Dynamics and Asymptotic Behavior of the Solutions of a Nonlinear Differential Equation

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DYNAMICS AND ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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Master of Science

By
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DYNAMICS AND ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS
OF A NONLINEAR DIFFERENTIAL EQUATION

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Initial value problems of the form
\[ \frac{dx}{dt} = t \ x^p, \quad x(a) = \beta \]
are examined, first when \( p = 2 \). Applying Euler's method, a numerical approximation technique, when \( p = 2 \) for certain initial conditions produces a numerical solution which resembles a bifurcation diagram very similar to that produced by the logistic map. Comparisons of such numerical solutions to the logistic map are made, and a partial explanation of such numerical solutions is given. Then, the exact solution of the initial value problem with \( p = 2 \), for which the software package Mathematica 3.0 determines an explicit formula, is analyzed to determine its uniqueness, range of existence, and dependency upon initial conditions. The long-term behavior of the solution is also determined. Solutions of the initial value problem are also analyzed when \( p \) is an integer greater than 2. Conclusions about the behavior of solutions to such initial value problems are made, and such conclusions depend in part upon whether \( p \) is even or odd. Mathematica Version 3.0 was unable to determine formulas for selected problems of this form.
Chapter 1
Introduction

The initial value problem (IVP) to be analyzed is
\[
\frac{dx}{dt} = t - x^2, \quad x(\alpha) = \beta, \quad \text{where}
\]
\[
\alpha \in [0, \infty) \quad \text{and} \quad \beta \in (-\infty, \infty).
\]

This IVP is of the general form
\[
\frac{dx}{dt} = f(t, x), \quad x(\alpha) = \beta.
\]

Some questions of interest about the initial value problem (1.1) which are considered here are the following:

1. What is the behavior of the numerical solution of the IVP for initial condition \( x(0) = 0 \) when Euler's method is applied?

2. What comparisons can be made between the behavior of this numerical solution and the behavior of the logistic map?

3. Can an exact solution of the differential equation of (1.1) be found?

4. Does the IVP (1.1) possess a unique solution which exists for all \( t \in [\alpha, \infty) \), does it fail to exist after a finite time, or does the solution fail to be unique?

5. In what way does the solution of (1.1) depend upon the initial condition \( x(\alpha) = \beta \)?

6. What is the long-term behavior of the solution or solutions produced?

7. Based upon results of the analysis of (1.1) or through other observations, can any conjectures or conclusions be made about initial value problems of the form
\frac{dx}{dt} = t - x^p, \quad x(\alpha) = \beta, \quad p > 2 \quad (1.3)

for which \( p \) is an integer?

Chapter 2 provides the necessary background of definitions, theorems, and techniques used in the analysis. In Chapter 3, the numerical solution to (1.1) for a particular initial condition is discussed and compared to the bifurcation diagram of the logistic map. The exact solution to (1.1) computed by Mathematica is given, and examples of the solution's dependence upon initial conditions are provided. Then the exact solution is analyzed to determine its existence and uniqueness, dependence upon initial conditions, and its long-term behavior. In Chapter 4, the general case of (1.3) is discussed. Existence and uniqueness of solutions to (1.3) are determined, and conclusions about the long-term behavior of these solutions are made, which differ between even and odd \( p \).
Chapter 2
General Theorems and Definitions

• Gamma Function

The Gamma function for some complex independent variable $x$ is defined to be

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} \, ds, \quad \Re x > 0$$

(Lebedev, 1972, p. 1).

• Bessel's Equation and Cylinder Functions

The second order linear differential equation

$$u'' + \frac{i}{z} u' + \left(1 - \frac{\nu^2}{z^2}\right) u = 0, \quad (2.1)$$

where $z$ is a complex independent variable and $\nu$ is a parameter which can take arbitrary real or complex values, is known as Bessel's Equation, and its solutions are known as cylinder functions, which take the form of infinite power series (Lebedev, 1972, p. 98).

• "Big Oh" Notation

We say that $f(z)$ is of order $\varphi(z)$ as $z \to z_0$ and write $f(z) = O(\varphi(z))$ as $z \to z_0$ if the inequality $|f(z)| \leq A |\varphi(z)|$ holds in a neighborhood of $z_0$, where $A$ is some constant. If $z_0$ is not explicitly mentioned, then $z_0 = \infty$. (Lebedev, 1972, p. 11.)

• Airy Functions

Airy functions are solutions to the linear, second-order differential equation
\[ u'' - zu = 0, \quad (2.2) \]

namely,
\[ \text{AiryAi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{3^{2k+1} \Gamma(k+\frac{1}{3})} \sum_{k=1}^{\infty} \frac{z^k}{3^{2k+2} \Gamma(k+\frac{4}{3})}, \quad (2.3) \]

and
\[ \text{AiryBi}(z) = 3^{\frac{1}{2}} \left[ \sum_{k=0}^{\infty} \frac{z^{3k}}{3^{2k+2} \Gamma(k+\frac{4}{3})} + \sum_{k=1}^{\infty} \frac{z^{3k+1}}{3^{2k+3} \Gamma(k+\frac{7}{3})} \right], \quad (2.4) \]

where \(|z| < \infty\) and \(z\) is complex. Each is a solution for specific initial conditions (Lebedev, 1972, p. 137).

The Airy functions can also be given in terms of cylinder functions (Lebedev, 1972, p. 137) in the form
\[ \text{AiryAi}(z) = \frac{z^{\frac{3}{2}}}{\pi} \left[ I_{-\frac{1}{3}} \left( \frac{2z^{\frac{3}{2}}}{3} \right) - I_{\frac{1}{3}} \left( \frac{2z^{\frac{3}{2}}}{3} \right) \right], \quad |\arg z| < \frac{2\pi}{3}, \quad (2.5) \]
or
\[ \text{AiryAi}(z) = \frac{1}{\pi} \left( \frac{z}{3} \right)^{\frac{1}{2}} K_{\frac{1}{3}} \left( \frac{2z^{\frac{3}{2}}}{3} \right), \quad |\arg z| < \frac{2\pi}{3}, \quad (2.6) \]

and
\[ \text{AiryBi}(z) = \left( \frac{z}{3} \right)^{\frac{1}{2}} \left[ I_{-\frac{1}{3}} \left( \frac{2z^{\frac{3}{2}}}{3} \right) + I_{\frac{1}{3}} \left( \frac{2z^{\frac{3}{2}}}{3} \right) \right], \quad |\arg z| < \frac{2\pi}{3}, \quad (2.7) \]

where \(I_\nu(y)\) and \(K_\nu(y)\) are Bessel functions and are solutions to the equation
\[ u'' + \frac{1}{z} u' \left( 1 + \frac{z^2}{y^2} \right) u = 0 \]
(Lebedev, 1972, p. 110), where
\[ I_\nu(y) = \sum_{k=0}^{\infty} \frac{(\frac{y}{3})^{\nu+2k}}{\Gamma(k+1) \Gamma(k+\nu+1)}, \quad |y| < \infty, \quad |\arg y| < \pi, \quad (2.8) \]

and
\[ K_y(y) = \frac{\pi}{2} \frac{I_y(y) - I_{y+1}(y)}{\sin \pi y}, \quad |\arg y| < \pi, \quad y \neq \ldots, -2, -1, 0, 1, 2, \ldots \]  
(2.9)

(Lebedev, 1972, p. 108).

Also, we will use the notation
\[ \text{AiryAiPrime}[z] = \frac{d}{dz} (\text{AiryAi}[z]), \]  
(2.10)

and
\[ \text{AiryBiPrime}[z] = \frac{d}{dz} (\text{AiryBi}[z]). \]  
(2.11)

- Asymptotic representations of \( I_\nu(y) \) and \( K_\nu(y) \).

The following equations give asymptotic representations of \( I_\nu(y) \) and \( K_\nu(y) \) (Lebedev, 1972, p. 121 - 123):

\[ I_\nu(y) = e^y (2\pi y)^{-\frac{1}{2}} \left[ \sum_{k=0}^{n} (-1)^k (\nu, k) (2y)^{-k} + O(\sqrt{-n}) \right], \quad |\arg y| \leq \frac{\pi}{4} - \delta, \]  
(2.12)

and

\[ K_\nu(y) = (\frac{e}{2y})^{\frac{1}{2}} e^{-y} \left[ \sum_{k=0}^{n} (\nu, k) (2y)^{-k} + O(\sqrt{-n}) \right], \quad |\arg y| \leq \pi - \delta, \]  
(2.13)

where \( \delta \) is an arbitrarily small positive number, and

\( (\nu, k) = \frac{(-1)^k}{k!} (\frac{1}{2} - \nu) (\frac{1}{2} + \nu)_k, \quad (\nu, 0) = 1, \) and \( (\nu, k) = (-\nu, k). \)

Also,

\[ (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)}. \]  
(2.14)
\textbf{Lipschitz Condition}

A function \( f(t, y) \) is said to satisfy a \textit{Lipschitz condition} in the variable \( y \) on a set \( D \subseteq \mathbb{R}^2 \) if a constant \( L > 0 \) exists with the property that
\[
|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2|
\]
whenever \((t, y_1), (t, y_2) \in D\). The constant \( L \) is called a \textit{Lipschitz constant} for \( f \) (Burden and Faires, 1997, p. 254).

\textbf{Right Maximal Interval of Existence}

Let \( f(t, x) \) be continuous on a \((t, x)\) - set \( E \), and let \( x = x(t) \) be a solution of
\[
\frac{dx}{dt} = f(t, x)
\]
on an interval \( J \). The interval \( J \) is called a \textit{right maximal interval of existence} for \( x \), if there does not exist an extension of \( x(t) \) over an interval \( J_1 \), where \( J \) is a proper subset of \( J_1 \), and \( J \) and \( J_1 \) have different right endpoints, so that \( x = x(t) \) remains a solution of the differential equation (Hartman, 1973, p.12).

