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Diving Into Reliable Numerical Observability and Stabilization of the One-dimensional Wave Equation

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DIVING INTO RELIABLE NUMERICAL OBSERVABILITY AND STABILIZATION OF THE ONE-DIMENSIONAL WAVE EQUATION

A Capstone Experience/Thesis Project Presented in Partial Fulfillment of the Requirements for the Degree Bachelor of Arts with Mahurin Honors College Graduate Distinction at Western Kentucky University

By

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Abstract

In this project, a one-dimensional wave equation, which is a partial differential equation (PDE) describing vibrations on a string, is considered. It is known that the PDE model is exactly observable and exponentially stabilizable. The main goal of this project is to construct a numerical approximation technique, so-called the direct filtering technique, to prove that the Finite Difference and Finite Element space-discretized 1-D wave equations (i) with homogeneous Dirichlet boundary conditions are uniformly observable, (ii) with controlled boundary conditions are uniformly exponentially stable, as the approximation parameters tend to zero. It is crucial to develop reliable numerical approximation techniques for the controlled systems modeled by wave equations so that engineers can design reliable and robust controllers and sensors.

I dedicate this thesis to my parents, Joe and Angela Moore, and my sister, Evelyn, for always supporting me in everything I do. I also dedicate it to my boyfriend, Jacob, for all his support, patience, and advice over the last few years. I couldn't have done it without you all.

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D.J. Price, E. Moore, A.O. Ozer, Boundary Control of a 1D Wave Equation by the Filtered Finite-Difference Method, Wolfram Demonstrations Project. Published: June 3, 2020.

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1 Introduction

Consider a vibrating string of length L clamped at both ends, as in Fig. 1. The equation of motion is described by a partial differential equation of the one-dimensional wave equation with Dirichlet boundary conditions

$$\begin{cases} w_{tt} - c^2 w_{xx} = f(x,t), & (x,t) \in (0,L) \times \mathbb{R}^+ \\ w(0,t) = 0, \quad w(L,t) = u(t), & t \in \mathbb{R}^+ \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in (0,L) \end{cases} \tag{1}$$

where $u(t)$ is a boundary controller, $f(x,t)$ is a distributed controller, and c is the speed of the wave propagations of vibrations on the string. The initial and boundary value problem (1) is often used to model the control/stabilization of vibrations on a string of length L which is fixed on both ends, see Figure 1.

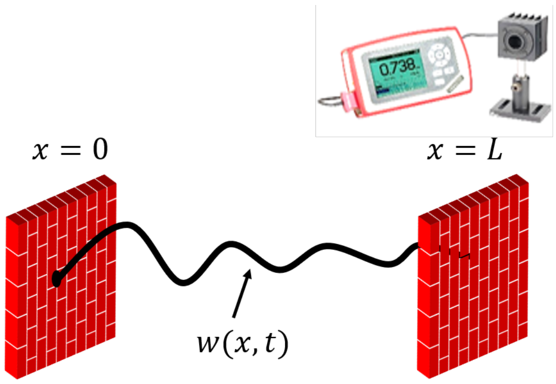

Figure 1: Waves on the string can be controlled based on the observer's measurement at the right end. For the choice of boundary conditions in (1), the slope of the displacement at the tip, i.e., $u_x(L,t)$, is measured in real-time and fed back into the controller to suppress vibrations on the string.

The investigation of exact observability/stabilization of (1) by a boundary $u(t)$ or a distributed $f(x,t)$ observer/controller aids understanding other mathematical physics problems using the same wave equation model (1). For example, longitudinal vibrations on a piezoelectric beam [9] without magnetic effects, which is a smart beam that can be used as both an actuator and sensor, can be modeled exactly the same as (1). Additionally, sound vibrations inside a perfectly insulated duct also follow the same model (1).

Exact controllability of systems is simply to apply an external force (or forces), a controller, to steer the system from any initial state to a final state in a finite amount of time. Exactly controlling the sound or mechanical vibrations propagating on a finite medium has been a major research problem in the field of control theory of partial differential equations. *Exact observability* of systems in control theory is measuring a certain system property for a finite time to be able to distinguish two different initial states. It is well-known that exact controllability and exact observability are dual problems of each other [6, 7]. In other words, proving exact observability is the same as proving exact controllability, and vice versa. Therefore, proving the exact boundary controllability problem (1), i.e. $f(x,t) \equiv 0$, is equivalent to proving the exact boundary observability inequality:

$$\int_0^T |z_x(L,t)|^2 dt \geq C(T) \int_0^L (|z_0(x)|^2 + |z_1(x)|^2) dx \tag{2}$$

for the control-free initial and boundary problem

$$\begin{cases} z_{tt} - c^2 z_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ z(0,t) = 0, \quad z(L,t) = 0, & t \in \mathbb{R}^+ \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x), & x \in (0,L) \end{cases} \tag{3}$$

where $T > 0$ is the observation time, $z_x(L,t)$ is the observed quantity, and $C(T)$ is the observability constant depending on T. For the wave-type equations, T is supposed to be large enough, i.e., $T > \frac{2L}{c}$, which is twice the total distance traveled from one end to the other divided by the speed at which the wave propagates.

Additionally, in control theory, *exponential stabilizability* is an important property, where one looks for a boundary-type state controller, i.e., $u(t) = -\alpha w_t(L,t)$ with $\alpha > 0$:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ w(0,t) = 0, \quad w_x(L,t) = -\alpha w_t(L,t), & t \in \mathbb{R}^+ \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in (0,L) \end{cases} \tag{4}$$

to steer any initial state to decay exponentially to equilibrium state (rest position) in finite time. This is only obtained if the infinitely many eigenvalues have strictly negative real parts, which means that a finite input or non-zero initial condition will not cause the system to ”blow up” (i.e., give an unbounded output). Moreover, for a fixed, finite input, oscillations in the output will decay at an exponential rate and the output will tend asymptotically to a different final, steady-state value. In order to prove that a system is exponentially stable, it must first be proved to be exactly observable, and therefore, exactly controllable.

Partial differential equations such as (1) are infinitely dimensional systems because they have infinitely many eigenvalues [9], so exact controllability and observability of vibrations and exponential stabilization are all infinite-dimensional control problems. In practice, sensors (or observers) work through algorithms on the chip, and therefore, they are doing calculations in finite dimensions (i.e., the computer world). However, sensors only observe a finite number of vibrational modes. For that reason, the sensor, designed for the observed quantity $z_x(L,t)$ in (2) at the tip of the string, must have an algorithm in its chip based on a finite-dimensional numerical approximation of (2). This project will be focusing solely on the semi-discretization of the space variable x in (1), not the time variable t. The exact observability inequality (2) is proved rigorously in Section 2, mimicking the steps in [6]. However, this does not hold for the well-known numerical approximations of (1) such as Finite Difference Method, Finite Element Method, or Finite Volumes Method [4, 10, 13]. The major issue in showing the observability of the discretized system is losing the uniform gap between two consecutive eigenvalues that exists naturally for the infinite dimensional system. The existence of the uniform gap allows us to rely on vibrational observations measured by the sensors. When this gap is not present due to the numerical approximation, the sensor cannot distinguish one vibrational mode from another. When this happens, the system is not observable.

A previous undergraduate student member of Dr. Özer’s research group, investigated the issues that arise from blindly applying the Finite Element Method (FEM) [3] via a senior thesis project. A spectral analysis is performed on the FEM discretization of (1); that is, the eigenvalues and eigenfunctions of the approximated system are precisely described. In doing so, it is found that, despite holding up slightly better than the Finite Difference Method, applying the Finite Element Method by itself still does not uphold the uniform gap condition among any two consecutive eigenvalues, as can been seen in Figure 2. This project picks up where [3] left off. The lack of the uniform gap condition among the all eigenvalues, which implies lack of observability, is shown in [2, 3], but the lack of observability is not proved. This requires proving several non-trivial energy estimates. Moreover, the exponential stabilizability is also investigated completely.

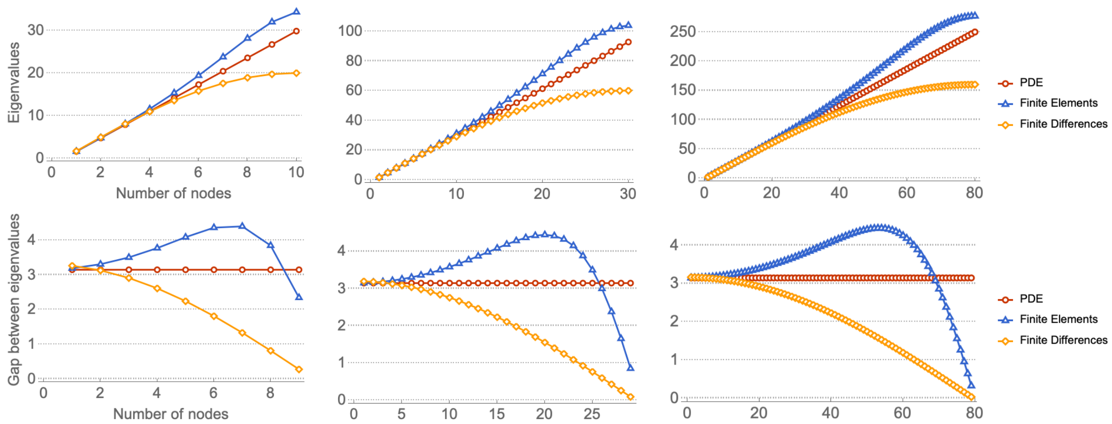

Figure 2: (a) Eigenvalues $\lambda_1, ..., \lambda_k, ..., \lambda_N$ of (1), Finite Differences, and Finite Elements for $N = 10, 30, 80$. (b) The gap between two consecutive eigenvalues $|\lambda_{k+1} - \lambda_k| \to 0$ as $N \to \infty$.

In other words, this project will prove that systems of equations modelling the vibrations on a string retain the observability/controllability properties, meaning the vibrations can be controlled in a finite time, after discretization only when a filtering technique is used to eliminate some of the eigenvalues. It will also show that the system discretized using the Finite Difference Method has the exponential stabilizability property, meaning that, with the addition of a control term in the boundary conditions, the initial energy decays exponentially to an equilibrium.

To illustrate the necessity of the filtering in the Finite-Difference discretization of (1), consider a string of length 1 attached to a wall on the left end $x = 0$, and free at the other end $x = L$. The boundary damping-type observed quantity, $v_t(L,t)$ is fed back to system as in (4). Once, the string is exposed to an initial conditions comprised of high-frequency sinusoidal waves, which is the worst kind of scenarios, the overall vibrations are simulated for $T = 10$ seconds and two main cases are observed as follows

- Filtering vs. non-filtering with the boundary damping.
- Boundary damping vs. non-boundary damping under the effect of filtering.

The simulations of these two cases are implemented via the recently published Wolfram Demonstration Project [12], and the references therein.

In contrast to the exponential decay results for the solutions of (4), Figures 3 and 5 show that the existence of the boundary controller at the tip of the string alone is not enough to stabilize the system to the equilibrium. The vibrations still stays on the string after $T = 10$ seconds if there is no filtering. This is due to the previously mentioned loss of the gap condition and lack of observability of the system. Figure 5 proved that the exponential stabilizability of (4) can only be achieved with the implementation of the filtering.

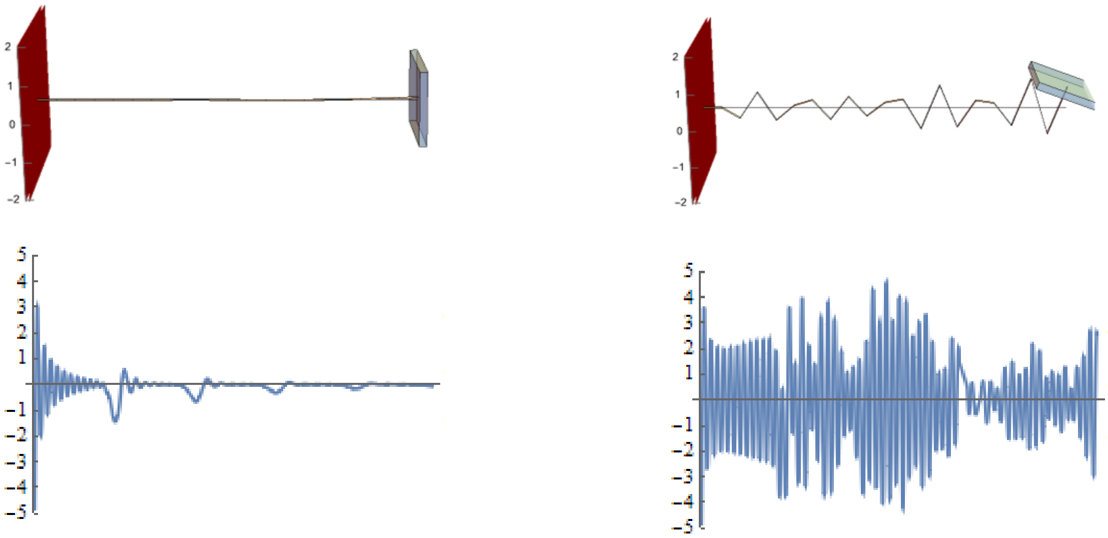

Figure 3: [Column 1] Vibrations on the string (top) and tip velocity (bottom) with indirect filtering and boundary damping. [Column 2] Vibrations on the string (top) and tip velocity (bottom) with boundary damping and no filtering.

