Calculus of Variations on Time Scales and Its Applications to Economics

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CALCULUS OF VARIATIONS ON TIME SCALES
AND ITS APPLICATIONS TO ECONOMICS

By
Christopher Steven McMahan

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
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To my loving family.
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The sum of the accomplishments of any one person can be found, term by term, in the faces of those who have driven them on.
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Abstract

CALCULUS OF VARIATIONS ON TIME SCALES AND ITS APPLICATIONS TO ECONOMICS

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The goal of time scale research is to progress the development of a harmonized theory that is all encompassing of the more commonly known specialized forms. The main results of this paper is the presentation of the Ramsey model which can be written using both the Δ and ∇ operators, and solved using the two separate theories of the calculus of variations on time scales. The next presentation will be of the solution of an adjustment model, for a specific form of a time scale, whose functional can only be optimized, using the existing theory, when written with the Δ operator. We will also develop certain elements of stochastic time scale calculus, in order to lay the groundwork necessary to develop the theory of stochastic calculus of variations on time scales.
Chapter 1

Introduction

For the convenience of the reader a few basic definitions and background theorems will be presented here, which can be found in [7,8]. First let \( T \) be a time scale (which is a nonempty closed subset of \( \mathbb{R} \)), and let \([a, b]\) be a closed and bounded interval in \( T \). Then we will define the forward and backward jump operators as,

\[
\sigma(t) = \inf\{s \in T : s > t\} \quad \rho(t) = \sup\{s \in T : s < t\}.
\]

Note that in this definition, \( \inf\{\emptyset\} = \sup T \) and \( \sup\{\emptyset\} = \inf T \), where \( \emptyset \) denotes the empty set. Let the set \( T^k \) be the set derived from \( T \), such that if \( T \) has a left scattered maximum \( t_1 \) then \( T^k = T - \{t_1\} \), otherwise \( T^k = T \). Also let the set \( T_k \) be the set derived from \( T \), such that if \( T \) has a right scattered minimum \( t_2 \) then \( T_k = T - \{t_2\} \), otherwise \( T_k = T \).

**Definition 1.0.1.** Assume \( f : T \to \mathbb{R} \) is a function and let \( t \in T^k \). Then we define \( f^\Delta(t) \) to be the number (provided that it exists) with the property that given any \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) such that
\[ |f(\sigma(t)) - f(s)| - f^\Delta(t)|\sigma(t) - s| \leq \varepsilon|\sigma(t) - s| \]

for all \( s \in U \). We call \( f^\Delta(t) \) the delta (or Hilger) derivative of \( f \) at \( t \).

**Definition 1.0.2.** Assume \( f : T \rightarrow \mathbb{R} \) is a function and let \( t \in T_k \). Then we define \( f^\nabla(t) \) to be the number (provided that it exists) with the property that given any \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) such that

\[ |f(\rho(t)) - f(s)| - f^\nabla(t)|\rho(t) - s| \leq \varepsilon|\rho(t) - s| \]

for all \( s \in U \). We call \( f^\nabla(t) \) the nabla derivative of \( f \) at \( t \).

**Theorem 1.0.1.** Assume that \( f : T \rightarrow K \) is a function, and let \( t \in T^k \) then if \( f \) is differentiable at \( t \) then the following hold,

1. \( f(\sigma(t)) = (\sigma(t) - t)f^\Delta(t) + f(t) \)

2. If \( f \) is differentiable at \( t \), then \( f \) is continuous at \( t \).

**Theorem 1.0.2.** Assume that \( f : T \rightarrow K \) is a function, and let \( t \in T^k \) then if \( f \) is differentiable at \( t \) then the following hold,

1. \( f(\rho(t)) = f(t) - (t - \rho(t))f^\nabla(t) \)

2. If \( f \) is differentiable at \( t \), then \( f \) is continuous at \( t \).

**Definition 1.0.3.** A function \( f : T \rightarrow \mathbb{R} \) is called rd-continuous provided it is continuous at right dense points in \( T \) and its left sided limit exists and is finite at left dense points in \( T \).

The set of rd-continuous functions will be denoted by \( C_{rd} \).
Definition 1.0.4. A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left dense points in $\mathbb{T}$ and its right sided limit exists and is finite at right dense points in $\mathbb{T}$. The set of ld-continuous functions will be denoted by $C_{ld}$.

Theorem 1.0.3. Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then

1. The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ with
   $$(f + g)^\triangle(t) = f^\triangle(t) + g^\triangle(t).$$

2. The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ with
   $$(fg)^\triangle(t) = f^\triangle(t)g(t) + f(\sigma(t))g^\triangle(t) = f^\triangle(t)g(\sigma(t)) + f(t)g^\triangle(t).$$

3. If $g(t)g(\sigma(t)) \neq 0$ then $\frac{f}{g}$ is differentiable at $t$ and
   $$\left(\frac{f}{g}\right)^\triangle(t) = \frac{f^\triangle(t)g(t) - f(t)g^\triangle(t)}{g(t)g(\sigma(t))}.$$
3. If \( g(t)g(\rho(t)) \neq 0 \) then \( \frac{f}{g} \) is differentiable at \( t \) and

\[
\left( \frac{f}{g} \right)'(t) = \frac{f'(t)g(t) - f(t)g'(t)}{g(t)g(\rho(t))}.
\]

**Definition 1.0.5.** The function \( p : T \to \mathbb{R} \) is \( \mu \) - regressive provided

\[
1 + \mu(t)p(t) \neq 0 \text{ for all } t \in T^\kappa
\]

holds. The set of all \( \mu \) - regressive and rd-continuous functions will be denoted by

\[
\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).
\]

**Definition 1.0.6.** The function \( p : T \to \mathbb{R} \) is \( \nu \) - regressive provided

\[
1 - \nu(t)p(t) \neq 0 \text{ for all } t \in T^\kappa
\]

holds. The set of all \( \nu \) - regressive and ld-continuous functions will be denoted by

\[
\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}) = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R}).
\]

**Definition 1.0.7.** If \( p \in \mathcal{R} \), then the exponential function is given by

\[
e_p(t, s) := \exp \left( \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau \right) \text{ for } s, t \in \mathbb{T}
\]

where \( \xi_h(z) \) is defined by

\[
\xi_h(z) = \frac{1}{h} \text{Log}(1 + hz)
\]

for all \( h > 0 \) and define \( \xi_0(z) = z \) for \( h = 0 \).
**Definition 1.0.8.** If \( p \in \mathcal{R}_\nu \), then the nabla exponential function is given by

\[
\hat{e}_p(t, s) := \exp \left( \int_s^t \hat{\xi}_\nu(\tau)p(\tau) \nabla \tau \right) \text{ for } s, t \in \mathbb{T}
\]

where \( \hat{\xi}_h(z) \) is defined by

\[
\hat{\xi}_h(z) = -\frac{1}{h} \log(1 - hz)
\]

for all \( h > 0 \) and define \( \hat{\xi}_0(z) = z \) for \( h = 0 \).

**Theorem 1.0.5.** Suppose that \( p \in \mathcal{R} \) and fix \( t_0 \in \mathbb{T} \). Then \( \hat{e}_p(\cdot, t_0) \) is a solution of the initial value problem

\[
y^\Delta = p(t)y, \quad y(t_0) = 1.
\]

**Theorem 1.0.6.** Suppose that \( q \in \mathcal{R}_\nu \) and fix \( t_0 \in \mathbb{T} \). Then \( \hat{e}_q(\cdot, t_0) \) is a solution of the initial value problem

\[
y^\nabla = q(t)y, \quad y(t_0) = 1.
\]

**Theorem 1.0.7.** If \( p, q \in \mathcal{R} \), then

1. \( e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s) \)
2. \( e_p^\Delta(t, s) = p(t)e_p(t, s) \)
3. \( \left( \frac{1}{e_p(t, s)} \right)^\Delta = -\frac{p(t)}{e_p(\sigma(t), s)} \)
4. \( e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s) \) where \( p(t) \oplus q(t) = p(t) + q(t) + p(t)q(t)\mu(t) \)
5. \( \frac{\hat{e}_p(t, s)}{\hat{e}_q(t, s)} = e_{\frac{p-\nu}{1+p\nu}}(t, s). \)

**Theorem 1.0.8.** If \( p, q \in \mathcal{R}_\omega \), then

1. \( \hat{e}_p(\rho(t), s) = (1 - \nu(t)p(t))\hat{e}_p(t, s) \)

2. \( \hat{e}_p^n(t, s) = p(t)\hat{e}_p(t, s) \)

3. \( \left( \frac{1}{\hat{e}_p(t, s)} \right)^n = \frac{-p(t)}{\hat{e}_p(\rho(t), s)} \)

4. \( \hat{e}_p(t, s)\hat{e}_q(t, s) = \hat{e}_{p \oplus q}(t, s) \) where \( p(t) \oplus q(t) = p(t) + q(t) - p(t)q(t)\nu(t) \)

5. \( \frac{\hat{e}_p(t, s)}{\hat{e}_q(t, s)} = e_{\frac{p-\nu}{1+p\nu}}(t, s). \)

**Theorem 1.0.9.** (Equivalence of Delta and Nabla Exponential Functions) If \( p \) is continuous and \( \mu \) — regressive, then

\[ e_p(t, t_0) = \hat{e}_{\frac{p-\nu}{1+p\nu}}(t, t_0) = \hat{e}_{\oplus \nu}(p)(t, t_0) \]

if \( q \) is continuous and \( \nu \) — regressive, then

\[ \hat{e}_q(t, t_0) = e_{\frac{\nu}{1-\nu\mu}}(t, t_0) = e_{\ominus \nu}(q)(t, t_0). \]

**Theorem 1.0.10.** If \( p \) is continuous and \( \mu \) — regressive, then

\[ e_p^n(t, t_0) = \frac{p^n}{1 + p^n\nu}e_p(t, t_0) \]

if \( q \) is continuous and \( \nu \) — regressive, then

\[ \hat{e}_q^n(t, t_0) = \frac{q^n}{1 - q^n\mu}\hat{e}_q(t, t_0). \]
Proof. If $p$ is continuous and $\mu - regressive$, then
\[ e_p^\nu(t, t_0) = e_p^\nu\mu(t, t_0) = \frac{p^\rho}{1 + p^\rho \nu} e_p^\rho(t, t_0) = \frac{p^\rho}{1 + p^\rho \nu} e_p(t, t_0) \]
and similarly if $q$ is continuous and $\nu - regressive$, then
\[ e_q^\Delta(t, t_0) = e_q^\Delta\nu(t, t_0) = \frac{q^\sigma}{1 - q^\sigma \mu} e_q^\sigma(t, t_0) = \frac{q^\sigma}{1 - q^\sigma \mu} e_q(t, t_0). \]

\[ \square \]

Theorem 1.0.11. If $a, b, c \in T, \alpha \in \mathbb{R}$, and $f, g \in C_{td},$ then

1. $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t$
2. $\int_a^b \alpha f(t) \Delta t = \alpha \int_a^b f(t) \Delta t$
3. $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_b^c f(t) \Delta t$
4. $\int_a^b (f(\sigma(t)))g^\Delta(t) \Delta t = (fg)^{(b)} - (fg)^{(a)} - \int_a^b f^\Delta(t)g(t) \Delta t$
5. $\int_a^b f(t)g^\Delta(t) \Delta t = (fg)^{(b)} - (fg)^{(a)} - \int_a^b f^\Delta(t)g(\sigma(t)) \Delta t$
6. $\int_a^b f(t) \Delta t = 0$
7. If $t \in T^\kappa$, then $\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu(t) f(t)$.

Theorem 1.0.12. If $a, b, c \in T, \alpha \in \mathbb{R}$, and $f, g \in C_{td},$ then

1. $\int_a^b [f(t) + g(t)] \nabla t = \int_a^b f(t) \nabla t + \int_a^b g(t) \nabla t$
2. $\int_a^b \alpha f(t) \nabla t = \alpha \int_a^b f(t) \nabla t$
3. $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$

4. $\int_a^b f(\rho(t))g(\rho(t))\nabla t = (fg)(b) - (fg)(a) - \int_a^b f\nabla(t)g(t)\nabla t$

5. $\int_a^b f(t)g(\gamma(\rho(t)))\nabla t = (fg)(b) - (fg)(a) - \int_a^b f\nabla(t)g(\rho(t))\Delta t$

6. $\int_a^a f(t)\nabla t = 0$

7. If $t \in T_\kappa$, then $\int_{\rho(t)}^t f(\tau)\Delta \tau = \nu(t)f(t)$.

**Definition 1.0.9.** For two $\Delta$-differentiable functions $y_1$ and $y_2$ we define the Wronskian $W = W(y_1, y_2)$ by

$$W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}.$$
Chapter 2

Calculus of Variations on Time Scales

The goal of this chapter is to present the development and theory associated with the Calculus of Variations on Time Scales as it was attained in [2,3,4,5,9]. We will first develop the theory for the \( \Delta \) operator in 2.1 and then for the \( \nabla \) operator in 2.2.

2.1 Calculus of Variations with the \( \Delta \) Operator

The following lemma is crucial to the proof of the main theorems that build the theory of Calculus of Variations with the delta operator.

Lemma 2.1.1. Let \( f \) be a continuous function on \( [a, b] \). If

\[
\int_a^b f(t)g(t)\Delta t = 0
\]

for every function \( g(t) \in C_{\text{rd}}[a, b] \) with \( g(a) = g(b) = 0 \), then \( f(t) = 0 \) for \( t \in [a, b]_\mathbb{T} \).
Proof. Assume to the contrary, there exists $t_0 \in [a, b]_\kappa$ s.t. $f(t_0) > 0$,

Case 1.) $t_0$ is right scattered, choose $g(t_0) = 1$ and $g(t) = 0$ otherwise, then

$$
\int_{t_0}^{\sigma(t_0)} f(t)g(t)\Delta t = (\sigma(t_0) - t_0)f(t_0)g(t_0) = (\sigma(t_0) - t_0)f(t_0) > 0.
$$

Case 2.) $t_0$ is right dense, then by continuity of $f$ there exists $\delta > 0$ s.t. $f(t) > 0$ on $[t_0, t_0 + \delta)$. Then define $g(t) = ((t_0 + \delta) - t)(t - t_0)$ for $t \in (t_0, t_0 + \delta)$ and $g(t) = 0$ otherwise. Then

$$
\int_{t_0}^{t_0+\delta} f(t)g(t)\Delta t = \int_{t_0}^{t_0+\delta} ((t_0 + \delta) - t)(t - t_0)f(t)\Delta t > 0.
$$

Case 3.) If $t_0 = a$ and $t_0 \in [a, b]_\kappa$, then $t_0$ must be a right dense point. Again by continuity of $f$, there exists $\delta > 0$ such that $f(t) > 0$ on $[a, a + \delta)$. Define $g(t) = (t - a)(a + \delta - t)$ for $t \in (a, a + \delta)$ and $g(t) = 0$ otherwise. Then

$$
\int_{a}^{a+\delta} f(t)g(t)\Delta t = \int_{a}^{a+\delta} ((a + \delta) - t)(t - a)f(t)\Delta t > 0.
$$

Case 4.) If $t_0 = b$ and $t_0 \in [a, b]_\kappa$ then $t_0$ must be a left dense point. Then by continuity of $f$, there exists $\delta > 0$ such that $f(t) > 0$ on $(b - \delta, b]$. Define $g(t) = (t - (b - \delta))(b - t)$ for $t \in (b - \delta, b)$. Then

$$
\int_{b-\delta}^{b} f(t)g(t)\Delta t = \int_{b-\delta}^{b} (t - (b - \delta))(b - t)f(t)\Delta t > 0
$$

which is a contradiction. So we have $f(t) \equiv 0$ on $[a, b]_\kappa$. \qed

The following Theorem is obtained by Bohner in the paper [5].
Theorem 2.1.1. (Euler's Necessary Condition) If a function $y(t)$ provides a local extremum to the functional,

$$J[y] = \int_{\sigma(a)}^{\sigma(b)} L(t, y(\sigma(t)), y^\Delta(t)) \Delta t$$

where $y \in C^2[a, \sigma^2(b)]$ and $y(\sigma(a)) = A$ and $y(\sigma^2(b)) = B$, then $y$ must satisfy the Euler-Lagrange equation,

$$L_{y'}(t, y, y^\Delta) - L_{y^\Delta}(t, y, y^\Delta) = 0$$

for $t \in [a, \sigma(b)]^\ast$.

Proof. Assume that $L(t, u, v)$ for each $t \in [\sigma(a), \sigma^2(b)] \subseteq \mathbb{T}$ is a class $C^2$ function of $(u, v)$. Let $y \in C^1[a, \sigma^2(b)]$ with $y(\sigma(a)) = A, y(\sigma^2(b)) = B$, where

$$C^1[a, \sigma^2(b)] = \{y : [a, \sigma^2(b)] \to \mathbb{R} | y^\Delta \text{ is continuous on } [a, \sigma^2(b)]^\ast\}.$$

The simplest variational problem is to extremize (maximize or minimize)

$$J[y] := \int_{\sigma(a)}^{\sigma(b)} L(t, y(\sigma(t)), y^\Delta(t)) \Delta t.$$

The optimization problem is then formulated as minimize (or maximize) $J[y]$. We say that $y_0 \in C^1[a, \sigma^2(b)]$ minimizes (or maximizes) the simplest variation problem if

$$J[y_0] \leq J[y]$$

for all $y \in C^1[a, \sigma^2(b)]$. We say $J$ has a local minimum (or maximum) at $y_0$ provided that there is a $\delta > 0$ such that
\[ J[y_0] \leq J[y] \]

for all \( y \in C^1[a, \sigma^2(b)] \) with \( \|y - y_0\| < \delta \). Here we consider the norm

\[
\|y\| = \max_{t \in [a, \sigma^2(b)]} |y(t)| + \max_{t \in [a, \sigma^2(b)]} |y^\Delta(t)|.
\]

Now let \( h : [\sigma(a), \sigma^2(b)] \to \mathbb{R} \) be any admissible variation, i.e., \( h \in C^1[\sigma(a), \sigma^2(b)] \) with \( h(\sigma(a)) = h(\sigma^2(b)) = 0 \). Assume that the simplest variational problem has a local extremum at \( y = y(t) \). Then we define

\[
\varphi(\epsilon) := J[y(t) + \epsilon h(t)],
\]

where \(-\infty < \epsilon < \infty\). Since \( \varphi \) has a local extremum at \( \epsilon = 0 \), we have that

\[
\varphi'(0) = 0
\]

\[
\varphi''(0) \geq 0 \quad (\leq 0)
\]

in the local minimum (maximum) case. Next we consider

\[
\varphi(\epsilon) = \int_{\sigma(a)}^{\sigma^2(b)} L(t, y(\sigma(t)) + \epsilon h(\sigma(t)), y^\Delta(t) + \epsilon h^\Delta(t)) \Delta t.
\]

Differentiating with respect to \( \epsilon \), we have

\[
\varphi'(\epsilon) = \int_{\sigma(a)}^{\sigma^2(b)} \frac{d}{d\epsilon} L(t, y(\sigma(t)) + \epsilon h(\sigma(t)), y^\Delta(t) + \epsilon h^\Delta(t)) \Delta t
\]

\[
= \int_{\sigma(a)}^{\sigma^2(b)} \{ L_u(t, y(\sigma(t)) + \epsilon h(\sigma(t)), y^\Delta(t) + \epsilon h^\Delta(t)) h(\sigma(t))
\]

\[
+ L_v(t, y(\sigma(t)) + \epsilon h(\sigma(t)), y^\Delta(t) + \epsilon h^\Delta(t)) h^\Delta(t) \} \Delta t.
\]

Hence, we obtain
\[ v'(0) = \int_{\sigma(a)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t. \]

The integral
\[ \int_{\sigma(a)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t \]
gives the first variation of \( J[y] \), denoted by \( J_1[h] \). So a necessary condition for \( y(t) \) to be a local minimum (maximum) is
\[ J_1[h] = \int_{\sigma(a)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t = 0 \]
for all \( h \in C^1[\sigma(a), \sigma^2(b)] \) with \( h(\sigma(a)) = h(\sigma^2(b)) = 0 \). Using the property of \( \triangle \)-integral [1], we have
\[ J_1[h] = \int_{\sigma(a)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t \]
\[ + \int_{\sigma(b)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t \]
\[ = \int_{\sigma(a)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t \]
\[ + (\sigma^2(b) - \sigma(b)) \{ L_{y^\sigma}(\sigma(b), y(\sigma^2(b)), y^\Delta(\sigma(b))) h(\sigma^2(b)) \]
\[ + L_{y^\Delta}(\sigma(b), y(\sigma^2(b)), y^\Delta(\sigma(b))) h^\Delta(\sigma(b)) \} \]
\[ = \int_{\sigma(a)}^{\sigma^2(b)} \{ L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)) h^\sigma(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)) h^\Delta(t) \} \Delta t \]
\[ + (\sigma^2(b) - \sigma(b)) L_{y^\Delta}(\sigma(b), y(\sigma^2(b)), y^\Delta(\sigma(b))) h^\Delta(\sigma(b)) \]
using the equality \( (\sigma^2(b) - \sigma(b)) h^\Delta(\sigma(b)) = h(\sigma^2(b)) - h(\sigma(b)) \), one can obtain
\[ J_1[h] = \int_{\sigma(a)}^{\sigma(b)} \left\{ L_{y^\sigma} (t, y^\sigma, y^{\Delta}) h^\sigma + L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) h^{\Delta} \right\} \Delta t \]

\[ - L_{y^{\Delta}} (\sigma(b), y(\sigma^2(b)), y^{\Delta}(\sigma(b))) h(\sigma(b)). \]

Integration by parts gives

\[ = \int_{\sigma(a)}^{\sigma(b)} L_{y^\sigma} (t, y^\sigma, y^{\Delta}) h^\sigma \Delta t + (L_{y^{\Delta}} h)(\sigma(b)) - (L_{y^{\Delta}} h)(\sigma(a)) \]

\[ - \int_{\sigma(a)}^{\sigma(b)} L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) h^\sigma \Delta t - (L_{y^{\Delta}} h)(\sigma(b)) \]

\[ = \int_{\sigma(a)}^{\sigma(b)} \left\{ L_{y^\sigma} (t, y^\sigma, y^{\Delta}) h^\sigma - L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) h^{\Delta} \right\} \Delta t. \]

From the integral

\[ \int_{\sigma(a)}^{\sigma(b)} \{ L_{y^\sigma} (t, y^\sigma, y^{\Delta}) - L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) \} h^\sigma \Delta t = (\sigma(a) - \sigma(b)) (\{ L_{y^\sigma} - L_{y^{\Delta}} \} h)(\sigma(a)) = 0 \]

we get

\[ \int_{\sigma(a)}^{\sigma(b)} \{ L_{y^\sigma} (t, y^\sigma, y^{\Delta}) - L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) \} h^\sigma \Delta t = 0 \]

for all \( h \in C^1[\sigma(a), \sigma^2(b)] \) with \( h(\sigma(a)) = h(\sigma^2(b)) = 0 \). The necessary condition for \( J[y] \) to have an extremum for \( y = y(t) \) is that

\[ \varphi'(0) = \int_{\sigma(a)}^{\sigma(b)} \{ L_{y^\sigma} (t, y^\sigma, y^{\Delta}) - L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) \} h^\sigma \Delta t = 0 \] (2.1.1)

for all admissible \( h \). According to Lemma 2.1.1, (2.1.1) implies that

\[ L_{y^\sigma} (t, y^\sigma, y^{\Delta}) - L_{y^{\Delta}} (t, y^\sigma, y^{\Delta}) = 0 \] (2.1.2)

a result known as a Euler-Lagrange equation.
In the development of Theorem 2.1.1, if the initial setup is relaxed such that one or even both of the boundary conditions are not given then the following Theorems can be obtained.