\textbf{Taylor's Theorem}

Let \( f \) be a function and suppose \( f \in C^n [a, b] \), that \( f^{(n+1)} \) exists on \([a, b]\), and that \( x_0 \in [a, b] \). For every \( x \in [a, b] \) there exists a number \( \xi(x) \) between \( x_0 \) and \( x \) with
\[
 f(x) = P_n(x) + R_n(x),
\]
where
\[
P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,
\]
and
\[
 R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x - x_0)^{n+1},
\]
where $R_n(x)$ is the error term (Burden and Faires, 1997, p. 10).

- **Theorem for Satisfaction of Lipschitz Conditions**

  Suppose $f(t, x)$ is defined on a convex set $D \subseteq \mathbb{R}^2$. If a constant $L > 0$ exists with

  $$| \partial_x f(t, x) | \leq L \text{ for all } (t, x) \in D,$$

  then $f$ satisfies a Lipschitz condition on $D$ in the variable $x$ with Lipschitz constant $L$. (Burden and Faires, 1997, p. 255.)

- **Uniqueness Theorem 1**

  Suppose that $D = \{(t, x) \mid a \leq t \leq b, -\infty < x < \infty\}$ and that $f(t, x)$ is continuous on $D$. If $f$ satisfies a Lipschitz condition on $D$ in the variable $x$, then the initial-value problem

  $$\frac{dx}{dt} = f(t, x), \quad a \leq t \leq b, \quad x(a) = \beta, \quad (2.15)$$

  has a unique solution $x(t)$ for $a \leq t \leq b$. (Burden and Faires, 1997, p. 255.)

- **Uniqueness Theorem 2**

  Consider the initial value problem

  $$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0. \quad (2.16)$$

  Assume that $f$ and $\partial_x f(t, x)$ are continuous functions on the rectangular region $R: a < t < b, \ c < x < d$ containing the point $(t_0, x_0)$. Then there exists an interval $|t - t_0| < h$ centered at $t_0$ on which there exists one and only one solution to the differential equation that satisfies the initial condition. (Abell and Brazelton, 1996, p. 85.)
- **Existence Theorem 1**

Given the initial value problem (2.16), suppose \( f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous and \( t_0, x_0 \in \mathbb{R} \). Then, there exists some \( T > t_0 \) such that the IVP has a solution on \([t_0, T)\). (Peano, 1885 – 1886, p. 677 – 685.)

- **Existence Theorem 2**

Let \( f(t, x) \) be continuous on the closure \( \overline{E} \) of an open \((t, x)\)-set and let (2.16) possess a solution \( x = x(t) \) on a right maximal interval \( J \). Then either \( J = [t_0, \infty) \), or \( J = [t_0, \delta) \) with \( \delta < \infty \) and \((\delta, x(\delta)) \in \partial E\), or \( J = [t_0, \delta) \) with \( \delta < \infty \) and \( |x(t)| \rightarrow \infty \) as \( t \rightarrow \delta \). (Hartman, 1973, p. 14.)

**Note:** \( \partial E \) is the boundary of the set \( E \).

- **Intermediate Value Theorem**

Let \( f \) be a continuous function on a domain containing \([a, b]\), with say \( f(a) < f(b) \). Then for any \( y \) in between, \( f(a) < y < f(b) \), there exists \( x \) in \((a, b)\) with \( f(x) = y \). (Strichartz, 1995, p. 130.)

- **Euler's Method**

Burden and Faires (1997, p. 259 – 260) describe Euler's method. The objective of this method is to obtain an approximation to the well-posed initial value problem (2.15) by iterating the difference equation

\[
\begin{align*}
    w_0 &= \beta, \\
    \frac{w_{i+1} - w_i}{h} &= f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1,
\end{align*}
\]

where \( w \) is the iteration variable, \( h \) is the step size used in the approximation, and \( N \) is the number of time steps to be taken in the interval. The step size \( h \) is related to \( N \) by

\[
h = \frac{b - a}{N}.
\]
This method produces a discrete approximation to the solution curve \( x(t) \) at the points \( t_i \), known as mesh points, in the interval \([a, b]\). The mesh points will be evenly distributed throughout the interval \([a, b]\) using a constant step method by choosing a positive integer \( N \) and selecting the mesh points \( \{t_0, t_1, \ldots, t_N\} \) where

\[
t = a + ih, \quad \text{for each } i = 0, 1, \ldots, N,
\]

and the distance between each of the mesh points is the step size.

Euler's Method is derived by using Taylor's Theorem with \( n = 1 \) as follows. Suppose that the unique solution to the initial value problem (2.15) has two continuous derivatives on \([a, b]\), so that for each \( i = 0, 1, \ldots, N - 1 \),

\[
x(t_{i+1}) = x(t_i) + h x'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} x''(\xi_i),
\]

where \( \xi_i \in [t_i, t_{i+1}] \). Since \( h = t_{i+1} - t_i \), then

\[
x(t_{i+1}) = x(t_i) + h x'(t_i) + \frac{h^2}{2} x''(\xi_i),
\]

and because \( x(t) \) satisfies the differential equation in (2.15),

\[
x(t_{i+1}) = x(t_i) + h f(t_i, x(t_i)) + \frac{h^2}{2} x''(\xi_i).
\]

Euler's method approximates the solution \( x(t_i) \) at \( t_i \) for each \( i = 1, \ldots, N \) by using the approximations \( w_i \) in which the error term is deleted. Therefore,

\[
w_0 = \beta,
\]

\[
w_{i+1} = w_i + h f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1.
\]
Chapter 3
Analysis of the Initial Value Problem

The Numerical Solution

The initial value problem (IVP) to be analyzed is

\[
\frac{dx}{dt} = t \quad x^2, \quad x(0) = 0,
\] (3.1)

where \(x\) is some function of the independent variable \(t\). This IVP is discussed briefly by Acheson (1997, p. 43-45). As can be seen, this is a first-order, non-autonomous differential equation with a single initial condition. This IVP is of the form of (2.15) and therefore can be approximated by Euler's method. When the IVP in (3.1) is approximated using Euler's Method with \(w_0 = 0\) and stepsize \(h = 0.05\) on the entire interval \([0, 900]\) as Acheson does (1997, p. 43-45), something unexpected happens. The approximated solution produces a bifurcation-like diagram, as seen in Figure 1.

Figure 1

This bifurcation-like pattern is strikingly similar to the pattern that is produced by the logistic map

\[ x_{n+1} = A \times_n (1 - x_n) \]
in which the asymptotic behavior of the iterates is graphed as the parameter $A$ is varied from 0 to 4, as seen in Figure 2.

Figure 2

This strange behavior seen in Figure 1 is not a characteristic of the solution $x(t)$, since a solution to the IVP in (3.1) can be found. Using Mathematica's differential equation solver \texttt{DSolve} (Wolfram, 1991, p. 783), the following solution to the initial value problem in (3.1) is found to be

$$x( t ) = \frac{\text{AiryAiPrime}[t] + \text{AiryBiPrime}[t]}{\text{AiryAi}[t] + \text{AiryBi}[t]}. \quad (3.2)$$

Also, Mathematica provides a general solution to the differential equation of (1.1), which is

$$x( t ) = \frac{\text{AiryAiPrime}[t] + C \text{AiryBiPrime}[t]}{\text{AiryAi}[t] + C \text{AiryBi}[t]}, \quad (3.3)$$

where $C$ is a constant to be determined by the initial condition.

This function can be confirmed as a solution to the differential equation with initial condition $x(0) = 0$ by differentiating it with respect to $t$ using Mathematica, which gives the result

$$x'( t ) = \frac{t \text{AiryAiPrime}[t] + t \text{AiryBiPrime}[t]}{\text{AiryAi}[t] + \text{AiryBi}[t]} - \frac{\left(\text{AiryAiPrime}[t] + \text{AiryBiPrime}[t]\right)^2}{\left(\text{AiryAi}[t] + \text{AiryBi}[t]\right)^2}. \quad (3.3)$$

After simplification, it is clearly seen that $x'(t)$ is of the form $t - (x(t))^2$. 
The exact solution to this IVP also demonstrates another type of chaotic behavior in its dependence upon initial conditions. Using Mathematica’s `DSolve` procedure with initial conditions \( x(0) = -0.729 \) and \( x(0) = -0.7295 \) shows this behavior.

With \( x(0) = -0.729 \), the solution to the IVP is

\[
x(t) = \frac{\text{AiryAiPrime}(t) + \frac{\text{AiryBiPrime}(t) \left( -729 \frac{3}{4} + 1000 \frac{3}{4} \right)}{729 \frac{3}{4} + 3000 \frac{3}{4}}}{\text{AiryAi}(t) + \frac{\text{AiryBi}(t) \left( -729 \frac{3}{4} + 1000 \frac{3}{4} \right)}{729 \frac{3}{4} + 3000 \frac{3}{4}}},
\]

which after an initial time period, appears to demonstrate asymptotically the same behavior as the solution of the original IVP (see Figure 3).

Figure 3

![Figure 3](image)

However, with \( x(0) = -0.7295 \), the solution is

\[
x(t) = \frac{\text{AiryAiPrime}(t) + \frac{\text{AiryBiPrime}(t) \left( -1459 \frac{3}{4} + 2000 \frac{3}{4} \right)}{1459 \frac{3}{4} + 6000 \frac{3}{4}}}{\text{AiryAi}(t) + \frac{\text{AiryBi}(t) \left( -1459 \frac{3}{4} + 2000 \frac{3}{4} \right)}{1459 \frac{3}{4} + 6000 \frac{3}{4}}},
\]

whose behavior is very different, as can be seen in Figure 4.

Figure 4
The graph of the function has a vertical asymptote, which indicates that the solution to the IVP with initial condition $x(0) = -0.7295$ only exists for finite time. Although the function which provides the solution of the IVP is defined for values of $t$ which are greater than that of the point at which the asymptote occurs, the solution itself does not exist past this point.