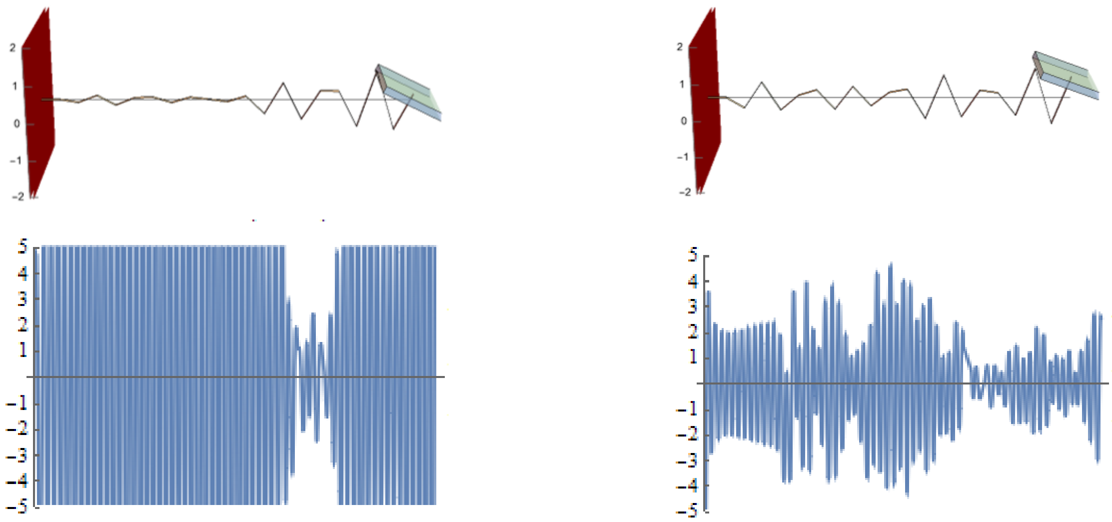

Figure 4: [Column 1] Vibrations on the string at time T (top) and tip velocity (bottom) without filtering and boundary damping. [Column 2] Vibrations on the string at time T (top) and tip velocity (bottom) with no filtering but boundary damping.

Figure 4 demonstrates that filtering by itself allows the system to be controlled to the equilibrium, and that a controller on the boundary hastens the process.

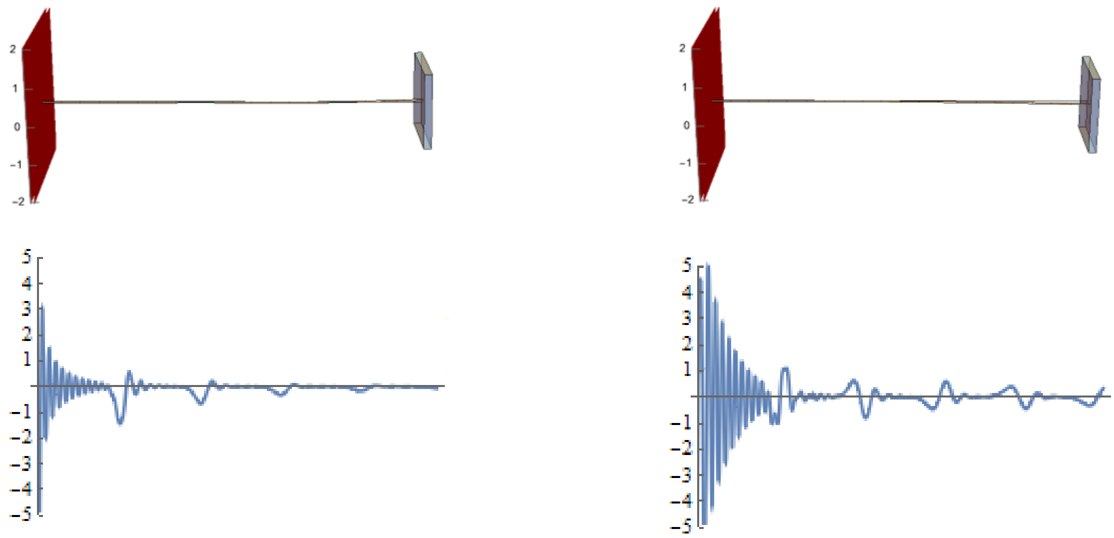

Figure 5: [Column 1] Vibrations on the string at time T (top) and tip velocity (bottom) with indirect filtering and boundary damping. [Column 2] Vibrations on the string at time T (top) and tip velocity (bottom) with indirect filtering and no boundary damping.

The results proved in this project are already published in two papers by the pioneers in the field [4, 1, 13]. Our major task here is to understand and decipher the complicated proofs of the results. In particular, the results of interest from these papers are the lack of observability of the semi-discrete Finite Difference and Finite Element approximations of (1) [4], the application of the direct filtering technique to these approximations and resulting proof of exact observability [4], and the proof of exponential stability with two boundary controllers. [1, 13]:

Here is the outline of the project. In Section 2, the exact observability inequality is established for 3. In Section 3, the semi-discretized Finite Difference approximations are introduced together with the finite-dimensional spectral analysis. Then, discrete energy estimates are proved to show that the discretized system lacks uniform observability as the discretization parameter tends to zero. This discrepancy is removed by introducing the direct filtering of spurious eigenvalues so that the discretized system retains the exact observability. We let the filtering space of numerical solutions be

$$\mathcal{C}_h(\gamma) := \left\{ u = \sum_{h^2\lambda_k(h)\leq\gamma} \left[a_k \sin\left(\sqrt{\lambda_k(h)}t\right) + b_k \cos\left(\sqrt{\lambda_k(h)}t\right)\right] \psi^k, a_k, b_k \in \mathbb{R} \right\}. \quad (5)$$

In Section 4, the semi-discretized Finite Element approximations are introduced instead together with the finite-dimensional spectral analysis. Then, similar to the Finite Difference-discretized model, discrete energy estimates are proved to show that the discretized system lacks uniform observability as the discretization parameter tends to zero. This discrepancy is removed by introducing the direct filtering of spurious eigenvalues so that the discretized system retains the exact observability. Finally, in Section 5, the exponential stability of the Finite Difference discretizations of the PDE model (4) after the application of the direct filtering.

2 Exact observability of the wave equation

Consider the control-free PDE model for the wave equation, $u(t) = 0, f(x,t) \equiv 0$ in (1). For simplicity, take also $c = 1$ in (1):

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0, \quad t \in \mathbb{R}^+ \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0,L). \end{cases} \quad (6)$$

The energy of the solutions of (6) is

$$E(t) = \underbrace{\frac{1}{2}\int_0^L |u_x(x,t)|^2 dx}_{\text{Potential Energy}} + \underbrace{\frac{1}{2}\int_0^L |u_t(x,t)|^2 dx}_{\text{Kinetic Energy}}, \quad t \geq 0. \tag{7}$$

It is straightforward to show that $\frac{dE(t)}{dt} = 0$, and therefore $E(t) \equiv E(0)$ for all $t \geq 0$. The next result is the so-called exact observability result for the wave equation, which implies that the the measurement of $u_x(L,t)$ for $T > 2L$ seconds is enough to recover the initial data $(u_0(x), u_1(x)) \in \mathcal{H}$, where $\mathcal{H} = H_0^1(0,L) \times L^2(0,L)$, $H_0^1(0,L) = \{f(x) : f, f' \in L^2(0,L), f(0) = f(L) = 0\}$,
$L^2(0,L) = \left\{f(x) : \int_0^L |f(x)|^2 dx < \infty\right\}$. Note that $T = 2L$ is optimal.

Theorem 1 *Let $T > 2L$. Then there exists a constant $C(T) > 0$ (to be determined) such that for every $(u_0, u_1) \in \mathcal{H}$ the following inequality holds:*

$$\int_0^T |u_x(L,t)|^2 dt \geq C(T)E(0). \tag{8}$$

Proof: The multiplier technique is already used to prove (8) for higher-dimensional wave equations [6]. Now, we are going to adopt the same approach for the one-dimension (6). First, multiply the equation in (6) by the multiplier xu_x, and integrate, to obtain

$$\int_0^T \int_0^L (u_{tt} - u_{xx}) x u_x dx dt = 0$$

Next, integrate by parts and simplify to obtain

$$\begin{aligned} 0 &= \int_0^L x u_x u_t|_0^T dx - \int_0^T \int_0^L x u_t u_{xt} dx dt - \int_0^T \int_0^L x u_{xx} u_x dx dt, \\ 0 &= \int_0^L x u_x u_t|_0^T dx - \int_0^T \int_0^L \frac{x}{2}\frac{d}{dx}|u_t|^2 dx dt - \int_0^T \int_0^L \frac{x}{2}\frac{d}{dx}|u_x|^2 dx dt. \end{aligned} \tag{9}$$

By letting $X(t)|_0^T := \int x u_t u_x|_0^T dx$,

$$\begin{aligned} X(t)|_0^T - \int_0^T \frac{x}{2}|u_t|^2|_0^L dt + \int_0^T\int_0^L \frac{1}{2}|u_t|^2 dxdt - \int_0^T \frac{x}{2}|u_x|^2|_0^L dt \\ + \int_0^T\int_0^L \frac{1}{2}|u_x|^2 dxdt = 0. \end{aligned}$$

By the boundary conditions (6), this becomes

$$X(t)|_0^T - \frac{L}{2}\int_0^T |u_x(L,t)|^2 dt + \frac{1}{2}\int_0^T\int_0^L \left[|u_t|^2 + |u_x|^2\right] dxdt = 0, \tag{10}$$

and therefore,

$$\begin{aligned} \frac{L}{2}\int_0^T |u_x(L,t)|^2 dt &= \frac{1}{2}\int_0^T\int_0^L \left[|u_t|^2 + |u_x|^2\right] dxdt + X(t)|_0^T \\ &= \int_0^T E(t)dt + X(t)|_0^T \qquad (11) \\ &= TE(0) + X(t)|_0^T. \qquad (12) \end{aligned}$$

Next, an estimate is sought for $|X(t)|$:

$$\begin{aligned} \left|X(t)|_0^T\right| &\leq L\int_0^L |u_t||u_x|dx\Big|_0^T \\ &\leq \left|\frac{L}{2}\int_0^L \left[|u_t|^2 + |u_x|^2\right] dx\right|_{t=0}^T \\ &\leq \left|\frac{L}{2}\int_0^L \left[|u_t|^2 + |u_x|^2\right] dx\right|_{t=0} + \left|\frac{L}{2}\int_0^L \left[|u_t|^2 + |u_x|^2\right] dx\right|_{t=T} = 2LE(0) \qquad (13) \end{aligned}$$

where the Young's inequality (226) is used, i.e. $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$. Now, by (13), $X(t)|_0^T \geq -2LE(0)$ and (11) becomes

$$\frac{L}{2}\int_0^T |u_x(L,t)|^2 dt \geq C(T)E(0) \tag{14}$$

where $C(T) = \frac{2(T-2L)}{L}$. Here, $T > 2L$ for the inequality to holds true. It can also be shown that $T = 2L$ is optimal [7]. $\square$

2.1 First-order form and spectral analysis

Let $\vec{v} = (u, u_t)^T$. Then (6) can be re-written in the first-order form (in terms of the time derivative) as the following

$$\vec{v}_t = \mathscr{A}\vec{v} := \begin{bmatrix} 0 & I \\ D_x^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \tag{15}$$

where $D_x^2 = \frac{\partial^2}{\partial x^2}$, and $I = D_x^0$ is the identity operator.

Now consider the following eigenvalue problem corresponding to (15):

$$\mathscr{A}\vec{v} = \lambda\vec{v}. \tag{16}$$

A straightforward calculation, see i.e. [2], shows that the operator $\mathscr{A}$ has infinitely many and purely imaginary distinct eigenvalues

$$\lambda_n = \frac{n\pi i}{L}, \quad n \in \mathbb{Z} - \{0\} \tag{17}$$

with $\lambda_{-n} = -\lambda_n$, and the eigenvectors

$$\psi_n = \begin{pmatrix} \sin \lambda_n x \\ \lambda_n \sin \lambda_n x \end{pmatrix}, \quad n \in \mathbb{Z} - \{0\} \tag{18}$$

It is observed that the gap between two consecutive eigenvalues is $|\lambda_{n+1} - \lambda_n| = \frac{\pi}{L}$ for all $n \in \mathbb{Z} - \{0\}$, and this uniform gap is independent of frequency n. It is crucial to point out that the existence of a uniform gap is strongly related to the exact observability result in Theorem 1.