**Theorem 2.1.2.** If

\[ J[y] = \int_{\sigma(a)}^{\sigma(b)} L(t, y(\sigma(t)), y'\Delta(t)) \Delta t, \]

where \( y \in C^2[a, \sigma^2(b)] \) and \( y(\sigma(a)) = A \), has a local extremum at \( y(t) \), then \( y(t) \) satisfies the Euler-Lagrange equation for \( t \in [\sigma(a), \sigma^2(b)] \), \( y(\sigma(a)) = A \) and \( y(t) \) satisfies the condition

\[ (\sigma^2(b) - \sigma(b)) L_{y\sigma}(\sigma(b), y(\sigma^2(b)), y'\Delta(\sigma(b))) + L_{y\sigma}(\sigma(b), y(\sigma^2(b)), y'\Delta(\sigma(b))) = 0. \]

(2.1.3)

**Proof.** As in the proof of Theorem 2.1.1, \( J[h] = 0 \) for all \( h \in C^1[\sigma(a), \sigma^2(b)] \) with \( h(\sigma(a)) = 0 \). In solving the simplest variational problem in order to obtain the Euler-Lagrange equation, if we use \( h(\sigma(a)) = 0 \), we get

\[
\int_{\sigma(a)}^{\sigma(b)} \{ L_{y\sigma}(t, y', y'\Delta) - L_{y\sigma}(t, y', y'\Delta) \} \Delta(t) \Delta t \\
+ \{(\sigma^2(b) - \sigma(b)) L_{y\sigma}(\sigma(b), y(\sigma^2(b)), y'\Delta(\sigma(b))) + L_{y\sigma}(\sigma(b), y(\sigma^2(b)), y'\Delta(\sigma(b))) \} h(\sigma^2(b)) = 0
\]

for all \( h \in C^1[\sigma(a), \sigma^2(b)] \). The conclusion of the theorem follows easily. \( \square \)

**Theorem 2.1.3.** If
\( J[y] = \int_{\sigma(a)}^{\sigma(b)} L(t, y(\sigma(t)), y^\Delta(t)) \Delta t \),

where \( y \in C^2[a, \sigma^2(b)] \) and \( y(\sigma^2(b)) = B \), has a local extremum at \( y(t) \), then \( y(t) \) satisfies the Euler-Lagrange equation for \( t \in [\sigma(a), \sigma^2(b)] \), \( y(\sigma^2(b)) = B \) and \( y(t) \) satisfies the condition

\[
L_{y^\Delta} \left( \sigma(a), y(\sigma^2(a)), y^\Delta(\sigma(a)) \right) = 0 \tag{2.1.4}
\]

Proof: As in the proof of Theorem 2.1.1, \( J_1[h] = 0 \) for all \( h \in C^1[\sigma(a), \sigma^2(b)] \) with \( h(\sigma^2(b)) = 0 \). In solving the simplest variational problem in order to obtain the Euler-Lagrange equation, if we use \( h(\sigma^2(b)) = 0 \), the following is obtained

\[
\int_{\sigma(a)}^{\sigma(b)} \left\{ L_{y^\sigma}(t, y(\sigma(t)), y^\Delta(t)) - L_{y^\Delta}(t, y(\sigma(t)), y^\Delta(t)) \right\} h(t) \Delta t
\]

\[
-L_{y^\Delta} \left( \sigma(a), y(\sigma^2(a)), y^\Delta(\sigma(a)) \right) h(\sigma(a)) = 0
\]

for all \( h \in C^1[\sigma(a), \sigma^2(b)] \). The conclusion of the theorem follows easily. \( \square \)

**Theorem 2.1.4.** If \( y(t) \) is a local extremum of \( J[y] \) where \( y \in C^2[a, \sigma^2(b)] \), then \( y(t) \) satisfies the Euler-Lagrange equation for \( t \in [\sigma(a), \sigma^2(b)] \) and the following conditions

\[
L_{y^\Delta} \left( \sigma(a), y(\sigma^2(a)), y^\Delta(\sigma(a)) \right) = 0
\]

\[
(\sigma^2(b) - \sigma(b))L_{y^\sigma} \left( \sigma(b), y(\sigma^2(b)), y^\Delta(\sigma(b)) \right) + L_{y^\Delta} \left( \sigma(b), y(\sigma^2(b)), y^\Delta(\sigma(b)) \right) = 0
\]

which are (2.1.3) and (2.1.4) in Theorems 2.1.2 and 2.1.3.
2.2 Calculus of Variations with the $\nabla$ Operator

The following lemma is crucial to the proof of the main theorems that build the theory of Calculus of Variations with the nabla operator.

Lemma 2.2.1. Let $f$ be a continuous function on $[a, b]$, if

$$\int_a^b f(t)g(t)\nabla t = 0$$

for every function $g(t) \in C_0[a, b]$ with $g(a) = g(b) = 0$, then $f(t) = 0$ for $t \in [a, b]$.\(^\ast^\ast\)

Proof. Assume to the contrary, there exists $t_0 \in [a, b]$. s.t. $f(t_0) > 0$.

Case 1.) $t_0$ is left scattered, choose $g(t_0) = 1$ and $g(t) = 0$ otherwise, then

$$\int_{t_0}^{t_0} f(t)g(t)\nabla t = (t_0 - \rho(t_0))f(t_0)g(t_0) = (t_0 - \rho(t_0))f(t_0) > 0.$$

Case 2.) $t_0$ is left dense, then by continuity of $f$ there exists $\delta > 0$ s.t. $f(t) > 0$ on $(t_0 - \delta, t_0]$. Then define $g(t) = (t - (t_0 - \delta))(t_0 - t)$ for $t \in (t_0 - \delta, t_0)$ and $g(t) = 0$ otherwise. Then we have

$$\int_{t_0-\delta}^{t_0} f(t)g(t)\nabla t = \int_{t_0-\delta}^{t_0} (t - (t_0 - \delta))(t_0 - t)f(t)\nabla t > 0.$$

Case 3.) If $t_0 = a$ and $t_0 \in [a, b]$, then $t_0$ must be a right dense point. Again by continuity of $f$, there exists $\delta > 0$ s.t. $f(t) > 0$ on $[a, a + \delta]$. Define $g(t) = (t - a)(a + \delta - t)$ for $t \in (a, a + \delta)$ and $g(t) = 0$ otherwise. Then

$$\int_a^{a+\delta} f(t)g(t)\nabla t = \int_a^{a+\delta} ((a + \delta) - t)(t - a)f(t)\nabla t > 0.$$
Case 4.) If $t_0 = b$ and $t_0 \in [a, b]_c^c$ then $t_0$ must be a left dense point. Then by continuity of $f$, there exists $\delta > 0$ s.t. $f(t) > 0$ on $(b - \delta, b]$. Define $g(t) = (t - (b - \delta))(b - t)$ for $t \in (b - \delta, b)$. Then

$$\int_{b-\delta}^{b} f(t)g(t)\,dt = \int_{b-\delta}^{b} (t - (b - \delta))(b - t) f(t)\,dt > 0$$

which is a contradiction. So we have $f(t) \equiv 0$ on $[a, b]_c^c$. 

The following Theorems were obtained by Atici, Biles, and Lebedinsky in the paper [2].

Theorem 2.2.1. (Euler’s Necessary Condition) If a function $y(t)$ provides a local extremum to the functional

$$J[y] = \int_{\rho^2(a)}^{\rho(b)} L(t, y(\rho(t)), y(\rho(t)))\,dt$$

where $y \in C^2[\rho^2(a), \rho(b)]$ and $y(\rho^2(a)) = A, y(\rho(b)) = B$, then $y$ must satisfy the Euler-Lagrange equation

$$L_{y^r}(t, y, y^r) - L_y(t, y, y^r) = 0$$

for $t \in [\rho(a), b]_c^c$.

Proof. Assume that $L(t, u, v)$ is a class $C^2$ function of $(u, v)$ for each $t \in [\rho^2(a), \rho(b)] \subseteq T$.

Let $y \in C^1[\rho^2(a), \rho(b)]$ with $y(\rho^2(a)) = A, y(\rho(b)) = B$, where

$$C^1[\rho^2(a), \rho(b)] = \{y : [\rho^2(a), \rho(b)] \to \mathbb{R} \mid y^r \text{ is continuous on } [\rho^2(a), \rho(b)]_c\}.$$ 

The simplest variational problem is to extremize (maximize or minimize)
\[ J[y] := \int_{\rho^2(a)}^{\rho^2(b)} L(t, y(\rho(t)), y^n(t)) \, \nabla t. \]

If \( y_0 \in C^1[\rho^2(a), \rho(b)] \) minimizes (or maximizes) the variation problem if

\[ J[y_0] \leq J[y] \]

for all \( y \in C^1[\rho^2(a), \rho(b)] \). Then \( J \) is said to have a local minimum (or maximum) at \( y_0 \) provided that there is a \( \delta > 0 \) such that

\[ J[y_0] \leq J[y] \]

for all \( y \in C^1[\rho^2(a), \rho(b)] \) with \( \| y - y_0 \| < \delta \). Here consider the norm to be

\[ \| y \| = \max_{t \in [\rho^2(a), \rho(b)]} |y(t)| + \max_{t \in [\rho^2(a), \rho(b)]} |y^n(t)|. \]

Now let \( h : [\rho^2(a), \rho(b)] \to \mathbb{R} \) be any admissible variation, i.e., \( h \in C^1[\rho^2(a), \rho(b)] \) with \( h(\rho^2(a)) = h(\rho(b)) = 0 \). Assume that this variational problem has a local extremum at \( y \).

Then define

\[ \varphi(\epsilon) := J[y(t) + \epsilon h(t)]. \]

where \(-\infty < \epsilon < \infty \). Since \( \varphi \) has a local extremum at \( \epsilon = 0 \), we have that

\[ \varphi'(0) = 0 \]

and

\[ \varphi''(0) \geq 0 \ (\leq 0) \]

in the local minimum (maximum) case. Now consider
\[ \varphi(e) = \int_{\rho(a)}^{\rho(b)} L(t, y(\rho(t)) + \epsilon h(\rho(t)), y^\nabla(t) + \epsilon h^\nabla(t)) \nabla t. \]

Differentiating with respect to \( \epsilon \), we have

\[ \varphi'(e) = \int_{\rho(a)}^{\rho(b)} \frac{d}{d\epsilon} L(t, y(\rho(t)) + \epsilon h(\rho(t)), y^\nabla(t) + \epsilon h^\nabla(t)) \nabla t \]

\[ = \int_{\rho(a)}^{\rho(b)} \left\{ L_u(t, y(\rho(t)) + \epsilon h(\rho(t)), y^\nabla(t) + \epsilon h^\nabla(t)) h(\rho(t)) \right\} \nabla t. \]

Hence we obtain

\[ \varphi'(0) = \int_{\rho(a)}^{\rho(b)} \left\{ L_u(t, y^\rho(t), y^\nabla(t)) h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^\nabla(t)) h^\nabla(t) \right\} \nabla t. \]

The Integral

\[ \int_{\rho(a)}^{\rho(b)} \left\{ L_u(t, y^\rho(t), y^\nabla(t)) h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^\nabla(t)) h^\nabla(t) \right\} \nabla t \]

gives the first variation of \( J[y] \), denoted by \( J_1[h] \). So a necessary condition for \( y(t) \) to be a local minimum (maximum) is

\[ J_1[h] = \int_{\rho(a)}^{\rho(b)} \left\{ L_u(t, y^\rho(t), y^\nabla(t)) h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^\nabla(t)) h^\nabla(t) \right\} \nabla t = 0 \]

for all \( h \in C^1[\rho(a), \rho(b)] \) with \( h(\rho(a)) = h(\rho(b)) = 0 \). Then using the properties of \( \nabla \)-integration, we have

\[ J_1[h] = \int_{\rho(a)}^{\rho(b)} \left\{ L_u(t, y^\rho(t), y^\nabla(t)) h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^\nabla(t)) h^\nabla(t) \right\} \nabla t \]

\[ + \int_{\rho(a)}^{\rho(b)} \left\{ L_u(t, y^\rho(t), y^\nabla(t)) h^\rho(t) + L_{y^\nabla}(t, y^\rho(t), y^\nabla(t)) h^\nabla(t) \right\} \nabla t \]
\[
= \int_{\rho(a)}^{\rho(b)} \{L_{y\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y\nabla}(t, y^\rho, y^\nabla)h^\nabla\} \nabla t + (\rho(a) - \rho^2(a))\{L_{y\rho}(\rho(a), y(\rho^2(a)), y(\rho(a)))h(\rho^2(a))

+ L_{y\nabla}(\rho(a), y(\rho^2(a)), y(\rho(b)))h(\rho(a))\}

= \int_{\rho(a)}^{\rho(b)} \{L_{y\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y\nabla}(t, y^\rho, y^\nabla)h^\nabla\} \nabla t + (\rho(a) - \rho^2(a))L_{y\nabla}(\rho(a), y(\rho^2(a)), y(\rho(a)))h(\rho(a))

\]

and, using the equality \((\rho(a) - \rho^2(a))h(\rho(a)) = h(\rho(a)) - h(\rho^2(a))\), one can obtain

\[
J_1[h] = \int_{\rho(a)}^{\rho(b)} \{L_{y\rho}(t, y^\rho, y^\nabla)h^\rho + L_{y\nabla}(t, y^\rho, y^\nabla)h^\nabla\} \nabla t + L_{y\nabla}(\rho(a), y(\rho^2(a)), y(\rho(a)))h(\rho(a)).

\]

Then integration by parts gives

\[
= \int_{\rho(a)}^{\rho(b)} L_{y\rho}(t, y^\rho, y^\nabla)h^\rho \nabla t + (L_{y\nabla}h)(\rho(b)) - (L_{y\nabla}h)(\rho(a))

- \int_{\rho(a)}^{\rho(b)} L_{y\nabla}(t, y^\rho, y^\nabla)h^\rho \nabla t + (L_{y\rho}h)(\rho(a))

= \int_{\rho(a)}^{\rho(b)} \{L_{y\rho}(t, y^\rho, y^\nabla)h^\rho - L_{y\nabla}(t, y^\rho, y^\nabla)h^\nabla\} \nabla t.

\]

Again, using the properties of \(\nabla\)-integration

\[
\int_{\rho(b)}^{b} \{L_{y\rho}(t, y^\rho, y^\nabla) - L_{y\nabla}(t, y^\rho, y^\nabla)\} h^\rho \nabla t = (\rho(b) - b)(\{L_{y\rho} - L_{y\nabla}\} h)(\rho(b)) = 0
\]

this implies that

\[
\int_{\rho(a)}^{\rho(b)} \{L_{y\rho}(t, y^\rho, y^\nabla) - L_{y\nabla}(t, y^\rho, y^\nabla)\} h^\rho \nabla t = 0
\]
for all $h \in C^2[\rho^2(a), \rho(b)]$ with $h(\rho^2(a)) = h(\rho(b)) = 0$. The necessary condition for $J[y]$ to have an extremum for $y = y(t)$ is that

$$\varphi'(0) \int_{\rho(a)}^{\rho(b)} \{L_{y'}(t, y^p, y^n) - L_{y'}(t, y^p, y^n)\} h^p \nabla t = 0 \quad (2.2.1)$$

for all admissible $h$. According to Lemma 2.2.1, (2.2.1) implies that

$$L_{y'}(t, y^p, y^n) - L_{y'}(t, y^p, y^n) = 0 \quad (2.2.2)$$

a result known as an Euler-Lagrange equation. □

In the development of Theorem 2.2.1, if the initial setup is relaxed such that one or even both of the boundary conditions are not given then the following Theorems can be obtained.

**Theorem 2.2.2.** If

$$J[y] = \int_{\rho(a)}^{\rho(b)} L(t, y(\rho(t)), y^n(t)) \nabla t,$$

where $y \in C^2[\rho^2(a), \rho(b)]$ and $y(\rho^2(a)) = A$, has a local extremum at $y(t)$, then $y(t)$ satisfies the Euler-Lagrange equation for $t \in [\rho^2(a), b_1, y(\rho^2(a)) = A$, and $y(t)$ satisfies the condition

$$L_{y'}(\rho(b), y(\rho^2(b)), y^n(\rho(b))) = 0. \quad (2.2.3)$$

**Proof.** As in the proof of Theorem 2.2.1 $J_1[h] = 0$ for all $h \in C^1[\rho^2(a), \rho(b)]$ with $h(\rho^2(a)) = 0$. In solving the simplest variational problem in order to obtain the Euler-Lagrange equation, if we use $h(\rho^2(a)) = 0$, the following is obtained

$$\int_{\rho(a)}^{\rho(b)} \{L_{y'}(t, y(\rho(t)), y^n(t)) - L_{y'}(t, y(\rho(t)), y^n(t))\} h(\rho(t)) \nabla t$$
+L_y^\varphi (\rho(b), y(\rho^2(b)), y^\varphi(\rho(b))) h(\rho^2(b)) = 0

for all $h \in C^1[\rho^2(a), \rho(b)]$. The conclusion of the theorem follows easily. \qed

**Theorem 2.2.3.** If

$$J[y] = \int_{\rho^2(a)}^{\rho(b)} L(t, y(\rho(t)), y^\varphi(\rho(t))) \, dt,$$

where $y \in C^2[\rho^2(a), \rho(b)]$ and $y(\rho(b)) = B$, has a local extremum at $y(t)$, then $y(t)$ satisfies the Euler-Lagrange equation for $t \in [\rho^2(a), b]$, $y(\rho(b)) = B$, and $y(t)$ satisfies the condition

$$(\rho(a) - \rho^2(a)) L_y^\varphi (\rho(a), y(\rho^2(a)), y^\varphi(\rho(a))) + L_y^\varphi (\rho(a), y(\rho^2(a)), y^\varphi(\rho(a))) = 0.$$

(2.2.4)

**Proof.** As in the proof of Theorem 2.2.1 $J[y] = 0$ for all $h \in C^1[\rho^2(a), \rho(b)]$ with $h(\rho(b)) = 0$. In solving the simplest variational problem in order to obtain the Euler-Lagrange equation, if we use $h(\rho(b)) = 0$, the following is obtained

$$\int_{\rho^2(a)}^{\rho(b)} \left\{ L_y^\varphi (t, y(\rho(t)), y^\varphi(t)) - L_y^\varphi (t, y(\rho(t)), y^\varphi(t)) \right\} h(\rho(t)) \, dt$$

$$+ \{(\rho(a) - \rho^2(a)) L_y^\varphi (\rho(a), y(\rho^2(a)), y^\varphi(\rho(a))) +$$

$$L_y^\varphi (\rho(a), y(\rho^2(a)), y^\varphi(\rho(a))) \right\} h(\rho^2(b)) = 0$$

for all $h \in C^1[\rho^2(a), \rho(b)]$. The conclusion of the theorem follows easily. \qed
**Theorem 2.2.4.** If \( y(t) \) is a local extremum of \( J[y] \) where \( y \in C^2[\rho^2(a), \rho(b)] \), then \( y(t) \) satisfies the Euler-Lagrange equation for \( t \in [\rho^2(a), b] \) and the following conditions

\[
L_y \nabla \rho(b), y(\rho^2(b)), y^\nabla (\rho(b)) = 0
\]

\[
(\rho(a) - \rho^2(a)) L_y \rho(a), y(\rho^2(a)), y^\nabla (\rho(a)) + L_y \rho(a), y(\rho^2(a)), y^\nabla (\rho(a)) = 0
\]

which are (2.2.3) and (2.2.4) in Theorems 2.2.2 and 2.2.3.

### 2.3 Future Developments

A few notes on work that needs to be accomplished in order to help refine the Calculus of Variations on time scales. First try to redefine the \( \Delta \) and \( \nabla \) versions of the theory in order to develop other forms that the functional may take so that the theory can be applied more broadly to other models that cannot be written in the form of the functional that currently exists. The next step would be to find an application of the Double Integral Calculus of Variations, which is written with the \( \Delta \) operator, found in [9], and also develop the free boundary conditions for this theory. Then generate the before mentioned theory for the \( \nabla \) operator.
Chapter 3

Ramsey Model

In this chapter, we will study the Ramsey model attained from [6] which determines the behavior of saving/consumption as the result of optimal intertemporal choices by individual households. Before writing the model on time scales we will present its discrete and continuous versions so that one can see how the time scale model unifies its discrete and continuous counterparts.