The reason for the divergence of these solutions can be understood by considering the direction field graph, shown in Figure 5.
The region in which solutions to the differential equation will have positive slope is the region bounded by the functions $x(t) = t^{\frac{1}{2}}$ and $x(t) = -t^{\frac{1}{2}}$, and as can be seen by the direction field graph, $x(t) = t^{\frac{1}{2}}$ acts as an attractor to solutions of the differential equation because any solution with initial conditions above $x(t) = -t^{\frac{1}{2}}$ will be asymptotically drawn to $x(t) = t^{\frac{1}{2}}$. The function $x(t) = -t^{\frac{1}{2}}$ acts as a repellor, because solutions with initial conditions in a small neighborhood of $x(t) = -t^{\frac{1}{2}}$ will diverge from it. The point of divergence of solutions with initial condition $x(0) = \beta$ appears to exist somewhere between $-0.729$ and $-0.7295$, based upon information presented by the direction field graph and from these observations.

The direction field is also useful in determining the conditions that allow a bifurcation such as that in Figure 1 to take place. In order for the numerical iterations to oscillate with increasing amplitude around a function such as $x(t) = t^{\frac{1}{2}}$, the slopes at all points within a sufficiently small neighborhood must be directed toward the function and these slopes must be steep enough to cause successive iterates to move between regions whose points have negative slopes and regions whose points have positive slopes. As Figure 5 indicates, these conditions are satisfied in sufficiently small
neighborhoods around points which satisfy \( x(t) = t^{\frac{1}{2}} \), and therefore the bifurcation-like pattern in Figure 1 occurs. Using Euler's method with step size 0.05, an iterate which is a sufficient vertical distance from \( x(t) = t^{\frac{1}{2}} \) will have a specific positive or negative slope at its location in the \((t, x)\)-plane. Euler's Method will generate the next iteration by moving in a straight line with the current slope a horizontal distance of 0.05. If this slope is large enough in absolute value, then this next iterate is generated on the opposite side of \( x(t) = t^{\frac{1}{2}} \) where the slopes have the opposite sign. If the magnitude of the slope of this new iterate is large enough, then the next iterate is then produced on the opposite side of \( x(t) = t^{\frac{1}{2}} \), and the process can begin all over again. This process occurs in Figure 1 after the bifurcation takes place and the numerical solution to the initial value problem appears to be two separate curves. It can also be seen that such oscillations could not occur around \( x(t) = -t^{\frac{1}{2}} \) because the slopes above and below this function are directed away from it. Therefore, successive iterates near \( x(t) = -t^{\frac{1}{2}} \) are repelled from the function and cannot oscillate around it.

The bifurcation-like pattern of Figure 1 ends when an iterate above \( x(t) = t^{\frac{1}{2}} \) has a negative slope of sufficiently large magnitude such that the next iterate is produced below \( x(t) = -t^{\frac{1}{2}} \). When such an iterate is generated, then all following iterations diverge towards negative infinity since the slopes of all points below \( x(t) = -t^{\frac{1}{2}} \) are negative.

Using Euler's Method, the numerical solution to the initial value problem can be represented as an iterated function as follows.

Euler's Method applied to the IVP is

\[
x_{n+1} = x_n + h f(t_n, x_n), \quad x_0 = 0,
\]

where

\[
f(t_n, x_n) = t_n - x_n^2.
\]

Then, Euler's Method yields

\[
x_{n+1} = x_n + h (t_n, x_n^2)
\]
\[ x_n + h t_n = h x_n^2. \]

Since \( t_n = h n \), it follows that

\[ x_{n+1} = n h^2 + x_n - h x_n^2 \]

\[ = n h^2 + x_n (1 - h x_n). \]

Comparing the result to the iteration function of the logistic map

\[ x_{n+1} = A x_n (1 - x_n), \]

it can be seen that the iterative functions are very similar, except that the function for the IVP has an explicit dependence upon the number of steps taken \((n)\), which the logistic map does not. This dependency could be part of the reason why the numerical solution to the IVP does not approach a horizontal asymptote, since the presence of the increasing \( n \) value at each time step may prevent this.

### Existence and Uniqueness of the Solution

The first step in the analysis of the initial value problem in (1.2) where

\[ f(t, x) = t x^2, \]

is to determine its range of existence and uniqueness.

Existence Theorem 1 guarantees that the initial value problem has a solution which exists on an interval \([a, T)\) for some \( T > a \) because \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( a, b \in \mathbb{R} \).

By Existence Theorem 2, the initial value problem possesses a solution \( x = x(t) \) on a maximal right interval \( J \), where either \( J = [a, \infty) \), or \( J = [\alpha, \delta) \) with \( \delta < \infty \), and \( |x(t)| \rightarrow \infty \) as \( t \rightarrow \delta^- \).

Therefore, either a solution exists for all time, starting at \( \alpha \), or the solution approaches an asymptote and exists for only a finite time, which will be shown to happen for certain initial conditions.
If \( f(t, x) \) satisfies a Lipschitz condition on a certain region, then it can easily be shown that the initial value problem has a unique solution for some interval \( \alpha \leq t \leq b \).

For this IVP

\[
| \partial_x f(t, x) | = 2x,
\]

which cannot be bounded by any constant for all \((t, x) \in D\), where

\[
D = \{(t, x) | \alpha \leq t \leq b, -\infty < x < \infty\},
\]

and therefore \( f \) does not satisfy a Lipschitz condition. So, uniqueness cannot be shown by Uniqueness Theorem 1. However, Uniqueness Theorem 2 can be used to show that a unique solution exists locally. Since \( f \) and \( \partial_x f(t, x) \) are continuous functions in a rectangle \( R: \alpha < t < b, c < x < d \) containing the initial point \((t_0, x_0)\), then there exists an interval \(|t - t_0| < h\), where \( h \) is some positive number, centered at \( t_0 \), on which there exists a unique solution to the differential equation that satisfies the initial condition.

Uniqueness Theorem 2 can be used to show that the solution is unique as long as it exists as well. The results of the theorem can be extended by examining the solution to the initial value problem at some arbitrary point \((\alpha_1, \beta_1)\) which exists on the solution curve for the initial condition. Since existence of the solution with initial condition \( x(\alpha) = \beta \) is proven on some interval \( J \) for which \( J = [\alpha, \infty) \), or \( J = [\alpha, \delta) \) with \( \delta < \infty \), then the only way in which the solution can fail to be unique on its domain of existence is for the solution to split into two or more solutions. Therefore, to prove that the solution is unique as long as it exists, a proof by contradiction is given as follows.
Assume that the solution \( x(t) \) splits into two or more solutions at an arbitrary point \((\alpha_1, \beta_1)\) on the solution curve. Then two distinct solutions with initial condition \( x(\alpha_1) = \beta_1 \) exist. The existence of two such solutions is a contradiction to Uniqueness Theorem 2, however. Therefore, the assumption that the solution can split into two or more solutions at some arbitrary point \((\alpha_1, \beta_1)\) is incorrect, and so, the solution \( x(t) \) must be unique as long as the solution exists.

**Characteristics of the Solution**

The direction field graph indicates that the solution is dependent upon the initial conditions. Some initial \( x \) value exists, called the point of divergence and denoted \( \beta_d \), for which each solution with initial condition \( x(\alpha) = \beta \) and \( \beta > \beta_d \) approaches \( x(t) = t^{\frac{1}{2}} \) as \( t \) approaches infinity, and each solution with initial condition \( x(\alpha) = \beta \) and \( \beta < \beta_d \) approaches some vertical asymptote.

First, the point of divergence for an initial condition \( x(0) = \beta \), denoted \( \beta_d \), is found, then \( \beta_d \) is determined for an arbitrary initial condition \( x(\alpha) = \beta \).

**Point of Divergence for \( x(0) = \beta \)**

Our goal is to try to find a point on the \( x \)-axis, \( \beta_d \), for which solutions with initial conditions \( x(0) = \beta > \beta_d \) exhibit vastly different behavior than solutions with initial conditions \( x(0) = \beta < \beta_d \), where the general solution is given by (3.3).

A solution for the constant of integration in terms of the initial condition \( x(0) = \beta \), \( C_0 \), is found to be

\[
C_0 = \frac{-3^{\frac{1}{2}} \, \Gamma\left(\frac{2}{3}\right) - \beta \, 3^{\frac{1}{2}} \, \Gamma\left(\frac{1}{3}\right)}{\beta \, 3^{\frac{1}{2}} \, \Gamma\left(\frac{1}{3}\right)} - 3 \, \Gamma\left(\frac{2}{3}\right). \tag{3.4}
\]

Now a general solution to the initial value problem

\[
\frac{dx}{dt} = t \cdot x^2, \quad x(0) = \beta
\]

is...
\[
X(t) = \frac{\text{AiryAiPrime}[t] + \frac{3^\frac{1}{3} \Gamma\left(\frac{2}{3}\right)}{\beta \frac{3^\frac{2}{3} \Gamma\left(\frac{1}{3}\right)}{\text{AiryBiPrime}[t]} \text{AiryBi}[t]}}{\text{AiryAi}[t] + \frac{3^\frac{2}{3} \Gamma\left(\frac{1}{3}\right)}{\beta \frac{3^\frac{2}{3} \Gamma\left(\frac{1}{3}\right)}{\text{AiryBi}[t]}} \text{AiryAi}[t]}.
\] 

(3.5)

After analyzing the solution for \( C_0 \), it was found that a particular value of \( \beta \) causes \( C_0 \) to equal zero and another value makes \( C_0 \) undefined.

The value that causes \( C_0 \) to equal zero turns out to be \( x_d \), where

\[
x_d = -\frac{3^\frac{1}{3} \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \approx -0.729011.
\]

(3.6)

The solution to the initial value problem with \( x(0) = x_c \) is

\[
x(t) = \frac{\text{AiryAiPrime}[t]}{\text{AiryAi}[t]},
\]

and is graphed in Figure 6.