3 Finite Difference semi-discretized wave equation

Consider the Finite Difference space semi-discretization of (6) such that $u(x_j,t) \approx u_j(t)$, where $u(x_j,t)$ is the approximations of $u(x,t)$ at $x = x_j$. So, given $N \in \mathbb{N}$, we set $h = \frac{L}{N+1}$ to discretize the interval $[0,L]$ as follows:

$$x_0 = 0 < x_1 = h < ... < x_N = Nh < x_{N+1} = L, \tag{19}$$

where $x_j = jh, j = 0, ..., N+1$.

Then, use the central difference formula $u_{xx}(x_j,t) \approx \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2}$ to generate the following finite-difference semi-discretization of (6):

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0, \quad 0 < t < T, j = 1, ..., N \\ u_0(t) = u_{N+1}(t) = 0, \quad 0 < t < T \\ u_j(0) = u_j^0, u_j'(0) = u_j^1, \quad j = 0, ..., N+1, \end{cases} \tag{20}$$

where primes denote derivation with respect to time. Analogously, the energy (7) of the solutions of the PDE model (6) is also discretized as the following

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(t)|^2 + \left|\frac{u_{j+1}(t)-u_j(t)}{h}\right|^2\right]. \tag{21}$$

3.1 First-order form and spectral analysis

First, consider the auxiliary eigenvalue problem corresponding to the finite difference-discretized second order differential operator $\frac{\partial^2}{\partial x^2}$:

$$\begin{cases} -\frac{\psi_{j+1}+\psi_{j-1}-2\psi_j}{h^2} = \lambda\psi_j, \quad j = 1, ..., N, \\ \psi_0 = \psi_{N+1} = 0. \end{cases} \tag{22}$$

The eigenvalues, $\lambda_j(h), j = 1, ..., N$, are explicitly found as follows (see [4, 2]):

$$\lambda_k(h) = \frac{4}{h^2} \sin^2 \frac{\pi k h}{2L}, \quad k = 1, ..., N. \tag{23}$$

Likewise, the eigenvectors $\vec{\psi}_k = (\psi_{k,1}, ..., \psi_{k,N})^T$ are

$$\psi_{k,j} = \sin \frac{\pi h k j}{L}, \quad j = 1, ..., N, \quad k = 1, ..., N. \tag{24}$$

Here, i.e., $\vec{\psi}_N$, would be of the form $\vec{\psi}_N = [\sin \frac{N\pi h}{L}, \sin \frac{2N\pi h}{L}, \ldots, \sin \frac{N^2 \pi h}{L}]$.

Let $\vec{u} = [u_1, u_2, ..., u_N]$ so that the discretized model (22) can be re-written in the first-order form

$$\vec{u}_t = \begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix} \vec{u}, \tag{25}$$

where the matrix A_h is defined by

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{N \times N.} \tag{26}$$

Now, consider the eigenvalue problem corresponding to (25)

$$\begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix} \begin{bmatrix} \vec{u_1} \\ \vec{u_2} \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} \vec{u_1} \\ \vec{u_2} \end{bmatrix}. \tag{27}$$

Theorem 2 (St.Clair-Ozer [2]) *The solutions of (27) are*

$$\tilde{\lambda}_k(h) = \pm i\sqrt{\lambda_k(h)}, \tag{28}$$

$$\begin{bmatrix} \vec{u_{1,k}} \\ \vec{u_{2,k}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\tilde{\lambda}_k(h)} \sin(\frac{j\pi kh}{L}) \\ \sin(\frac{j\pi kh}{L}) \end{bmatrix}, \quad k, j = \pm 1, \pm 2, ..., \pm N., \tag{29}$$

where $\lambda_{-k} = \lambda_k$, $\lambda_k(h)$, and $\psi_{k,j}$ are defined in (23)-(24).

3.2 Technical results

Now, the results of the following lemmas are essential to prove one of the main results, the lack of observability. Compacter versions of these proofs are already provided in [4].

Lemma 1 *For any eigenvector $\vec{\psi}$ with eigenvalue λ of (22), the following identity holds:*

$$\sum_{j=0}^{N} \left| \frac{\psi_j - \psi_{j+1}}{h} \right|^2 = \lambda \sum_{j=1}^{N} |\psi_j|^2. \tag{30}$$

If $\vec{\psi}^k$ and $\vec{\psi}^l$ are eigenvectors associated to eigenvalues $\lambda_k \neq \lambda_l$, it follows that

$$\sum_{j=0}^{N} (\psi_{k,j} - \psi_{k,j+1})(\psi_{l,j} - \psi_{l,j+1}) = 0. \tag{31}$$

Proof: Multiply the equation (22) by ψ_j and take the sum for $j = 1, \ldots, N$:

$$\sum_{j=1}^{N} - \left[\frac{\psi_{j+1} + \psi_{j-1} - 2\psi_j}{h^2} \right] \psi_j = \sum_{j=1}^{N} \lambda |\psi_j|^2. \tag{32}$$

The left-hand side of (32) becomes

$$\sum_{j=1}^{N} - \left[\frac{\psi_{j+1} + \psi_{j-1} - 2\psi_j}{h^2} \right] \psi_j = \frac{1}{h^2} \sum_{j=1}^{N} (2|\psi_j|^2 - \psi_j \psi_{j+1} - \psi_j \psi_{j-1}).$$

In order to prove (31), we use the fact that, since A_h is a symmetric matrix, its eigenvectors associated with distinct eigenvalues are orthogonal, and in particular,

A_h-orthogonal, i.e., $(\psi^k, \psi^l) = 0$, and $(A_h\psi^k, \psi^l) = 0$, respectively. Therefore,

$$\sum_{j=1}^{N} \psi_{k,j}\psi_{l,j} = 0, \tag{33}$$

and

$$0 = \sum_{j=1}^{N}(\psi_{k,j+1} + \psi_{k,j-1} - 2\psi_{k,j})\psi_{l,j} = \sum_{j=1}^{N}(\psi_{k,j+1} + \psi_{k,j-1})\psi_{l,j}. \tag{34}$$

Therefore,

$$\sum_{j=1}^{N} \psi_{k,j+1}\psi_{l,j} = -\sum_{j=1}^{N} \psi_{k,j-1}\psi_{j,l} = -\sum_{j=1}^{N} \psi_{k,j}\psi_{l,j+1}. \tag{35}$$

In other words,

$$\sum_{j=1}^{N}[\psi_{k,j+1}\psi_{l,j} + \psi_{k,j}\psi_{l,j+1}] = 0, \tag{36}$$

which, considering (33), is equivalent to (31). □

Lemma 2 *For any eigenvector $\vec{\psi} = (\psi_1, \ldots, \psi_N)^T$ of the system (22), the following identity holds:*

$$h\sum_{j=0}^{N}\left|\frac{\psi_{j+1} - \psi_j}{h}\right|^2 = \frac{2L}{4 - \lambda h^2}\left|\frac{\psi_N}{h}\right|^2. \tag{37}$$

Proof: Start with the identity (30) in Lemma 1 for a normalized eigenvector $\vec{\psi}$ so that

$$h\sum_{j=1}^{N}|\psi_j|^2 = 1, \tag{38}$$

Then, (30) becomes

$$h\sum_{j=0}^{N}\left|\frac{\psi_j - \psi_{j+1}}{h}\right|^2 = \lambda. \tag{39}$$

Thus,

$$\lambda = \frac{1}{h}\sum_{j=0}^{N}(|\psi_j|^2 + |\psi_{j+1}|^2 - 2\psi_j\psi_{j+1}) = \frac{2}{h^2} - \frac{2}{h}\sum_{j=0}^{N}\psi_j\psi_{j+1}, \tag{40}$$

and thus,

$$\sum_{j=1}^{N}\psi_j\psi_{j+1} = \frac{1}{h} - \frac{\lambda h}{2}. \tag{41}$$

Next, multiply both sides of (22) by $j\frac{\psi_{j+1}-\psi_{j-1}}{2}$, and take the sum for $j = 1, \dots, N$ (note that this is a discrete version of the multiplier $x\psi_x$). On the left-hand side, use (38),(41), and $j \leq N+1$ to obtain

$$\begin{aligned}
&-\frac{1}{h^2}\sum_{j=1}^{N}(\psi_{j+1} + \psi_{j-1} - 2\psi_j)j\frac{\psi_{j+1}-\psi_{j-1}}{2} \\
&= -\frac{1}{h^2}\sum_{j=1}^{N}\frac{j}{2}[|\psi_{j+1}|^2 - |\psi_{j-1}|^2 - 2\psi_j(\psi_{j+1} - \psi_{j-1})] \\
&= -\frac{1}{h^2}\sum_{j=1}^{N}\frac{j}{2}[|\psi_{j+1}|^2 - |\psi_{j-1}|^2] + \frac{1}{h^2}\sum_{j=1}^{N} j\psi_j(\psi_{j+1} - \psi_{j-1}) \\
&= -\frac{1}{h^2}\left[\frac{1}{2}(|\psi_2|^2 - |\psi_0|^2) + (|\psi_3|^2 - |\psi_1|^2) + \frac{3}{2}(|\psi_4|^2 - |\psi_2|^2) + \dots\right. \\
&\quad \left. + \frac{N-1}{2}(|\psi_{N-2}|^2 - |\psi_N|^2) + \frac{N}{2}(|\psi_{N-1}|^2 - |\psi_{N+1}|^2)\right] + \frac{1}{h^2}[(\psi_2 - \psi_0)\psi_1 \\
&\quad + 2(\psi_3 - \psi_1)\psi_2 + \dots + (N-1)(\psi_N - \psi_{N-2})\psi_{N-1} + N(\psi_{N-1} - \psi_{N+1})\psi_N] \\
&= -\frac{1}{h^2}\left[|\psi_1|^2(-1) + |\psi_2|^2\left(\frac{1}{2} - \frac{3}{2}\right) + |\psi_3|^2(1-2) + \dots + |\psi_{N-1}|^2\left(-\frac{N-2}{2}\right.\right. \\
&\quad \left.\left. + \frac{N}{2}\right) + |\psi_N|^2\left(-\frac{N-1}{2}\right)\right] + \frac{1}{h^2}[\psi_1\psi_2 - 2\psi_1\psi_2 + 2\psi_2\psi_3 - 3\psi_2\psi_3 + \dots \\
&\quad + (N-1)\psi_{N-1}\psi_N + N\psi_{N-1}\psi_N] \\
&= \frac{1}{h^2}\sum_{j=1}^{N}|\psi_j|^2 - \frac{N+1}{2h^2}|\psi_N|^2 - \frac{1}{h^2}\sum_{j=1}^{N}\psi_j\psi_{j+1} \\
&= \frac{1}{h^2}\left(\frac{1}{h}\right) - \frac{N+1}{2}\left|\frac{\psi_N}{h}\right|^2 - \frac{1}{h^2}\left(\frac{1}{h} - \frac{\lambda h}{2}\right) = \frac{\lambda}{2h} - \frac{N+1}{2}\left|\frac{\psi_N}{h}\right|^2.
\end{aligned}$$

On the right-hand side,

$$\lambda \sum_{j=1}^{N} j\psi_j \left(\frac{\psi_{j+1}-\psi_{j-1}}{2}\right) = \frac{\lambda}{2}\sum_{j=1}^{N}(j\psi_j\psi_{j+1} - j\psi_j\psi_{j-1})$$
$$= \frac{\lambda}{2}[(\psi_1\psi_2 - \psi_1\psi_0) + (2\psi_2\psi_3 - 2\psi_2\psi_1) + \cdots + ((N-1)\psi_{N-1}\psi_N$$
$$-(N-1)\psi_{N-1}\psi_{N-2}) + (N\psi_N\psi_{N+1} - N\psi_N\psi_{N-1})]$$
$$= -\frac{\lambda}{2}\sum_{j=1}^{N}\psi_j\psi_{j+1} = -\frac{\lambda}{2}\left(\frac{1}{h} - \frac{\lambda h}{2}\right).$$

Therefore, $\frac{\lambda}{2h} - \frac{N+1}{2}\left|\frac{\psi_N}{h}\right|^2 = -\frac{\lambda}{2}\left(\frac{1}{h} - \frac{\lambda h}{2}\right)$. In other words,

$$\frac{L}{2}\left|\frac{\psi_N}{h}\right|^2 = \frac{(N+1)h}{2}\left|\frac{\psi_N}{h}\right|^2 = \frac{\lambda}{2} + \frac{\lambda}{2} - \frac{\lambda^2 h^2}{4} = \lambda\left(1 - \frac{\lambda h^2}{4}\right). \tag{42}$$

Combining (39) and (42), we find that

$$h\sum_{j=0}^{N}\left|\frac{\psi_j - \psi_{j+1}}{h}\right|^2 = \lambda = \frac{L}{2\left(1-\frac{\lambda h^2}{4}\right)}\left|\frac{\psi_N}{h}\right|^2 = \frac{2L}{4-\lambda h^2}\left|\frac{\psi_N}{h}\right|^2. \quad \square \tag{43}$$