**Discrete Model:** We want to maximize the Ramsey model which is,

$$\max_{t=0}^{T-1} (1 + p)^{-t} U[C_t],$$

subject to initial wealth $W_0$ that can always be invested for an exogeneously-given certain rate of yield $r$; or subject to the constraint,

$$C_t = W_t - \frac{W_{t+1}}{1 + r},$$

or

$$\max_{W_t} \sum_{t=0}^{T-1} (1 + p)^{-t} U \left[ W_t - \frac{W_{t+1}}{1 + r} \right].$$
where the quantities are defined as

\[ C_t \] consumption
\[ p \] discount rate
\[ U_t \] instantaneous utility function
\[ W_t \] production function.

The Euler-Lagrange equation for the discrete model is as follows,

\[ \frac{r - p}{1 + r} U'[C(t)] + \Delta[U'[C(t)]] = 0. \]

**Continuous Model:** We want to maximize the Ramsey model which is,

\[ \text{Max } \int_0^T e^{-pt} U[C(t)] dt, \]

subject to,

\[ C(t) = rW(t) - W'(t) \]

or,

\[ \text{Max}_{[W(t)]} \int_0^T e^{-pt} U[rW(t) - W'(t)] dt. \]

The Euler-Lagrange equation then becomes,

\[ (r - p)U'[C(t)] + [U'[C(t)]]' = 0. \]

We will now develop two formulations of the time scale Ramsey model, in order to employ both the nabla and delta calculus of variations.
3.1 The Ramsey Model with the $\nabla$ Derivative

Consider the constraint for the continuous case,

$$C(t) = rW(t) - W'(t)$$

which can be written as follows by factoring out $-e^{rt}$,

$$C(t) = -e^{rt}[-re^{-rt}W(t) + e^{-rt}W'(t)].$$

The previous equation can be obtained by taking the indicated derivative with respect to $t$ in the following:

$$C(t) = -e^{rt}[e^{-rt}W(t)]'.$$ \hspace{1cm} (3.1.1)

Now consider the discrete constraint,

$$C_{t-1} = W_{t-1} - \frac{W_i}{1 + r},$$

which can be rewritten as follows by factoring out $-(1 + r)^{t-1}$,

$$C_{t-1} = -(1 + r)^{t-1} \left[ \frac{W_i}{(1 + r)^t} - \frac{W_{t-1}}{(1 + r)^{t-1}} \right].$$

The previous equation can be obtained by taking the indicated difference with the respect to $t$ in the following,

$$C_{t-1} = -(1 + r)^{t-1} \nabla \left[ \frac{W_i}{(1 + r)^t} \right].$$ \hspace{1cm} (3.1.2)

Using the new formulations (3.1.1) and (3.1.2) of the continuous and discrete constraint, a generalization can be made in order to develop the time scale constraint with the nabla operator, which is as follows,
\[ C(\rho(t)) = -[\hat{e}_{-r}(\rho(t), 0)]^{-1} [\hat{e}_{-r}(t, 0)W(t)]^\nabla. \]

Then by taking the nabla derivative of \([\hat{e}_{-r}(t, 0)W(t)]\) the following is obtained,

\[
C(\rho(t)) = -[\hat{e}_{-r}(\rho(t), 0)]^{-1} [\hat{e}_{-r}(t, 0)W(\rho(t)) + \hat{e}_{-r}(t, 0)W^\nabla(t)]
\]

\[
= - [(1 + \nu(t)r)\hat{e}_{-r}(t, 0)]^{-1} [-r\hat{e}_{-r}(t, 0)W(\rho(t)) + \hat{e}_{-r}(t, 0)W^\nabla(t)].
\]

Then by distributing through by \(- [(1 + \nu(t)r)\hat{e}_{-r}(t, 0)]^{-1}\) the constraint for the nabla version of the Ramsey model is obtained,

\[
C(\rho(t)) = \frac{rW(\rho(t))}{1 + \nu(t)r} - \frac{W^\nabla(t)}{1 + \nu(t)r}.
\]

**The Ramsey Model with the Nabla Derivative:** The Ramsey model with the nabla derivative is

\[
\text{Max}_{[W(t)]} \int_{\rho^2(0)}^{\rho^2(t)} \hat{e}_{-p}(\rho(t), 0)U [\frac{rW(\rho(t))}{1 + \nu(t)r} - \frac{W^\nabla(t)}{1 + \nu(t)r}] \nabla t.
\]

Note that this model includes the discrete and continuous models as special cases. First let's derive the Euler-Lagrange equation using Theorem 2.2.1 in Chapter 2. In this model,

\[
L(t, W^p, W^\nabla) = \hat{e}_{-p}(\rho(t), 0)U \frac{rW(\rho(t))}{1 + \nu(t)r} - \frac{W^\nabla(t)}{1 + \nu(t)r}
\]

so we obtain the following Euler-Lagrange equation

\[
\hat{e}_{-p}(\rho(t), 0)U' \left[ \frac{rW(\rho(t))}{1 + \nu(t)r} - \frac{W^\nabla(t)}{1 + \nu(t)r} \right] \left( \frac{r}{1 + \nu(t)r} \right) + \left[ \hat{e}_{-p}(\rho(t), 0)U' \left[ \frac{rW(\rho(t))}{1 + \nu(t)r} - \frac{W^\nabla(t)}{1 + \nu(t)r} \right] \left( \frac{1}{1 + \nu(t)r} \right) \right]^\nabla = 0.
\]
Then by substituting \( C(\rho(t)) \) in for \( \frac{rW(\rho(t))}{1 + \nu(t)r} - \frac{W^\nabla(\rho(t))}{1 + \nu(t)r} \) the following is obtained,

\[
\dot{\epsilon}_p(\rho(t), 0)U'(C(\rho(t))) \left( \frac{r}{1 + \nu(t)r} \right) + \left[ \dot{\epsilon}_p(\rho(t), 0)U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right) \right]^{\nabla} = 0.
\]

Then using the product rule the previous equation becomes

\[
\dot{\epsilon}_p(\rho(t), 0)U'(C(\rho(t))) \left( \frac{r}{1 + \nu(t)r} \right) + \dot{\epsilon}_p(\rho(t), 0) \left[ U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right) \right]^{\nabla}
\]

\[
+ \dot{\epsilon}_p(\rho(t), 0)U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right) = 0.
\]

Then by taking the nabla derivative of the nabla exponential and reducing \( \dot{\epsilon}_p(\rho(t), 0) \) the above equation becomes

\[
\dot{\epsilon}_p(\rho(t), 0)U'(C(\rho(t))) \left( \frac{r}{1 + \nu(t)r} \right) - p\dot{\epsilon}_p(\rho(t), 0)U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right)
\]

\[
+ (1 + \nu(t)r)\dot{\epsilon}_p(\rho(t), 0) \left[ U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right) \right]^{\nabla} = 0.
\]

Then dividing through by \( \dot{\epsilon}_p(\rho(t), 0) \) yields

\[
U'(C(\rho(t))) \left( \frac{r}{1 + \nu(t)r} \right) + (1 + \nu(t)r)p \left[ U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right) \right]^{\nabla}
\]

\[
+ U'(C(\rho(t))) \left( \frac{-p}{1 + \nu(t)r} \right) = 0,
\]

which is the same as

\[
\left[ U'(C(\rho(t))) \left( \frac{1}{1 + \nu(t)r} \right) \right]^{\nabla} = \frac{p - r}{(1 + \nu(t)r)(1 + \nu(t)r)p} U'(C(\rho(t))).
\]

Then assume that \( \alpha(t) = \frac{1}{1 + \nu(t)r} \) is nabla differentiable (note that \( \nu(t) \) is not nabla differentiable in general), then again using the product rule the following is obtained
\[ \alpha(p(t)) [U'(C(p(t)))] + \alpha^\ast(t) \cdot [U'(C(p(t)))] = \frac{p - r}{(1 + \nu(t)r)(1 + \nu(p(t))p)} U'(C(p(t))). \]

which is the same as

\[ \left[ \frac{U'(C(p(t)))}{U'(C(p(t)))} \right]^{\alpha} = \left( \frac{p - r - \alpha^\ast(t)(1 + \nu(t)r)(1 + \nu(p(t))p)}{(1 + \nu(t)r)(1 + \nu(p(t))p)\alpha(p(t))} \right) U'(C(p(t))). \]

Then by substituting \( \frac{1}{1 + \nu(p(t))r} \) in for \( \alpha(p(t)) \) and rearranging the following is obtained

\[ \left[ \frac{U'(C(p(t)))}{U'(C(p(t)))} \right]^{\alpha} = \left( \frac{p - r - \alpha^\ast(t)(1 + \nu(t)r)(1 + \nu(p(t))p)}{(1 + \nu(t)r)(1 + \nu(p(t))p)} \right). \]

for \( t \in [\rho^2(0), \rho^2(T)] \).

### 3.2 The Ramsey Model with the \( \Delta \) Derivative

Consider the constraint for the continuous case,

\[ C(t) = rW(t) - W'(t) \]

which can be written as follows by factoring out \(-e^{rt}\),

\[ C(t) = -e^{rt}[-re^{-rt}W(t) + e^{-rt}W'(t)]. \]

The previous can be obtained by taking the indicated derivative with respect to \( t \) in the following,

\[ C(t) = -e^{rt}[e^{-rt}W(t)]'. \]  \hspace{1cm} (3.2.1)

Now consider the discrete constraint,
\[ C_t = W_t - \frac{W_{t+1}}{1+r}, \]

which can be rewritten as follows by factoring out \(-(1+r)^{t-1}\),

\[ C_t = -(1 + r)^{-1} \left( \frac{W_{t+1}}{(1+r)^t} - \frac{W_t}{(1+r)^{t-1}} \right). \]

The previous can be obtained by taking the indicated difference with the respect to \(t\) in the following,

\[ C_t = -(1 + r)^{t-1} \Delta \left[ \frac{W_t}{(1+r)^{t-1}} \right]. \quad (3.2.2) \]

Using the two new formulations (3.2.1) and (3.2.2), of the continuous and discrete constraint, a generalization can be made in order to develop the time scale constraint with the delta operator, which is as follows,

\[ C(t) = -[\hat{e}_{-r}(\rho(t), 0)]^{-1}[\hat{e}_{-r}(\rho(t), 0)w(t)]^\Delta. \]

Then by taking the delta derivative of \([\hat{e}_{-r}(\rho(t), 0)w(t)]^\Delta\) the following is obtained,

\[ C(t) = -[\hat{e}_{-r}(\rho(t), 0)]^{-1}\left[ \frac{-r}{1 + \mu(\rho(t))r} \hat{e}_{-r}(\rho(t), 0)W(\sigma(t)) + \hat{e}_{-r}(\rho(t), 0)W^\Delta(t) \right]. \]

Then by distributing through by \(-[\hat{e}_{-r}(\rho(t), 0)]^{-1}\) the constraint for the delta version of the Ramsey model is obtained,

\[ C(t) = \left[ \frac{-r}{1 + \mu(\rho(t))r} W(\sigma(t)) - W^\Delta(t) \right]. \]

**The Ramsey Model with the Delta Derivative:** The Ramsey model with the delta derivative is,
Note that this model includes the discrete and continuous model as special cases. First we derive the Euler-Lagrange equation using Theorem 2.1.1 in Chapter 2. In this model,

$$L(t, W^\sigma, W^\Delta) = \dot{e}_{-p}(t, 0) U \left[ \frac{r}{1 + \mu(\rho(t))r} W(\sigma(t)) - W^\Delta(t) \right]$$

so we obtain the following Euler-Lagrange equation

$$\dot{e}_{-p}(t, 0) U' \left[ \frac{r}{1 + \mu(\rho(t))r} W^\sigma - W^\Delta \right] \left( \frac{1}{1 + \mu(\rho(t))r} \right) + \left[ \dot{e}_{-p}(t, 0) U' \left( \frac{r}{1 + \mu(\rho(t))r} W^\sigma - W^\Delta \right) \right] = 0.$$  

Then by substituting $C(t)$ in for $\frac{r}{1 + \mu(\rho(t))r} W(\sigma(t)) - W^\Delta(t)$ the following is obtained

$$\dot{e}_{-p}(t, 0) U' (C(t)) \left( \frac{r}{1 + \mu(\rho(t))r} \right) + \left[ \dot{e}_{-p}(t, 0) U' (C(t)) \right]^\Delta = 0.$$  

Then using the product rule the previous equation becomes

$$\dot{e}_{-p}(t, 0) U' (C(t)) \left( \frac{r}{1 + \mu(\rho(t))r} \right) + \dot{e}_{-p}(t, 0) U' (C(t)) + \dot{e}_{-p}(\sigma(t), 0) [U' (C(t))] = 0.$$  

Then by taking the delta derivative of the nabla exponential and reducing $\dot{e}_{-p}(\sigma(t), 0)$ the above equation becomes

$$\dot{e}_{-p}(t, 0) U' (C(t)) \left( \frac{r}{1 + \mu(\rho(t))r} \right) + \dot{e}_{-p}(t, 0) U' (C(t)) - \frac{p}{1 + \mu(t)p} \dot{e}_{-p}(t, 0) U' (C(t))$$

$$\frac{1}{1 + \nu(\sigma(t))p} \dot{e}_{-p}(t, 0) [U' (C(t))]^\Delta = 0.$$  

Then by dividing through by $\dot{e}_{-p}(t, 0)$ yields
\[ U''(C(t)) \left( \frac{r}{1 + \mu(\rho(t))r} \right) - \frac{p}{1 + \mu(t)p} U'(C(t)) + \frac{1}{1 + \nu(\sigma(t))p} [U'(C(t))]^\Delta = 0, \]

which is the same as

\[ \frac{1}{1 + \nu(\sigma(t))p} [U'(C(t))]^\Delta = \frac{p(1 + \mu(\rho(t))r) - r(1 + \mu(t)p)}{(1 + \mu(\rho(t))r)(1 + \mu(t)p)} U''(C(t)). \]

Then by multiplying through by \(1 + \nu(\sigma(t))p\) and simplifying the following is obtained

\[ [U'(C(t))]^\Delta = \frac{(p - r) + (\mu(\rho(t)) - \mu(t))pr}{(1 + \mu(t)r)(1 + \mu(t)p)} U'(C(t)), \]

which can be written as follows

\[ \frac{[U'(C(t))]^\Delta}{U''(C(t))} = \frac{(p - r) + (\mu(\rho(t)) - \mu(t))pr}{(1 + \mu(t)r)(1 + \mu(t)p)} \]

for \(t \in [0, T]^\kappa\).

### 3.3 A Comparison of the Two Solutions of the Ramsey Model

The first comparison of the two solutions obtained in Sections 3.2 and 3.3 will be made where \(T = \mathbb{R}\). The solution obtained from the Ramsey model with the nabla derivative is

\[ \frac{[U'(C(\rho(t)))]}{U''(C(\rho(t)))} = \left( \frac{(p - r - \alpha(\rho(t))(1 + \nu(t)r)(1 + \nu(\rho(t))p)}{(1 + \nu(t)r)(1 + \nu(\rho(t))p)} \right) \]

then taking the time scale to be \(T = \mathbb{R}\) the previous becomes
\[
\frac{[U'(C(t))]'}{U''(C(t))} = p - r
\]

for \( t \in [0, T] \). The solution obtained from the Ramsey model with the delta derivative is,

\[
\frac{[U'(C(t))]^\Delta}{U'(C(t))} = \frac{(p - r) + ((\mu(p(t)) - \mu(t))pr)(1 + \nu\sigma(t)p)}{(1 + \mu(t)r)(1 + \mu(t)p)}
\]

then taking the time scale to be \( T = \mathbb{R} \) the previous becomes

\[
\frac{[U'(C(t))]'}{U''(C(t))} = p - r,
\]

for \( t \in [0, T] \). So when \( T = \mathbb{R} \) the two solutions obtained using the nabla and delta calculus of variations are the same. It is also worth noting that the before mentioned solutions are also the same as the optimized solution obtained from the continuous model using the regular calculus of variations, given in section 3.1. The next comparison will be made where \( T = h\mathbb{Z} \). The solution to the Ramsey model with the nabla derivative on \( T = h\mathbb{Z} \) is as follows

\[
\nabla\frac{[U'(C(\rho(t)))]}{U''(C(\rho(t)))} = \frac{p - r}{1 + hp}.
\]

Then by taking the indicated backward difference the following is obtained

\[
\frac{U'(C(\rho(t))) - U'(C(\rho(t))))}{h} U'(C(\rho(t))) = \frac{p - r}{1 + hp}
\]

which is the same as

\[
U'(C(\rho(\rho(t)))) = \left[ \frac{-h(p - r)}{1 + hp} + 1 \right] U'(C(\rho(t)));
\]

which yields the following recursive relation
\[ U'(C(\rho(t))) = \frac{1 + hp}{1 + hr} U'(C(\rho(t))) \]

for \( t \in [-\frac{1}{h}, T - \frac{2}{h}] \). The solution to the Ramsey model with the delta derivative on \( T = h\mathbb{Z} \) is as follows

\[ \frac{\Delta[U'(C(t))]}{U'(C(t))} = \frac{p - r}{1 + hr}. \]

Then by taking the indicated difference the following is obtained

\[ \frac{U'(C(\sigma(t))) - U'(C(t))}{U'(C(t))} = \frac{p - r}{1 + hp} \]

which is the same as

\[ U'(C(\sigma(t))) = \left[ \frac{h(p - r)}{1 + hr} + 1 \right] U'(C(t)), \]

which yields the following recursive relation

\[ U'(C(\sigma(t))) = \frac{1 + hp}{1 + hr} U'(C(t)) \]

for \( t \in [\frac{1}{h}, T - \frac{1}{h}] \). It is worth noting that the recursive relations obtained from the two optimization methods are the same but are defined on two separate intervals of the time scale \( h\mathbb{Z} \).
Chapter 4

Model of Adjustment

In this chapter, we will form an adjustment model with the $\Delta$ derivative on time scales. The adjustment model, attained from [13], unlike the Ramsey model does not have a functional that matches the theory when formulated with the $\nabla$ derivative and therefore can only be optimized using the Calculus of variations with the $\Delta$ operator. The adjustment model will first be solved using the assumption that the boundary conditions are known to be the values given by the target function at the endpoints and again solved with the assumption that nothing is known about the boundary conditions.

The Discrete Model: We want to minimize the dynamic model of adjustment which is,

$$\sum_{t=1}^{T} r^t [\alpha(y(t) - \bar{y}(t))^2 + (y(t) - y(t - 1))^2],$$

where $y(t)$ is the output state variable, $r > 1$ is the exogeneous rate of discount, and $\bar{y}$ is the desired target level (which for the purposes of this paper we will consider three cases
which are $\bar{y}$ is constant, linear, and exponential) and $T$ is the horizon. The first component of the loss function above is the disequilibrium cost due to deviations from the desired target and the second component characterizes the agent’s aversion to output fluctuations.