Figure 6

The value that makes \( C_0 \) undefined, denoted by \( x_u \), is

\[
x_u = \frac{3^\frac{1}{3} \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \approx 0.729011.
\]

(3.8)

The general solution of the differential equation found by Mathematica is not valid for the initial condition \( x(0) = x_u \). Even though the general form of the solution does not apply for the initial condition \( x(0) = x_u \), the characteristics of the solution to the initial value problem with this initial condition can be determined.
Consider two initial value problems with initial conditions \( x(0) = x_u + \epsilon \) and \( x(0) = x_u - \epsilon \), for \( \epsilon > 0 \) and \( \epsilon << 1 \). For the solutions with each of these initial conditions, the constant \( C_0 \) will not be undefined, and the general form will be valid as a solution. Both these solutions will asymptotically approach \( x(t) = t^{\frac{1}{2}} \), as is proven later in the chapter when the long term behavior of the solutions is considered, and therefore it can be shown that the solution to the IVP with initial condition \( x(0) = x_u \) must also approach \( x(t) = t^{\frac{1}{2}} \).

This is proven by contradiction. Assume that the solution to the IVP with initial condition \( x(0) = x_u \) does not approach \( x(t) = t^{\frac{1}{2}} \). Therefore, because the solutions above and below it do approach \( x(t) = t^{\frac{1}{2}} \), then it must intersect one of the solutions above or below it. But this is a contradiction to Uniqueness Theorem 2. So the solution to the IVP with initial condition \( x(0) = x_u \) must approach \( x(t) = t^{\frac{1}{2}} \).

### Point of Divergence for \( x(\alpha) = \beta \)

We will now determine the point of divergence for an arbitrary initial condition \( x(\alpha) = \beta \).

First, a solution for the constant of integration, \( C_G \), in terms of the initial condition \( x(\alpha) = \beta \) is

\[
C_G = \frac{\text{AiryAiPrime}[\alpha] - \beta \text{AiryAi}[\alpha]}{\beta \text{AiryBi}[\alpha] - \text{AiryBiPrime}[\alpha]}.
\]

The general solution to the initial value problem given by (1.1) is

\[
x(t) = \frac{\text{AiryAiPrime}[t] + \beta \text{AiryAi}[t]}{\beta \text{AiryBi}[t] + \text{AiryBiPrime}[t]}.
\]

The value that causes \( C_G \) to equal zero for a particular initial \( \alpha \) is a value, denoted by \( \beta_d \), which satisfies

\[
\beta_d = \frac{\text{AiryAiPrime}[\alpha]}{\text{AiryAi}[\alpha]}.
\]

This equation, which expresses \( \beta_d \) as a function of \( \alpha \), defines a function which is identical to the solution of the initial value problem with initial condition \( x(0) = x_d \). Therefore, the function given in (3.7),

\[
x_f(t) = \frac{\text{AiryAiPrime}[t]}{\text{AiryAi}[t]},
\]
not only gives the point of divergence for any value of $t$ greater than or
equal to zero but is also the solution to (1.1) with initial condition $x(\alpha) = \beta_d$
because $(\alpha, \beta_d)$ exists on the solution curve (3.7).

The value that makes $C_G$ undefined for a particular initial $\alpha$ is a value,
denoted by $\beta_u$, which satisfies

$$\beta_u = \frac{\text{AiryBiPrime}[a]}{\text{AiryBi}[a]}.$$  \hspace{1cm} (3.13)

This function determines the value which makes $C_G$ undefined for each
initial $\alpha$. For each such initial condition $x(\alpha) = \beta_u$, the general solution is
not valid because $C_G$ is undefined.

Considering the function of (3.13) as a function of time gives

$$x_2(t) = \frac{\text{AiryBiPrime}[t]}{\text{AiryBi}[t]}.$$ \hspace{1cm} (3.14)

This function is interesting because its derivative is

$$x_2'(t) = t \frac{\text{AiryBiPrime}[t]^2}{\text{AiryBi}[t]^2},$$

which satisfies the differential equation in (1.1). Also,

$$\frac{\text{AiryBiPrime}[0]}{\text{AiryBi}[0]} = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{4})}.$$  \hspace{1cm} (3.15)

Therefore equation (3.14), which is graphed in Figure 7,
gives the solution to the initial value problem

\[ \frac{dx}{dt} = t \cdot x^2, \quad x(0) = \frac{3^{\frac{3}{2}} \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} = x_0, \quad (3.16) \]

and gives the solution to (1.1) with initial condition \( x(\alpha) = \beta_0 \).

**Long-Term Behavior of the Solution**

The graphs of solutions to (1.1) for various initial conditions suggest that the solutions have particular characteristics as \( t \) increases. Solutions with initial \( x \) values greater than the before mentioned point of divergence, \( \beta_d \), appear to approach \( x = t^{\frac{1}{2}} \), while solutions with initial \( x \) values less than the point of divergence appear to approach vertical asymptotes and fail to exist after some finite time. Graphs of the solution with initial condition precisely at \( \beta_d \) indicate that this solution asymptotically approaches \( x = -t^{\frac{1}{2}} \). The long-term behavior of the solutions as the variable \( t \) approaches infinity for each of these situations is determined in the following analysis.
Asymptotic Behavior When $x(\alpha) = \beta_u$

We will now consider the asymptotic behavior of the singular solution, given by equation (3.14), which is the solution to (1.1) with initial condition $x(\alpha) = \beta_u$, the value which makes $C_\alpha$ undefined. Lebedev (1972, p. 138) provides the following asymptotic representation for $\text{AiryBi}[t]$:

$$\text{AiryBi}[t] = \frac{t}{\pi^\frac{1}{3}} e^{\left(\frac{2}{3} t^{\frac{3}{2}}\right)} \left[ 1 + O(t^{-\frac{3}{2}}) \right]. \quad (3.17)$$

Now, an asymptotic representation for $\text{AiryBiPrime}[t]$ can be derived as follows: Equation (2.7) gives $\text{AiryBi}[t]$ in terms of cylinder functions. When this expression is differentiated, the following form for $\text{AiryBiPrime}[t]$ is obtained:

$$\text{AiryBiPrime}[t] = \left(\frac{1}{3}\right)^{\frac{1}{3}} \left[ I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) t^{\frac{1}{3}} + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) t^{\frac{1}{3}} \right] + \frac{1}{2(3t)^{\frac{1}{3}}} \left[ I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) \right].$$

$$\quad (3.18)$$

Using the identity $I_{\nu-1}(y) + I_{\nu+1}(y) = 2 I_\nu'(y)$ (Lebedev, 1972, p. 110), where $\nu = \frac{1}{3}$ and $\nu = -\frac{1}{3}$, the derivative of $I_\nu(y)$ can be expressed in terms of the same cylinder functions of different parameters, and the above derivatives of the cylinder functions become

$$I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) = I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) \quad \text{and}$$

$$I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) = I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right).$$

So, using these relations, $\text{AiryBiPrime}[t]$ becomes

$$\text{AiryBiPrime}[t] = \frac{t}{2(3t)^{\frac{1}{3}}} \left[ I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) \right]$$

$$+ \frac{1}{2(3t)^{\frac{1}{3}}} \left[ I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) + I_{\frac{1}{3}} \left(\frac{2i^{\frac{1}{3}}}{3}\right) \right].$$

Replacing the cylinder functions with their asymptotic representations given by equation (2.12) with $n = 0$, the above expression becomes
AiryBiPrime[t] = \frac{t}{2(3)^{\frac{1}{2}}} \left[ \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[1 + O(\ t^{-\frac{3}{2}})\right] \right. \\
+ \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[1 + O(\ t^{-\frac{3}{2}})\right] \\
+ \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[1 + O(\ t^{-\frac{3}{2}})\right] \right. \\
+ \frac{1}{2(3)^{\frac{1}{2}}} \left( \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[1 + O(\ t^{-\frac{3}{2}})\right] \right. \\
+ \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[1 + O(\ t^{-\frac{3}{2}})\right] \right].

This form can be simplified to

AiryBiPrime[t] = \frac{t}{2(3)^{\frac{1}{2}}} \left[ \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[4 + O(\ t^{-\frac{3}{2}})\right] \right]

+ \frac{1}{2(3)^{\frac{1}{2}}} \left[ \left( \frac{3}{4\pi t^2} \right)^{\frac{1}{2}} e^{\left(\frac{2}{3} t^\frac{3}{2}\right)} \left[2 + O(\ t^{-\frac{3}{2}})\right] \right],

because \( O(\ t^{-\frac{3}{2}}) + O(\ t^{-\frac{3}{2}}) = O(\ t^{-\frac{3}{2}}). \)

Further simplification produces

AiryBiPrime[t] = \frac{t^{\frac{1}{2}}}{4(\pi)^{\frac{3}{2}}} e^{\left(\frac{3}{2} t^\frac{3}{2}\right)} \left[4 + O(\ t^{-\frac{3}{2}})\right]

+ \frac{1}{4(\pi)^{\frac{3}{2}}} e^{\left(\frac{3}{2} t^\frac{3}{2}\right)} \left[2 + O(\ t^{-\frac{3}{2}})\right].