Lemma 3 *(Conservation of Energy) For any $h > 0$ and u being solution of (20), the energy is convserved*

$$E_h(t) = E_h(0), \quad \forall t \in [0, T]. \tag{44}$$

Proof: First, multiply (20) by $\bar{u}_j'(t)$ and take the sum for $j = 1, ..., N$ to obtain

$$\sum_{j=1}^{N} u_j''\bar{u}_j' = \frac{1}{h^2}\sum_{j=1}^{N}(u_{j+1} + u_{j-1} - 2u_j)\bar{u}_j'. \tag{45}$$

Let $z = u_j''\bar{u}_j'$ so that $\bar{z} = \bar{u}_j''u_j'$. Then,

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{N}|u_j'|^2 = \frac{1}{2}\frac{d}{dt}\sum_{j=1}^{N}(u_j'\bar{u}_j') = \frac{1}{2}\sum_{j=1}^{N}(u_j''\bar{u}_j' + u_j\bar{u}_j'') = \mathrm{Re}\left(\sum_{j=1}^{N}u_j''\bar{u}_j'\right).$$

Moreover,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N}|u_j-u_{j+1}|^2 &= \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N}(u_j-u_{j+1})(\bar{u}_j-\bar{u}_{j+1})\\
&= \frac{1}{2}\sum_{j=0}^{N}[(u_j'-u_{j+1}')(\bar{u}_j-\bar{u}_{j+1})+(u_j-u_{j+1})(\bar{u}_j'-\bar{u}_{j+1}')]\\
&= \frac{1}{2}\sum_{j=0}^{N}[u_j'\bar{u}_j-u_j'\bar{u}_{j+1}-u_{j+1}'\bar{u}_j+u_{j+1}'\bar{u}_{j+1}+u_j\bar{u}_j'-u_j\bar{u}_{j+1}'\\
&\qquad -u_{j+1}\bar{u}_j'+u_{j+1}\bar{u}_{j+1}']\\
&= \mathrm{Re}\left(\sum_{j=0}^{N}u_j\bar{u}_j'-u_j\bar{u}_{j+1}'-u_{j+1}\bar{u}_j'+u_{j+1}\bar{u}_{j+1}'\right)\\
&= \mathrm{Re}[(u_0\bar{u}_0'-u_0\bar{u}_1'-u_1\bar{u}_0'+u_1\bar{u}_1')+(u_1\bar{u}_1'-u_1\bar{u}_2'-u_2\bar{u}_1'+u_2\bar{u}_2')+\ldots\\
&= \mathrm{Re}\left[\sum_{j=1}^{N}(-u_{j+1}\bar{u}_j'-u_{j-1}\bar{u}_j'+2u_j\bar{u}_j')\right]\\
&= \mathrm{Re}\left[-\sum_{j=1}^{N}(u_{j+1}+u_{j-1}-2u_j)\bar{u}_j'\right]. \qquad (46)
\end{aligned}$$

Now, (45) and (46) leads to

$$\begin{aligned}
&\frac{d}{dt}\sum_{j=0}^{N}\left[|u_j'|^2+\left|\frac{u_j-u_{j+1}}{h}\right|^2\right] = \frac{d}{dt}\sum_{j=0}^{N}|u_j'|^2+\frac{d}{dt}\frac{1}{h^2}\sum_{j=0}^{N}|u_j-u_{j+1}|^2\\
&= 2\,\mathrm{Re}\left(\sum_{j=1}^{N}u_j''\bar{u}_j'\right)+\left(-2\,\mathrm{Re}\left[\frac{1}{h^2}\sum_{j=1}^{N}(u_{j+1}+u_{j-1}-2u_j)\bar{u}_j'\right]\right)\\
&= 2\,\mathrm{Re}\left[\frac{1}{h^2}\sum_{j=1}^{N}(u_{j+1}+u_{j-1}-2u_j)\bar{u}_j'\right]-2\,\mathrm{Re}\left[\frac{1}{h^2}\sum_{j=1}^{N}(u_{j+1}+u_{j-1}-2u_j)\bar{u}_j'\right]\\
&= 0,
\end{aligned}$$

which is equivalent to (44). □

Lemma 4 *Let* $X_h(t) = h\sum_{j=1}^{N} j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)u_j'$. *For any* $h>0$ *and* u *being a solution of (20), the following identity holds*

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T\left[u_j'u_{j+1}'+\left|\frac{u_{j+1}-u_j}{h}\right|^2\right]dt+X_h(t)|_0^T = \frac{L}{2}\int_0^T\left|\frac{u_N(t)}{h}\right|^2 dt. \qquad (47)$$

Proof: First, multiply (20) by $j\frac{u_{j+1}-u_{j-1}}{2}$, which is a discrete version of the classical multiplier xu_x for the wave equation (6), to obtain

$$\sum_{j=1}^{N}\int_0^T u_j'' j\frac{u_{j+1}-u_{j-1}}{2}dt = \frac{1}{h^2}\sum_{j=1}^{N}\int_0^T (u_{j+1}+u_{j-1}-2u_j) j\frac{u_{j+1}-u_{j-1}}{2}dt. \quad (48)$$

The left-hand side, after integrating by parts, becomes

$$\sum_{j=1}^{N}\int_0^T u_j'' j\frac{u_{j+1}-u_{j-1}}{2}dt = \sum_{j=1}^{N} u_j' j\frac{u_{j+1}-u_{j-1}}{2}\Bigg|_0^T - \sum_{j=1}^{N}\int_0^T u_j' j\frac{u_{j+1}'-u_{j-1}'}{2}dt,$$

and since

$$\begin{aligned}
&\sum_{j=1}^{N} u_j' j(u_{j+1}'-u_{j-1}') = \sum_{j=1}^{N} u_j' j u_{j+1}' - u_j' j u_{j-1}' \\
&= (u_1'u_2' - u_1'u_0') + (2u_2'u_3' - 2u_2'u_1') + (3u_3'u_4' - 3u_3'u_2') + \dots \\
&\quad +((N-1)u_{N-1}'u_N' - (N-1)u_{N-1}'u_{N-2}') + (Nu_N'u_{N+1}' - Nu_N'u_{N-1}') \\
&= -u_1'u_2' - u_2'u_3' - \cdots - u_{N-2}'u_{N-1}' - u_{N-1}'u_N' \\
&= -\sum_{j=1}^{N} u_j'u_{j+1}',
\end{aligned}$$

the left-hand side of (48) becomes

$$\sum_{j=1}^{N}\int_0^T u_j'' j\frac{u_{j+1}-u_{j-1}}{2}dt = \sum_{j=1}^{N} u_j' j\frac{u_{j+1}'-u_{j-1}'}{2}\Bigg|_0^T + \frac{1}{2}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt. \quad (49)$$

The right-hand side of (48) can be also simplified to

$$\begin{aligned}
&\sum_{j=1}^{N}\int_0^T (u_{j+1}+u_{j-1}-2u_j) j\frac{(u_{j+1}-u_{j-1})}{2}dt = \sum_{j=1}^{N}\int_0^T \frac{1}{2} j[u_{j+1}u_{j+1} - u_{j+1}u_{j-1} \\
&\qquad +u_{j+1}u_{j-1} - u_{j-1}u_{j-1} - 2u_j(u_{j+1}-u_{j-1})]dt \\
&= \frac{1}{2}\sum_{j=1}^{N}\int_0^T j[|u_{j+1}|^2 - |u_{j-1}|^2]dt - \sum_{j=1}^{N}\int_0^T j u_j(u_{j+1}-u_{j-1})dt.
\end{aligned}$$

We use the following facts that

$$\sum_{j=1}^{N} ju_j(u_{j+1}-u_{j-1}) = u_1(u_2-u_0)+2u_2(u_3-u_1)+3u_3(u_4-u_2)$$
$$+\cdots+(N-1)u_{N-1}(u_N-u_{N-2})+Nu_N(u_{N+1}-u_{N-1})$$
$$= u_1u_2+2u_2u_3-2u_1u_2+3u_3u_4-3u_3u_2+\cdots+(N-1)u_{N-1}u_N$$
$$-(N-1)u_{N-1}u_{N-2}+Nu_Nu_{N-1}$$
$$= -u_1u_2-u_2u_3-\cdots-u_Nu_{N-1}$$
$$= -\sum_{j=1}^{N} u_ju_{j+1},$$

and

$$\frac{1}{2}\sum_{j=1}^{N} j[|u_{j+1}|^2-|u_{j-1}|^2] = \frac{1}{2}[(u_2^2-u_0^2)+2(u_3^2-u_1^2)+3(u_4^2-u_2^2)+\dots$$
$$+(N-1)(u_N^2-u_{N-2}^2)+N(u_{N+1}^2-u_{N-1}^2)] = \frac{1}{2}(u_2^2+2u_3^2-2u_1^2$$
$$+3u_4^2-3u_2^2+\cdots+(N-1)u_N^2-(N-1)u_{N-2}^2-Nu_{N-1}^2)$$
$$= -u_1^2-u_2^2-u_3^2-\cdots-u_{N-1}^2+\frac{(N-1)}{2}u_N^2$$
$$= -u_1^2-u_2^2-u_3^2-\cdots-u_{N-1}^2-u_N^2+u_N^2+\frac{(N-1)}{2}u_N^2$$
$$= -\sum_{j=1}^{N} u_j^2+\left(\frac{N-1}{2}+1\right)u_N^2 = -\sum_{j=1}^{N} u_j^2+\frac{N+1}{2}u_N^2$$

to find

$$\frac{1}{2}\sum_{j=1}^{N}\int_0^T j[|u_{j+1}|^2-|u_{j-1}|^2]dt-\sum_{j=1}^{N}\int_0^T ju_j(u_{j+1}-u_{j-1})dt$$
$$= -\sum_{j=1}^{N}\int_0^T |u_j|^2dt+\frac{N+1}{2}\int_0^T |u_N|^2dt+\sum_{j=1}^{N}\int_0^T u_ju_{j+1}dt$$
$$= \frac{N+1}{2}\int_0^T |u_N|^2dt-\frac{1}{2}\sum_{j=0}^{N}\int_0^T \left[u_j^2+u_{j+1}^2-2u_ju_{j+1}\right]$$
$$= \frac{N+1}{2}\int_0^T |u_N|^2dt-\frac{1}{2}\sum_{j=0}^{N}\int_0^T |u_j-u_{j+1}|^2dt. \tag{50}$$

Finally, combining (48-50) yields the desired result

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T\left[u_j'u_{j+1}'+\frac{|u_j-u_{j+1}|^2}{h^2}\right]dt+X_h(t)|_0^T$$
$$=\frac{h}{2}\sum_{j=0}^{N}\int_0^T u_j'u_{j+1}'dt+\frac{1}{2h}\sum_{j=0}^{N}\int_0^T|u_j-u_{j+1}|^2dt+h\sum_{j=1}^{N}j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)u_j'\Bigg|_0^T$$
$$=h\left(-\sum_{j=1}^{N}u_j'j\frac{u_{j+1}-u_{j-1}}{2}\Bigg|_0^T+\sum_{j=1}^{N}\int_0^T u_j''j\frac{u_{j+1}-u_{j-1}}{2}dt\right)$$
$$-\frac{1}{h}\left(-\frac{N+1}{2}\int_0^T|u_N|^2dt+\sum_{j=1}^{N}\int_0^T(u_{j=1}+u_{j-1}-2u_j)j\frac{u_{j+1}-u_{j-1}}{2}dt\right)$$

$$+h\sum_{j=1}^{N}u_jj\frac{u_{j+1}-u_{j-1}}{2}\Bigg|_0^T$$
$$=h\left(\frac{1}{h^2}\sum_{j=1}^{N}\int_0^T(u_{j+1}+u_{j-1}-2u_j)j\frac{u_{j+1}-u_{j-1}}{2}dt\right)$$
$$-\frac{1}{h}\left(-\frac{N+1}{2}\int_0^T|u_N|^2dt+\sum_{j+1}^{N}\int_0^T(u_{j+1}+u_{j-1}-2u_j)j\frac{u_{j+1}-u_{j-1}}{2}dt\right)$$
$$=\frac{N+1}{2h}\int_0^T|u_N|^2dt=\frac{(N+1)h}{2}\int_0^T\left|\frac{u_N(t)}{h}\right|^2dt=\frac{L}{2}\int_0^T\left|\frac{u_N(t)}{h}\right|^2dt.\quad\square$$

Lemma 5 (Equipartition of Energy) *Let $Y_h(t)=h\sum_{j=1}^{N}u_j'u_j$. For any $h>0$ and u being a solution of (20), the following identity holds:*

$$-h\sum_{j=1}^{N}\int_0^T|u_j'|^2dt+h\sum_{j=0}^{N}\int_0^T\left|\frac{u_j-u_{j+1}}{h}\right|^2dt+Y_h(t)|_0^T=0. \tag{51}$$