The Euler-Lagrange equation for the discrete model is as follows,

$$ry(t + 1) - (r + \alpha + 1)y(t) + y(t - 1) + \alpha \bar{y}(t) = 0.$$ 

**The Continuous Model:** We want to minimize the dynamic model of adjustment which is,

$$\int_0^T e^{(r-1)t} [\alpha(y(t) - \bar{y}(t))^2 + (y'(t))^2]dt. $$

The Euler-Lagrange equation becomes

$$y''(t) + (r - 1)y'(t) - \alpha y(t) + \alpha \bar{y}(t) = 0.$$

### 4.1 Model of Adjustment with the $\Delta$ Operator

**The Time Scales Model:** The time scales model which we wish to minimize is

$$\int_0^{\rho(T)} e_{(r-1)}(\sigma(t), 0)[\alpha(y(\sigma(t)) - \bar{y}(\sigma(t)))^2 + (y^\Delta(t))^2] \Delta t.$$ 

Note that this model includes the discrete and continuous models as special cases. First we derive the Euler-Lagrange equation using Theorem 2.1.1 in Chapter 2. In this model,

$$L(t, y(\sigma(t)), y^\Delta(t)) = e_{(r-1)}(\sigma(t), 0)[\alpha(y(\sigma(t)) - \bar{y}(\sigma(t)))^2 + y^\Delta(t))^2]$$
so we obtain the following Euler-Lagrange equation

\[ e_{(r-1)}(\sigma(t), 0)[2\alpha(y(\sigma(t)) - \overline{y}(\sigma(t))] - 2[e_{(r-1)}(\sigma(t), 0)]^\Delta y^\Delta(\sigma(t)) \]

\[-2e_{(r-1)}(\sigma(t), 0)y^\Delta(\sigma(t)) = 0.\]

Then using the property \( e_{(r-1)}(\sigma(t)) = (r - 1)e_{(r-1)}(\sigma(t)) \) for the exponential function \( e_{(r-1)} \) we have

\[ e_{(r-1)}(\sigma(t), 0)[2\alpha(y(\sigma(t)) - \overline{y}(\sigma(t))] - 2(r - 1)e_{(r-1)}(\sigma(t), 0)y^\Delta(\sigma(t)) \]

\[-2e_{(r-1)}(\sigma(t), 0)y^\Delta(\sigma(t)) = 0.\]

Then by dividing through the previous equation by \(-2e_{(r-1)}(\sigma(t), 0)\) we obtain

\[ y^\Delta(\sigma(t)) + (r - 1)y^\Delta(\sigma(t)) - \alpha y(\sigma(t)) + \alpha \overline{y}(\sigma(t)) = 0, \]

which has no closed solution in the literature. For further examination of this model consider the time scale where \( \mu(t) = h \) is a constant. Now rearranging the Euler-Lagrange equation using the property \( f(\sigma(t)) = (\sigma(t) - t)f^\Delta(t) + f(t) \) the following is obtained

\[ y^\Delta(t) + (r - 1)\left[ y^\Delta(t) + \mu(t)y^\Delta\right] - \alpha \left[ y(t) + \mu(t)y^\Delta(t)\right] + \alpha \left[ \overline{y}(t) + \mu(t)\overline{y}^\Delta(t)\right] = 0 \]

which simplifies to,

\[ [1 + \mu(t)(r - 1)]y^\Delta(t) + [r - 1 - \alpha \mu(t)]y^\Delta(t) - \alpha y(t) + \alpha \left[ \overline{y}(t) + \mu(t)\overline{y}^\Delta(t)\right] = 0. \]

Then, by substituting \( \mu(t) = h \) and rearranging, the following is obtained
\[ [1 + h(r - 1)] y^\Delta(t) + [r - 1 - \alpha h] y^\Delta(t) - \alpha y(t) + \alpha \left[ \bar{y}(t) + h\bar{y}^\Delta(t) \right] = 0 \]

which then by dividing through the previous equation by \( 1 + h(r - 1) \) the equation becomes

\[ y^\Delta(t) + \left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right] y^\Delta(t) - \frac{\alpha}{1 + hr - h} y(t) + \frac{\alpha}{1 + hr - h} \left[ \bar{y}(t) + h\bar{y}^\Delta(t) \right] = 0. \]

The previous is a non-homogeneous equation, so we will first need to solve the homogeneous equation,

\[ y^\Delta(t) + \left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right] y^\Delta(t) - \frac{\alpha}{1 + hr - h} y(t) = 0 \]

which has the following characteristic equation,

\[ \lambda^2 + \left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right] \lambda - \frac{\alpha}{1 + hr - h} = 0. \]

The solutions to the characteristic equation are

\[ \lambda_1 = -\left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right] + \sqrt{\left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right]^2 + 4 \frac{\alpha}{1 + hr - h}} \]

and

\[ \lambda_2 = -\left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right] - \sqrt{\left[ \frac{r - 1 - \alpha h}{1 + hr - h} \right]^2 + 4 \frac{\alpha}{1 + hr - h}}. \]

So the general solution to the homogeneous equation is

\[ y(t) = C_1 e_{\lambda_1}(t, 0) + C_2 e_{\lambda_2}(t, 0), \]

were \( e_{\alpha}(t, t_0) = (1 + \alpha h)^{(t-t_0)/h} \) for \( \mathbb{T} = h\mathbb{Z} \). Then variation of parameters is used to obtain the particular solution to the non-homogeneous equation, which will be of the form
\[ y(t) = C_1(t)e_{\lambda_1}(t,0) + C_2(t)e_{\lambda_2}(t,0), \]

where \( C_1(t) \) and \( C_2(t) \) are functions that satisfy the following system of equations,

\[
\begin{bmatrix}
  e_{\lambda_1}(\sigma(t),0) & e_{\lambda_2}(\sigma(t),0) \\
  \lambda_1e_{\lambda_1}(\sigma(t),0) & \lambda_2e_{\lambda_2}(\sigma(t),0)
\end{bmatrix}
\begin{bmatrix}
  C_1^\Delta(t) \\
  C_2^\Delta(t)
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  -\frac{\sigma}{1+hr-h}[\bar{y}(t) + h\bar{y}\Delta(t)]
\end{bmatrix}.
\]

Solving for \( \begin{bmatrix} C_1^\Delta(t) \\ C_2^\Delta(t) \end{bmatrix} \) the following is obtained,

\[
\begin{bmatrix}
  C_1^\Delta(t) \\
  C_2^\Delta(t)
\end{bmatrix} = \frac{1}{W^{\sigma}}
\begin{bmatrix}
  \lambda_2e_{\lambda_2}(\sigma(t),0) & -e_{\lambda_2}(\sigma(t),0) \\
  -\lambda_1e_{\lambda_1}(\sigma(t),0) & e_{\lambda_1}(\sigma(t),0)
\end{bmatrix}
\begin{bmatrix}
  0 \\
  -\frac{\sigma}{1+hr-h}[\bar{y}(t) + h\bar{y}\Delta(t)]
\end{bmatrix},
\]

were \( W^{\sigma} = W^{\sigma}(e_{\lambda_1}(t,0),e_{\lambda_2}(t,0))(t) \). Which implies that,

\[
C_1^\Delta(t) = \frac{e_{\lambda_2}(\sigma(t),0)}{W^{\sigma}(e_{\lambda_1}(t,0),e_{\lambda_2}(t,0))(t)} \frac{\sigma}{1+hr-h}[\bar{y}(t) + h\bar{y}\Delta(t)]
\]

\[
C_2^\Delta(t) = \frac{-e_{\lambda_1}(\sigma(t),0)}{W^{\sigma}(e_{\lambda_1}(t,0),e_{\lambda_2}(t,0))(t)} \frac{\sigma}{1+hr-h}[\bar{y}(t) + h\bar{y}\Delta(t)].
\]

Then, since \( e_{\lambda_1}(t,0) \) and \( e_{\lambda_2}(t,0) \) are both differentiable, the Wronskian is given by

\[
W^{\sigma}(e_{\lambda_1}(t,0),e_{\lambda_2}(t,0))(t) = (\lambda_2 - \lambda_1)e_{\lambda_1\oplus\lambda_2}(\sigma(t),0).
\]

Then by substituting the previous expression into \( C_1^\Delta(t) \) and \( C_2^\Delta(t) \) we obtain,

\[
C_1^\Delta(t) = \frac{e_{\lambda_2}(\sigma(t),0)}{(\lambda_2 - \lambda_1)e_{\lambda_1\oplus\lambda_2}(\sigma(t),0)} \frac{\sigma}{1+hr-h}[\bar{y}(t) + h\bar{y}\Delta(t)]
\]

\[
C_2^\Delta(t) = \frac{-e_{\lambda_1}(\sigma(t),0)}{(\lambda_2 - \lambda_1)e_{\lambda_1\oplus\lambda_2}(\sigma(t),0)} \frac{\sigma}{1+hr-h}[\bar{y}(t) + h\bar{y}\Delta(t)].
\]

which simplifies to,
\[ C^\Delta_1(t) = \frac{\alpha}{1 + h\tau - h} \left[ \bar{y}(t) + h\bar{y}^\Delta(t) \right] \frac{1 + h\tau - h}{(\lambda_2 - \lambda_1)e_{\lambda_2}(\sigma(t), 0)} \]
\[ C^\Delta_2(t) = \frac{\alpha}{1 + h\tau - h} \left[ \bar{y}(t) + h\bar{y}^\Delta(t) \right] \frac{1 + h\tau - h}{(\lambda_2 - \lambda_1)e_{\lambda_2}(\sigma(t), 0)} \]

Now consider the three target functions,
\[ \bar{y}(t) = \text{constant} = \beta \]
\[ \bar{y}(t) = \text{linear} = vt + \beta \]
\[ \bar{y}(t) = \text{exponential} = e_\beta(t, 0) \]

### 4.2 Constant Target Function

Let \( \bar{y}(t) = \beta \), then \( C^\Delta_1(t) \) and \( C^\Delta_2(t) \) simplify to

\[ C^\Delta_1(t) = \frac{\alpha}{1 + h\tau - h} \beta \frac{1 + h\tau - h}{(\lambda_2 - \lambda_1)e_{\lambda_2}(\sigma(t), 0)} \]
\[ C^\Delta_2(t) = \frac{-\alpha}{1 + h\tau - h} \beta \frac{1 + h\tau - h}{(\lambda_2 - \lambda_1)e_{\lambda_2}(\sigma(t), 0)} \]

Then by integrating both sides of \( C^\Delta_1(t) \) and \( C^\Delta_2(t) \) the following is obtained

\[ C_1(t) = \frac{-\frac{\alpha}{1 + h\tau - h}\beta}{\lambda_1(\lambda_2 - \lambda_1)e_{\lambda_1}(t, 0)} \]
\[ C_2(t) = \frac{\frac{\alpha}{1 + h\tau - h}\beta}{\lambda_2(\lambda_2 - \lambda_1)e_{\lambda_2}(t, 0)} \]

Then the particular solution is

\[ y(t) = C_1 e_{\lambda_1}(t, 0) + C_2 e_{\lambda_2}(t, 0) + C_1(t)e_{\lambda_1}(t, 0) + C_2(t)e_{\lambda_2}(t, 0) \]
\[ y(t) = C_1 e_{\lambda_1}(t, 0) + C_2 e_{\lambda_2}(t, 0) + \frac{-\frac{\alpha}{1 + h\tau - h}\beta}{\lambda_1(\lambda_2 - \lambda_1)} + \frac{\frac{\alpha}{1 + h\tau - h}\beta}{\lambda_2(\lambda_2 - \lambda_1)} \]
\[ y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{-\alpha \beta}{\lambda_1 \lambda_2 (1 + h \tau - h)}. \]

Assuming that the boundary conditions are given by the values of the target function at the endpoints we have that \( y(0) = \beta \) and \( y(\rho(T)) = \beta \). In figure 4.1 a series of graphs are shown to illustrate the optimized particular solution. The first three graphs are found taking the time scales to be such that \( \mu(t) = 1, 0.5, \) and \( 0.001 \) for all \( t \in \mathbb{T} \). The fourth graph illustrates the optimal solution for a mixed time scale where \( \mathbb{T} = \{[0, 6) \cap h_1 \mathbb{Z}\} \cup \{[6, 14) \cap h_2 \mathbb{Z}\} \cup \{[14, 30] \cap h_3 \mathbb{Z}\} \) where \( h_1 = 1, h_2 = 0.5, \) and \( h_3 = 0.001 \). Note that for all four graphs \( \beta = 1.75, \alpha = 4, T = 30, \) and \( r = 1.9, \) which are arbitrarily chosen.

![Figure 4.1: Optimized Solutions for Constant Target Function with Fixed Endpoints.](image)

Assuming that the boundary conditions are not given, Theorems 2.1.2.-2.1.4. give that the boundary conditions are \( y^\Delta(0) = 0 \) and \( y^\Delta(\rho^2(T)) = h \alpha [\bar{y}(\rho(T)) - y(\rho(T))] \).
In figure 4.2 a series of graphs are shown to illustrate the optimized particular solution obtained using the free boundary conditions. The first three graphs are found taking the time scales to be such that \( \mu(t) = 1, 0.5, \) and 0.001 for all \( t \in T \). The fourth graph illustrates a mixed time scale where \( T = \{[0, 6] \cap h_1 Z\} \cup \{[6, 14] \cap h_2 Z\} \cup \{[14, 30] \cap h_3 Z\} \) where \( h_1 = 1, h_2 = 0.5, \) and \( h_3 = 0.001 \). Note that for all four graphs \( \beta = 1.75, \alpha = 4, T = 30, \) and \( \tau = 1.9, \) which are arbitrarily chosen.

![Figure 4.2: Optimized Solutions for Constant Target Function with Free Endpoints.](image)

The graphs generated for section 4.2 were created using Mathematica, the code used to solve for \( C_1 \) and \( C_2 \) and graph the optimized solution can be found in Appendix (A.2).
4.3 Linear Target Function

Let \( y(t) = vt + \beta \), then \( C_1^\Delta(t) \) and \( C_2^\Delta(t) \) simplify to

\[
C_1^\Delta(t) = \frac{-\alpha}{1 + hr - h}(vt + \beta) \frac{e_{\lambda_1}(\sigma(t), 0)}{(\lambda_2 - \lambda_1)}
\]

\[
C_2^\Delta(t) = \frac{-\alpha}{1 + hr - h}(vt + \beta) \frac{e_{\lambda_2}(\sigma(t), 0)}{(\lambda_2 - \lambda_1)}
\]

then by integrating both sides of \( C_1^\Delta(t) \) and \( C_2^\Delta(t) \) the following is obtained

\[
C_1(t) = \frac{-\alpha}{1 + hr - h}(vt + \beta) \frac{e_{\lambda_1}(t, 0)}{\lambda_1(\lambda_2 - \lambda_1)} + \frac{-\alpha}{1 + hr - h}(1 + h\lambda_1)v \frac{e_{\lambda_1}(t, 0)}{\lambda_1^2(\lambda_2 - \lambda_1)}
\]

\[
C_2(t) = \frac{-\alpha}{1 + hr - h}(vt + \beta) \frac{e_{\lambda_2}(t, 0)}{\lambda_2(\lambda_2 - \lambda_1)} + \frac{-\alpha}{1 + hr - h}(1 + h\lambda_2)v \frac{e_{\lambda_2}(t, 0)}{\lambda_2^2(\lambda_2 - \lambda_1)}
\]

Then the particular solution is,

\[
y(t) = C_1e_{\lambda_1}(t, 0) + C_2e_{\lambda_2}(t, 0) + C_1(t)e_{\lambda_1}(t, 0) + C_2(t)e_{\lambda_2}(t, 0)
\]

\[
y(t) = C_1e_{\lambda_1}(t, 0) + C_2e_{\lambda_2}(t, 0) + \frac{-\alpha}{1 + hr - h}(vt + \beta) \frac{e_{\lambda_1}(t, 0)}{\lambda_1(\lambda_2 - \lambda_1)} + \frac{-\alpha}{1 + hr - h}(1 + h\lambda_1)v \frac{e_{\lambda_1}(t, 0)}{\lambda_1^2(\lambda_2 - \lambda_1)}
\]

\[
+ \frac{-\alpha}{1 + hr - h}(vt + \beta) \frac{e_{\lambda_2}(t, 0)}{\lambda_2(\lambda_2 - \lambda_1)} + \frac{-\alpha}{1 + hr - h}(1 + h\lambda_2)v \frac{e_{\lambda_2}(t, 0)}{\lambda_2^2(\lambda_2 - \lambda_1)}
\]

Assuming the boundary conditions are given by the values of the target function at the endpoints we have that \( y(0) = \beta \) and \( y(\rho(T)) = v(\rho(T)) + \beta \). In figure 4.3 a series of graphs are shown to illustrate the optimized particular solution. The first three graphs are found taking the time scales to be such that \( \mu(t) = 1, 0.5, \) and \( 0.001 \) for all \( t \in T \). The fourth graph illustrates the optimal solution for a mixed time scale where
\[ T = \{ [0, 6) \cap h_1 \mathbb{Z} \} \cup \{ [6, 14) \cap h_2 \mathbb{Z} \} \cup \{ [14, 30] \cap h_3 \mathbb{Z} \} \] where \( h_1 = 1, \ h_2 = 0.5, \) and \( h_3 = 0.001. \) Note that for all four graphs \( v = 4, \ \beta = 0.25, \ \alpha = 4, \ T = 30, \) and \( r = 1.9, \) which are arbitrarily chosen.

Figure 4.3: Optimized Solutions for Linear Target Function with Fixed Endpoints.

Assuming that the boundary conditions are not given, Theorems 2.1.2.-2.1.4. give that the boundary conditions are \( y^A(0) = 0 \) and \( y^A(\rho^2(T)) = h\alpha[y(\rho(T)) - y(\rho(T))] \).

In figure 4.4 a series of graphs are shown to illustrate the optimized particular solution. The first three graphs are found taking the time scales to be such that \( \mu(t) = 1, \ 0.5, \) and 0.001 for all \( t \in T. \) The fourth graph illustrates a mixed time scale where \( T = \{ [0, 6) \cap h_1 \mathbb{Z} \} \cup \{ [6, 14) \cap h_2 \mathbb{Z} \} \cup \{ [14, 30] \cap h_3 \mathbb{Z} \} \) where \( h_1 = 1, \ h_2 = 0.5, \) and \( h_3 = 0.001. \) Note that for all four graphs \( v = 4, \ \beta = 0.25, \ \alpha = 4, \ T = 30, \) and \( r = 1.9. \)
The graphs generated for section 4.3 were created using Mathematica, the code used to solve for $C_1$ and $C_2$ and graph the optimized solution can be found in Appendix (A.3).

### 4.4 Exponential Target Function

Let $y(t) = e^t$ then $C^\Delta_1(t)$ and $C^\Delta_2(t)$ simplify to,

$$C^\Delta_1(t) = \frac{\alpha}{1 + h\sigma - h} (1 + h\beta)e_\beta(t, 0)$$

$$C^\Delta_2(t) = \frac{-\beta}{1 + h\sigma - h} (1 + h\beta)e_\beta(t, 0)$$

then using the properties of the $\Delta$-exponential the previous simplifies to the following
\[
C_1^\Delta(t) = \frac{-\alpha}{1+h\beta-h} \left( \frac{1}{\lambda_2 - \lambda_1}(1 + h\lambda_2) \right) e^{\frac{\beta - \lambda_1}{1+h\lambda_1}} (t, 0)
\]
\[
C_2^\Delta(t) = \frac{-\alpha}{1+h\beta-h} \left( \frac{1}{\lambda_2 - \lambda_1}(1 + h\lambda_2) \right) e^{\frac{\beta - \lambda_2}{1+h\lambda_2}} (t, 0)
\]

then by integrating both sides of \( C_1^\Delta(t) \) and \( C_2^\Delta(t) \) the following is obtained

\[
C_1(t) = \frac{-\alpha}{1+h\beta-h} \left( \frac{1}{\lambda_2 - \lambda_1}(1 + h\lambda_2) \right) e^{\frac{\beta - \lambda_1}{1+h\lambda_1}} (t, 0)
\]
\[
C_2(t) = \frac{-\alpha}{1+h\beta-h} \left( \frac{1}{\lambda_2 - \lambda_1}(1 + h\lambda_2) \right) e^{\frac{\beta - \lambda_2}{1+h\lambda_2}} (t, 0)
\]

Then the particular solution is,

\[
y(t) = C_1 e_{\lambda_1}(t, 0) + C_2 e_{\lambda_2}(t, 0) + C_1(t) e_{\lambda_1}(t, 0) + C_2(t) e_{\lambda_2}(t, 0)
\]

\[
y(t) = C_1 e_{\lambda_1}(t, 0) + C_2 e_{\lambda_2}(t, 0) + \frac{-\alpha}{1+h\beta-h} \left( \frac{1}{\lambda_2 - \lambda_1}(1 + h\lambda_2) \right) e_{\beta}(t, 0) + \frac{-\alpha}{1+h\beta-h} \left( \frac{1}{\lambda_2 - \lambda_1}(1 + h\lambda_2) \right) e_{\beta}(t, 0)
\]

Assuming the boundary conditions are given by the values of the target function at the endpoints we have that \( y(0) = 1 \) and \( y(\rho(T)) = e_\beta(\rho(T), 0) \). In figure 4.5 a series of graphs are shown to illustrate the optimized particular solution. The first three graphs are found taking the time scales to be such that \( \mu(t) = 1, 0.5, \) and 0.001 for all \( t \in T \). The fourth graph illustrates the optimal solution for a mixed time scale where \( T = \{0, 6) \cap h_1Z \} \cup \{6, 14) \cap h_2Z \} \cup \{14, 30] \cap h_3Z \} \) where \( h_1 = 1, h_2 = 0.5, \) and \( h_3 = 0.001 \). Note that for all four graphs \( \beta = 0.25, \alpha = 4, T = 30, \) and \( r = 1.9, \) which are arbitrarily chosen.
Assuming that the boundary conditions are not given, Theorems 2.1.2.-2.1.4. give that the boundary conditions are $y^A(0) = 0$ and $y^A(\rho^2(T)) = h\alpha[y(\rho(T)) - y(\rho(T))]$.

In figure 4.6 a series of graphs are shown to illustrate the optimized particular solution. The first three graphs are found taking the time scales to be such that $\mu(t) = 1, 0.5,$ and $0.001$ for all $t \in T$. The fourth graph illustrates a mixed time scale where $T = \{[0, 6) \cap h_1Z \} \cup \{[6, 14) \cap h_2Z \} \cup \{[14, 30) \cap h_3Z \}$ where $h_1 = 1, h_2 = 0.5,$ and $h_3 = 0.001$.

Note that for all four graphs $\beta = 0.25, \alpha = 4, T = 30,$ and $r = 1.9$. 

Figure 4.5: Optimized Solutions for Exponential Target Function with Fixed Endpoints.
Figure 4.6: Optimized Solutions for Exponential Target Function with Free Endpoints.

The graphs generated for section 4.4 were created using Mathematica, the code used to solve for $C_1$ and $C_2$ and graph the optimized solution can be found in Appendix (A.4).
Chapter 5

Stochastic Calculus on Time Scales

Many processes such as the motion of a diffusing particle and the prices of stocks are often modeled by a stochastic process. In application most stochastic processes to date have used either a continuous sampling time or a discrete sampling time which is scaled to suite the occasion. Examples of these ideas can be seen in [1,10,11]. The goal of this work will be to introduce Brownian motion as a process that allows for stochastic modeling of events on any time scale. We will also begin to develop the Ito-Integral on time scales by presenting a form that works on any non-dense time scale.

5.1 Brownian Motion Defined on Time Scales

Definition 5.1.1. Brownian motion $B(t)$, where $t \in T$, is a stochastic process with the following properties.
• (Independence of Increments) $B(t) - B(s)$, for $t > s$, is independent of the past, that is of $B(u), 0 \leq u \leq s$, or of the $\sigma$-field generated by $B(u), u \leq s$, for all $t, s, u \in \mathbb{T}$.

• (Normal Increments) For all $t, s \in \mathbb{T}$, $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t - s$. This implies that taking $s = 0$ that $B(t) - B(0)$ has $N(0, t)$ distribution, given that $0 \in \mathbb{T}$.

• (Continuity of Paths) $B(t)$ is a continuous function of $t$.

Remark 5.1.1. The existence of Brownian motion on time scales can be verified by construction, but is much easier to note that $\mathbb{T} \subseteq \mathbb{R}$, and since Brownian Motion exists on $\mathbb{R}$ then it also exists on any subset of $\mathbb{R}$ and furthermore retains its original properties.