(3.19)

Now, using the asymptotic forms of AiryBiPrime[t] and AiryBi[t] given by (3.17) and (3.19), \( x_2(t) \) becomes

\[
x_2(t) = \frac{\frac{t}{4(\pi)^{\frac{3}{2}}} e^{\left(\frac{3}{2} t^\frac{3}{2}\right)} \left[4 + O(\ t^{-\frac{3}{2}})\right] + \frac{t}{4(\pi)^{\frac{3}{2}}} e^{\left(\frac{3}{2} t^\frac{3}{2}\right)} \left[2 + O(\ t^{-\frac{3}{2}})\right]}{\epsilon^{\frac{3}{4}}}.
\]

Simplifying, \( x_2(t) \) reduces to

\[
x_2(t) = \frac{\frac{t}{4(\pi)^{\frac{3}{2}}} \left[4 + O(\ t^{-\frac{3}{2}})\right] + \frac{t}{4(\pi)^{\frac{3}{2}}} \left[2 + O(\ t^{-\frac{3}{2}})\right]}{\epsilon^{\frac{3}{4}}} \left[1 + O(\ t^{-\frac{3}{2}})\right] \left[1 + O(\ t^{-\frac{3}{2}})\right],
\]

and to
Therefore,
\[ x_2(t) = \frac{\frac{1}{i} \left[ 4 + O \left( t^{-\frac{1}{2}} \right) \right] - \frac{1}{2i} \left[ 2 + O \left( t^{-\frac{1}{2}} \right) \right]}{\left[ 1 + O \left( t^{-\frac{1}{2}} \right) \right]}. \]

Finally, taking the limit as \( t \) approaches infinity of (3.20) gives
\[ \lim_{t \to \infty} (x_2(t) \cdot t^{\frac{i}{2}}) = \lim_{t \to \infty} \left( \frac{\frac{1}{i} \left[ 4 + O \left( t^{-\frac{1}{2}} \right) \right] - \frac{1}{2i} \left[ 2 + O \left( t^{-\frac{1}{2}} \right) \right]}{\left[ 1 + O \left( t^{-\frac{1}{2}} \right) \right]} \right) \cdot t^{\frac{i}{2}}. \]

Since \( \lim_{t \to \infty} B \cdot t^{\frac{i}{2}} \cdot O \left( t^{-\frac{3}{2}} \right) = 0 \) for any real constant \( B \), \( x_2(t) \) asymptotically approaches \( x = t^{\frac{1}{2}} \) as \( t \) approaches infinity.

Asymptotic Behavior When \( x(\alpha) = \beta_d \)

Next, it is shown that the solution to the IVP in (1.1) with initial condition \( x(\alpha) = \beta_d \) (the value which makes \( C_G = 0 \)) asymptotically approaches \( x = -t^{\frac{1}{2}} \). The solution is given by equation (3.7). Lebedev (1972, p. 138) provides the following asymptotic representation for \( \text{AiryAi}[t] \):
\[ \text{AiryAi}[t] = \frac{t^{\frac{i}{4}}}{2 \pi^{\frac{3}{4}}} e^{-\left( \frac{2}{3} t^{\frac{1}{2}} \right)} \left[ 1 + O \left( t^{-\frac{1}{2}} \right) \right]. \]

An asymptotic representation for \( \text{AiryAiPrime}[t] \) is derived as follows. Equation (2.6) gives \( \text{AiryAi}[t] \) in terms of cylinder functions. When this expression is differentiated, the following form for \( \text{AiryAiPrime}[t] \) is obtained:
AiryAiPrime[t] = \frac{t}{\pi (3)^{\frac{1}{2}}} K_{\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right) + \frac{l}{2 \pi (3)^{\frac{1}{2}}} K_{\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right). \quad (3.25)

The identity \( K_{\nu-1}(y) + K_{\nu+1}(y) = -2 K_\nu'(y) \) (Lebedev, 1972, p. 110), where \( \nu = \frac{1}{3} \), gives the following form of the derivative of the cylinder function:

\[ K_{\frac{1}{3}}' \left( \frac{2t^{\frac{3}{2}}}{3} \right) = \frac{K_{-\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right)}{2} + \frac{K_{\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right)}{2}. \]

Using this identity, \( \text{AiryAiPrime}[t] \) becomes

\[ \text{AiryAiPrime}[t] = \frac{t}{2 \pi (3)^{\frac{1}{2}}} \left[ K_{-\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right) + K_{\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right) \right] + \frac{l}{2 \pi (3)^{\frac{1}{2}}} \left[ K_{\frac{1}{3}} \left( \frac{2t^{\frac{3}{2}}}{3} \right) \right]. \]

Replacing the cylinder functions with their asymptotic representations given by equation (2.13) with \( n = 0 \), the above expression becomes

\[ \text{AiryAiPrime}[t] = -\frac{t}{2 \pi (3)^{\frac{1}{2}}} \left[ \left( \frac{3\pi}{4t^2} \right)^\frac{1}{2} e^{-\left(\frac{3i}{t}\right)^\frac{1}{2}} [1 + O \left( t^{-\frac{1}{2}} \right)] \right] + \frac{l}{2 \pi (3)^{\frac{1}{2}}} \left[ \left( \frac{3\pi}{4t^2} \right)^\frac{1}{2} e^{-\left(\frac{3i}{t}\right)^\frac{1}{2}} [1 + O \left( t^{-\frac{1}{2}} \right)] \right]. \]

This equation can be simplified to produce

\[ \text{AiryAiPrime}[t] = -\frac{t}{4 (\pi)^{\frac{1}{2}}} e^{-\left(\frac{3i}{t}\right)} [2 + O \left( t^{-\frac{1}{2}} \right)] + \frac{l}{4 (\pi)^{\frac{1}{2}}} e^{-\left(\frac{3i}{t}\right)} [1 + O \left( t^{-\frac{1}{2}} \right)]. \]

\( (3.26) \)

Using these asymptotic forms of \( \text{AiryAi}[t] \) and \( \text{AiryAiPrime}[t] \) given in (3.24) and (3.26), \( x_1(t) \) becomes

\[ x_1(t) = -\frac{t}{4 (\pi)^{\frac{1}{2}}} e^{-\left(\frac{3i}{t}\right)} [2 + O \left( t^{-\frac{1}{2}} \right)] + \frac{l}{4 (\pi)^{\frac{1}{2}}} e^{-\left(\frac{3i}{t}\right)} [1 + O \left( t^{-\frac{1}{2}} \right)]. \]

Simplifying, \( x_1(t) \) reduces to

\[ x_1(t) = -\frac{t}{2} \left[ 2 + O \left( t^{-\frac{1}{2}} \right) \right] + \frac{l}{2} \left[ 1 + O \left( t^{-\frac{1}{2}} \right) \right], \]

and to
Therefore,

\[ x_1(t) = \frac{-t^{1/2} \left[ 2 + O\left( t^{-1/2} \right) \right] + \frac{1}{2t} \left[ 1 + O\left( t^{-1/2} \right) \right]}{\left[ 1 + O\left( t^{-1/2} \right) \right]} . \]

Therefore,

\[ x_1(t) - (- t^{1/2}) = \frac{-t^{1/2} \left[ 2 + O\left( t^{-1/2} \right) \right] + \frac{1}{2t} \left[ 1 + O\left( t^{-1/2} \right) \right]}{\left[ 1 + O\left( t^{-1/2} \right) \right]} \quad (- t^{1/2}). \quad (3.27) \]

Taking the limit as \( t \) approaches infinity of (3.27) gives

\[
\lim_{t \to \infty} (x_2(t) \ (- t^{1/2})) = \lim_{t \to \infty} \left( \frac{-t^{1/2} \left[ 2 + O\left( t^{-1/2} \right) \right] + \frac{1}{2t} \left[ 1 + O\left( t^{-1/2} \right) \right]}{\left[ 1 + O\left( t^{-1/2} \right) \right]} \right) = (- t^{1/4}) ,
\]

(3.28)

\[
= \lim_{t \to \infty} \left( \frac{-t^{1/2} \left[ 2 + O\left( t^{-1/2} \right) \right] + \frac{1}{2t} \left[ 1 + O\left( t^{-1/2} \right) \right]}{\left[ 1 + O\left( t^{-1/2} \right) \right]} \right) + t^{1/2}.
\]

(3.29)

Comparing (3.29) to (3.21), it can be seen that the limit in (3.29) also equals zero. Therefore, \( x_1(t) \) approaches \( x = - t^{1/2} \) asymptotically as \( t \) approaches infinity.

**Asymptotic Behavior When \( C_G \neq 0 \).**

Finally, the asymptotic behavior of the general solution, given by (3.3), for the case in which \( C_G \) is defined and nonzero can be determined.

Equation numbers are given after the corresponding equations in this section. Using the asymptotic representations of all the Airy functions provided in (3.17), (3.19), (3.24), and (3.26), \( x(t) \) becomes
\[
X(t) = \frac{-i^\frac{1}{2}}{4(\pi)^2} \frac{e^{\left(\frac{i}{t}\right)}}{t} \left[ 2 + O(t^{-\frac{1}{2}}) \right] + \frac{i^\frac{1}{2}}{4(\pi)^2} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] \]
\[
+ \frac{\tfrac{i}{2} \pi^\frac{1}{4}}{2(\pi)^\frac{1}{2}} \frac{e^{\left(\frac{i}{t^2}\right)}}{t} \left[ 1 + O(t^{-\frac{1}{2}}) \right] + C \left( \frac{i^\frac{1}{2}}{4(\pi)^2} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] \right)
\]
\]

(3.30)

Multiplying numerators and denominators by \(4 (\pi)^\frac{1}{2} \frac{i}{2} \pi^\frac{1}{4} \frac{e^{(\frac{1}{t^2})}}{t} \) gives

\[
X(t) = \frac{-i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{i}{t^2}\right)}}{t^2} \left[ 2 + O(t^{-\frac{1}{2}}) \right] + \frac{i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] + C \left( \frac{i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] \right) + \frac{C(i^\frac{1}{2} \frac{e^{\left(\frac{i}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right])}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] + C \left( \frac{i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] \right)
\]

Therefore,

\[
X(t) = \frac{-i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{i}{t^2}\right)}}{t^2} \left[ 2 + O(t^{-\frac{1}{2}}) \right] + \frac{i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] + C \left( \frac{i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] \right) + \frac{C(i^\frac{1}{2} \frac{e^{\left(\frac{i}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right])}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] + C \left( \frac{i^\frac{1}{2}}{2 \pi^\frac{1}{2}} \frac{e^{\left(\frac{1}{t^2}\right)}}{t^2} \left[ 1 + O(t^{-\frac{1}{2}}) \right] \right)
\]

(3.31)

