used, then there is a non-uniqueness of representation (e.g. $\sqrt{2, 6, 6, \ldots} = \sqrt{3, 2, 2, \ldots})$ similar to the non-uniqueness of a given real number's representation (e.g., $1.3\overline{999} = 1.4\overline{000}$). It was discovered that continued radicals whose constituents are periodic have a special property similar to the special property that periodic decimal representations have (i.e., every rational number has a periodic decimal expansion); this property is described in the following theorem.

Theorem 6.1. *Suppose $M = \{m_1, m_2, \ldots, m_n\}$ satisfies the "minimal duplications" conditions of Theorem 3.4 or of Theorem 4.3. Then the continued radical representation of a rational number $x \geq 0$ using terms from M is periodic if and only if x is an integer in $S(M) = [\varphi_{m_1}, \varphi_{m_n}]$.*

Proof. Suppose x is an integer in $S(M) = [\varphi_{m_1}, \varphi_{m_n}]$. If x has a non-unique representation, then Theorem 3.4 implies that the representation of x is periodic. If x has a unique representation, then the algorithm will generate a unique set $\{a_1, a_2, \ldots\}$ such that $x = \sqrt{a_1, a_2, \ldots}$.

The algorithm chooses the unique $a_1 \in M$ as large as possible so that

$$x = \sqrt{a_1 + b_1},$$

where $b_1 = x^2 - a_1$ is an integer in $[\varphi_{m_1}, \varphi_{m_n}] \cap \mathbb{Z}$.

Repeating the algorithm (to find the representation of b_1) yields

$$x = \sqrt{a_1, a_2 + b_2}$$

where $b_2 = {b_1}^2 - a_2$ is an integer in $[\varphi_{m_1}, \varphi_{m_n}] \cap \mathbb{Z}$.

This process is repeated ad infinitum, but each b_i is chosen from $[\varphi_{m_1}, \varphi_{m_n}] \cap \mathbb{Z}$, a finite set; thus, there must exist some $b_i = b_{i-k}$, and b_i is therefore recursively defined, with period k. Uniqueness of representation now implies that the a_i's are periodic.

Conversely, suppose $x \in \mathbb{Q}$, and $x = \sqrt{a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots}$, then x is a solution to

$$(\ldots(((x^2 - a_1)^2 - a_2)^2 - a_3)^2 - \ldots a_{k-1})^2 - a_k = 0,$$

a monic polynomial in x with a constant term that is an integer. Thus, the Rational Root Theorem implies that $x \in \mathbb{Z}$.

If $x \in \mathbb{Q}$, and $x = \sqrt{b_1, b_2, \ldots, b_l, a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots}$, then

$$(\ldots(((x^2 - b_1)^2 - b_2)^2 - b_3)^2 - \ldots b_{l-1})^2 - b_l - \sqrt{a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots} = 0.$$

Since

$$\sqrt{b_1, b_2, \ldots, b_l, a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots} \in \mathbb{Q}$$

implies that

$$\sqrt{a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots} \in \mathbb{Q},$$

we have, by the previous case,

$$\sqrt{a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots} \in \mathbb{Z}.$$

Thus

$$(\ldots(((x^2 - b_1)^2 - b_2)^2 - b_3)^2 - \ldots b_{l-1})^2 - b_l - \sqrt{a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots}$$

is a monic polynomial in x with a constant term that is an integer. Thus, the Rational Root Theorem again implies that $x \in \mathbb{Z}$.

$\square$

This result is reminiscent of Lagrange's result that a real number is represented by an infinite periodic continued fraction if and only if it is an irrational solution to a quadratic equation with integer coefficients [**6**].

6.2. A Pattern

A point of potential further research is the exploration of the possibility of the existence a binary operation that could be applied to given continued radical representations that is analogous to the addition or subtraction of the numbers being represented. A natural extension of this search would be to determine whether or not a group structure could be imposed on a given set of representable numbers and its respective binary operation.

In exploring this possibility an interesting potential pattern was discovered, the ramifications of which have not been fully explored. This pattern is best described by an example.

EXAMPLE 6.2. *According to the algorithm presented in Chapter 3, let*

$$M = \{6, 13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90\}$$

Then, the set, $S(M)$, of numbers representable by the elements of M is $[3, 10]$. Then the algorithm provides the following representations of the integers in $S(M)$.

$$3 = \sqrt{6, 6, \ldots}$$

$$4 = \sqrt{13, 6, 6, \ldots}$$

$$5 = \sqrt{20, 20, \ldots}$$

$$6 = \sqrt{27, 76, 20, 20, \ldots}$$

$$7 = \sqrt{41, 55, 76, 20, 20, \ldots}$$

$$8 = \sqrt{55, 76, 20, 20, \ldots}$$

$$9 = \sqrt{76, 20, 20, \ldots}$$

$$10 = \sqrt{90, 90, \ldots}$$

One may note in the previous example that, for each representation, the number of elements preceding the periodic elements appears to obey a symmetric pattern. Since other examples seem to generally conform to this pattern, there appears to be a kind of symmetry to the "length" of these representations.