5.1.1 Quadratic Variation of Brownian Motion on Time Scales

The first thing we wish to do is to define a $\Delta$ partition of a set which can be obtained from [9].

Definition 5.1.2. Let $\delta > 0$ be given. A partition of maximum length $\delta$ on the interval $[a, b] \cap \mathbb{T}$ (where $a, b \in \mathbb{T}, a < b$) is defined by

$$a = t_0 < t_1 < \ldots < t_n = b$$

where $t_i \in \mathbb{T}$ for $i = 0, 1, 2, \ldots, n$, and for each $i = 1, 2, \ldots, n$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$. 
Remark 5.1.2. Every left scattered point is "eventually" in the sequence of partitions $P_n$, i.e. for all left scattered points $\bar{t} \in \mathbb{T}$ there exists $N$ such that for all $n \geq N$, $\bar{t} \in P_n$.

The following definition of Quadratic Variation of a stochastic process is obtained from [10].

Definition 5.1.3. (Quadratic Variation) Let $F$ be a stochastic process on $\mathbb{R}$. The quadratic variation of $F : \mathbb{R} \to \mathbb{R}$ over the $[0, t]$ is defined as follows. Let $P_n$ be a sequence of partitions for which the maximum lengths $\delta_n \to 0$ as $n \to \infty$. Then, the quadratic variation is given by

$$[F, F](t) = F^2(0) + \lim_{n \to \infty} \sum_{t \in P_n} (F(t_i) - F(t_{i-1}))^2.$$

Now for a comparison we will look at the optional quadratic variation and the predictable quadratic variation, found in [1], defined on discrete time sets.

Definition 5.1.4. Let $F(n)$ be a stochastic process on $\mathbb{N} \cup \{0\}$.

- (Optional Quadratic Variation) The process defined by

$$[F, F](t) = F^2(0) + \sum_{i=1}^{t} (F(i) - F(i - 1))^2$$

is called the optional quadratic variation of $F$ on $\{0, 1, ..., n\}$.

- (Predictable Quadratic Variation) The predictable quadratic variation of $F$ on $\{0, 1, ..., n\}$ is defined by
\[
\langle F, F \rangle(t) = E[F^2(0)] + \sum_{i=1}^{t} E[(F(i) - F(i-1))^2 | \mathcal{F}_{i-1}]
\]

where \( \mathcal{F}_{i} \) is the information given on the process up to the time \( i \).

Now consider the new definition below for the optional and predictable quadratic variation on any finite set.

**Definition 5.1.5.** Let \( F \) be a finite time stochastic process. i.e. \( F \) takes values on a finite number of elements \( 0 = t_0 < t_1 < ... < t_n = t \).

- (Optional Quadratic Variation) The process defined by

\[
[F, F](t) = F^2(0) + \sum_{i=1}^{n} (F(t_i) - F(t_{i-1}))^2
\]

is called the optional quadratic variation of \( F \).

- (Predictable Quadratic Variation) The predictable quadratic variation of \( F \) is defined by

\[
\langle F, F \rangle(t) = E[F^2(0)] + \sum_{i=1}^{n} E[(F(t_i) - F(t_{i-1}))^2 | \mathcal{F}_{t_{i-1}}]
\]

where \( \mathcal{F}_{t_i} \) is the information given on the process up to the time \( t_i \).

**Remark 5.1.3.** If we consider the set of partitions \( P_n \) to be the finite set that the optional and predictable quadratic variation is being defined on and take the limit as \( n \to \infty \) then we get the following definition, which holds the definition of quadratic variation of a stochastic process on \( \mathbb{R} \) as a subcase.
Definition 5.1.6. (Optional) Quadratic Variation on Time Scales) Let $0, t \in T$ with $0 < t$ be given. Let $F$ be a stochastic process on $T$. The quadratic variation of $F : T \to \mathbb{R}$ over the $[0, t]$ is defined as follows. Let $P_n$ be a sequence of partitions for which the maximum lengths $\delta_n \to 0$ as $n \to \infty$. Then, the quadratic variation is given by

$$[F, F](t) = F^2(0) + \lim_{n \to \infty} \sum_{t_i \in P_n} (F(t_i) - F(t_{i-1}))^2.$$

Definition 5.1.7. (Predictable Quadratic Variation on Time Scales) Let $0, t \in T$ with $0 < t$ be given. Let $F$ be a stochastic process on $T$. The predictable quadratic variation of $F : T \to \mathbb{R}$ over the $[0, t]$ is defined as follows. Let $P_n$ be a sequence of partitions for which the maximum lengths $\delta_n \to 0$ as $n \to \infty$. Then, the predictable quadratic variation relative to the filtration $\mathcal{F}$ is given by

$$\langle F, F \rangle(t) = E[F^2(0)] + \lim_{n \to \infty} \sum_{t_i \in P_n} E[(F(t_i) - F(t_{i-1}))^2|\mathcal{F}_{t_{i-1}}].$$

where $\mathcal{F}_{t_i}$ is the information given on the process up to the time $t_i$.

Remark 5.1.4. When $[0, t] \cap T = [0, t] \cap \mathbb{R}$ then we have that the standard quadratic variation of $F$ is the same as the optional quadratic variation of $F$. Also when the time scale has a finite number of elements in the interval $[0, t]$ then the time scale quadratic variation of $F$ is equal to the optional quadratic variation of $F$.

For the convenience of the reader the following well-known theorem and its proof are given,

Theorem 5.1.1. The quadratic variation of Brownian motion over $[0, t]$ is $t$, when $T = \mathbb{R}$.
Proof. The proof is given for a sequence of partitions $P_n = \{t^n_i = \frac{t^n_i}{2^n}, i = 0, 1, \ldots, 2^n\}$ which implies that $\delta_n = \max_i(t^n_i - t^n_{i-1}) = \frac{1}{2^{n-1}}$ we know that $\sum_n \delta_n < \infty$. Then let $T_n = \sum_i (B(t^n_i) - B(t^n_{i-1}))^2$ and notice that as $n \to \infty$ this converges to the quadratic variation of Brownian motion.

$$E[T_n] = E \sum_i (B(t^n_i) - B(t^n_{i-1}))^2 = \sum_i E(B(t^n_i) - B(t^n_{i-1}))^2$$

$$= \sum_i (t^n_i - t^n_{i-1}) = t(1)$$

and

$$Var[T_n] = Var \sum_i (B(t^n_i) - B(t^n_{i-1}))^2 = \sum_i Var(B(t^n_i) - B(t^n_{i-1}))^2$$

then using the fourth moment of the normal distribution the previous becomes

$$= \sum_i 2(t^n_i - t^n_{i-1})^2 \leq 2 \max_i(t^n_i - t^n_{i-1})t = 2t\delta_n. (2)$$

This implies then that $\sum_{n=1}^{\infty} Var(T_n) < \infty$. The using the monotone convergence theorem we find that $E \sum_{n=1}^{\infty} (T_n - ET_n)^2 < \infty$. This implies that the series inside the expectation converges almost surely. Hence its terms converge to zero, and $T_n - ET_n \to 0$ almost surely. Consequently this shows that $T_n \to t$ almost surely. (Note that this proof is done for a specific sequence of partitions but it can be shown to be true for any sequence of partitions which are successive refinements with the condition that $\delta_n \to 0$ as $n \to \infty$) □

Remark 5.1.5. The quadratic variation of Brownian motion on the interval $[0, t] \cap \mathbb{T}$ for any given time scale $\mathbb{T}$ is not necessarily non-random. Consider $\mathbb{T} = \mathbb{Z}$ and calculate the
quadratic variation of Brownian motion over the interval [0, 1] and we see that \([B, B](1)\) is a random variable with expected value 1 and variance 2.

**Theorem 5.1.2.** The quadratic variation of Brownian motion on the interval \([0, t] \cap \mathbb{T}\) is a random variable if there exists \(\bar{t} \in [0, t] \cap \mathbb{T}\) such that \(\rho(\bar{t}) < t\).

**Proof.** Assume that the interval \([0, t] \cap \mathbb{T}\) has one left scattered point \(\bar{t}\) and define the partitioning scheme where \(\delta_n = \frac{1}{2^n-1}\) then since we know that \(\bar{t}\) is left scattered then there is some \(\delta_N < \bar{t} - \rho(\bar{t})\). So \(\bar{t} = t_j^N \in P_n\) for all \(n \geq N\). Then from the proof of quadratic variation on \(\mathbb{R}\) we see that there is no change in (1) but (2) becomes

\[
\text{Var}(T_n) = \sum_i 2(t_i^n - t_{i-1}^n)^2 = \sum_{i \neq j} 2(t_i^n - t_{i-1}^n)^2 + 2(t_j^n - t_{j-1}^n)^2 \geq 2(t_j^n - t_{j-1}^n)^2
\]

and we know that \(\lim_{n \to \infty} T_n \to [B, B](t)\) and that the \(\lim_{n \to \infty} E[T_n] \to t\) but \(\lim_{n \to \infty} \text{Var}[T_n] \to 2(\nu(\bar{t}))^2 > 0\) so the random variable \(T_n\) converges to a random variable (note the criteria for convergence to a non-random variable would be that \(\lim_{n \to \infty} \text{Var}[T_n] \to 0\)). \(\Box\)

### 5.1.2 Properties of Brownian Motion on Time Scales

**Theorem 5.1.3.** A Brownian motion on a time scale \(\mathbb{T}\) is a Gaussian process with zero mean function, and covariance function \(\min(t, s)\), where \(t, s \in \mathbb{T}\). Conversely, a Gaussian process with zero mean function and covariance function \(\min(t, s)\) is a Brownian motion.

**Proof.** The proof of this theorem is the same as the continuous case found in [13] with the additional assumption that \(t, s \in \mathbb{T}\). \(\Box\)
5.2 The Itô-Integral on Non-Dense Time Scales

The following definition of a discrete stochastic integral was obtained from [10].

**Definition 5.2.1.** The forward looking stochastic integral in discrete time of a process $H$ with respect to a process $S$ is defined by

$$
\langle \tilde{F} \rangle (H \cdot S)_t := H_0 S_0 + \sum_{n=1}^{t-1} H_n (S_{n+1} - S_n).
$$

Now consider the process $H$ to be some (simple) random process $X(t)$ and the process $S$ to be Brownian motion. Then we have the following stochastic integral

$$
\langle \tilde{F} \rangle (X \cdot B)_t := X(0)B(0) + \sum_{i=1}^{t-1} X(i)(B(i+1) - B(i)) = \sum_{i=1}^{t-1} X(i)(B(i+1) - B(i))
$$

(3)

and say that (3) is the Itô-integral in discrete time of the process $X(t)$.

**Definition 5.2.2.** The Itô-integral of any (simple) random process on any finite interval of a time scale $[0, t] \cap \mathbb{T}$, with the assumption that $0, t \in \mathbb{T}$, is given by

$$
\int_0^t X(t) \Delta_t B(t) = \sum_{t_i \in [0, t] \cap \mathbb{T}} X(t_i)(B(\sigma(t_i)) - B(t_i)).
$$

Now to illustrate the difference in the traditional Itô-integral of Brownian motion on the interval $[0, T] \cap \mathbb{R}$ which is known to be

$$
\int_0^T B(t) dB(t) = \frac{1}{2} B^2(t) - \frac{1}{2} T.
$$

We will calculate the finite time scale Itô-integral for any time scale such that $[0, T] \cap \mathbb{T}$ is finite and such that $0, T \in \mathbb{T}$. First construct $P_n$ as previously discussed such that if
\[ t \in [0, T] \cap T \text{ then } t \in P_n. \text{ This would yield that } P_n = \{0 = t_0 < t_1 < \ldots < t_n = T\} = [0, T] \cap T. \text{ Then the integral is} \]
\[
\int_0^T B(t) \Delta_t B(t) = \sum_{t_i \in \{0, T\} \cap T} B(t_i) (B(\sigma(t_i)) - B(t_i))
\]
\[= \sum_{i=0}^{n-1} B(t_i) (B(\sigma(t_i)) - B(t_i))
\]
then consider each summand
\[
B(t_i)[B(\sigma(t_i)) - B(t_i)] = \frac{1}{2} (B(t_i) B(\sigma(t_i)) - 2 B^2(t_i) + B^2(\sigma(t_i)) - B^2(\sigma(t_i))) = \frac{1}{2} (B^2(\sigma(t_i)) - B^2(t_i) - (B(\sigma(t_i)) - B(t_i))^2)
\]
so
\[
\sum_{i=0}^{n-1} B(t_i) (B(\sigma(t_i)) - B(t_i)) = \frac{1}{2} \sum_{i=0}^{n-1} (B^2(\sigma(t_i)) - B^2(t_i)) - \frac{1}{2} \sum_{i=0}^{n-1} (B(\sigma(t_i)) - B(t_i))^2.
\]
Note that the first sum is a telescoping series and the last sum is the time scale quadratic variation of Brownian motion, so we have that
\[
\int_0^T B(t) \Delta_t B(t) = \frac{1}{2} B^2(t) - \frac{1}{2} [B, B](T).
\]
Note that the time scale quadratic variation \([B, B](T)\) is a random variable whose expectation is \(T\), but has some non-zero variance which is dependent upon the time scale \(T\).

### 5.3 Future Developments

The further derivation of stochastic calculus on time scale is an extremely important task. Most of the modeling done in economics and in other fields are done with respect
to some stochastic process. After a more complete time scale theoretical development has been accomplished for stochastic calculus then the development of the theory of Stochastic Time Scale Calculus of Variations could begin which would allow for the optimization of many of the before mentioned models. To start this process a general time scale Itô-integral needs to be formulated, along with the surrounding theory. It would be significant if this theory could be developed for both the $\Delta$ and $\nabla$ operators, since as seen earlier there exist cases where one can only write a model in one form or the other. Another step will be to try to generate a full understanding of Brownian Motion and develop the definition of a Poisson process on time scales, as these are the more commonly used stochastic processes.
Appendix A

Mathematica Code

A.1 Basic Variable Declaration

h := 1
h05 := .5
h01 := .001
r := 1.9
a := 4
b := .25
bl := 1.75
v := 4
T1 := 6
A.2 Code for Constant Target Function

Constant Target Function for $hZ$ Time Scales with Fixed Boundary Conditions

For $h=1$

$\text{NBC19}_h := \frac{-a+b_1}{h_1+h_2(1+h_3-r-h)}$

$\text{NBCD1}_h := \frac{-a+b_1}{h_1+h_2(1+h_3-r-h)}$

$\text{NBCe19}_h := (1 + h_1 \cdot h)^{(T_3 - h)/h}$

$\text{NBCe219}_h := (1 + h_2 \cdot h)^{(T_3 - h)/h}$
NBCM1:= 
\[
\begin{pmatrix}
1 & 1 \\
\text{NBCe19h} & \text{NBCe29h}
\end{pmatrix}
\]

BBCM1:=
\[
\begin{pmatrix}
-NBCD1 + b1 \\
-NBC19 + b1
\end{pmatrix}
\]

NBCCL1:=LinearSolve[NBCM1, BBCM1]

NBCcL1:=NBCCL1[[1,1]]

NBCcL2:=NBCCL1[[2,1]]

NBCGL1:=Table[
\[(i - 1) \cdot h, NBCcL1 \cdot (1 + h1 \cdot h)^{(i - 1) \cdot h/h} +
NBCcL2 \cdot (1 + h2 \cdot h)^{(i - h) \cdot h/h} + \frac{-a+b1}{h1+b2\ast(1+h\ast r-h)}\], \{i, T3/h\}\]

NBCGL11 = ListPlot[NBCGL1, PlotRange \rightarrow \{-2, T3\}, \{0, 3\},
PlotStyle \rightarrow \{RGBColor[.8, 0, .2], PointSize[.018]\}]

Export["ConstantLongFixedh1.eps", NBCGL11, "EPS"]

For h=.5

NBC29:=
\[
\frac{-a+b1}{h3+b4\ast(1+h05\ast r-h05)}
\]

NBCD2:=
\[
\frac{-a+b1}{h3+b4\ast(1+h05\ast r-h05)}
\]

NBCe129h:=(1 + h3 \cdot h05)^{(T3 - h05)/h05}

NBCe229h:=(1 + h4 \ast h05)^{(T3 - h05)/h05}
NBCM2 := \( \begin{pmatrix} 1 & 1 \\ NBCe129h & NBCe229h \end{pmatrix} \)

BBCM2 := \( \begin{pmatrix} NBCD2 - b1 \\ b1 - NBC29 \end{pmatrix} \)

NBCCL2 := LinearSolve[NBCM2, BBCM2]

NBCcL3 := NBCCL2[[1, 1]]

NBCcL4 := NBCCL2[[2, 1]]

NBCGL2 := Table[{(i - 1) * h05, NBCcL3 * (1 + h3 * h05)\( (h05 * (i - 1))/h05 \) + NBCcL4 * (1 + h4 * h05)\( (i - 1) * h05)/h05 \) + \(-a*b1\) \( h3*a*b4+1+05+r-h05 \)}, \{i, T3/h05\}]

NBCGL12 := ListPlot[NBCGL2, PlotRange -> \{\{-2, T3\}, \{0, 3\}\}, PlotStyle -> \{RGBColor[0, .8, .2], PointSize[.018]\}]

Export["ConstantLongFixedh5.eps", NBCGL12, "EPS"]

For \(h=.001\)

NBC39 := \( \frac{-a*b1}{h5*h6*(1+h01+r-h01)} \)

NBCD3 := \( \frac{-a*b1}{h5*h6*(1+h01+r-h01)} \)

NBCe139h := \((1 + h5 * h01)^\( (T3 - h01)/h01 \)\)

NBCe239h := \((1 + h6 * h01)^\( (T3 - h01)/h01 \)\)
NBCM3 := \begin{pmatrix}
1 & 1 \\
NCe139h & NCe239h \\
\end{pmatrix}

BBCM3 := \begin{pmatrix}
b1 - NCBD3 \\
b1 - NC39 \\
\end{pmatrix}

NBCCL3 := \text{LinearSolve}[NBCM3, BBCM3]

NBCcL5 := NBCCL3[[1, 1]]

NBCcL6 := NBCCL3[[2, 1]]

NBCGL3 := \text{Table}\left[\left\{\left(i - 1\right) * h01, \left(1 + h5 * h01\right)^{(\left(\left(i - 1\right) * h01\right)/h01)} + \frac{-a*b1}{h5+h6*(1+h01*r-h01)}\right\}, \left\{i, T3/h01\right\}\right]

NBCGL13 = \text{ListPlot}[NBCGL3, \text{PlotRange} \rightarrow \{-2, T3\}, \{0, 3\}, \text{PlotStyle} \rightarrow \{\text{RGBColor}[0.2, 0.8], \text{PointSize}[0.018]\}]

\text{Export}["ConstantLongFixedh001.eps", NBCGL13, "EPS"]

For a Mixed Time Scale

CA10 := \frac{-a*b1}{h1+h2*(1+h*r-h)}

CA13 := \frac{-a*b1}{h1+h2*(1+h*r-h)}

CAD1 := 0

CA23 := \frac{-a*b1}{h3+h4*(1+h05*r-h05)}

CA26 := \frac{-a*b1}{h3+h4*(1+h05*r-h05)}

CAD2 := 0
\[
\begin{align*}
\text{CA36:} &= \frac{-a+b_1}{h_5 h_6 s (1 + h_0_1 + r - h_0_1)} \\
\text{CA39:} &= \frac{-a+b_1}{h_5 h_6 s (1 + h_0_1 + r - h_0_1)} \\
\text{CAD3:} &= 0 \\
\text{Ce113:} &= (1 + h_1 + h)^{(T_1/h)} \\
\text{Ce213:} &= (1 + h_2 + h)^{(T_1/h)} \\
\text{Ce123:} &= (1 + h_3 + h_0_5)^{(T_1/h_0_5)} \\
\text{Ce223:} &= (1 + h_4 + h_0_5)^{(T_1/h_0_5)} \\
\text{Ce126:} &= (1 + h_3 + h_0_5)^{(T_2/h_0_5)} \\
\text{Ce226:} &= (1 + h_4 + h_0_5)^{(T_2/h_0_5)} \\
\text{Ce136:} &= (1 + h_5 + h_0_1)^{(T_2/h_0_1)} \\
\text{Ce236:} &= (1 + h_6 + h_0_1)^{(T_2/h_0_1)} \\
\text{Ce139h:} &= (1 + h_5 + h_0_1)^{(T_3 - h_0_1)/h_0_1} \\
\text{Ce239h:} &= (1 + h_6 + h_0_1)^{(T_3 - h_0_1)/h_0_1} \\
\end{align*}
\]

\[
\text{CMT:} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{Ce113} & \text{Ce213} & -\text{Ce123} & -\text{Ce223} & 0 & 0 & 0 \\
\text{h1 * Ce113} & \text{h2 * Ce213} & -\text{h3 * Ce123} & -\text{h4 * Ce223} & 0 & 0 & 0 \\
0 & 0 & \text{Ce126} & \text{Ce226} & -\text{Ce136} & -\text{Ce236} & 0 \\
0 & 0 & \text{h3 * Ce126} & \text{h4 * Ce226} & -\text{h5 * Ce136} & -\text{h6 * Ce236} & 0 \\
0 & 0 & 0 & 0 & \text{Ce139h} & \text{Ce239h} & 0
\end{pmatrix}
\]
\[ \begin{pmatrix} b_1 - CA_{10} \\ CA_{23} - CA_{13} \\ CA_{D2} - CA_{D1} \\ CA_{36} - CA_{26} \\ CA_{D3} - CA_{D2} \\ b_1 - CA_{39} \end{pmatrix} \]

\[
CCLT := \text{LinearSolve} [CMT, CBT];
\]

\[
CcLT1 := CCLT[[1, 1]]
\]

\[
CcLT2 := CCLT[[2, 1]]
\]

\[
CcLT3 := CCLT[[3, 1]]
\]

\[
CcLT4 := CCLT[[4, 1]]
\]

\[
CcLT5 := CCLT[[5, 1]]
\]

\[
CcLT6 := CCLT[[6, 1]]
\]

\[
GCLT1 := \text{Table}[\{(i - 1) \cdot h, CcLT1 \cdot (1 + h1 \cdot h)^\alpha ((i - 1) \cdot h/h) + CcLT2 \cdot (1 + h2 \cdot h)^\alpha ((i - 1) \cdot h/h) + \frac{-\alpha h}{h1 + h2 \cdot (1 + h - h)}\}, \{i, (T1/h) + 1\}]
\]

\[
GCLT2 := \text{Table}[\{i \cdot h05 + T1, CcLT3 \cdot (1 + h3 \cdot h05)^\alpha ((i \cdot h05 + T1)/h05) + CcLT4 \cdot (1 + h4 \cdot h05)^\alpha ((i \cdot h05 + T1)/h05) + \frac{-\alpha h}{h3 + h4 \cdot (1 + h05 / h05)}\}, \{i, (T2 - T1)/h05\}]
\]

\[
GCLT3 := \text{Table}[\{i \cdot h01 + T2, CcLT5 \cdot (1 + h5 \cdot h01)^\alpha ((i \cdot h01 + T2)/h01) + CcLT6 \cdot (1 + h6 \cdot h01)^\alpha ((i \cdot h01 + T2)/h01) + \frac{-\alpha h}{h5 + h6 \cdot (1 + h01 / h01)}\}, \{i, ((T3 - T2)/h01 - 1)\}]
\]

\[
GCLT11 = \text{ListPlot}[GCLT1, \text{PlotRange} \to \{-2, T3\}, \{0, 3\}]
\]
Constant Target Function for $hZ$ Time Scales with Free Boundary Conditions