Proof: First, we multiply in (20) by u_j, the discrete version of the classical multiplier u for the wave equation. This gives us

$$\sum_{j=1}^{N}\int_0^T u_j''u_jdt-\sum_{j=1}^{N}\int_0^T\frac{(u_{j+1}+u_{j-1}-2u_j)}{h^2}u_jdt=0. \tag{52}$$

By the integration by parts for the left-hand side,

$$\sum_{j=1}^{N}\int_0^T u_j''u_j dt = \sum_{j=1}^{N} u_j u_j'\Big|_0^T - \sum_{j=1}^{N}\int_0^T |u_j'|^2 dt. \tag{53}$$

Moroever,

$$\begin{aligned}
&\sum_{j=1}^{N}(u_{j+1}+u_{j-1}-2u_j)u_j = \sum_{j=1}^{N} u_{j+1}u_j + u_{j-1}u_j - 2u_ju_j \\
&= (u_2u_1 + u_0u_1 - 2u_1u_1) + (u_3u_2 + u_1u_2 - 2u_2u_2) \\
&\quad + \dots + (u_Nu_{N-1} + u_{N-2}u_{N-1} - 2u_{N-1}u_{N-1}) \\
&\quad + (u_{N+1}u_N + u_{N-1}u_N - 2u_Nu_N) \\
&= -[(u_0u_0 - u_0u_1 - u_1u_0 + u_1u_1) + (u_1u_1 - u_1u_2 - u_2u_1 + u_2u_2) \\
&\quad + (u_2u_2 - u_2u_3 - u_3u_2 + u_3u_3) + \dots \\
&\quad + (u_{N-1}u_{N-1} - u_{N-1}u_N - u_Nu_{N-1} + u_Nu_N) \\
&\quad + (u_Nu_N - u_Nu_{N+1} - u_{N+1}u_N + u_{N+1}u_{N+1})] \\
&= -\sum_{j=0}^{N}(u_ju_j - u_ju_{j+1} - u_{j+1}u_j + u_{j+1}u_{j+1}) \\
&= -\sum_{j=0}^{N}(u_j - u_{j+1})(u_j - u_{j+1}) = -\sum_{j=0}^{N}|u_j - u_{j+1}|^2.
\end{aligned} \tag{54}$$

Now, combining (52-54) lead to

$$\begin{aligned}
&\sum_{j=1}^{N}\int_0^T u_j''u_j dt - \sum_{j=1}^{N}\int_0^T \frac{(u_{j+1}+u_{j-1}-2u_j)}{h^2}u_j dt \\
&= -\sum_{j=1}^{N}\int_0^T |u_j'|^2 dt + \sum_{j=1}^{N} u_j'u_j\Big|_0^T + \frac{1}{h^2}\sum_{j=0}^{N}\int_0^T |u_j - u_{j+1}|^2 dt \\
&\quad -h\sum_{j=1}^{N}\int_0^T |u_j'|^2 dt + h\sum_{j=0}^{N}\int_0^T \left|\frac{u_j - u_{j+1}}{h}\right|^2 dt + Y_h(t)|_0^T = 0.
\end{aligned}$$

Thus, we conclude that (51) holds. □

Let

$$Z_h(t) := h\sum_{j=1}^{N} u_j'\left[j\frac{u_{j+1}-u_{j-1}}{2} - \frac{\Lambda h^2}{8}u_j\right]. \tag{55}$$

Lemma 6 *For any $h > 0$, $t \in [0, T]$ and u being a solution of (20) in which Λ is the upper bound on the eigenvalues entering in its Fourier development, it follows that*

$$|Z_h(t)| \leq \sqrt{L^2 - \frac{\Lambda h^4}{16} + \frac{3\Lambda h^2}{16\lambda_1}} E_h(0). \tag{56}$$

Proof: Let $\eta = \frac{-\Lambda h^2}{8}$. Then,

$$Z_h(t) = h \sum_{j=1}^{N} u'_j \left[j \frac{(u_{j+1} - u_{j-1})}{2} + \eta u_j \right]. \tag{57}$$

Using the Cauchy-Schwartz Inequality (223), (57) becomes

$$|Z_h(t)| \leq h \left[\sum_{j=1}^{N} |u'_j|^2 \right]^{\frac{1}{2}} \left[\sum_{j=1}^{N} \left| j \frac{(u_{j+1} - u_{j-1})}{2} + \eta u_j \right|^2 \right]^{\frac{1}{2}} \tag{58}$$

Using the Triangle inequality (222) we also have

$$\begin{aligned} h \sum_{j=1}^{N} \left| j \frac{(u_{j+1} - u_{j-1})}{2} + \eta u_j \right|^2 &= h \sum_{j=1}^{N} \left[\frac{j^2}{4} |u_{j+1} - u_{j-1}|^2 + \eta^2 u_j^2 \right. \\ &\quad \left. + \eta j (u_{j+1} - u_{j-1}) u_j \right] \\ &= h \sum_{j=1}^{N} \left[\frac{j^2}{4} |u_{j+1} - u_j + u_j - u_{j-1}|^2 + \eta^2 u_j^2 + \eta j (u_{j+1} - u_{j-1}) u_j \right] \\ &\leq h \sum_{j=1}^{N} \left[\frac{j^2}{2} |u_{j+1} - u_j|^2 + \frac{j^2}{2} |u_j - u_{j-1}|^2 \right] + \eta^2 u_j^2 + \eta j (u_{j+1} - u_{j-1}) u_j. \end{aligned} \tag{59}$$

And since

$$\begin{aligned} h \sum_{j=1}^{N} \eta j (u_{j+1} - u_{j-1}) u_j &= h\eta [(u_2 - u_0) u_1 + 2(u_3 - u_1) u_2 \\ &\quad + \cdots + (N-1)(u_N - u_{N-2}) u_{N-1} + N(u_{N+1} - u_{N-1}) u_N] \\ &= h\eta [u_1 u_2 + 2u_2 u_3 - 2u_1 u_2 + \cdots + (N-1) u_{N-1} u_N \\ &\quad - (N-1) u_{N-2} u_{N-1} - N u_{N-1} u_N] \\ &= h\eta [-u_1 u_2 - u_2 u_3 - \cdots - u_{N-1} u_N] = -h\eta \sum_{j=1}^{N} u_j u_{j+1}, \end{aligned}$$

the following is immediate

$$h\sum_{j=1}^{N}\left|j\frac{(u_{j+1}-u_{j-1})}{2}+\eta u_j\right|^2 \;\; \leq \;\; h\sum_{j=0}^{N}\left[\frac{j^2}{2}|u_{j+1}-u_j|^2+\frac{j^2}{2}|u_j-u_{j-1}|^2+\eta^2u_j^2 \right. $$
$$\left. -\eta u_j u_{j+1}\right]. \tag{60}$$

Since $h=\frac{L}{N+1}$ and $j\leq N+1$, $j^2\leq (N+1)^2=\frac{L^2}{h^2}$, we proceed by

$$\begin{aligned}
h\sum_{j=1}^{N}\left|j\frac{(u_{j+1}-u_{j-1})}{2}+\eta u_j\right|^2 &\leq h\sum_{j=0}^{N}\left[\frac{j^2}{2}|u_{j+1}-u_j|^2+\frac{j^2}{2}|u_j-u_{j-1}|^2\right.\\
&\quad \left.+\eta^2u_j^2-\eta u_ju_{j+1}\right]\\
&\leq L^2h\sum_{j=0}^{N}\left[\frac{|u_j-u_{j-1}|^2}{h^2}+\eta^2u_j^2-\eta u_ju_{j+1}+|\eta|u_j^2-|\eta|u_j^2\right]\\
&= L^2h\sum_{j=0}^{N}\frac{|u_j-u_{j-1}|^2}{h^2}-|\eta|h\sum_{j=1}^{N}(u_j^2-u_ju_{j+1})+[\eta^2+|\eta|]h\sum_{j=1}^{N}u_j^2\\
&= L^2h\sum_{j=0}^{N}\frac{|u_j-u_{j-1}|^2}{h^2}-\frac{|\eta|h}{2}\sum_{j=0}^{N}(u_j^2+u_{j+1}^2-2u_ju_{j+1})+[\eta^2+|\eta|]h\sum_{j=1}^{N}u_j^2\\
&= \left(L^2-\frac{|\eta|h}{2}\right)h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2+[\eta^2+|\eta|]h\sum_{j=1}^{N}u_j^2.
\end{aligned}\tag{61}$$

From here, we use the discrete form of Poincare's inequality (225)

$$h\sum_{j=1}^{N}u_j^2\leq\frac{1}{\lambda_1}h\sum_{j=0}\left|\frac{u_j-u_{j-1}}{h}\right|^2,$$

and the fact that $\eta^2+|\eta|\leq\frac{3}{2}|\eta|$, which is due to the fact that $|\eta|=\frac{\Lambda h^2}{8}\leq\frac{1}{2}$, as can be seen from (23), to find that

$$\begin{aligned}
h\sum_{j=1}^{N}\left|j\frac{(u_{j+1}-u_{j-1})}{2}+\eta u_j\right|^2 &\leq \left(L^2-\frac{|\eta|h}{2}\right)h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2\\
+[\eta^2+|\eta|]h\sum_{j=1}^{N}u_j^2 &= \left[L^2-\frac{|\eta|h^2}{2}+\frac{(\eta^2+|\eta|)}{\lambda_1}\right]h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2\\
&\leq \left[L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}\right]h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2
\end{aligned}\tag{62}$$

Next, combine (58) and (62) to obtain

$$
\begin{aligned}
|Z_h| \;&\le\; h\left[\sum_{j=1}^{N}|u_j'|^2\right]^{\frac{1}{2}}\left[\sum_{j=1}^{N}\left|j\frac{(u_{j+1}-u_{j-1})}{2}+\eta u_j\right|^2\right]^{\frac{1}{2}} \\
&\le \left(h\sum_{j=1}^{N}|u_j'|^2\right)^{\frac{1}{2}}\left(\left[L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}\right]h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2\right)^{\frac{1}{2}} \\
&= \sqrt{L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}}\left(h\sum_{j=1}^{N}|u_j'|^2\right)^{\frac{1}{2}}\left(h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2\right)^{\frac{1}{2}}. \quad (63)
\end{aligned}
$$

By the Triangle inequality (222) again,

$$
\begin{aligned}
&\frac{1}{2}\left(h\sum_{j=1}^{N}|u_j'|^2\right)^{\frac{1}{2}}\left(h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\left(h\sum_{j=1}^{N}|u_j'|^2\right)\left(h\sum_{j=0}^{N}\left|\frac{u_j-u_{j-1}}{h}\right|^2\right) = E_h(0),
\end{aligned}
$$

and therefore

$$
|Z_h| \le \sqrt{L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}}E_h(0). \quad \square \tag{64}
$$

Theorem 3 *(Lack of Observability)*

For any $T>0$, *we have*

$$
\sup_{\vec{u} \text{ solves } (20)}\left[\frac{E_h(0)}{\int_0^T |u_N(t)/h|^2 dt}\right] \to \infty \quad as \quad h \to 0. \tag{65}
$$

Proof: Every solution $\vec{u}=(u_1,...,u_N)$ of (20) can be written as

$$
u(t) = \sum_{k=1}^{N}\left[a_k \sin\sqrt{\lambda_k(h)}t + b_k\cos\sqrt{\lambda_k(h)}t\right]\psi_k, \tag{66}
$$

which can also be written as

$$
\vec{u}(t) = \sum_{k=1}^{N} c_k e^{i\sqrt{\lambda_k(h)}t}\vec{\psi}_k. \tag{67}
$$

Since each eigenvector solves (20), we consider the N^{th} eigenvector just for $\vec{u}$

$$\vec{u}(t) = e^{i\sqrt{\lambda_N}t}\vec{\psi}_N, \tag{68}$$

where $\vec{u} = (u_1, ..., u_N)^T$ and $\vec{\psi}_N = (\psi_{N,1}, ..., \psi_{N,N})^T$. The discretized version of observability (8) is of the following form

$$E_h(0) \leq C(T,h) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt. \tag{69}$$

So, looking at (69) and using Lemma 2, we have

$$\begin{aligned}
\int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt &= \int_0^T \frac{|\psi_{N,N}|^2}{h^2} dt \\
&= T\frac{|\psi_{N,N}|^2}{h^2} \\
&= T\left[\frac{(4-\lambda_N(h)h^2)h}{2L} \sum_{j=0}^{N} \left| \frac{\psi_{N,j} - \psi_{N,j+1}}{h} \right|^2\right] \\
&= \frac{Th}{2L}(4-\lambda_N(h)h^2) \sum_{j=0}^{N} \left| \frac{\psi_{N,j} - \psi_{N,j+1}}{h} \right|^2 .
\end{aligned} \tag{70}$$