6.3. Computer Generated Partial Representations

6.3.1. Generation of Representations. It is possible to use the computer algebra system *Mathematica* to produce partial continued radical representations for specified numbers. The following program, based on the algorithm presented in Chapter 3, allows the user to specify the number to be represented, b, and the accuracy to which the partial continued radical representation will approximate the specified number, ε.

In this program, the set of nonzero positive integers, M, used by the algorithm to represent the user-specified number, b, is itself based on the value of b, and is automatically generated by the program. This set M is provided in the program's output.

```
Clear[f, ϕ, x, ϵ, m, n, M, a, i, k];
$MaxExtraPrecision = Infinity;
f[a_] := If[Length[a] > 0, Sqrt[a[[1]] + f[Drop[a, 1]]], 0];
(* f takes as a parameter a list of non−negative integers a=
   {a1,a2,... an} and returns the value of the radical √(a1+√(a2+...+√an)) *)

ϕ[x_] := (1 + Sqrt[1 + 4 * x]) / 2; (* ϕ[x] gives the value of the continued radical √(x+√(x+√(x+... *)

ϵ = 10^-10;
(* ϵ determines how well we want the continued radical to approximate b;
  it should be expressed as 10^−n for some natural number, n *)
b = π; (* b, the number to be represented, must be a real number greater than or equal to 2 *)

n = Floor[b] - 1; (* n*(n+1) will be the smallest number used in the continued radical *)
j = Floor[b]; (* j*(j+1) will be the largest number used in the continued radical *)

M = {};
For[i = 0, i < j + n + 2, i++, M = Append[M, n * (n + 1) + i (j - n)]];
(* M contains all of the numbers that will be used in the continued radical *)
Print["The set of numbers used in this nested radical: M = ", M];
p = Length[M]; (* p is the index of the last element of M *)
a = {}; (* a will hold the elements of the continued radical that approximate b *)

(* The following algorithm chooses the first number in the continued radical *)
If[b >= √(M[[p]] + ϕ[M[[1]]]), a = Append[a, M[[p]]],
      (For[k = p - 1, k >= 1, k--,
                        If[b >= √(M[[k]] + ϕ[M[[1]]]) &&
                                  b < √(M[[k + 1]] + ϕ[M[[1]]]),
    a = Append[a, M[[k]]]]]
      )];
```

```
(* The following algorithm chooses numbers for the continued radical until the continued radical is within ε of b *)
While[b - f[a] > ε,
                    (For[k = p, k >= 1, k--,
                            If[b >= f[Append[a, M[[k]] + ϕ[M[[1]]]]],
    (a = Append[a, M[[k]]]; k = 1)]]) ];

Print[f[a]];
Print["Continued Radical = ", N[f[a], 1 / Abs[Log[ε, 10]] + 5]];
Print["                    b = ", N[b, 1 / Abs[Log[ε, 10]] + 5]];
Print["The difference: ", N[b - f[a], 1 / Abs[Log[ε, 10]] + 5]];
```

The set of numbers used in this nested radical: M = {6,7,8,9,10,11,12}

$$\sqrt{6+\sqrt{11+\sqrt{12+\sqrt{11+\sqrt{8+\sqrt{8+\sqrt{9+\sqrt{6+\sqrt{10+\sqrt{8+\sqrt{12+2\sqrt{3}}}}}}}}}}}}$$

Continued Radical = 3.14159265355899

b = 3.14159265358979

The difference: $3.08033598628299 \times 10^{-11}$

6.3.2. Representations of Integers. The following program, based on the code presented in Section 6.3.1, takes as input the "seed values," n and j. These "seed values" generate the set $M = \{n(n+1), \ldots, n(n+1)+k(j-n), \ldots, j(j+1)\}$, used by the algorithm presented in Chapter 3 to create the partial continued radical representations. The user also provides the program with the values ε and $tMax$, which determine the accuracy of, and number of elements used in the partial continued radical representations, respectively.