For $h=1$

\[
\begin{align*}
CNA19 &= \frac{-a+b_1}{h_1 + b_2 (1 + h \sigma - h)} \\
CNAD1 &= 0 \\
CNe119 &= (1 + h_1 \ast h) \cdot ((T3 - h)/h) \\
CNe219 &= (1 + h_2 \ast h) \cdot ((T3 - h)/h) \\
CNe119h &= (1 + h_1 \ast h) \cdot ((T3 - 2 \ast h)/h) \\
CNe219h &= (1 + h_2 \ast h) \cdot ((T3 - 2 \ast h)/h) \\
\end{align*}
\]

\[
CNM1 := \begin{pmatrix}
    \frac{h1}{h1 + CNe119h + h \ast a \ast CNe119} & \frac{h2}{h2 + CNe219h + h \ast a \ast CNe219}
\end{pmatrix}
\]
CBM1 := \begin{pmatrix}
  -CNAD1 \\
  h \cdot a((b1) - CNA19) - CNAD1
\end{pmatrix}

NCCL1 := \text{LinearSolve}[CNM1, CBM1]

NCcL1 := NCCL1[[1, 1]]

NCcL2 := NCCL1[[2, 1]]

NGCL1 := \text{Table}\{(i - 1) \cdot h, NCcL1 \cdot (1 + h1 \cdot h)^((i - 1) \cdot h/h)
+ NCcL2 \cdot (1 + h2 \cdot h)^((i - 1) \cdot h/h) + \frac{-a\cdot h}{h1 + h2 \cdot (1 + h\cdot r - h)}\}, \{i, T3/h\}\}

NGCL11 = \text{ListPlot}[NGCL1, \text{PlotRange} \to \{\{-2, T3\}, \{0, 3\}\},
\text{PlotStyle} \to \{\text{RGBColor}[.8, 0, .2], \text{PointSize} [.018]\}]

Export["ConstantLongh1.eps", NGCL11, "EPS"]

For h=.5

CNA29 := \frac{-a\cdot h}{h3 + h4 \cdot (1 + h5 \cdot r - h5)}

CNAD2 := 0

CNe129 := (1 + h3 \cdot h05)^((T3 - h05)/h05)

CNe229 := (1 + h4 \cdot h05)^((T3 - h05)/h05)

CNe129h := (1 + h3 \cdot h05)^((T3 - 2 \cdot h05)/h05)

CNe229h := (1 + h4 \cdot h05)^((T3 - 2 \cdot h05)/h05)

\text{CNM2 := }
\begin{pmatrix}
  h3 \\
  h4 \\
  h3 \cdot CNe129h + h05 \cdot a \cdot CNe129 \\
  h4 \cdot CNe229h + h05 \cdot a \cdot CNe229
\end{pmatrix}
CBM2 := 
\[
\begin{pmatrix}
-CNAD2 \\
h05 * a((bl) - CNA29) - CNAD2
\end{pmatrix}
\]

NCCL2 := LinearSolve[CNM2, CBM2]

NCcL3 := NCCL2[[1, 1]]

NCcL4 := NCCL2[[2, 1]]

NGCL2 := Table[
{(i - 1) * h05, NCcL3 * (1 + h3 * h05)^((i - 1) * h05/h05) + NCcL4 * (1 + h4 * h05)^((i - 1) * h05/h05) + 6555^1^=505)}, 
{i, T3/h05}]

NGCL12 = ListPlot[NGCL2, PlotRange -> {{-2, T3}, {0, 3}},
PlotStyle -> {RGBColor[0, .8, .2], PointSize[.018]}]

Export["ConstantLongh05.eps", NGCL12, "EPS"]

For h=.001

CNA39 := \[-a*b1 h5*h6*(l+h01»r-b0l)\]

CNAD3 := 0

CNe139 := (1 + h5 * h01)^((T3 - h01)/h01)

CNe239 := (1 + h6 * h01)^((T3 - h01)/h01)

CNe139h := (1 + h5 * h01)^((T3 - 2 * h01)/h01)

CNe239h := (1 + h6 * h01)^((T3 - 2 * h01)/h01)

CNM3 := 
\[
\begin{pmatrix}
h5 & h6 \\
h5 * CNe139h + h01 * a * CNe139 & h6 * CNe239h + h01 * a * CNe239
\end{pmatrix}
\]
For a Mixed Time Scale

\[ CA_{10} = \frac{-a * b_1}{h_1 + h_2 * (1 + h_3 * r - h)} \]

\[ CA_{13} = \frac{-a * b_1}{h_1 + h_2 * (1 + h_3 * r - h)} \]

\[ CAD_1 = 0 \]

\[ CA_{23} = \frac{-a * b_1}{h_3 + h_4 * (1 + h_5 * r - h_0)} \]

\[ CA_{26} = \frac{-a * b_1}{h_3 + h_4 * (1 + h_5 * r - h_0)} \]

\[ CAD_2 = 0 \]

\[ CA_{36} = \frac{-a * b_1}{h_5 + h_6 * (1 + h_0 * r - h_0)} \]

\[ CA_{39} = \frac{-a * b_1}{h_5 + h_6 * (1 + h_0 * r - h_0)} \]

\[ CAD_3 = 0 \]
\[
\text{Ce113} := (1 + h1 * h)^{(T1/h)}
\]
\[
\text{Ce213} := (1 + h2 * h)^{(T1/h)}
\]
\[
\text{Ce123} := (1 + h3 * h05)^{(T1/h05)}
\]
\[
\text{Ce223} := (1 + h4 * h05)^{(T1/h05)}
\]
\[
\text{Ce126} := (1 + h3 * h05)^{(T2/h05)}
\]
\[
\text{Ce226} := (1 + h4 * h05)^{(T2/h05)}
\]
\[
\text{Ce136} := (1 + h5 * h01)^{(T2/h01)}
\]
\[
\text{Ce236} := (1 + h6 * h01)^{(T2/h01)}
\]
\[
\text{Ce139h} := (1 + h5 * h01)^{(T3 - h01)/h01)}
\]
\[
\text{Ce239h} := (1 + h6 * h01)^{(T3 - h01)/h01)}
\]

\[
\text{NCMT} :=
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
\text{Ce113} & \text{Ce213} & \text{Ce123} & \text{Ce223} & 0 & 0 \\
h1 * \text{Ce113} & h2 * \text{Ce213} & -h3 * \text{Ce123} & -h4 * \text{Ce223} & 0 & 0 \\
0 & 0 & \text{Ce126} & \text{Ce226} & \text{Ce136} & \text{Ce236} \\
0 & 0 & h3 * \text{Ce126} & h4 * \text{Ce226} & -h5 * \text{Ce136} & -h6 * \text{Ce236} \\
0 & 0 & 0 & 0 & \text{Ce139h} & \text{Ce239h}
\end{pmatrix}
\]

\[
\text{NCBT} :=
\begin{pmatrix}
-\text{CAD1} \\
\text{CA23} - \text{CA13} \\
\text{CAD2} - \text{CAD1} \\
\text{CA36} - \text{CA26} \\
\text{CAD3} - \text{CAD2} \\
h01 * a * (b1 - \text{CA39}) - \text{CAD3}
\end{pmatrix}
\]
NCCLT := LinearSolve[NCMT, NCBT];

NCcLT1 := NCCLT[[1, 1]]

NCcLT2 := NCCLT[[2, 1]]

NCcLT3 := NCCLT[[3, 1]]

NCcLT4 := NCCLT[[4, 1]]

NCcLT5 := NCCLT[[5, 1]]

NCcLT6 := NCCLT[[6, 1]]

NGCLT1 := Table[{(i - 1) * h, NCcLT1 * (1 + h1 * h)^((i - 1) * h/h) + NCcLT2 * (1 + h2 * h)^((i - 1) * h/h) + \frac{-a*b*l}{b0+h2*(1+h0*(h-h0))}, {i, (T1/h) + 1}}

NGCLT2 := Table[{i * h05 + T1, NCcLT3 * (1 + h3 * h05)^((i * h05 + T1)/h05) + NCcLT4 * (1 + h4 * h05)^((i * h05 + T1)/h05) + \frac{-a*b*l}{b3+h4*(1+h05*(h-h05))}, {i, (T2 - T1)/h05}]

NGCLT3 := Table[{i * h01 + T2, NCcLT5 * (1 + h5 * h01)^((i * h01 + T2)/h01) + NCcLT6 * (1 + h6 * h01)^((i * h01 + T2)/h01) + \frac{-a*b*l}{b5+h6*(1+h01*(h-h01))}, {i, (T3 - T2)/h01 - 1}]

NGCLT11 = ListPlot[NGCLT1, PlotRange -> {{-2, T3}, {0, 3}}, PlotStyle -> {RGBColor[.8, 0, .2], PointSize[.018]}];

NGCLT12 = ListPlot[NGCLT2, PlotRange -> {{-2, T3}, {0, 3}}, PlotStyle -> {RGBColor[0, .8, .2], PointSize[.018]}];

NGCLT13 = ListPlot[NGCLT3, PlotRange -> {{-2, T3}, {0, 3}},
A.3 Code for Linear Target Function

Linear Target Function for $hZ$ Time Scales with Fixed Boundary Conditions

For $h=1$

\begin{align*}
\text{NBA19} &:= -\frac{a}{b_1(b_2-h_1)} (w(T_3-h)+b) + \frac{a}{h_2(b_2-h_1)} (1+h+h_1)w + \frac{a}{k_2(b_2-h_1)} (w(T_3-h)+b) + \frac{a}{h_2^2(b_2-h_1)} (1+h+h_2)w \\
\text{NBA19} &:= -\frac{a}{h_1(h_2-h_1)} (1+h+h_1)w + \frac{a}{h_2(h_2-h_1)} (1+h+h_2)w + \frac{a}{h_1^2(h_2-h_1)} (1+h+h_1)w + \frac{a}{h_2^2(h_2-h_1)} (1+h+h_2)w \\
\text{NBe119h} &:= (1+h_1+h)w((T_3-h)/h) \\
\text{NBe219h} &:= (1+h_2+h)w((T_3-h)/h) \\
\text{NBM1} &:= \begin{pmatrix} 1 \\ \text{NBe119h} \end{pmatrix} \\
\text{BBM1} &:= \begin{pmatrix} -\text{NBA19} + b \\ -\text{NBA19} + (T_3-h) + b \end{pmatrix} \\
\text{NBCL1} &:= \text{LinearSolve}[\text{NBM1, BBM1}] \\
\text{NBcL1} &:= \text{NBCL1}[[1, 1]]
\end{align*}
NBCL2 := NBCL1[[2, 1]]

NBGL1 := Table[{i * h - h, NBCL1 * (1 + h1 * h)^(i * h - h)/h} + NBCL2 * (1 + h2 * h)^(i * h - h)/h + \[\frac{h1}{h2} (1 + h1) (1 + h2) + h1 (1 + h1) (1 + h2) + \frac{h1}{h2} (1 + h1) (1 + h2) + h1 (1 + h1) (1 + h2)}{h2 (h2 - h1)}\], {i, T3/h}]

NBGL11 = ListPlot[NBGL1, PlotRange \[\to \{(-2, T3), (-10, (v * T3 + b))\}], PlotStyle \[\to \{RGBColor[.8, 0, .2], PointSize[.018]\\}\]], Export["LinearLongFixed1.eps", NBGL11, "EPS"]

For \(h = .5\)

NBA29 := \(\frac{1}{h3 (h4 - h3)} (v(T3 - h05) + b) + \frac{1}{h3 (h4 - h3)} (1 + h05 + b) + \frac{1}{b4 (b4 - h3)} (1 + h05 + b)\)

NBAD2 := \(\frac{1}{h3 (h4 - h3)} (b) + \frac{1}{h3 (h4 - h3)} (1 + h05 + b) + \frac{1}{b4 (b4 - h3)} (1 + h05 + b)\)

NBe129h := \((1 + h3 * h05)^n ((T3 - h05)/h05)\)

NBe229h := \((1 + h4 * h05)^n ((T3 - h05)/h05)\)

NBM2 := \(\begin{pmatrix} 1 & 1 \\ NBe129h & NBe229h \end{pmatrix}\)

BBM2 := \(\begin{pmatrix} -NBAD2 + b \\ v(T3 - h05) + b - NBA29 \end{pmatrix}\)
NBCL2 := LinearSolve[NBM2, BBM2]

NBcL3 := NBCL2[[1, 1]]

NBcL4 := NBCL2[[2, 1]]

NBGL2 := Table[
  {(z - 1) * h05, NBcL3 * (1 + h3 * h05) * ((h05 * (i - 1))/h05) +
   NBcL4 * (1 + h4 * h05) * ((i - 1) * h05)/h05 +
   (1 + h05 * h3) * ((v * ((i - 1) * h05) + b))/h3 * (b4 - h3) +
   (1 + h05 * h4) * ((v * ((i - 1) * h05) + b))/h4 * (b4 - h3) +
   h3^2 * (b4 - h3) * h4 * (b4 - h3) * (1 + h05 * h4) * v}
  , {i, T3/h05}]

NBGL12 = ListPlot[NBGL2, PlotRange -> {{-2, T3}, {-10, (v * T3 + b)}},
  PlotStyle -> {RGBColor[0, .8, .2], PointSize[.018]}]

Export[*LinearLongFixedh5.eps", NBGL12, "EPS"]

For h = 0.001

NBA39 :=

NBAD3 :=

NBe139h := (1 + h5 * h01) * ((T3 - h01)/h01)

NBe239h := (1 + h6 * h01) * ((T3 - h01)/h01)

NBM3 :=

\[
\begin{pmatrix}
1 & 1 \\
NBe139h & NBe239h
\end{pmatrix}
\]
BBM3:= \[
\begin{pmatrix}
 b - \text{NBAD3} \\
 v \times (T3 - h01) + b - \text{NBA39}
\end{pmatrix}
\]

NBCL3:=LinearSolve[NBM3, BBM3]

NBcL5:=NBCL3[[1,1]]

NBcL6:=NBCL3[[2,1]]

NBGL3:=Table[\{
(i - 1) \times h01, 
NBcL5 \times (1 + h5 \times h01)^{((i - 1) \times h01)/h01}
\} + 
NBcL6 \times (1 + h6 \times h01)^{((i - 1) \times h01)/h01} + 
\frac{-\frac{\theta}{1 + h01 + h5}}{h5(h6-h5)} + 
\frac{\frac{\theta}{1 + h01 + h5}}{h6(h6-h5)} + 
\frac{\frac{\theta}{1 + h01 + h6}}{h6(h6-h5)} \}, 
\{i, T3/h01\}\}

NBGL13 = ListPlot[NBGL3, PlotRange -> \{\{-2, T3\}, \{-10, (v \times T3 + b)\}\},
PlotStyle -> \{RGBColor[.2,0,.8], PointSize[.018]\}]

Export["LinearLongFixedh001.eps", NBGL13, "EPS"]

For a Mixed Time Scale

A10:= \frac{\theta}{h1(h2-h1)}(b) + \frac{\theta}{h1(h2-h1)}(1+h+h1)u + \frac{\theta}{h1(h2-h1)}(1+h+h2)u + \frac{\theta}{h1(h2-h1)}(1+h+h2)u

A13:= \frac{\theta}{h1(h2-h1)}(uT1+b) + \frac{\theta}{h1(h2-h1)}(1+h+h1)u + \frac{\theta}{h1(h2-h1)}(uT1+b) + \frac{\theta}{h1(h2-h1)}(1+h+h2)u

AD1:= \frac{\theta}{h1(h2-h1)}(v) + \frac{\theta}{h1(h2-h1)}(v)

A23:= \frac{\theta}{h3(h4-h3)}(uT1+b) + \frac{\theta}{h3(h4-h3)}(1+h05+h1)u + \frac{\theta}{h3(h4-h3)}(uT1+b) + \frac{\theta}{h3(h4-h3)}(1+h05+h4)u

\frac{\theta}{h4(h4-h3)}(uT1+b)
A26:= \frac{a}{b_3(b_4-h_3)}(v(T_2+b)) + \frac{-a}{b_4(b_4-h_3)}(1+b_05+h_3)u + \frac{-a}{b_4(b_4-h_3)}(vT_2+b)

+ \frac{a}{b_4(b_4-h_3)}(v(T_0+b_05+h_4)u)

AD2:= \frac{a}{b_3(h_4-h_3)}(v) + \frac{-a}{b_4(h_4-h_3)}(v)

A36:= \frac{-a}{b_3(h_4-h_3)}(v(T_2+b)) + \frac{-a}{b_3(h_4-h_3)}(1+b_01+h_4)u + \frac{-a}{b_3(h_4-h_3)}(vT_2+b)

+ \frac{a}{b_4(h_4-h_3)}(v(T_0+b_01+h_6)u)

A39:= \frac{-a}{b_5(h_6-h_5)}(v(T_3-b_01)+b) + \frac{-a}{b_5(h_6-h_5)}(1+b_01+h_5)u + \frac{-a}{b_5(h_6-h_5)}(v(T_3-b_01)+b)

+ \frac{a}{b_6(h_6-h_5)}(v(T_3-b_01)+b)

AD3:= \frac{-a}{b_5(h_6-h_5)}(v(T_3-b_01)+b) + \frac{-a}{b_6(h_6-h_5)}(v(T_3-b_01)+b)

e113:=(1 + h_1 * h)^/(T_1/h)

e213:=(1 + h_2 * h)^/(T_1/h)

e123:=(1 + h_3 * h_05)^/(T_1/h_05)

e223:=(1 + h_4 * h_05)^/(T_1/h_05)

e126:=(1 + h_3 * h_05)^/(T_2/h_05)

e226:=(1 + h_4 * h_05)^/(T_2/h_05)

e136:=(1 + h_5 * h_01)^/(T_2/h_01)

e236:=(1 + h_6 * h_01)^/(T_2/h_01)

e139h:=(1 + h_5 * h_01)^/(T_3-h_01)/h_01)

e239h:=(1 + h_6 * h_01)^/(T_3-h_01)/h_01)

e139:=(1 + h_5 * h_01)^/(T_3/h_01)

e239:=(1 + h_6 * h_01)^/(T_3/h_01)
\[
\begin{align*}
\text{MT:=} & \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
\text{e113} & \text{e213} & -\text{e123} & -\text{e223} & 0 & 0 \\
\text{h1}*\text{e113} & \text{h2}*\text{e213} & -\text{h3}*\text{e123} & -\text{h4}*\text{e223} & 0 & 0 \\
0 & 0 & \text{e126} & \text{e226} & -\text{e136} & -\text{e236} \\
0 & 0 & \text{h3}*\text{e126} & \text{h4}*\text{e226} & -\text{h5}*\text{e136} & -\text{h6}*\text{e236} \\
0 & 0 & 0 & 0 & \text{e139h} & \text{e239h}
\end{pmatrix} \\
\text{BT:=} & \begin{pmatrix}
b - \text{A10} \\
\text{A23} - \text{A13} \\
\text{AD2} - \text{AD1} \\
\text{A36} - \text{A26} \\
\text{AD3} - \text{AD2} \\
\text{v}*(\text{T3} - \text{h01}) + b - \text{A39}
\end{pmatrix}
\end{align*}
\]

\[
\text{CLT:=LinearSolve[MT, BT];}
\]

\[
\begin{align*}
\text{cLT1}:= & \text{CLT[[1, 1]]} \\
\text{cLT2}:= & \text{CLT[[2, 1]]} \\
\text{cLT3}:= & \text{CLT[[3, 1]]} \\
\text{cLT4}:= & \text{CLT[[4, 1]]} \\
\text{cLT5}:= & \text{CLT[[5, 1]]} \\
\text{cLT6}:= & \text{CLT[[6, 1]]}
\end{align*}
\]

\[
\begin{align*}
\text{GLT1}:= & \text{Table}[(i-1)*h, \text{cLT1}* (1 + \text{h1} * h)^((i-1) * h/h) \\
& + \text{cLT2} * (1 + \text{h2} * h)^((i-1) * h/h) + \frac{\text{h2} (1 + \text{h2} * h)}{\text{h1} (\text{h2} - \text{h1})} \\
& + \frac{\text{h2} (1 + \text{h2} * h)}{\text{h1} (\text{h2} - \text{h1})} + \frac{\text{h2} (1 + \text{h2} * h)}{\text{h2} (\text{h2} - \text{h1})} + \frac{\text{h2} (1 + \text{h2} * h)}{\text{h2} (\text{h2} - \text{h1})} ,)
\end{align*}
\]
\{i, (T1/h) + 1\}\]