The energy for the solutions $\vec{u}$ is

$$\begin{aligned}
E_h(t) &= \frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(t)|^2 + \left| \frac{\psi_j(t) - \psi_{j+1}(t)}{h} \right|^2\right] \\
&= \frac{h}{2}\sum_{j=0}^{N}\left[\left|\sqrt{\lambda_N(h)}u_N\right|^2 + \left| \frac{\psi_j(t) - \psi_{j+1}(t)}{h} \right|^2\right] \\
&= \frac{h}{2}\sum_{j=0}^{N}\left[\lambda_N(h)|\psi_{N,j}|^2 + \left| \frac{\psi_j(t) - \psi_{j+1}(t)}{h} \right|^2\right].
\end{aligned} \tag{71}$$

From Lemma 1,

$$\sum_{j=0}^{N} \left| \frac{\psi_j(t) - \psi_{j+1}(t)}{h} \right|^2 = \lambda \sum_{j=1}^{N} |\psi_j|^2 \tag{72}$$

so that (71) becomes

$$E_h(t) = h \sum_{j=0}^{N} \left[\frac{|\psi_{N,j} - \psi_{N,j+1}|^2}{h^2} \right]. \tag{73}$$

Using the results from (70) and (73) leads to

$$E_h(0) = h \left[\frac{2L \int_0^T |\frac{u_N(t)}{h}|^2 dt}{Th(4 - \lambda_N(h)h^2)} \right]. \tag{74}$$

So,

$$\frac{E_h(0)}{\int_0^T |\frac{u_N(t)}{h}|^2 dt} = \frac{2L}{T(4 - \lambda_N(h)h^2)}. \tag{75}$$

Moreover, in view of (23),

$$\begin{aligned}
\lambda_N(h)h^2 &= \left(\frac{4}{h^2} \sin^2 \left(\frac{\pi N h}{2L} \right) \right) h^2 \\
&= 4 \sin^2 \left(\frac{\pi N h}{2L} \right) = 4 sin^2 \left(\frac{\pi(L-h)}{2L} \right) \\
&= 4 \sin^2 \left(\frac{\pi}{2} - \frac{h\pi}{2L} \right) \\
&= 4 \left[\sin \frac{\pi}{2} \cos \frac{h\pi}{2L} - \cos \frac{\pi}{2} \sin \frac{h\pi}{2L} \right]^2 \\
&= 4 \cos^2 \left(\frac{h\pi}{2L} \right),
\end{aligned} \tag{76}$$

and therefore,

$$\lim_{h \to 0} 4 \cos^2 \left(\frac{h\pi}{2L} \right) = 4 \lim_{h \to 0} \cos^2 \left(\frac{h\pi}{2L} \right) = 4 \lim_{h \to 0} \cos^2(0) = 4 \lim_{h \to 0} 1 = 4. \tag{77}$$

Finally, combining (75) and (77) results in

$$\lim_{h \to 0} \left(\frac{E_h(0)}{\int_0^T |\frac{u_N(t)}{h}|^2 dt} \right) = \lim_{h \to 0} \left(\frac{2L}{T(4 - \lambda_N(h)h^2)} \right). \tag{78}$$

As $h \to 0$, $T(4 - \lambda_N(h)h^2) \to 0$, so

$$\lim_{h\to 0}\left(\frac{2L}{T(4-\lambda_N(h)h^2)}\right) \to \infty. \tag{79}$$

3.3 Filtering the numerical scheme and numerical observability

Theorem 4 *Assume that $0 < \gamma < 4$. Then, there exists $T(\gamma) \geq 2L$ such that, for all $T > T(\gamma)$, there exists $C = C(T,\gamma)$ such that the observability inequality*

$$E_h(0) \leq C(T,\gamma)\int_0^T \left|\frac{u_N(t)}{h}\right| \, dt \tag{80}$$

holds for all solutions of (20) in the class $\mathcal{C}_h(\gamma)$ as in (5)*, uniformly as $h \to 0$. Moreover,*

(a) $T(\gamma) \nearrow \infty$ as $\gamma \nearrow 4$ and $T(\gamma) \searrow 2L$ as $\gamma \searrow 0$.

(b) $C = C(T,\gamma) \searrow \frac{L}{2(T-2L)}$ as $\gamma \searrow 0$.

Proof: With the conservation of energy result in Lemma 3

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T \left[u_j' u_{j+1}' + \left|\frac{u_{j+1}-u_j}{h}\right|^2\right] dt + X_h(t)|_0^T = \frac{L}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt,$$

where $X_h(t) = h\sum_{j=1}^{N} j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)u_j'$, becomes

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T [u_j'u_{j+1}' + (E_h(0) - |u_j'|^2)]dt + X_h(t)|_0^T$$
$$= TE_h(0) + \frac{h}{2}\sum_{j=0}^{N}\int_0^T [u_j'u_{j+1}' - |u_j'|^2)]dt + X_h(t)|_0^T = \frac{L}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt.$$

We also have

$$
\begin{aligned}
-\frac{1}{2}\sum_{j=0}^{N}\int_0^T |u_j' - u_{j+1}'|^2 dt &= 1\frac{1}{2}\sum_{j=0}^{N}\int_0^T (|u_j'|^2 + |u_{j+1}'|^2 - 2u_j'u_{j+1}')dt \\
= -\frac{1}{2}\Big[\int_0^T (|u_0'|^2 + |u_1'|^2 - 2u_0'u_1')dt + \int_0^T (|u_1'|^2 + |u_2'|^2 - 2u_1'u_2')dt \\
+ \cdots + \int_0^T (|u_{N-1}'|^2 + |u_N'|^2 - 2u_{N-1}'u_N')dt + \int_0^T (|u_N'|^2 + |u_{N+1}'|^2 \\
-2u_N'u_{N+1}')dt\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^T (|u_1'|^2 + |u_2'|^2 + \cdots + |u_{N-1}'|^2 + |u_N'|^2 - u_1'u_2' - \cdots - u_{N-1}'u_N')dt \\
&= \sum_{j=0}^{N}\int_0^T (u_j'u_{j+1}' - |u_j'|^2)dt. \qquad (81)
\end{aligned}
$$

The left hand side of (81) can be estimated as follows, with Λ as the largest eigenvalue in the Fourier development of u:

$$
u = \sum_{|\mu_k| \le \sqrt{\Lambda}} a_k e^{i\mu_k t}\psi^k, \qquad (82)
$$

with $\mu_k = \sqrt{\Lambda}$ for $k > 0$ and $\mu_{-k} = -\mu_k$. Therefore,

$$
u' = i \sum_{|\mu_k| \le \sqrt{\Lambda}} a_k \mu_k e^{i\mu_k t}\psi^k \qquad (83)
$$

So,

$$\sum_{j=0}^{N}|u_j'-u_{j+1}'|^2=\sum_{j=0}^{N}\left|\sum_{|\mu_k|\le\sqrt{\Lambda}}a_k\mu_k e^{i\mu_k t}\psi_{k,j}-\sum_{|\mu_k|\le\sqrt{\Lambda}}a_k\mu_k e^{i\mu_k t}\psi_{k,j+1}\right|^2$$
$$=\sum_{j=0}^{N}\left|\sum_{|\mu_k|\le\sqrt{\Lambda}}a_k\mu_k e^{i\mu_k t}(\psi_{k,j}-\psi_{k,j+1})\right|^2$$
$$=\sum_{j=0}^{N}\left|\sum_{|\mu_k|\le\sqrt{\Lambda}}a_k\mu_k e^{i\mu_k t}(\psi_{k,j}-\psi_{k,j+1})\right|\left|\sum_{|\mu_k|\le\sqrt{\Lambda}}a_k\mu_k e^{-i\mu_k t}(\psi_{k,j}-\psi_{k,j+1})\right|$$
$$=\sum_{j=0}^{N}\sum_{|\mu_k|\le\sqrt{\Lambda}}|a_k|^2\mu_k^2|\psi_{k,j}-\psi_{k,j+1}|^2$$
$$+\sum_{j=0}^{N}\sum_{\substack{|\mu_k|\le\sqrt{\Lambda}\\|\mu_l|\le\sqrt{\Lambda}\\\mu_k\neq\mu_l}}\mu_k\mu_l a_k\bar{a}_l e^{i(\mu_k-\mu_l)t}(\psi_{k,j}-\psi_{k,j+1})(\psi_{l,j}-\psi_{l,j+1}) \qquad (84)$$

Using the identities (30) and (31) from Lemma 1, (84) can be rewritten as

$$\sum_{j=0}^{N}|u_j'-u_{j+1}'|^2=\sum_{|\mu_k|\le\sqrt{\Lambda}}|a_k|^2\mu_k^2\left(h^2\lambda_k\sum_{j=1}^{N}|\psi_{k,j}|^2\right)$$
$$=\sum_{|\mu_k|\le\sqrt{\Lambda}}|a_k|^2\lambda_k^2h^2\sum_{j=1}^{N}|\psi_{k,j}|^2=\mu_k^2\sum_{|\mu_k|\le\sqrt{\Lambda}}|a_k|^2\lambda_k h^2\sum_{j=1}^{N}|\psi_{k,j}|^2$$
$$\le\Lambda\sum_{|\mu_k|\le\sqrt{\Lambda}}|a_k|^2\lambda_k h^2\sum_{j=1}^{N}|\psi_{k,j}|^2=\Lambda h^2\sum_{|\mu_k|\le\sqrt{\Lambda}}|a_k|^2\mu_k^2\sum_{j=1}^{N}|\psi_{k,j}|^2$$
$$=\Lambda h^2\sum_{j=1}^{N}\left|i\sum_{|\mu_k|\le\sqrt{\Lambda}}a_k\mu_k e^{i\mu_k t}\psi_{k,j}\right|^2=\Lambda h^2\sum_{j=1}^{N}|u_j'|^2. \qquad (85)$$

Therefore,

$$\sum_{j=0}^{N}\int_0^T[u_j'u_{j+1}-|u_j'|^2]dt=-\frac{1}{2}\sum_{j=0}^{N}\int_0^T|u_j'-u_{j+1}'|^2dt$$
$$\ge-\frac{1}{2}\int_0^T\left(\Lambda h^2\sum_{j=1}^{N}|u_j'|^2\right)dt=-\frac{\Lambda h^2}{2}]\int_0^T|u_j'|^2dt. \qquad (86)$$

Combining (81) and (86), we find

$$\begin{aligned}\frac{L}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \quad &= TE_h(0) + \frac{h}{2}\sum_{j=0}^N \int_0^T [u_j' u_{j+1}' - |u_j'|^2]dt + X_h(t)|_0^T \\ &\geq TE_h(0) + \frac{h}{2}\left(-\frac{\Lambda h^2}{2}\sum_{j=0}^N \int_0^T |u_j'|^2 dt\right) + X_h(t)|_0^T \\ &= TE_h(0) - \frac{\Lambda h^3}{4}\sum_{j=0}^N \int_0^T |u_j'|^2 dt + X_h(t)|_0^T. \end{aligned} \tag{87}$$

Using (51), the equipartition of energy identity, we find

$$\begin{aligned} h\sum_{j=0}^N \int_0^T |u_j'|^2 dt &= h\sum_{j=0}^N \int_0^T \left|\frac{u_j - u_{j+1}}{h}\right|^2 dt + Y_h(t)|_0^T \\ &= h\int_0^T \left(E_h(t) - \sum_{j=0}^N |u_j'|^2\right) dt + Y_h(t)_0^T \\ &= \int_0^T E_h(t)dt - h\sum_{j=0}^N \int_0^T |u_j|^2 dt + Y_h(t)|_0^T \\ &= \int_0^T E_h(0)dt + \frac{1}{2}Y_h(t)|_0^T = TE_h(0) + \frac{1}{2}Y_h(t)|_0^T. \end{aligned} \tag{88}$$

By combining (87) and (88), we deduce that

$$\begin{aligned} \frac{L}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \quad &\geq \quad TE_h(0) - \frac{\Lambda h^3}{4}\sum_{j=0}^N \int_0^T |u_j'|^2 dt + X_h(t)|_0^T \\ &= \quad TE_h(0) - \frac{\Lambda h^2}{4}\left(TE_h(0) + \frac{1}{2}Y_h(t)\Big|_0^T\right) + X_h(t)|_0^T \\ &= \quad \left(1 - \frac{\Lambda h^2}{4}\right) TE_h(0) - \frac{\Lambda h^2}{8} Y_h(t)\Big|_0^T + X_h(t)|_0^T \\ &= \quad T\left(1 - \frac{\Lambda h^2}{4}\right) E_h(0) + Z_h(t)|_0^T, \end{aligned} \tag{89}$$

where

$$\begin{aligned} Z_h(t) \quad &= \quad X_h(t) - \frac{\Lambda h^2}{8} Y_h(t) \\ &= \quad h\sum_{j=1}^N j\left(\frac{u_{j+1} - u_{j-1}}{2}\right) u_j' - \frac{\Lambda h^2}{8} h\sum_{j=1}^N u_j' u_j \\ &= \quad h\sum_{j=1}^N u_j' \left[j\left(\frac{u_{j+1} - u_{j-1}}{2}\right) - \frac{\Lambda h^2}{8} u_j\right] \end{aligned} \tag{90}$$

for every solution of (20) in which Λ is the largest eigenvalue entering its Fourier development.