Taking the limit of (3.31) as \(t\) approaches infinity gives

\[
\lim_{t \to \infty} \left( X(t) - t^\frac{1}{2} \right)
\]
\[
\lim_{t \to \infty} \left( \frac{-t^{\frac{1}{2}} e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 2 + O\left( t^{-\frac{1}{2}} \right) \right] + e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right]}{2 t e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right] + C4t \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right]} \right) + \frac{C\left( t^{\frac{3}{2}} \left[ 4 + O\left( t^{-\frac{1}{2}} \right) \right] + 2 + O\left( t^{-\frac{1}{2}} \right) \right)}{2 t e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right] + C4t \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right]}
\]

(3.32)

\[
\lim_{t \to \infty} \left( \frac{-t^{\frac{1}{2}} e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 2 + O\left( t^{-\frac{1}{2}} \right) \right] + e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right]}{2 t e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right] + C4t \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right]} \right) + \frac{C\left( t^{\frac{3}{2}} \left[ 4 + O\left( t^{-\frac{1}{2}} \right) \right] + 2 + O\left( t^{-\frac{1}{2}} \right) \right)}{2 t e^{\left(\frac{\pi}{4} t^{\frac{1}{2}}\right)} \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right] + C4t \left[ 1 + O\left( t^{-\frac{1}{2}} \right) \right]}
\]

(3.33)
30

\[
\lim_{t \to -\infty} \left( \frac{-t^{\frac{1}{2}} e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[2 + O\left(t^{-\frac{1}{2}}\right)\right] + e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right]}{2 t e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C4t [1 + O\left(t^{-\frac{1}{2}}\right)]} \right)
\]

\[
+ \frac{4 C t^{\frac{3}{2}} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right]}{2 t e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C4t [1 + O\left(t^{-\frac{1}{2}}\right)]}
\]

\[
+ \frac{-2 t^{\frac{1}{2}} e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] - C4t^{\frac{3}{2}} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right]}{2 t e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C4t [1 + O\left(t^{-\frac{1}{2}}\right)]}
\]

(3.34)

In (3.34), the terms \(4C t^{\frac{3}{2}}\) and \(-4C t^{\frac{3}{2}}\) are in bold type and have a zero sum, so

\[
\lim_{t \to \infty} (x(t) \frac{t^{\frac{1}{2}}}{t^{\frac{1}{2}}}) = \lim_{t \to \infty} \left( \frac{-t^{\frac{1}{2}} e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[2 + O\left(t^{-\frac{1}{2}}\right)\right] + e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right]}{2 t e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C4t [1 + O\left(t^{-\frac{1}{2}}\right)]} \right)
\]

\[
+ \frac{C t^{\frac{3}{2}} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C \left[2 + O\left(t^{-\frac{1}{2}}\right)\right]}{2 t e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C4t [1 + O\left(t^{-\frac{1}{2}}\right)]}
\]

\[
+ \frac{-2 t^{\frac{1}{2}} e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] - C4t^{\frac{3}{2}} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right]}{2 t e^{\left(\frac{1}{2} t^{\frac{1}{2}}\right)} \left[1 + O\left(t^{-\frac{1}{2}}\right)\right] + C4t [1 + O\left(t^{-\frac{1}{2}}\right)]}
\]

(3.35)

The denominators in the three fractions in (3.35) are the same, and it can be seen that all terms in the denominators approach zero in the limit except for \(4 \ C t\), which approaches infinity or negative infinity. Therefore, the common denominator has no finite limit. A constant (the number \(2 \ C\)) exists in one of the numerators. Also, all the other terms in the numerators in (3.35)
approach zero in the limit except for the $t^{\frac{1}{2}} O(t^{-\frac{1}{2}})$ terms. Such terms approach either a nonzero constant or zero. Therefore, the numerators in (3.35) either sum to zero or approach constants in the limit. So,

$$\lim_{t \to \infty} (x(t) \ t^{\frac{1}{2}}) = \frac{0}{\infty} = 0,$$

or

$$\lim_{t \to \infty} (x(t) \ t^{\frac{1}{2}}) = \frac{D}{\infty} = 0$$

for some nonzero constant $D$. In either case, the limit equals zero, which indicates that the formula for the solution $x(t)$ approaches $x = t^{\frac{1}{2}}$ as $t$ approaches infinity.

It is critical to note that the preceding results indicate that the function of $t$ given by (3.10) approaches $x = t^{\frac{1}{2}}$ as $t$ approaches infinity. As a function of $t$, (3.10) can be undefined at some point and still asymptotically approach $x = t^{\frac{1}{2}}$, but as a solution to the IVP in (1.1), if (3.10) is undefined at some value $t_0$, then the solution cannot exist past $t_0$. The cases in which the solution to (1.1) does not exist for all $t \in [0, \infty)$ are considered next.

**Solutions Which Fail to Exist After Some Finite Time**

Now it is proven that any solution to (1.1) fails to exist after finite time for all $\beta < \beta_d$ by first showing that the denominator of (3.10) must equal zero at some finite value of $t$ at which the numerator is nonzero.

Equation (3.9) gives the general form of the constant, $C_G$, for arbitrary initial conditions of (1.1). The value which makes $C_G$ zero, $\beta_d$, is given by (3.11), and the value which makes $C_G$ undefined, $\beta_u$, is given by (3.13). It can be easily determined that

$$C_G \geq 0 \text{ for } \beta \in [\beta_d, \beta_u], \text{ and } C_G < 0 \text{ for } \beta \in (-\infty, \beta_d) \cup (\beta_u, \infty).$$

Figure 8 provides the graph of (3.9) as a function of $\beta$ for $\alpha = 0$. 
Therefore, for all initial conditions chosen with \( \beta < \beta_d \), the general solution given by (3.10) will have the form

\[
x(t) = \frac{\text{AiryAiPrime}(t) + C_G \text{AiryBiPrime}(t)}{\text{AiryAi}(t) + C_G \text{AiryBi}(t)}, \quad \text{where } C_G < 0.
\]  

(3.36)

Consider the denominator of (3.36), and let

\[ j(t) = \text{AiryAi}(t) + C_G \text{AiryBi}(t), \quad \text{where } C_G < 0. \]  

(3.37)

At the initial value \( t = \alpha \),

\[ j(\alpha) = \text{AiryAi}(\alpha) + C_G \text{AiryBi}(\alpha). \]  

(3.38)

From (3.38) it can be determined that \( j(\alpha) > 0 \) when \( C_G > -\frac{\text{AiryAi}(\alpha)}{\text{AiryBi}(\alpha)} \), and taking the limit of (3.9) as \( \beta \) approaches negative infinity produces

\[
\lim_{\beta \to -\infty} C_G = \lim_{\beta \to -\infty} \beta \left( \frac{\text{AiryAiPrime}(\alpha) - \beta \text{AiryAi}(\alpha)}{\beta \text{AiryBi}(\alpha) - \text{AiryBiPrime}(\alpha)} \right) = \frac{\text{AiryAi}(\alpha)}{\text{AiryBi}(\alpha)}. \]  

(3.39)

Therefore, for all \( \beta \) chosen satisfying \( \beta < \beta_d \), \( j(\alpha) > 0 \). Now considering \( j(t) \) for \( t > \alpha \), and taking the limit of (3.37) as \( t \) approaches infinity produces

\[
\lim_{t \to \infty} j(t) = \lim_{t \to \infty} (\text{AiryAi}(t) + C_G \text{AiryBi}(t)) = -\infty,
\]  

(3.40)

because, as \( t \) approaches infinity, \( \text{AiryAi}(t) \) approaches zero, and \( \text{AiryBi}(t) \) approaches positive infinity, which are indicated by their asymptotic expansions discussed previously. Also, \( j(t) \) is continuous since both
AiryAi[t] and AiryBi[t] are continuous. Therefore, since \( j(t) \) is initially positive and approaches negative infinity as \( t \) approaches infinity, then by the Intermediate Value Theorem, \( j(t) = 0 \) for some finite \( t_j > 0 \).

Also, the numerator of (3.36) can be shown to be nonzero for all \( t \in [0, \infty) \) as follows: AiryAi[t] can be written as equation (2.6), which is in terms of the \( K_v(z) \) Bessel function, where \( z \) is a complex variable. According to Lebedev (1972, p. 127), \( K_v(z) \) has no zeroes in the region \( |\arg z| < \frac{\pi}{2} \), which contains all \( t \) such that \( t \in (0, \infty) \). Therefore, (2.6) indicates that \( \text{AiryAi}[t] \neq 0 \) for \( t > 0 \), and since \( \text{AiryAi}[0] > 0 \), then \( \text{AiryAi}[t] > 0 \) for all \( t \in [0, \infty) \). Since \( \text{AiryAi}[t] \) is a solution to (2.2), then the second derivative of \( \text{AiryAi}[t] \) is \( t \text{AiryAi}[t] \), which is also positive for all \( t \in (0, \infty) \). Since \( t \text{AiryAi}[t] \) is the first derivative of \( \text{AiryAiPrime}[t] \), then \( \text{AiryAiPrime}[t] \) is strictly increasing for all \( t \in (0, \infty) \), and since \( \text{AiryAiPrime}[0] < 0 \) and \( \text{AiryAiPrime}[t] \) approaches zero as \( t \) approaches infinity, which can be seen by equation (3.26), then \( \text{AiryAiPrime}[t] < 0 \) for all \( t \in (0, \infty) \). Similarly, it can be shown that \( \text{AiryBiPrime}[t] \) is always less than zero, so an indeterminate form does not occur where \( j(t) = 0 \). Therefore, since \( j(t) = 0 \) at some finite time, the solution to (1.1) with \( \beta < \beta_d \) will approach a vertical asymptote and fail to exist wherever \( j(t) = 0 \).

It can also be shown that any solution to (1.1) with \( \beta \) satisfying \( \beta > \beta_d \) cannot approach a vertical asymptote because \( x_1(t) \), the solution to (1.1) with \( \beta = \beta_d \), given by (3.12), exists for all \( t \in [0, \infty) \). This is true because \( \text{AiryAi}[t] \) and \( \text{AiryAiPrime}[t] \) are never zero for all \( t \in [0, \infty) \), as was shown earlier. Therefore, since \( x_1(t) \) is defined for all \( t \in [0, \infty) \), then any solution to (1.1) which begins above \( x_1(t) \) can never cross \( x_1(t) \), since this would be a violation of the uniqueness of the solutions to (1.1). Therefore, any solution to (1.1) with \( \beta \) satisfying \( \beta > \beta_d \) cannot approach a vertical asymptote and therefore exists for all \( t \in [0, \infty) \).