GLT2 := Table[\{i * h05 + T1, cLT3 * (1 + h3 * h05)^((i * h05 + T1)/h05) + cLT4 * (1 + h4 * h05)^((i * h05 + T1)/h05) + \frac{i * h05 * h05}{h3 (h4 - h3)} + \frac{i * h05 + T1}{h3 (h4 - h3)}\}, {i, (T2 - T1)/h05}]

GLT3 := Table[\{i * h01 + T2, cLT5 * (1 + h5 * h01)^((i * h01 + T2)/h01) + cLT6 * (1 + h6 * h01)^((i * h01 + T2)/h01) + \frac{i * h01 * h01}{h5 (h6 - h5)} + \frac{i * h01 + T2}{h5 (h6 - h5)}\}, {i, ((T3 - T2)/h01 - 1)}]

GLT11 = ListPlot[GLT1, PlotRange \rightarrow \{\{-2, T3\}, \{-10, (v * T3 + b)\}\}, PlotStyle \rightarrow \{RGBColor[.8, 0, .2], PointSize[.018]\}];

GLT12 = ListPlot[GLT2, PlotRange \rightarrow \{\{-2, T3\}, \{-10, (v * T3 + b)\}\}, PlotStyle \rightarrow \{RGBColor[0, .8, 2], PointSize[.018]\}];

GLT13 = ListPlot[GLT3, PlotRange \rightarrow \{\{-2, T3\}, \{-10, (v * T3 + b)\}\}, PlotStyle \rightarrow \{RGBColor[.2, 0, .8], PointSize[.018]\}];

GLT14 = Show[GLT11, GLT12, GLT13]

Export["MixedLinearLongFixed.eps", GLT14, "EPS"]

Linear Target Function for hZ Time Scales with Free Boundary Conditions
For $h=1$

$\text{NA19} := \frac{-\frac{\delta}{h} \exp((v * (T3 - h) + b))}{h1(h2-h1)} + \frac{-\frac{\delta}{h} \exp((1+h*h1))}{h1^2(h2-h1)} + \frac{-\frac{\delta}{h} \exp((v * (T3 - h) + b))}{h2(h2-h1)} + \frac{-\frac{\delta}{h} \exp((1+h*h2))}{h2^2(h2-h1)}$

$\text{NAD1} := \frac{-\frac{\delta}{h} \exp(v)}{h1(h2-h1)} + \frac{-\frac{\delta}{h} \exp(v)}{h2(h2-h1)}$

$\text{Ne119} := (1 + h1 * h)^((T3 - h)/h)$

$\text{Ne219} := (1 + h2 * h)^((T3 - h)/h)$

$\text{Ne119h} := (1 + h1 * h)^((T3 - 2 * h)/h)$

$\text{Ne219h} := (1 + h2 * h)^((T3 - 2 * h)/h)$

$\begin{pmatrix}
  h1 & h2 \\
  h1 * \text{Ne119h} + h * a * \text{Ne119} & h2 * \text{Ne219h} + h * a * \text{Ne219}
\end{pmatrix}$

$\text{NM1} := \begin{pmatrix}
  h1 \\
  h1 * \text{Ne119h} + h * a * \text{Ne119} \\
  h2 \\
  h2 * \text{Ne219h} + h * a * \text{Ne219}
\end{pmatrix}$

$\text{BM1} := \begin{pmatrix}
  h * a ((v * (T3 - h) + b) - \text{NA19}) - \text{NAD1}
\end{pmatrix}$

$\text{NCL1} := \text{LinearSolve}[\text{NM1}, \text{BM1}]$

$\text{NcL1} := \text{NCL1}[1,1]$  

$\text{NcL2} := \text{NCL1}[2,1]$  

$\text{NGL1} := \text{Table}[\{(i - 1) * h, \text{NeL1} * (1 + h1 * h)^((i - 1) * h/h) + \text{NeL2} * (1 + h2 * h)^((i - 1) * h/h) + \frac{-\frac{\delta}{h} \exp((v * (i - 1) * h + b))}{h1^2(h2-h1)} + \frac{-\frac{\delta}{h} \exp((i - 1) * h + b))}{h2(h2-h1)} + \frac{-\frac{\delta}{h} \exp((1+h*h2))}{h2^2(h2-h1)}\}, \{i, T3/h\}]]$

$\text{NGL11} = \text{ListPlot}[\text{NGL1}, \text{PlotRange} \to \{\{-2, T3\}, \{-10, (v * T3 + b)\} \}, \text{PlotStyle} \to \{\text{RGBColor}[.8, 0, .2], \text{PointSize} [.018]\}]$
For $h=0.5$

$$\text{NA29} := \frac{1 + h_{0.5} * (v * (T3 - h_{0.5}) + b)}{h_{3} (b_{4} - h_{3})} + \frac{1 + h_{0.5} * (1 + h_{0.5} * h_{3})}{h_{3} (b_{4} - h_{3})} + \frac{1 + h_{0.5} * (v * (T3 - h_{0.5}) + b)}{h_{4} (b_{4} - h_{3})}$$

$$\text{NAD2} := -\frac{1 + h_{0.5} * (v) - h_{0.5}}{h_{3} (b_{4} - h_{3})}$$

$$\text{Ne129} := (1 + h_{3} * h_{0.5})^{((T3 - h_{0.5})/h_{0.5})}$$

$$\text{Ne229} := (1 + h_{4} * h_{0.5})^{((T3 - h_{0.5})/h_{0.5})}$$

$$\text{Ne129h} := (1 + h_{3} * h_{0.5})^{((T3 - 2 * h_{0.5})/h_{0.5})}$$

$$\text{Ne229h} := (1 + h_{4} * h_{0.5})^{((T3 - 2 * h_{0.5})/h_{0.5})}$$

$$\text{NM2} := \begin{pmatrix} h_{3} & h_{4} \\ h_{3} * \text{Ne129h} + h_{0.5} * a * \text{Ne129} & h_{4} * \text{Ne229h} + h_{0.5} * a * \text{Ne229} \end{pmatrix}$$

$$\text{BM2} := \begin{pmatrix} -\text{NA29} \\ h_{0.5} * a((v * (T3 - h_{0.5}) + b)) - \text{NA29} - \text{NAD2} \end{pmatrix}$$

$$\text{NCL2} := \text{LinearSolve}[\text{NM2}, \text{BM2}]$$

$$\text{NcL3} := \text{NCL2}[[1, 1]]$$

$$\text{NcL4} := \text{NCL2}[[2, 1]]$$

$$\text{NGL2} := \text{Table}[(i - 1) * h_{0.5}, (1 + h_{3} * h_{0.5})^{(((i - 1) * h_{0.5})/h_{0.5})}]$$
\[ + \text{NcL4} \times (1 + h4 \times h05)^h((i - 1) \times h05)/h05) + \frac{1 + h6 \times h05 \times ((i - 1) \times h01 + b)}{h3(h4 - h3)} \]

\[ + \frac{1 + h01 \times h05 \times (1 + h05 + h3)}{h3^2(h4 - h3)} + \frac{h4(h4 - h3)}{h4(h4 - h3)} \]

\{i, T3/h05\}]

NGL12 = ListPlot[NGL2, PlotRange \to \{\{-2, T3\}, \{-10, (v \times T3 + b)\}\},
PlotStyle \to \{RGBColor[0, .8, .2], PointSize[.018]\}]
Export["LinearLongh5.eps", NGL12, "EPS"]

For \(h=0.01\)

\[ \text{NA39} := -\frac{1 + h01 \times h00 \times ((T3 - h01) + b)}{h3(b6 - h5)} + \frac{1 + h01 \times h05 \times (1 + h01 \times h5)}{h5(h6 - h3)} + \frac{1 + h01 \times h05 \times (1 + h01 \times h5)}{h6(b6 - h5)} \]

\[ \text{NAD3} := -\frac{1 + h01 \times h00 \times ((T3 - h01) + b)}{h5(b6 - h5)} + \frac{1 + h01 \times h00 \times (v)}{h6(b6 - h5)} \]

\[ \text{Ne139h} := (1 + h5 \times h01)^h((T3 - 2 \times h01)/h01) \]

\[ \text{Ne239h} := (1 + h6 \times h01)^h((T3 - 2 \times h01)/h01) \]

\[ \text{Ne139} := (1 + h5 \times h01)^h((T3 - h01)/h01) \]

\[ \text{Ne239} := (1 + h6 \times h01)^h((T3 - h01)/h01) \]

\[ \text{NM3} := \begin{pmatrix} \text{h5} & \text{h6} \\ \text{h5} \times \text{Ne139h} + \text{h01} \times a \times \text{Ne139} & \text{h6} \times \text{Ne239h} + \text{h01} \times a \times \text{Ne239} \end{pmatrix} \]

\[ \text{BM3} := \begin{pmatrix} \text{h01} \times a((v \times (T3 - h01) + b) - \text{NA39}) - \text{NAD3} \end{pmatrix} \]
NCL3 := LinearSolve[NM3, BM3]

NcL5 := NCL3[[1, 1]]

NcL6 := NCL3[[2, 1]]

NGL3 := Table[(i * h01, NcL5 * (1 + h5 * h01) + A((z * hOl)/hOl) + NcL6 * (1 + h6 * h01)A(i * hOl/hOl) + hOl2 - hQ1*(1 + h01*h5)), {i, T3/h01}]

NGL13 = ListPlot[NGL3, PlotRange → {{-2, T3}, {-10, (v * T3 + 6)}}, PlotStyle → {RGBColor[.2, 0, .8], PointSize[.018]}]

Export["LinearLongh001.eps", NGL13, "EPS"]

For a Mixed Time Scale

A13 := \(-\frac{a_u u_{T1} + b}{h_1(h_2 - h_1)} + \frac{a_u u_{T1} + b}{h_2(h_2 - h_1)} + \frac{a_u u_{T1} + b}{h_2(h_2 - h_1)} + \frac{a_u u_{T1} + b}{h_2(h_2 - h_1)}\)

AD1 := \(-\frac{a_u u_{T1} + b}{h_2(h_2 - h_1)} + \frac{a_u u_{T1} + b}{h_2(h_2 - h_1)}\)

A23 := \(-\frac{u_{T1} + b}{h_3(h_4 - h_3)} + \frac{u_{T1} + b}{h_3(h_4 - h_3)} + \frac{u_{T1} + b}{h_3(h_4 - h_3)} + \frac{u_{T1} + b}{h_3(h_4 - h_3)}\)

A26 := \(-\frac{u_{T2} + b}{h_3(h_4 - h_3)} + \frac{u_{T2} + b}{h_3(h_4 - h_3)} + \frac{u_{T2} + b}{h_3(h_4 - h_3)} + \frac{u_{T2} + b}{h_3(h_4 - h_3)}\)

AD2 := \(-\frac{u_{T2} + b}{h_3(h_4 - h_3)} + \frac{u_{T2} + b}{h_3(h_4 - h_3)}\)
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\[ A39 := \frac{1 + \beta_0 \gamma \alpha}{h_5 (h_6 - h_5)} \left( (1 + \beta_0 \gamma \alpha) u \right) + \frac{1 + \beta_0 \gamma \alpha}{h_3 (h_6 - h_5)} \left( (1 + \beta_0 \gamma \alpha) u \right) \]

\[ AD3 := \frac{1 + \beta_0 \gamma \alpha}{h_5 (h_6 - h_5)} \left( (1 + \beta_0 \gamma \alpha) u \right) + \frac{1 + \beta_0 \gamma \alpha}{h_5 (h_6 - h_5)} \left( (1 + \beta_0 \gamma \alpha) u \right) \]

\[ e_{113} := (1 + h_1 * h_0)^\gamma (T_1 / h_0) \]

\[ e_{213} := (1 + h_2 * h_0)^\gamma (T_1 / h_0) \]

\[ e_{123} := (1 + h_3 * h_05)^\gamma (T_1 / h_05) \]

\[ e_{223} := (1 + h_4 * h_05)^\gamma (T_1 / h_05) \]

\[ e_{126} := (1 + h_3 * h_05)^\gamma (T_2 / h_05) \]

\[ e_{226} := (1 + h_4 * h_05)^\gamma (T_2 / h_05) \]

\[ e_{136} := (1 + h_5 * h_01)^\gamma (T_2 / h_01) \]

\[ e_{236} := (1 + h_6 * h_01)^\gamma (T_2 / h_01) \]

\[ e_{139h} := (1 + h_5 * h_01)^\gamma ((T_3 - 2 * h_01) / h_01) \]

\[ e_{239h} := (1 + h_6 * h_01)^\gamma ((T_3 - 2 * h_01) / h_01) \]

\[ e_{139} := (1 + h_5 * h_01)^\gamma ((T_3 - h_01) / h_01) \]

\[ e_{239} := (1 + h_6 * h_01)^\gamma ((T_3 - h_01) / h_01) \]

\[ r_1 := h_5 * e_{139h} + h_01 * \alpha * e_{139} \]

\[ r_2 := h_6 * e_{239h} + h_01 * \alpha * e_{239} \]

\[ M := \]
\[
\begin{pmatrix}
h_1 & h_2 & 0 & 0 & 0 & 0 \\
e_{113} & e_{213} & -e_{123} & -e_{223} & 0 & 0 \\
h_1 \ast e_{113} & h_2 \ast e_{213} & -h_3 \ast e_{123} & -h_4 \ast e_{223} & 0 & 0 \\
0 & 0 & e_{126} & e_{226} & -e_{136} & -e_{236} \\
0 & 0 & h_3 \ast e_{126} & h_4 \ast e_{226} & -h_5 \ast e_{136} & -h_6 \ast e_{236} \\
0 & 0 & 0 & 0 & r_1 & r_2 
\end{pmatrix}
\]

\[
B := \begin{cases}
-AD_1 \\
A_{23} - A_{13} \\
0 \\
A_{36} - A_{26} \\
h_01 \ast a((v \ast (T_3 - h_01) + b) - A_{39}) - AD_3
\end{cases}
\]

\[
CL := \text{LinearSolve}[M, B]
\]

\[
cL_1 := CL[[1, 1]]
\]

\[
cL_2 := CL[[2, 1]]
\]

\[
cL_3 := CL[[3, 1]]
\]

\[
cL_4 := CL[[4, 1]]
\]

\[
cL_5 := CL[[5, 1]]
\]

\[
cL_6 := CL[[6, 1]]
\]

\[
GL_1 := \text{Table}[[ (i - 1) \ast h, cL_1 \ast (1 + h_1 \ast h)^\ast((i - 1) \ast h/h) \\
+ cL_2 \ast (1 + h_2 \ast h)^\ast((i - 1) \ast h/h) + \frac{\alpha \ast (1 + h) \ast u}{b_1(h_2 - h_1)} \\
+ \frac{\alpha \ast (1 + h) \ast u}{b_1(h_2 - h_1)} + \frac{\alpha \ast (1 + h) \ast u}{b_2(h_2 - h_1)} + \frac{\alpha \ast (1 + h) \ast u}{b_2(h_2 - h_1)} ]], \\
\{i, (T_1/h + 1)\}]
\]
A.4 Code for Exponential Target Function

Exponential Target Function for $hZ$ Time Scales with Fixed Boundary Conditions
For $h=1$

\[
\begin{align*}
FNB10 &:= \frac{x h (1 + h b)}{(b - h_1)(b_2 - h_1)} + \frac{x h (1 + h b)}{(b - h_2)(b_2 - h_1)}; \\
FNB19h &:= \frac{x h (1 + h b)}{(b - h_1)(b_2 - h_1)} (1 + b * h)^{(T3 - h)/h} \\
&+ \frac{x h (1 + h b)}{(b - h_2)(b_2 - h_1)} (1 + b * h)^{(T3 - h)/h}; \\
Feb0 &:= (1 + b * h)^{(T3 - h)/h}; \\
FNe119 &:= (1 + h_1 * h)^{(T3 - h)/h}; \\
FNe219 &:= (1 + h_2 * h)^{(T3 - h)/h}; \\
FBE1 &:= \begin{pmatrix} 1 & 1 \\
FNe119 & FNe219 \end{pmatrix}; \\
FNE1 &:= \begin{pmatrix} 1 - FNB10 \\
Feb0 - FNB19h \end{pmatrix}; \\
FNCe1 &:= \text{LinearSolve}[FBE1, FNE1]; \\
FNce1 &:=\text{FCe1}[[1, 1]]; \\
FNce2 &:= \text{FCe1}[[2, 1]]; \\
FNGe1 &:= \text{Table}\left\{\left\{i - 1, h, FNCe1 \cdot (1 + h_1 \cdot h)^{(i - 1) \cdot (i - 1)/b} \left(1 + b * h\right)^{(i - 1)/h}\right\}ight. \\
&+ \text{FNce2} \cdot (1 + h_2 \cdot h)^{(i - 1)/h} + \frac{x h (1 + h b)}{(b - h_1)(b_2 - h_1)} (1 + b * h)^{(i - 1)/h} \\
&+ \frac{x h (1 + h b)}{(b - h_2)(b_2 - h_1)} (1 + b * h)^{(i - 1)/h} \left\}, \left\{i, (T3/h)\right\}\right\}; \\
FNGe11 &:= \text{ListPlot}[FNGe1, \text{PlotRange} \to \{-2, T3\}, \{-100, 200\}]\right\};
PlotStyle\[RGBColor[.8,0,.2],PointSize[.018]]\]

Export["exponentialFixedh1.eps", FNGe11, "EPS"];

For \(h=.5\)

\[
FNB29:=\frac{1}{(b-h)(b-h)}(1 + b h05)^{(T3 - h05)/h05}
+ \frac{1}{b-h05}(1 + b h05)^{(T3 - h05)/h05}
\]

\[
FNB20:=\frac{1}{(b-h)(b-h)}(1 + b h05)^{(T3 - h05)/h05}
+ \frac{1}{(b-h)(b-h)}(1 + b h05)^{(T3 - h05)/h05}
\]

\[
FNB29h:=\frac{1}{(b-h)(b-h)}(1 + b h05)^{(T3 - h05)/h05}
+ \frac{1}{(b-h)(b-h)}(1 + b h05)^{(T3 - h05)/h05}
\]

\[
Fneb2:=(1 + b h05)^{(T3 - h05)/h05}
\]

\[
FNe129:=(1 + h3 h05)^{(T3 - h05)/h05}
\]

\[
FNe229:=(1 + h4 h05)^{(T3 - h05)/h05}
\]

\[
FBE2:=\begin{pmatrix} h3 & h4 \\ FNe129 & FNe229 \end{pmatrix}
\]

\[
FNE2:=\begin{pmatrix} 1 - FNB20 \\ Fneb2 - FNB29h \end{pmatrix}
\]

\[
FNCe2:=\text{LinearSolve}[FBE2,FNE2]
\]

\[
FNCe3:=FNCe2[[1,1]]
\]

\[
FNCe4:=FNCe2[[2,1]]
\]
FNGe2 := Table[
(1 + (i - 1) * h05 + h05)^(((i - 1) * h05)/h05)
+ Fncce4 * (1 + h4 * h05)^(((i - 1) * h05)/h05)
+ \frac{1}{(b-h3)}(1 + b * h05)^(((i - 1) * h05)/h05)
+ \frac{1}{(b-h4)}(1 + b * h05)^(((i - 1) * h05)/h05)
, {i, T3/h05}]

FNGe12 = ListPlot[FNGe2, PlotRange -> {{0, T3}, {-10, 200}},
PlotStyle -> {RGBColor[0, .8, .2], PointSize[.018]}]

Export[*exponentialFixedh5.eps*, FNGe12, *EPS*]

For h=.001

FNB30 := \frac{1}{(b-h5)(b-h6)}(1 + b * h01)^((T3 - h01)/h01)
+ \frac{1}{(b-h5)(b-h6)}(1 + b * h01)^((T3 - h01)/h01)
+ \frac{1}{(b-h6)}(1 + b * h01)^((T3 - h01)/h01)

FNB39h := \frac{1}{(b-h5)(b-h6)}(1 + b * h01)^((T3 - h01)/h01)
+ \frac{1}{(b-h6)}(1 + b * h01)^((T3 - h01)/h01)

Fneb3 := (1 + b * h01)^((T3 - h01)/h01)

FNe139 := (1 + h5 * h01)^((T3 - h01)/h01)

FNe239 := (1 + h6 * h01)^((T3 - h01)/h01)

FBE3 := \begin{pmatrix} 1 & 1 \\ FNe139 & FNe239 \end{pmatrix}
\[
\begin{align*}
\text{FNE3:}= & \begin{pmatrix}
1 - \text{FNB30} \\
\text{Fneb3} - \text{FNB39h}
\end{pmatrix} \\
\text{FNce5:}= & \text{LinearSolve}[\text{FBE3}, \text{FNE3}] \\
\text{FNce6:}= & \text{FNCE3}[[1, 1]] \\
\text{FNce6:}= & \text{FNCE3}[[2, 1]] \\
\text{FNGe3:}= & \text{Table}\left[\left\{\left((i - 1) \cdot h01, \text{FNE5} \cdot (1 + h5 \cdot h01)^{((i - 1) \cdot h01)/h01}\right) + \text{FNCE6} \cdot (1 + h6 \cdot h01)^{((i - 1) \cdot h01)/h01}\right.\right. \\
& \left.\left. + \frac{\text{FNE3} \cdot (1 + h01 b)}{(b-h5)(b-h5)} (1 + b \cdot h01)^{((i - 1) \cdot h01)/h01}\right)\right\}, \{i, T3/h01\}\right] \\
\text{FNGe13}= & \text{ListPlot}[\text{FNGe3}, \text{PlotRange} \rightarrow \{0, T3\}, \{-10, 200\}], \quad \text{PlotStyle} \rightarrow \text{RGBColor}[0.2, 0.8, 0.2], \text{PointSize}[0.018]
\end{align*}
\]