Next, we estimate the term Z_h using Lemma 6. So,

$$\begin{aligned} Z_h(t)|_0^T &= |Z_h(T) - Z_h(0)| \\ &\leq |Z_h(T)| + |Z_h(0)| \\ &\leq 2\sqrt{L^2 - \frac{\Lambda h^4}{16} + \frac{3\Lambda h^2}{16\lambda_1}} E_h(0). \end{aligned} \tag{91}$$

Therefore,

$$Z_h(T) - Z_h(0) \geq -2\sqrt{L^2 - \frac{\Lambda h^4}{16} + \frac{3\Lambda h^2}{16\lambda_1}} E_h(0). \tag{92}$$

Using (89) and (92), we deduce that

$$\left[T\left(1 - \frac{\Lambda h^2}{4}\right) - 2\sqrt{L^2 - \frac{\Lambda h^4}{16} + \frac{3\Lambda h^2}{16\lambda_1}}\right] E_h(0) \leq \frac{L}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt. \tag{93}$$

From (93) and considering that $\Lambda = \frac{\gamma}{h^2}$ in the class of solutions $\mathcal{C}_h(\gamma)$ of (20), we find that

$$E_h(0) \leq \frac{L}{\left[T\left(1 - \frac{\Lambda h^2}{4}\right) - 2\sqrt{L^2 + \frac{\gamma}{16}\left(\frac{3}{\lambda_1} - h^2\right)}\right]} \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt$$

provided that

$$T > \frac{2\sqrt{L^2 + \frac{\gamma}{16}\left(\frac{3}{\lambda_1} - h^2\right)}}{\left(1 - \frac{\Lambda h^2}{4}\right)}.$$

Considering that $\lambda_1 \geq \frac{\pi^2}{2L^2}$ for h sufficiently small, we have

$$T(\gamma) = \frac{2\sqrt{L^2\left(1 + \frac{3\gamma}{8\pi^2}\right) - \frac{\gamma h^2}{16}}}{1 - \frac{\gamma}{4}} \tag{94}$$

and

$$C(T,\gamma) = \frac{L}{2\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right) - \frac{\gamma h^2}{16}}\right]}. \tag{95}$$

So as $\gamma \searrow 0$,

$$\lim_{\gamma\to 0} T(\gamma) = \lim_{\gamma\to 0} \frac{2\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right) - \frac{\gamma h^2}{16}}}{1-\frac{\gamma}{4}} = \lim_{\gamma\to 0} 2\sqrt{L^2} = 2L \tag{96}$$

and

$$\lim_{\gamma\to 0} C(T,\gamma) = \lim_{\gamma\to 0} \frac{L}{2\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right) - \frac{\gamma h^2}{16}}\right]}$$
$$= \lim_{\gamma\to 0} \frac{L}{2(T-2L)}. \tag{97}$$

So, as $\gamma \searrow 0$, $T(\gamma) \searrow 2L$ and $C(T,\gamma) \searrow \frac{L}{2(T-2L)}$.

As $\gamma \nearrow 4$,

$$\lim_{\gamma\to 4} T(\gamma) = \lim_{\gamma\to 4} \frac{2\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right) - \frac{\gamma h^2}{16}}}{1-\frac{\gamma}{4}} \tag{98}$$

and as $\gamma \to 4$, $\left(1-\frac{\gamma}{4}\right) \to 0$, so $T(\gamma) \nearrow \infty$. As $\gamma \nearrow 4$,

$$\lim_{\gamma\to 4} C(T,\gamma) = \lim_{\gamma\to 4} \frac{L}{2\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right) - \frac{\gamma h^2}{16}}\right]} \tag{99}$$

$$= \lim_{\gamma\to 4} \frac{L}{-4\sqrt{L^2\left(1+\frac{3}{2\pi^2}\right) - \frac{h^2}{4}}}. \tag{100}$$

So the statements of Theorem 4 hold. □

4 Finite Element semi-discretized wave equation by linear splines

The idea of the Finite Element Method is to create a piecewise polynomial approximation of partial differential equations. In order to apply this method to the wave equation, the strong/continuous (PDE) form of the wave equation (6) is transformed into the weak formulation. This formulation allows to find different solutions that cannot arise from the strong form because solutions of the strong form must be twice differentiable over the interval of the string, $[0, L]$. To construct the weak form, we multiply both sides of the equation by $\phi(x)$, which is a continuously differentiable test function which satisfies the following boundary conditions:

$$\phi(0) = \phi(L) = 0 \text{ and } \phi(x) = 0, \text{ for } x < 0, \ x > L.$$

Then, both sides of the equality $(u_{tt} - u_{xx}) \cdot \phi = 0 \cdot \phi$ is integrated over $[0, L]$ to obtain the weak form of (6):

$$\int_0^L (u_{tt} - u_{xx}) \ \phi dx = 0. \tag{101}$$

After the integrating by parts and noting that $\phi(x)$ is zero at the endpoints, $x = 0$ and $x = L$, the weak form of the wave equation is as the following :

$$\int_0^L u_{tt}\phi \ dx + \int_0^L u_x\phi_x \ dx = 0. \tag{102}$$

Next, we partition the interval $[0, L]$ into finite elements and for each subinterval we define a basis function. Like with the Finite Difference method, we use the mesh parameter $h = \frac{L}{N+1}$. See Figure 6 for an image depicting these basis functions. Indeed, the basis functions are explicitly defined as hat functions, also known as

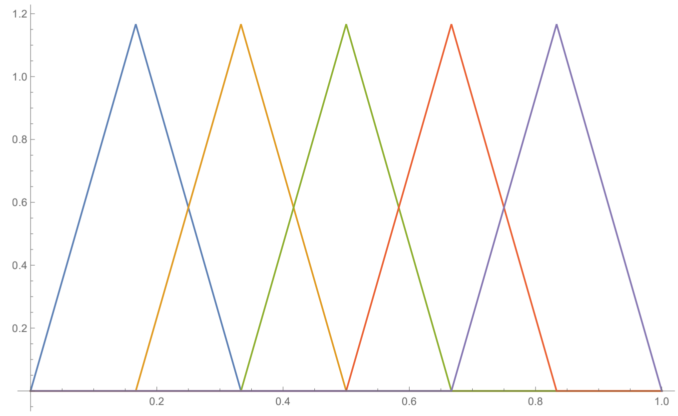

Figure 6: Basis equations $\{\phi_i(x)\}_{i=1}^{5}$ for the Finite Element Method.

piecewise linear splines. At each node x_i, the splines are defined as the following for i = 1, 2, ..., N:

$$\phi_i(x) = \begin{cases} 0, & x < x_{i-1} \\ \frac{1}{h}(x - x_{i-1}), & x_{i-1} < x < x_i \\ -\frac{1}{h}(x - x_{i+1}), & x_i < x < x_{i+1} \\ 0, & x > x_{i+1} \end{cases} \tag{103}$$

Piecewise polynomial basis functions, $\phi_i(x)$ for $i = 1, 2, \ldots, N$ are chosen such that they are linearly independent to one another. So the weak form can now be written for each type of hat function:

$$\int_0^L u_{tt}\phi_i \, dx + \int_0^L u_x \phi_{i,x} \, dx = 0, \text{ for } i = 1, 2, \ldots, N+1. \tag{104}$$

where $\phi_0 = \phi_N = 0$ due to the boundary conditions. The functions $\phi_i(x)$ form a basis of linearly independent functions on the interval $[0, L]$. Therefore, no linear combination of the functions in the basis can result in another function in the basis. This also means that all linear combinations in (104) will span the solution space of the weak formulation. Thus, the general solutions are sums of linear combinations of $\phi_i(x)$, seen below.

$$u(x,t) = \sum_i c_i(t)\phi_i(x),$$

such that the weak formulation can then be used to solve for all scalar coefficients, $c_i(t)$ for $i = 1, 2, \ldots, N+1$.

It follows that

$$u_{i,tt}^{N+1} = \sum_{i=0}^{N+1} c_{i,tt}(t)\phi_i(x), \tag{105}$$

and

$$u_{i,x}^{N+1} = \sum_{i=0}^{N+1} c_i(t)\phi_{i,x}(x). \tag{106}$$

Substituting the above summations (105) and (106) into (104) and letting $\phi_i(x)$ be equal to $\phi_j(x)$ for $j = 1, 2, , \ldots, N+1$ one at a time, (104) can be rewritten as:

$$\begin{aligned} 0 &= \int_0^L \left(\sum_{i=0}^{N+1} c_{i,tt}(t)\phi_i(x)\right)\phi_j(x)dx + \int_0^L \left(\sum_{i=0}^{N+1} c_i(t)\phi_{i,x}(x)\right)\phi_{j,x}(x)dx = 0 \\ &= \sum_{i=0}^{N+1} c_{i,tt}(t)\int_0^L \phi_i(x)\phi_j(x)dx + \sum_{i=0}^{N+1} c_i(t)\int_0^L \phi_{i,x}(x)\phi_{j,x}(x)dx. \end{aligned} \tag{107}$$

We must evaluate every instance of each integral in (107) to identify all of the coefficients of the functions of time $c_{i,tt}(t)$ and $c_i(t)$.

Note that, by the orthogonality of these "Galerkin" basis functions,

$$\int_0^L \phi_i(x)\phi_j(x)dx, \int_0^L \phi_{i,x}(x)\phi_{j,x}(x)dx = \begin{cases} nonzero, & |i-j| \leq 1 \\ 0, & |i-j| > 1 \end{cases}$$

Therefore, we need only solve the nonzero portion of these integrals where $\phi_i(x)$, $\phi_j(x)$ and $\phi_{i,x}(x)$, $\phi_{j,x}(x)$ are not orthogonal to each other. The details for this can be found in [3]. It shows the following:

$$\begin{cases} \int_0^L \phi_i^2 dx = \frac{2}{3}h, & j = i \\ \int_0^L \phi_i \cdot \phi_j dx = \frac{1}{6}h, & j = i \pm 1 \\ \int_0^L \phi_{i,x}^2 dx = \frac{2}{h}, & j = i \\ \int_0^L \phi_{i,x} \cdot \phi_{j,x} dx = -\frac{1}{h}, & j = i \pm 1 \end{cases} \tag{108}$$

From this, we can write the system (6) in the following discrete form:

$$\begin{cases} \frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}'' = \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = 0, \quad 0 < t < T, \quad j = 1, 2, \dots, N \\ u_0 = u_{N+1} = 0 \end{cases} \tag{109}$$

The discrete energy for the system (109) is also given by

$$E_h(t) = \frac{h}{6}\sum_{j=1}^{N} |u_j'|^2 + \frac{h}{12}\sum_{j=0}^{N} |u_j' + u_{j+1}'|^2 + \frac{h}{2}\sum_{j=0}^{N} \left|\frac{u_{j+1} - u_j}{h}\right|^2. \tag{110}$$

4.1 First-order form and spectral analysis

Letting the matrix M be

$$M := \begin{pmatrix} 2/3 & 1/6 & 0 & 0 & \dots & 0 \\ 1/6 & 2/3 & 1/6 & 0 & \dots & 0 \\ 0 & 1/6 & 2/3 & 1/6 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \\ 0 & 0 & \dots & 0 & 1/6 & 1/3 \end{pmatrix}_{(N+1)\times(N+1)} \tag{111}$$

and using A_h in (26), (109) can be written as the following matrix equation:

$$M \begin{pmatrix} c_{1,tt} \\ c_{2,tt} \\ \vdots \\ c_{N+1,tt} \end{pmatrix}_{(N+1)\times 1} + A_h \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N+1} \end{pmatrix}_{(N+1)\times 1} = 0. \tag{112}$$

Letting $\vec{c} = [c_1, c_2, \ldots, c_{N+1}, c_{1,t}, c_{2,t}, \ldots, c_{N+1,t}]^T$ we rewrite (112) in the first-order differential equation form

$$\vec{c_t} + \tilde{A}\vec{c} = \vec{c_t} + \begin{bmatrix} 0 & I \\ M^{-1}A_h & 0 \end{bmatrix} \vec{c} = 0. \tag{113}$$

Now, consider the eigenvalue problem $\tilde{A}\vec{z} = \tilde{\lambda}\vec{z}$, corresponding to (113)

$$\begin{bmatrix} 0 & I \\ M^{-1}A_h & 0 \end{bmatrix} \begin{bmatrix} \vec{z_1} \\ \vec{z_2} \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} \vec{z_1} \\ \vec{z_2} \end{bmatrix}. \tag{114}$$

Solving (114) is equivalent to solving

$$\begin{aligned} \vec{z_2} &= \tilde{\lambda}\vec{z_1}, \\ M^{-1}A_h\vec{z_1} &= \tilde{\lambda}^2 \tilde{z}_1. \end{aligned}$$

The details for this can be found in [3]:

Theorem 5 *The solutions of the eigenvalue problem (114) are given by*

$$\tilde{\lambda}_j = \frac{1}{h}\sqrt{\frac{6 - 6\cos\left(\frac{(2j-1)\pi h}{2(L-h)}\right)}{2 + \cos\left(\frac{(2j-1)\pi h}{2(L-h)}\right)}} \quad \textit{for } j = 1, 2, \ldots, N+1. \tag{115}$$

4.2 Technical results

Lemma 7 *For any $h > 0$ and any eigenvector of the system (109), the following identity holds:*

$$h\sum_{j=0}^{N}\left|\frac{\psi_{j+1}-\psi_j}{h}\right|^2 = \frac{6+\lambda h^2}{12-\lambda h^2}L\left|\frac{\psi_N}{h}\right|. \tag{116}$$

Moreover, $\lambda h^2 \leq 12h$ for any $h > 0$ and any eigenvalue and

$$\lambda_N(h)h^2 \to 12 \text{ as } h \to 0. \tag{117}$$

Proof: By (37) in Lemma 2, we have

$$h\sum_{j=0}^{N}\left|\frac{2L}{4-\rho h^2}\psi_{j+1}-\psi_j h\right|^2 = \left|\frac{\psi_N}{h}\right|, \tag{118}$$

where ρ is the eigenvalue of the matrix A associated to ψ.