It can now be concluded that solutions of (1.1) with initial conditions \( x(\alpha) = \beta \) when \( \beta < \beta_d \) fail to exist after finite times, solutions with initial conditions \( x(\alpha) = \beta_d \) asymptotically approach \( x = -t^{\frac{1}{2}} \) as \( t \) approaches infinity, and all other solutions asymptotically approach \( x = t^{\frac{3}{2}} \) as \( t \) approaches infinity.
Chapter 4

General case in which \( f(t,x) = t - x^p \)

The general case in which the initial value problem is of the form of (1.3) and \( p \) is an integer greater than two is considered. The \textit{DSolve} routine of \textit{Mathematica 3.0} is unable to find solutions to selected problems of this form, so the analysis is conducted without the explicit formulas of the solution. First, the existence and uniqueness of the solutions of (1.3) are determined.

Existence and Uniqueness of the General Case

Since \( f(t,x) = t - x^p \) is a continuous function of the form \( f : R \times R \rightarrow R \), and \( \alpha, \beta \in R \), then the solutions can be shown to exist on some interval \( J \) such that \( J = [\alpha, \infty) \), or \( J = [\alpha, \delta) \) with \( \delta < \infty \) as with solutions to (1.1). The solutions to (1.3) can also be shown to be unique on their entire domain of existence in a similar manner.

Characteristics of the solutions

First, it is shown that the solutions to (1.3) with \( f_2(t,x) = t - x^p \) and initial condition \( x(1) = 1 \) and appropriately chosen domains for the variables \( t \) and \( x \) are bounded above by the solution to the initial value problem given in (1.1), with \( f_1(t,x) = t - x^2 \), and \( x(1) = 1 \). We will also show the solutions are bounded below by the solution to

\[
\frac{dx}{dt} = f_3(t,x) \equiv 0, \quad \text{with} \quad x(1) = 1. \tag{4.1}
\]

Note that this solution is the constant function \( x = 1 \).

For the following argument, the points \( (t,x) \) such that \( t \in [1, \infty) \) and \( x \in [1, \infty) \) with \( x \) chosen such that \( x \leq t^{\frac{1}{p}} \) are considered. Thus, it follows that for all such \( (t,x) \),

\[
f_3(t,x) \leq f_2(t,x) \leq f_1(t,x), \quad \text{or} \quad 0 \leq t - x^p \leq t - x^2.
\]
Since a unique solution to (1.3) with initial condition \( x(1) = 1 \) can be shown to exist, then its solution must be bounded by the solutions of (1.1) with \( x(1) = 1 \) and (4.1) with the same initial condition. Such solutions to (1.3) cannot approach vertical asymptotes positively and fail to exist in finite times since the slope of the solution curve, \( f_2(t, x) \), is negative when \( x > t^{\frac{1}{p}} \), and solutions cannot approach vertical asymptotes negatively, since \( f_2(t, x) \) is non-negative in the region under consideration, which implies that such solutions cannot move downward. Therefore, each such solution to (1.3) exists for \( t \in [1, \infty) \) since the solutions to (1.1) and (4.1) both exist for \( t \in [1, \infty) \).

We can further show that solutions to (1.3) with appropriately chosen initial conditions approach positive infinity as \( t \) approaches infinity, but no more rapidly than \( x = t^{\frac{1}{p}} \) for \( t \in [0, \infty) \) and \( x \in [0, \infty) \).

Case 1: Odd \( p, t \in [0, \infty) \) and \( x \in [0, \infty) \)

We will first show this in the case when \( p \) is odd. Let \((t, x)\) be a point such that \( t \in [0, \infty) \) and \( x > t^{\frac{1}{p}} \). The solution of the differential equation in (1.3) passing through such points \((t, x)\) has slope \( f(t, x) = t - x^p < 0 \). This is also true for all sufficiently small open sets about such points. Therefore, the continuous solution to the differential equation in (1.3) passing through any such point \((t, x)\) is decreasing, and the solution is strictly decreasing if \( x > t^{\frac{1}{p}} \). Therefore, the solution to (1.3) must cross the curve \( x = t^{\frac{1}{p}} \). At a point on the curve \( x = t^{\frac{1}{p}} \), \( f(t, x) = 0 \), so the tangent line to the solution curve is horizontal, and the solution moves below \( x = t^{\frac{1}{p}} \).

Now, consider the points \((t, x)\) with \( t \in [0, \infty) \) and \( x < t^{\frac{1}{p}} \) and \( x \geq 0 \). The solution passing through each such point \((t, x)\) has slope \( f(t, x) = t - x^p > 0 \), and this is also true for all sufficiently small open sets about each such point. Therefore, the continuous solution to the differential equation in (1.3) passing through any such point \((t, x)\) is increasing. Such a solution is now bounded above by the curve \( x = t^{\frac{1}{p}} \), since the slope of the solution would be zero on this curve. It can be shown that the solution must approach positive infinity as \( t \) approaches infinity by first proving that it cannot be bounded above by any horizontal line.
Proving by contradiction, assume that the solution is bounded above by some horizontal line. The solution is strictly increasing for $0 \leq x < t^{\frac{1}{p}}$ because

$$f(t,x) = t - x^p = 0$$

only for $x = t^{\frac{1}{p}}$. So, if the strictly increasing solution is bounded above, then the solution must approach some horizontal line $x = c$ asymptotically as $t$ approaches infinity, and $f(t,x)$, which defines the slope of the tangent line to the solution curve at any point $(t,x)$ on the curve, must approach zero. However,

$$\lim_{t \to \infty} f(t,c) = \lim_{t \to \infty} (t - c^p) = \infty.$$  \hspace{1cm} (4.2)

The implication is that if the solution is approaching some constant $x = c$, then the slope of the tangent line to the solution is approaching infinity. Therefore, a contradiction exists. This contradiction implies that the assumption that the solution is bounded above is false, and therefore, the solution cannot be bounded above by any constant function. Therefore, a solution to (1.3) must approach positive infinity as $t$ approaches infinity, but no more rapidly than $x = t^{\frac{1}{p}}$ because the solution is bounded above by $x = t^{\frac{1}{p}}$.

Even better, it can be shown that a solution that exists below $x = t^{\frac{1}{p}}$ asymptotically approaches $x = t^{\frac{1}{p}}$ as $t$ approaches infinity. First we will show by contradiction that the solution curve cannot remain bounded above by any curve of the form $x = t^{\frac{1}{p}} - c$ for any positive constant $c$. So, by contradiction, we assume that some real, positive constant $c_1$ exists such that the solution is bounded above by $x = t^{\frac{1}{p}} - c_1$. This assumption implies that for some sufficiently large, positive constant $M$,

$$x(t) \leq t^{\frac{1}{p}} - c_1 \quad \text{for all} \quad t \geq M.$$ 

First, we note that the first derivative of $x = t^{\frac{1}{p}} - c$ for all real constants $c$ approaches zero as $t$ approaches infinity, and we consider $f(t,x) = t - x^p$ evaluated when $x = t^{\frac{1}{p}} - c_1$. After some time $t_0$, $t^{\frac{1}{p}} - c_1$ will be positive, so the following argument is made for $t > t_0$. Now,
\[ f(t, \frac{t^p}{t^p} - c_1) = t^p \left( \left( \frac{t^p}{t^p} - c_1 \right)^p. \right. \]

Using the binomial formula to simplify this produces
\[
f(t, \frac{t^p}{t^p} - c_1) = t - (t - c_1) + p \left( t - c_1 \right) + \frac{p!}{(p-2)!2!} (-c_1)^2 t^{p-2} + \ldots + (-c_1)^p,
\]
and
\[
f(t, \frac{t^p}{t^p} - c_1) = c_1 p t^{p-1} - \frac{p!}{(p-2)!2!} (-c_1)^2 t^{p-2} - \ldots - (-c_1)^p.
\]

Taking the limit as \( t \) approaches infinity produces
\[
\lim_{t \to \infty} f(t, \frac{t^p}{t^p} - c_1) = \lim_{t \to \infty} \left( c_1 p t^{p-1} - \frac{p!}{(p-2)!2!} (-c_1)^2 t^{p-2} - \ldots - (-c_1)^p \right) = \infty,
\]
(4.3)

because the leading term \( c_1 p t^{p-1} \) determines the behavior of \( f(t, x) \) in the limit. Therefore, the function \( f(t, x) = t - x^p \) approaches infinity along the curve \( x = t^{\frac{1}{p}} - c_1 \), which indicates that the slope of a solution curve \( x(t) \) below \( x = t^{\frac{1}{p}} - c_1 \) is also approaching infinity as \( t \) approaches infinity. Since the first derivative of \( x = t^{\frac{1}{p}} - c_1 \) approaches zero as \( t \) approaches infinity, the solution \( x(t) \) is forced to cross the curve \( x = t^{\frac{1}{p}} - c_1 \). Thus we have a contradiction since it is assumed that \( x = t^{\frac{1}{p}} - c_1 \) is an upper bound for \( x(t) \). Therefore, no \( c_1 \) exists such that \( x(t) \) is bounded above by a curve of the form \( x = t^{\frac{1}{p}} - c_1 \), which implies that the solution is not bounded above by any such curve of the form \( x = t^{\frac{1}{p}} - c \) where \( c \) is a real, positive constant.