For a Mixed Time Scale

\[
\begin{align*}
\text{B13:}= & \frac{\text{FNE3} \cdot (1 + h b)}{(b-h1)(b2-h1)} (1 + b \cdot h)^{T1/h} + \frac{\text{FNE3} \cdot (1 + h b)}{(b-h2)(b2-h1)} (1 + b \cdot h)^{T1/h} \\
\text{B10:}= & \frac{\text{FNE3} \cdot (1 + h b)}{(b-h1)(b2-h1)} + \frac{\text{FNE3} \cdot (1 + h b)}{(b-h2)(b2-h1)} \\
\text{BD13:}= & \frac{\text{FNE3} \cdot (1 + h b)}{(b-h1)(b2-h1)} (1 + b \cdot h)^{T1/h} + \frac{\text{FNE3} \cdot (1 + h b)}{(b-h2)(b2-h1)} (1 + b \cdot h)^{T1/h}
\end{align*}
\]
\[ B23 := \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T1/h05) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T1/h05) \]
\[ B26 := \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T2/h05) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T2/h05) \]
\[ BD23 := \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T1/h05) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T1/h05) \]
\[ BD26 := \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T2/h05) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h05)}{(b - h05)(b4 - h3)} (1 + b * h05)^{\gamma} (T2/h05) \]
\[ B36 := \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} (T2/h01) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} (T2/h01) \]
\[ B39 := \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} ((T3 - h01)/h01) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} ((T3 - h01)/h01) \]
\[ BD36 := \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} (T2/h01) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} (T2/h01) \]
\[ BD39h := \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} ((T3 - 2 * h01)/h01) \]
\[ + \frac{\alpha \cdot \beta \cdot (1 + b * h01)}{(b - h01)(b6 - h5)} (1 + b * h01)^{\gamma} ((T3 - 2 * h01)/h01) \]
\[ eb := (1 + b * h01)^{\gamma} ((T3 - h01)/h01) \]
\[ e113 := (1 + h1 * h)^{\gamma} (T1/h) \]
\[ e213 := (1 + h2 * h)^{\gamma} (T1/h) \]
\[ e123 := (1 + h3 * h05)^{\gamma} (T1/h05) \]
\[ e_{223} := (1 + h_4 \cdot h_0^5)^{(T_1/h_0^5)} \]
\[ e_{126} := (1 + h_3 \cdot h_0^5)^{(T_2/h_0^5)} \]
\[ e_{226} := (1 + h_4 \cdot h_0^5)^{(T_2/h_0^5)} \]
\[ e_{136} := (1 + h_5 \cdot h_0^1)^{(T_2/h_0^1)} \]
\[ e_{236} := (1 + h_6 \cdot h_0^1)^{(T_2/h_0^1)} \]
\[ e_{139} := (1 + h_5 \cdot h_0^1)^{((T_3 - h_0^1)/h_0^1)} \]
\[ e_{239} := (1 + h_6 \cdot h_0^1)^{((T_3 - h_0^1)/h_0^1)} \]

\[
\begin{pmatrix}
  h_1 & h_2 & 0 & 0 & 0 & 0 & 0 \\
  e_{113} & e_{213} & -e_{123} & -e_{223} & 0 & 0 & 0 \\
  h_1 \cdot e_{113} & h_2 \cdot e_{213} & -h_3 \cdot e_{123} & -h_4 \cdot e_{223} & 0 & 0 & 0 \\
  0 & 0 & e_{126} & e_{226} & -e_{136} & -e_{236} & 0 \\
  0 & 0 & h_3 \cdot e_{126} & h_4 \cdot e_{226} & -h_5 \cdot e_{136} & -h_6 \cdot e_{236} & 0 \\
  0 & 0 & 0 & 0 & e_{139} & e_{239} & 0 \\
\end{pmatrix}
\]

\[ F_{M1} := \]

\[
\begin{pmatrix}
  1 - B_{10} \\
  B_{23} - B_{13} \\
  B_{23} - B_{13} \\
  B_{36} - B_{26} \\
  B_{D36} - B_{D26} \\
  e_{b} - B_{39} \\
\end{pmatrix}
\]

\[ F_{B1} := \]

\[ F_{C1} := \text{LinearSolve}[F_{M1}, F_{B1}] \]

\[ F_{C1} := F_{C1}[[1, 1]] \]

\[ F_{C2} := F_{C1}[[2, 1]] \]

\[ F_{C3} := F_{C1}[[3, 1]] \]
Fce4:=FCe[[4, 1]]
Fce5:=FCe[[5, 1]]
Fce6:=FCe[[6, 1]]

FGel:=Table[{(z - 1) * ft, Feel * (1 + hi /h) + Fce2 * (1 + h2 * h)^((i - 1) * h/h) + b * hmi /i}, {i, T1/h + 1}]

FGe2:=Table[{z * h05 + Tl, Fce3 * (1 + h3 * h05)^((i * h05 + T1)/h05) + Fce4 * (1 + h4 * h05)^((i * h05 + T1)/h05) + b * h05 /((b-h2)/(b-h1)), (i - 1)*h/h), {i, (T2 - T1)/h05}}]

FGe3:=Table[{i * h01 + T2, Fce5 * (1 + h5 * h01)^((i * h01 + T2)/h01) + Fce6 * (1 + h6 * h01)^((i * h01 + T2)/h01) + b * h01 /((b-h0)/(b-h0)), (i - 1)*h/h), {i, (T3 - T2)/h01 - 1}]

FGel1 = ListPlot[FGel, PlotRange -> {{0, T3}, {-10, 200}}, PlotStyle -> {RGBColor[.8, 0, .2], PointSize[.018]}];

FGel2 = ListPlot[FGe2, PlotRange -> {{0, T3}, {-10, 200}}, PlotStyle -> {RGBColor[0, .8, .2], PointSize[.018]}];

FGel3 = ListPlot[FGe3, PlotRange -> {{0, T3}, {-10, 200}}, PlotStyle -> {RGBColor[.2, 0, .8], PointSize[.018]}];


\[ FGe14 = \text{Show}[FGe11, FGe12, FGe13] \]
\[ \text{Export["exponentialFixedMixed.eps", FGe14, "EPS"]}; \]

Exponential Target Function for \( hZ \) Time Scales with Free Boundary Conditions

For \( h=1 \)

\[ \text{NBD10} := \frac{\alpha b}{\Gamma(h)(b-h)} \left( 1 + h + b \right) + \frac{\alpha b}{(b-h)(b-h)} \left( 1 + h + b \right) ; \]
\[ \text{NBD19h} := \frac{\alpha b}{\Gamma(h)(b-h)} \left( 1 + h + b \right) \left( 1 + b * h \right)^{((T3 - 2 * h)/h)} \]
\[ + \frac{\alpha b}{(b-h)(b-h)} \left( 1 + b * h \right)^{((T3 - 2 * h)/h)} ; \]
\[ \text{nebl} := (1 + b * h)^{((T3 - h)/h)} ; \]
\[ \text{Nel19} := (1 + h1 * h)^{((T3 - h)/h)} ; \]
\[ \text{Ne219} := (1 + h2 * h)^{((T3 - h)/h)} ; \]
\[ \text{Ne119h} := (1 + h1 * h)^{((T3 - 2 * h)/h)} ; \]
\[ \text{Ne219h} := (1 + h2 * h)^{((T3 - 2 * h)/h)} ; \]

\[ \text{BE1} := \begin{pmatrix} h1 & h2 \\ h1 * \text{Nel19h} + h * a * \text{Ne119} & h2 * \text{Ne219h} + h * a * \text{Ne219} \end{pmatrix} ; \]
\[ \text{NE1} := \begin{pmatrix} -\text{NBD10} \\ h * a (\text{nebl} - \text{NB19}) - \text{NBD19h} \end{pmatrix} ; \]

\[ \text{NCe1} := \text{LinearSolve}[\text{BE1}, \text{NE1}] ; \]
\text{Nce1} := \text{Nce1}[1, 1];
\text{Nce2} := \text{Nce1}[2, 1];
\text{NGe1} := \text{Table}[[\{i - 1\} \ast h, \text{Nce1} \ast (1 \ast h1 \ast h)^{\ast (((i - 1) \ast h)/h)}
+ \text{Nce2} \ast (1 \ast h2 \ast h)^{\ast (((i - 1) \ast h)/h)} + \frac{A\ast h\ast (1 \ast h1 \ast h)}{(b - h1) \ast (b2 - h1)}(1 \ast b \ast h)^{\ast (((i - 1) \ast h)/h)} + \frac{A\ast h\ast (1 \ast h2 \ast h)}{(b - h2) \ast (b2 - h1)}(1 \ast b \ast h)^{\ast (((i - 1) \ast h)/h)}], \{i, (T3//>)}]
\text{NGe11} = \text{ListPlot}[\text{NGe1}, \text{PlotRange} \to \{0, T3\}, \{-10, 200\}]
\text{PlotStyle} \to \{\text{RGBColor}[.8, 0, .2], \text{PointSize}[.018]\}
\text{Export}["exponentialFreeh1.eps", \text{NGe11}, "EPS"]

\text{For} h = .5
\text{NB29} := \frac{A\ast h\ast (1 \ast h05 \ast h)}{(b - h3) \ast (b4 - h3)}(1 \ast b \ast h05)^{\ast ((T3 \ast h05)/h05)}
+ \frac{A\ast h\ast (1 \ast h05 \ast h)}{(b - h4) \ast (b4 - h3)}(1 \ast b \ast h05)^{\ast ((T3 \ast h05)/h05)}
\text{NBD20} := \frac{A\ast h\ast (1 \ast h05 \ast h)}{(b - h3) \ast (b4 - h3)} + \frac{A\ast h\ast (1 \ast h05 \ast h)}{(b - h4) \ast (b4 - h3)}
\text{NBD29h} := \frac{A\ast h\ast (1 \ast h05 \ast h)}{(b - h3) \ast (b4 - h3)}(1 \ast b \ast h05)^{\ast ((T3 \ast 2 \ast h05)/h05)}
+ \frac{A\ast h\ast (1 \ast h05 \ast h)}{(b - h4) \ast (b4 - h3)}(1 \ast b \ast h05)^{\ast ((T3 \ast 2 \ast h05)/h05)}
\text{neb2} := (1 \ast b \ast h05)^{\ast ((T3 \ast h05)/h05)}
\text{Ne129} := (1 \ast h3 \ast h05)^{\ast ((T3 \ast h05)/h05)}
\text{Ne229} := (1 \ast h4 \ast h05)^{\ast ((T3 \ast h05)/h05)}
\text{Ne129h} := (1 \ast h3 \ast h05)^{\ast ((T3 \ast 2 \ast h05)/h05)}
Ne229h := (1 + h4 * h05)^(T3 - 2 * h05)/h05

BE2 := 
\[
\begin{pmatrix}
  h3 & h4 \\
  h3 * Ne129h + h05 * a * Ne129 & h4 * Ne229h + h05 * a * Ne229
\end{pmatrix}
\]

NE2 := 
\[
\begin{pmatrix}
  -NBD20 \\
  h05 * a(neb2 - NB29) - NBD29h
\end{pmatrix}
\]

NCE2 := LinearSolve[BE2, NE2]

NCE3 := NCE2[[1, 1]]

NCE4 := NCE2[[2, 1]]

NGe2 := Table[{(i - 1) * h05, NCE3 * (1 + h3 * h05)^((i - 1) * h05)/h05 + NCE4 * (1 + h4 * h05)^((i - 1) * h05)/h05 + (1 + b * h05)^((i - 1) * h05)/h05, (i - 1) * h05}/h05], {i, T3/h05}]

NGe12 := ListPlot[NGe2, PlotRange -> {{0, T3}, {-10, 200}}, PlotStyle -> {RGBColor[0, .8, .2], PointSize[.018]}]

Export["exponentialFreeh5.eps", NGe12, "EPS"]

For h = .001

NB39 := \[
\frac{(1 + h01 + b)}{(b - h5)(b - h5)} (1 + b * h01)^((T3 - h01)/h01)
\]
\[ + \frac{\alpha h_5}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \frac{\alpha h_6}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \left( 1 + b \times h_01 \right)^{\left( T_3 - h_01 \right)/h_01} \]

\[ NBD30 := \frac{\alpha h_5}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \frac{\alpha h_6}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \left( 1 + b \times h_01 \right)^{\left( T_3 - h_01 \right)/h_01} \]

\[ NBD39h := \frac{\alpha h_5}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \frac{\alpha h_6}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \left( 1 + b \times h_01 \right)^{\left( T_3 - 2 \times h_01 \right)/h_01} \]

\[ neb3 := \left( 1 + b \times h_01 \right)^{\left( T_3 - h_01 \right)/h_01} \]

\[ Ne139h := \left( 1 + h_5 \times h_01 \right)^{\left( T_3 - 2 \times h_01 \right)/h_01} \]

\[ Ne239h := \left( 1 + h_6 \times h_01 \right)^{\left( T_3 - 2 \times h_01 \right)/h_01} \]

\[ Ne139 := \left( 1 + h_5 \times h_01 \right)^{\left( T_3 - h_01 \right)/h_01} \]

\[ Ne239 := \left( 1 + h_6 \times h_01 \right)^{\left( T_3 - h_01 \right)/h_01} \]

\[ BE3 := \begin{pmatrix} h_5 & h_6 \\ h_5 \times Ne139h + h_01 \times a \times Ne139 & h_6 \times Ne239h + h_01 \times a \times Ne239 \end{pmatrix} \]

\[ NE3 := \begin{pmatrix} -NBD30 \\ h_01 \times a \times (neb3 - NB39) - NBD39h \end{pmatrix} \]

\[ NCe3 := \text{LinearSolve}[BE3, NE3] \]

\[ Nce5 := NCe3[[1, 1]] \]

\[ Nce6 := NCe3[[2, 1]] \]

\[ NGe3 := \text{Table}[\{(i - 1) \times h_01, Nce5 \times \left( 1 + h_5 \times h_01 \right)^{\left( (i - 1) \times h_01 \right)/h_01} \}

\[ + Nce6 \times \left( 1 + h_6 \times h_01 \right)^{\left( (i - 1) \times h_01 \right)/h_01} \}

\[ + \frac{\alpha h_5}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \frac{\alpha h_6}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \left( 1 + b \times h_01 \right)^{\left( (i - 1) \times h_01 \right)/h_01} \}

\[ + \frac{\alpha h_5}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \frac{\alpha h_6}{(b - h_6)(b_6 - h_5)} \left( T_3 \right) \left( 1 + b \times h_01 \right)^{\left( (i - 1) \times h_01 \right)/h_01} \} , \{i, T3/h01\} \]
For a Mixed Time Scale

Clear[ce1, ce2, ce3, ce4, ce5, ce6]

\[\begin{align*}
B13: &= \frac{\alpha b}{(b-h_1)(b-h_2)} (1 + b \cdot h)^{(T_1/h)} \\
&+ \frac{\alpha b}{(b-h_2)(b-h_1)} (1 + b \cdot h)^{(T_1/h)} \\
BD10: &= \frac{\alpha b}{(b-h_1)(b-h_2)} (1 + b \cdot h)^{(T_1/h)} \\
&+ \frac{\alpha b}{(b-h_2)(b-h_1)} (1 + b \cdot h)^{(T_1/h)} \\
B23: &= \frac{\alpha b}{(b-h_3)(b-h_4)} (1 + b \cdot h_05)^{(T_1/h_05)} \\
&+ \frac{\alpha b}{(b-h_4)(b-h_3)} (1 + b \cdot h_05)^{(T_1/h_05)} \\
B26: &= \frac{\alpha b}{(b-h_3)(b-h_4)} (1 + b \cdot h_05)^{(T_2/h_05)} \\
&+ \frac{\alpha b}{(b-h_4)(b-h_3)} (1 + b \cdot h_05)^{(T_2/h_05)} \\
BD23: &= \frac{\alpha b}{(b-h_3)(b-h_4)} (1 + b \cdot h_05)^{(T_1/h_05)} \\
&+ \frac{\alpha b}{(b-h_4)(b-h_3)} (1 + b \cdot h_05)^{(T_1/h_05)} \\
BD26: &= \frac{\alpha b}{(b-h_3)(b-h_4)} (1 + b \cdot h_05)^{(T_2/h_05)} \\
&+ \frac{\alpha b}{(b-h_4)(b-h_3)} (1 + b \cdot h_05)^{(T_2/h_05)} \\
B36: &= \frac{\alpha b}{(b-h_3)(b-h_4)} (1 + b \cdot h_01)^{(T_2/h_01)} \\
\end{align*}\]
\[ + \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T2/h)} \]

\[ B39: \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T3-h)/h} \]

\[ + \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T3-h)/h} \]

\[ B36: \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T2/h)} \]

\[ + \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T2/h)} \]

\[ BD39: \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T3-2 \cdot h)/h} \]

\[ + \frac{a}{(b-h)(b-h)} (1 + b \cdot h)^{(T3-2 \cdot h)/h} \]

\[ eb: = (1 + b \cdot h)^{(T3-h)/h} \]

\[ e11: = (1 + h \cdot h)^{(T1/h)} \]

\[ e21: = (1 + h \cdot h)^{(T1/h)} \]

\[ e12: = (1 + h \cdot h)^{(T1/h)} \]

\[ e22: = (1 + h \cdot h)^{(T1/h)} \]

\[ e16: = (1 + h \cdot h)^{(T2/h)} \]

\[ e26: = (1 + h \cdot h)^{(T2/h)} \]

\[ e136: = (1 + h \cdot h)^{(T2/h)} \]

\[ e236: = (1 + h \cdot h)^{(T2/h)} \]

\[ e139: = (1 + h \cdot h)^{(T3-2 \cdot h)/h} \]

\[ e239: = (1 + h \cdot h)^{(T3-2 \cdot h)/h} \]

\[ e139: = (1 + h \cdot h)^{(T3-h)/h} \]

\[ e239: = (1 + h \cdot h)^{(T3-h)/h} \]
\[ \begin{align*} 
\text{s1} &= h5 \cdot e139h + h01 \cdot a \cdot e139 \\
\text{s2} &= h6 \cdot e239h + h01 \cdot a \cdot e239 \\
M1 &= \begin{pmatrix} 
    h1 & h2 & 0 & 0 & 0 & 0 \\
    e113 & e213 & -e123 & -e223 & 0 & 0 \\
    h1 \cdot e113 & h2 \cdot e213 & -h3 \cdot e123 & -h4 \cdot e223 & 0 & 0 \\
    0 & 0 & e126 & e226 & -e136 & -e236 \\
    0 & 0 & h3 \cdot e126 & h4 \cdot e226 & -h5 \cdot e136 & -h6 \cdot e236 \\
    0 & 0 & 0 & 0 & -el36 & -h5 \cdot el36 \\
\end{pmatrix} \\
B1 &= \begin{pmatrix} 
    -BD10 \\
    B23 - B13 \\
    BD23 - BD13 \\
    B36 - B26 \\
    BD36 - BD26 \\
    h01 \cdot a(\text{eb} - B39) - BD39h \\
\end{pmatrix} \\
C &:= \text{LinearSolve}[M1, B1] \\
\text{ce1} &= C[[1, 1]] \\
\text{ce2} &= C[[2, 1]] \\
\text{ce3} &= C[[3, 1]] \\
\text{ce4} &= C[[4, 1]] \\
\text{ce5} &= C[[5, 1]] \\
\text{ce6} &= C[[6, 1]] \\
Gc1 &= \text{Table}[(\{i - 1\} \cdot h, c1 \cdot (1 + h1 \cdot h)^n((i - 1) \cdot h/h) \\
&+ c2 \cdot (1 + h2 \cdot h)^n((i - 1) \cdot h/h) + \frac{a}{(h1+b)(h1+b)}(1 + b \cdot h)^n((i - 1) \cdot h/h)]
\[ Ge2 := \text{Table}\left\{\left(1 + b \ast h\right)^{\left(\frac{i \ast h \ast \left(1 + b \ast h\right)}{h}\right)} + \left(1 + b \ast h\right)^{\left(\frac{i \ast h \ast \left(1 + b \ast h\right)}{h}\right)}\right\}, \left\{i, T1/h + 1\right\}\right]\]

\[ Ge3 := \text{Table}\left\{\left(1 + b \ast h\right)^{\left(\frac{i \ast h \ast \left(1 + b \ast h\right)}{h}\right)} + \left(1 + b \ast h\right)^{\left(\frac{i \ast h \ast \left(1 + b \ast h\right)}{h}\right)}\right\}, \left\{i, \left(T2 - T1\right)/h05\right\}\right]\]

\[ Ge11 = \text{ListPlot}\left[\{\text{Ge1}, \text{PlotRange} \to \{\{0, T3\}, \{-10, 200\}\}\}, \text{PlotStyle} \to \{\text{RGBColor}[.8, 0, .2], \text{PointSize}[.018]\}\right]\]

\[ Ge12 = \text{ListPlot}\left[\{\text{Ge2}, \text{PlotRange} \to \{\{0, T3\}, \{-10, 200\}\}\}, \text{PlotStyle} \to \{\text{RGBColor}[.8, 0, .2], \text{PointSize}[.018]\}\right]\]

\[ Ge13 = \text{ListPlot}\left[\{\text{Ge3}, \text{PlotRange} \to \{\{0, T3\}, \{-10, 200\}\}\}, \text{PlotStyle} \to \{\text{RGBColor}[.2, 0, .8], \text{PointSize}[.018]\}\right]\]

\[ Ge14 = \text{Show}\left[\text{Ge11}, \text{Ge12}, \text{Ge13}\right]\]

\[ \text{Export}["exponentialFreeMixed.eps", \text{Ge14, "EPS"}]; \]
Bibliography