The program then generates as output, the partial continued radical representation of all of the integers in the interval of "representable" numbers; according to the algorithm, this interval is $[n+1, j+1]$. This set of representations is of the kind presented in Example 6.2.

```
(*Given positive integers n, j, this program generates all of the representations of the integers
  in the interval [φ[n(n+1)],φ[j(j+1)]]*)
Clear[f, ϕ, x, ϵ, m, n, M, a, i, k, t, tMAX];
$MaxExtraPrecision = Infinity;
$RecursionLimit = Infinity;
f[a_] := If[Length[a] > 0, Sqrt[a[[1]] + f[Drop[a, 1]]], 0];
(* f takes as a parameter a list of non-negative integers a=
  {a1,a2,... an} and returns the value of the radical √(a1+√(a2+...+√an)) *)

ϕ[x_] := (1 + Sqrt[1 + 4 * x]) / 2; (* ϕ[x] gives the value of the continued radical √(x+√(x+√(x+...))) *)

ϵ = 10^-10; (* ϵ determines how well we want the continued radical to approximate b
  it should be expressed as 10^-n for some natural number, n *)
tMax = 20; (* tMax is the maximum number of elements that the radical will contain *)
n = 2; (* n*(n+1) will be the smallest number used in the continued radical *)
j = 9;  (* j*(j+1) will be the largest number used in the continued radical *)

M = {};
For[i = 0, i < j + n + 2, i++, M = Append[M, n * (n + 1) + i (j - n)]];
(* M contains all of the numbers that will be used in the continued radical *)
Print["The set of numbers used in this nested radical: M = ", M];
For[b = (n + 1), b <= (j + 1), b++,
p = Length[M]; (* p is the index of the last element of M *)
a = {}; (* a will hold the elements of the continued radical that approximate b *)
```

```
(* The following algorithm chooses the first number in the continued radical *)
If[b >= √(M[[p]] + φ[M[[1]]]), a = Append[a, M[[p]]],
    (For[k = p - 1, k >= 1, k--,
                          If[b >= √(M[[k]] + φ[M[[1]]]) &&
                                b < √(M[[k + 1]] + φ[M[[1]]]),
      a = Append[a, M[[k]]]]
    )];

(* The following algorithm chooses numbers for the continued radical until the continued radical is within ε of b *)
t = 1;
  While[b - f[a] > ε && t < tMax,
                (For[k = p, k >= 1, k--,
                          If[b >= f[Append[a, M[[k]] + φ[M[[1]]]]],
      (a = Append[a, M[[k]]]; k = 1)]]
                 t ++;)];

Print[b , " = ", f[a]]];
```

The set of numbers used in this nested radical: M =
{6, 13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90}

$$3 = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}}}}}}}}}}}$$

$$4 = \sqrt{13+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}}}}}}}}}}}}$$

$$5 = \sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+2\sqrt{5}}}}}}}}}}}$$

$$6 = \sqrt{27+\sqrt{76+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+2\sqrt{5}}}}}}}}}}}$$

$$7 = \sqrt{41+\sqrt{55+\sqrt{76+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+2\sqrt{5}}}}}}}}}}}$$

$$8 = \sqrt{55+\sqrt{76+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+2\sqrt{5}}}}}}}}}}}$$

$$9 = \sqrt{76+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20+2\sqrt{5}}}}}}}}}}}$$

$$10 = \sqrt{90+\sqrt{90+\sqrt{90+\sqrt{90+\sqrt{90+\sqrt{90+\sqrt{90+\sqrt{90+3\sqrt{10}}}}}}}}}$$

Bibliography

[1] R. I. Andrushkiw, On the convergence of continued radicals with applications to polynomial equations, *J. Franklin Inst.*, 319 (4) 1985, 391–396.

[2] Jonathan M. Borwein and G. de Barra, Nested Radicals, *The American Mathematical Monthly*, 98 (8) Oct. 1991, 735–739.

[3] Calvin C. Clawson, *Mathematical Mysteries: The Beauty and Magic of Numbers*, Plenum Press, New York, 1996.

[4] Johan Karlsson and Hans Wallin, Continued fractions and iterated function systems, in *Continued Fractions and Orthogonal Functions (Loen, 1992)* , 191–210, Lecture Notes in Pure and Appl. Math., 154, Dekker, New York, 1994.

[5] D. Laugwitz, Kettenwurzeln und Kettenoperationen, *Elemente der Mathematik*, 45 (4) 1990, 89–98.

[6] A. Ya. Khinchin,*Continued Fractions*, Dover Publications, Minepla, NY, 1997 republication of the 1964 edition.

[7] R. Daniel Mauldin and Mariusz Urbański, Conformal iterated function systems with applications to the geometry of continued fractions, *Trans. Am. Math. Soc.* 351 (12) 1999, 4995–5025.

[8] Bettina Richmond and Thomas Richmond, *A Discrete Transition to Advanced Mathematics,* Brooks/Cole Series in Advanced Mathematics, Thomson Learning, 2004.

[9] Walter S. Sizer, Continued Roots, *Mathematics Magazine* 59 (1) Feb. 1986, 23–27

[10] Walter S. Sizer and E. K. Wiredu, Convergence of some continued roots in the complex plane. *Bull. Malaysian Math. Soc.* (2) 19 (1996), no. 1, 1–7.