Now, we show by contradiction that if a solution exists below \( x = t^{\frac{1}{p}} \), then it asymptotically approaches \( x = t^{\frac{1}{p}} \) as \( t \) approaches infinity. We want to show that for all \( \epsilon > 0 \), some sufficiently large, positive constant \( H \) exists such that for all \( t \geq H \),
\[
| t^{\frac{1}{p}} - x(t) | \leq \epsilon,
\]
(4.4)
\[ x(t) \geq \frac{1}{p} \bar{t} - \epsilon, \quad (4.5) \]

since \( \frac{1}{p} - x(t) \) positive because \( x(t) \) is below \( \frac{1}{p} \bar{t} \). By contradiction, we assume that some \( \epsilon > 0 \) exists such that for all \( H \), some value \( t_H \) exists such that \( t_H \geq H \), and

\[ | \frac{1}{p} t x(t) | > \epsilon, \quad \text{that is} \]
\[ x(t) < \frac{1}{p} \bar{t} \quad \epsilon. \quad (4.6) \]

Since it has been shown that \( x(t) \) cannot be bounded above by any curve of the form \( x = \frac{1}{p} t - c \), then it cannot be bounded above by \( x = \frac{1}{p} \bar{t} - \epsilon \).

Therefore, \( x(t) \) must cross \( x = \frac{1}{p} \bar{t} - \epsilon \) and move above it, and then by the assumption, \( x(t) \) must again cross \( x = \frac{1}{p} \bar{t} - \epsilon \) and move below \( x = \frac{1}{p} \bar{t} - \epsilon \).

Now consider for sufficiently large \( t \)

\[ \bar{t} = \sup \{ t \in [t^*, \hat{t}] : x(t) \geq \frac{1}{p} \bar{t} - \epsilon \}, \quad (4.8) \]

where \( t^* \) is the value of \( t \) at which \( x(t) = \frac{1}{p} \bar{t} - \epsilon \) and \( x(t) \) moves above \( \frac{1}{p} \bar{t} - \epsilon \), and \( \hat{t} \) is some value of \( t \) at which \( x(t) < \frac{1}{p} \bar{t} - \epsilon \). Now,

\[ x(\bar{t}) \geq \frac{1}{p} \bar{t} - \epsilon, \quad (4.9) \]

and for \( t \in (\bar{t}, \hat{t}] \),

\[ x(t) < \frac{1}{p} \bar{t} - \epsilon. \quad (4.10) \]

However,

\[ x'(t) > \frac{1}{p} t^{\frac{1}{p} - 1} \quad (4.11) \]

for all \( t \in (\bar{t}, \hat{t}] \) since \( \frac{1}{p} t^{\frac{1}{p} - 1} \) approaches zero for the large values of \( t \) which are being considered, and because \( f(t, x) = t - x^p \) increases as \( x \) approaches zero from above for fixed values of \( t \), and as shown before by (4.3), \( f(t, \frac{1}{p} \bar{t} - \epsilon) = t - \left( \frac{1}{p} \bar{t} - \epsilon \right)^p \) is becoming very large. When we integrate from \( \bar{t} \) to \( \hat{t} \),
\[ \int_0^t x'(t) \, dt > \int_0^t \frac{1}{p} t^{\frac{1}{p} - 1} \, dt , \]  
we get
\[ x(\bar{t}) - x(\bar{t}) > \bar{t}^{\frac{1}{p}} - \bar{t}^{\frac{1}{p}} . \]  
Also, from (4.9), we get
\[ x(\bar{t}) \bar{t}^{\frac{1}{p}} + \epsilon \geq x(\bar{t}) - x(\bar{t}) > \bar{t}^{\frac{1}{p}} - \bar{t}^{\frac{1}{p}} . \]  
And so,
\[ x(\bar{t}) \bar{t}^{\frac{1}{p}} + \epsilon > \bar{t}^{\frac{1}{p}} - \bar{t}^{\frac{1}{p}}, \]  
which implies that
\[ x(\bar{t}) > \bar{t}^{\frac{1}{p}} + \epsilon . \]  
This inequality however, is a contradiction. Therefore, \( x(t) \) asymptotically approaches \( x = \bar{t}^{\frac{1}{p}} \) as \( t \) approaches infinity.

**Case 2: Even \( p, t \in [0, \infty) \) and \( x \in [0, \infty) \)**

The preceding arguments are valid when \( p \) is even. When \( t \in [0, \infty) \) and \( x \in [0, \infty) \), solutions to (1.3) behave in the same manner. The difference between the cases arises when \( x < 0 \).

**Case 3: Odd \( p, t \in [0, \infty) \) and \( x < 0 \)**

When \( p \) is odd, then \( f(t, x) = t - x^p > 0 \) for all points \((t, x)\) where \( t \in [0, \infty) \) and \( x < 0 \). Solutions to (1.3) for odd \( p \) which exist in the \((t, x)\) plane for which \( t \geq 0 \) and \( x < 0 \) are strictly increasing, and must cross the \( t \)-axis since they cannot be bounded above by \( x = 0 \). After crossing the \( t \)-axis, the solutions behave as described before. Figure 9 provides the direction field for \( f(t, x) = t - x^p \), which is similar to the direction field of all such differential equations of the form given in (1.3) for odd \( p \). The solid line is the curve \( x = t^{\frac{1}{p}} \).

Figure 9
Case 4: Even $p$, $t \in [0, \infty)$ and $-t^{\frac{1}{p}} < x < 0$

When $p$ is even, then $f(t, x) = t - x^p = 0$ when $x = t^{\frac{1}{p}}$ or $x = -t^{\frac{1}{p}}$. For all points $(t, x)$ between these curves, $f(t, x) = t - x^p > 0$, and any solutions which exist at the points $(t, x)$ satisfying $t \in [0, \infty)$ and $-t^{\frac{1}{p}} \leq x \leq t^{\frac{1}{p}}$ are strictly increasing and then behave in the way described by preceding arguments when $p$ is even. At the points $(t, x)$ which do not satisfy $t \in [0, \infty)$ and $-t^{\frac{1}{p}} \leq x \leq t^{\frac{1}{p}}$, $f(t, x) = t - x^p < 0$. Solutions that pass through such points are discussed in the following case.
Case 5: Even $p, t \in [0, \infty)$ and $x < -t^{\frac{1}{p}}$

The behavior of solutions existing beneath $x = -t^{\frac{1}{p}}$ are undetermined. If initial conditions of (1.3) with even $p$ are chosen in which $\beta < 0$ and $|\beta| >> \alpha$, then

$$f(t, x) = t - x^p \approx -x^p.$$  

The differential equation

$$\frac{dx}{dt} = -x^p, \quad \text{with even } p \text{ and } p > 2,$$  

has a solution

$$x(t) = \left(\frac{1}{(p-1)(t+\omega)}\right)^{\frac{1}{p-1}},$$  

where $\omega$ is the constant of integration. This solution to (4.17) approaches a vertical asymptote and fails to exist after a finite time for any initial conditions of the form $x(\alpha) = \beta$ where $\alpha \geq 0$ and $\beta < 0$ because

$$\omega = \left(\frac{1}{(p-1)\beta^{p-1}}\right)^\alpha < 0$$  

for such initial conditions and because (4.17) is autonomous. Therefore, a solution to (1.3) with even $p$ and $\beta < 0$ and $|\beta| >> \alpha$ behaves much like the function in (4.18), approaches a vertical asymptote, and fails to exist after some finite time. Figure 10 provides the direction field for $f(t, x) = t - x^6$, which is similar to the direction field of all such differential equations of the form given in (1.3) for even $p$. The solid lines are the curves $x = t^\frac{1}{6}$ and $x = -t^\frac{1}{6}$. 
Therefore, when $p$ is odd, all solutions to (1.3) which exist in the $(t, x)$ - plane for which $t \geq 0$ must asymptotically approach $x = t^{1/p}$ as $t$ approaches infinity. When $p$ is even, solutions to (1.3) which exist on or above the curve $x = -t^{1/p}$ must also asymptotically approach $x = t^{1/p}$ as $t$ approaches infinity, and certain solutions below $x = -t^{1/p}$ must fail to exist after finite time.
Appendix

The following Mathematica codes are used in the text.

- Euler's Method Code

The following code produces the bifurcation-like diagram of Figure 1.

\[
f[t_, x_] := t - x^2;
\]

\[
h = 0.05;
\]

\[
step[{t_, x_}] := {t + h, x + h f[t, x]};
\]

\[
data = NestList[step, {0., 0.}, 18000];
\]

\[
ListPlot[data]
\]

(Skeel and Keiper, 1993, p. 329-330.)

- Logistic Map Code

The following code produces the bifurcation diagram of the logistic map in Figure 2.
\[fa[x_] := a x (1 - x);\]

Clear[a];
ListPlot[
 Flatten[Table[
 Transpose[{Table[a, {17}],
   NestList[fa, Nest[fa, 0.5, 10], 16}]
 }],
 {a, 0, 4, 0.01}
 ], 1],
 PlotStyle -> PointSize[0.001];

(Gray and Glynn, 1991, p. 101-104.)

- **Direction Field Plot Statement**

The following Mathematica statements are used to produce the direction field graph for \(f(t,x) = t - x^2\). The `PlotVectorField` statement can easily be modified to produce the direction field graphs of other functions.

\[<< \text{Graphics`PlotField`;}\]

\[\text{PlotVectorField[}\{1, t - x^2\}, \{t, 0, 4\}, \{x, -2, 2\},\]
\[\text{Axes} \rightarrow \text{Automatic, ScaleFunction} \rightarrow (1&),\]
\[\text{HeadLength} \rightarrow 0, \text{HeadWidth} \rightarrow 0,\]
\[\text{PlotPoints} \rightarrow 25]\]

- **Mathematica's DSolve Statement**

Wolfram (1991, p.783) provides the following form for the `DSolve` statement:

\[\text{DSolve[eqn, y[x], x].}\]

Initial conditions are given as equations and the `DSolve` statement to solve an initial value problem is
For example, the DSolve statement which gives the solution of the differential equation in (1.1) is the following command.

\[
\text{DSolve} [x'[t] == t - x[t]^2, x[t], x]
\]

The DSolve statement which gives the solution of the initial value problem (3.1) is the following command.

\[
\text{DSolve} \{x'[t] == t - x[t]^2, x[0] == 0\}, x[t], x]
\]
References