The eigenvalues and eigenvectors of the system

$$\begin{cases} -\left[\frac{\psi_{j+1}-2\psi_j+\psi_{j-1}}{h^2}\right] = \lambda\left[\frac{2}{3}\psi_j+\frac{1}{6}\psi_{j+1}+\frac{1}{6}\psi_{j-1}\right], \quad j=1,2,\ldots,N \\ \psi_0=\psi_{N+1}=0 \end{cases} \tag{119}$$

are the eigenvalues and eigenvectors of the matrix $M^{-1}A_h$. From [3], these are

$$\lambda_n(M^{-1}A_h) = \frac{\lambda_k(A_h)}{1-\frac{h^2}{6}\lambda_k(A_h)}, \tag{120}$$

so $\rho = \frac{\lambda}{1+\frac{h^2}{6}\lambda}$. Therefore,

$$\begin{aligned}\frac{2L}{4-\rho h^2} &= \frac{2L}{4-\left(\frac{\lambda}{1+\frac{h^2}{6}\lambda}\right)h^2} = \frac{2L}{\frac{4\left(1+\frac{h^2}{6}\lambda\right)-\lambda h^2}{\left(1+\frac{h^2}{6}\lambda\right)}} = \frac{2L\left(1+\frac{h^2}{6}\lambda\right)}{4\left(1+\frac{h^2}{6}\lambda\right)-\lambda h^2} \\ &= \frac{\left(2-\frac{2h^2}{6}\lambda\right)L}{4+\frac{4h^2}{6}\lambda-\lambda h^2} = \frac{(12+2h^2\lambda)L}{24+4h^2\lambda-6\lambda h^2} = \frac{6+h^2\lambda}{12-\lambda h^2}L.\end{aligned} \tag{121}$$

In addition, using $h = \frac{L}{N+1}$,

$$\lim_{h\to 0} \lambda_N(h)h^2 = \lim_{h\to 0} 6 \left[\frac{1-\cos(N\pi h/L)}{2+\cos(N\pi h/L)}\right] = \lim_{h\to 0} 6 \left[\frac{1-\cos(\pi - h\pi/L)}{2+\cos(\pi - h\pi/L)}\right]$$
$$= \lim_{h\to 0} 6 \left[\frac{1+\cos(h\pi/L)}{2-\cos(h\pi/L)}\right] = \lim_{h\to 0} 6 \left[\frac{2}{1}\right] = 12 \tag{122}$$

Lemma 8 *For any eigenvector $\vec{\psi}$ with eigenvalue of λ of (119), the following identity holds:*

$$\begin{aligned}\sum_{j=0}^{N} \left|\frac{\psi_j - \psi_{j+1}}{h}\right|^2 &= \lambda \sum_{j=1}^{N} \left(\frac{2}{3}|\psi_j|^2 + \frac{1}{3}\psi_j\psi_{j+1}\right) \\ &= \lambda \sum_{j=1}^{N} \left(\frac{1}{3}|\psi_j|^2 + \frac{1}{6}|\psi_j + \psi_{j+1}|^2\right).\end{aligned} \tag{123}$$

Proof: We multiply in (119) by ψ_j and add for $j = 1, \ldots, N$ to find

$$\sum_{j=1}^{N} - \left[\frac{\psi_{j+1} + \psi_{j-1} - 2\psi_j}{h^2}\right]\psi_j = \sum_{j=1}^{N} \lambda \left[\frac{2}{3}\psi_j + \frac{1}{6}\psi_{j+1} + \frac{1}{6}\psi_{j-1}\right]\psi_j. \tag{124}$$

The left-hand side is the same as in the proof of Lemma 1, so we only need to check the right-hand side:

$$\lambda \sum_{j=1}^{N} \left[\frac{2}{3}\psi_j + \frac{1}{6}\psi_{j+1} + \frac{1}{6}\psi_{j-1}\right] \psi_j = \lambda \sum_{j=1}^{N} \left[\frac{2}{3}|\psi_j|^2 + \frac{1}{6}\psi_j\psi_{j+1} + \frac{1}{6}\psi_j\psi_{j-1}\right]$$
$$= \lambda \sum_{j=1}^{N} \frac{1}{3}|\psi_j|^2 + \lambda \sum_{j=1}^{N} \left[\frac{1}{3}|\psi_j|^2 + \frac{1}{6}\psi_j\psi_{j+1} + \frac{1}{6}\psi_j\psi_{j-1}\right]$$
$$= \lambda \sum_{j=1}^{N} \frac{1}{3}|\psi_j|^2 + \lambda \left[\left(\frac{1}{3}|\psi_1|^2 + \frac{1}{6}\psi_1\psi_2 + \frac{1}{6}\psi_1\psi_0\right) + \left(\frac{1}{3}|\psi_2|^2 + \frac{1}{6}\psi_2\psi_3\right.\right.$$
$$\left. + \frac{1}{6}\psi_2\psi_1\right) + \cdots + (\frac{1}{3}|\psi_{N-1}|^2 + \frac{1}{6}\psi_{N-1}\psi_N + \frac{1}{6}\psi_{N-1}\psi_{N-2}\Big)$$
$$\left. + \left(\frac{1}{3}|\psi_N|^2 + \frac{1}{6}\psi_N\psi_{N+1} + \frac{1}{6}\psi_N\psi_{N-1}\right)\right]$$
$$= \lambda \sum_{j=1}^{N} \frac{1}{3}|\psi_j|^2 + \lambda \left[\frac{1}{3}(|\psi_1|^2 + \cdots + |\psi_N|^2) + \frac{1}{3}(\psi_1\psi_2 + \psi_2\psi_3 + \cdots + \psi_{N-1}\psi_N)\right]$$
$$= \lambda \sum_{j=1}^{N} \frac{1}{3}|\psi_j|^2 + \lambda \sum_{j=1}^{N} \left[\frac{1}{3}|\psi_j|^2 + \frac{1}{3}\psi_j\psi_{j+1}\right]$$
$$= \lambda \sum_{j=1}^{N} \frac{1}{3}|\psi_j|^2 + \lambda \sum_{j=1}^{N} \left[\frac{1}{6}|\psi_j|^2 + \frac{1}{6}|\psi_{j+1}|^2 + \frac{1}{3}\psi_j\psi_{j+1}\right]$$
$$= \lambda \sum_{j=1}^{N} \frac{1}{3}|\psi_j|^2 + \lambda \sum_{j=1}^{N} \frac{1}{6}|\psi_j + \psi_{j+1}||\psi_j + \psi_{j+1}|$$
$$= \lambda \sum_{j=1}^{N} \left(\frac{1}{3}|\psi_j|^2 + \frac{1}{6}|\psi_j + \psi_{j+1}|^2\right),$$

as needed. □

Lemma 9 *(Conservation of Energy)*

For any $h > 0$ and any solution of (109), it follows that $\frac{dE(t)}{dt} = 0$, i.e.,

$$E_h(t) = E_h(0), \quad \forall t \in (0, T). \tag{125}$$

Proof: We multiply in (109) by u'_j and add for $j = 1, \ldots, N$ to find

$$\sum_{j=1}^{N} \left(\frac{2}{3}u''_j + \frac{1}{6}u''_{j+1} + \frac{1}{6}u''_{j-1}\right) u'_j - \sum_{j=1}^{N} \frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} u'_j = 0. \tag{126}$$

The right-hand side of (126) is the same as in the proof of Lemma 3, so we know

$$\sum_{j=1}^{N}\left(-\frac{u_{j+1}+u_{j-1}-2u_j}{h^2}\right)u_j' = \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N}\left|\frac{u_j-u_{j+1}}{h}\right|^2. \tag{127}$$

We also know from the proof of Lemma 3 that

$$2\sum_{j=1}^{N}u_j''u_j' = \frac{d}{dt}\sum_{j=1}^{N}|u_j'|^2. \tag{128}$$

Additionally, we find

$$\begin{aligned}
\frac{1}{6}\sum_{j=1}^{N}(u_{j+1}''u_j' + u_{j-1}''u_j') &= \frac{1}{6}[(u_2''u_1' + u_0''u_1') + (u_3''u_2' + u_1''u_2') + \dots \\
&\quad +(u_N''u_{N-1}' + u_{N-2}''u_{N-1}') + (u_{N+1}''u_N' + u_{N-1}''u_N')] \\
&= \frac{1}{6}[(u_1''u_2' + u_1'u_2'') + (u_2''u_3' + u_2'u_3'') + \dots + (u_{N-1}''u_N' + u_{N-1}'u_N'')] \\
&= \frac{1}{6}\sum_{j=1}^{N}(u_j''u_{j+1}' + u_j'u_{j+1}'') = \frac{1}{6}\frac{d}{dt}\sum_{j=1}^{N}u_j'u_{j+1}'
\end{aligned} \tag{129}$$

So combining (128) and (129), the left-hand side becomes

$$\begin{aligned}
\sum_{j=1}^{N}\left(\frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}''\right)u_j' &= \frac{2}{3}\sum_{j=1}^{N}u_j''u_j' + \frac{1}{6}\sum_{j=1}^{N}\left(u_{j+1}''u_j' + u_{j-1}''u_j'\right) \\
= \frac{1}{3}\frac{d}{dt}\sum_{j=1}^{N}|u_j'|^2 + \frac{1}{6}\frac{d}{dt}\sum_{j=1}^{N}u_j'u_{j+1}' &= \frac{1}{6}\frac{d}{dt}\sum_{j=1}^{N}|u_j'|^2 + \frac{1}{12}\frac{d}{dt}\sum_{j=1}^{N}|u_j' + u_{j+1}'|^2.
\end{aligned}$$

Recalling (110) and combining (126), (127), and (130), we have

$$\frac{d}{dt}\left(\frac{h}{6}\sum_{j=1}^{N}|u_j'|^2 + \frac{h}{12}\sum_{j=0}^{N}|u_j' + u_{j+1}'|^2 + \frac{h}{2}\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2\right) = 0. \tag{130}$$

Therefore, $\frac{dE(t)}{dt} = 0.\square$

Theorem 6 *(Lack of Observability)*

For any $T > 0$, we have

$$\sup_{\vec{u}\ \textit{solves}\ (20)}\left[\frac{E_h(0)}{\int_0^T |u_N(t)/h|^2 dt}\right] \to \infty \quad \textit{as} \quad h \to 0. \tag{131}$$

Proof: Like with the proof of Theorem 3, we consider the Nth eigensolution just for $\vec{u}$

$$\vec{u} = e^{i\sqrt{\lambda_N}t}\vec{\psi}_N, \tag{132}$$

where $\vec{u} = (u_1, ..., u_N)^T$ and $\vec{\psi}_N = (\psi_{N,1}, ..., \psi_{N,N})^T$. The observability inequality is

$$E_h(0) \leq C(T,h) \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt, \tag{133}$$

and our goal is to prove the non-existence of $C(T)$.

The energy for the system (109), (110) can be written as follows for the Nth eigensolution:

$$E_h(0) = \frac{h}{2}\left(\frac{1}{3}\sum_{j=1}^{N}|\psi'_{N,j}|^2 + \frac{1}{6}\sum_{j=0}^{N}|\psi'_{N,j} + \psi'_{N,j+1}|^2\right) + \frac{h}{2}\sum_{j=0}^{N}\left|\frac{\psi_{N,j+1} - \psi_{N,j}}{h}\right|^2. \tag{134}$$

By (134), Lemma 8, and $u'_{N,j} = \tilde{\lambda}_N u_N$, the discretized energy of (109) is

$$\begin{aligned} E_h(0) &= \frac{\lambda_N h}{2}\left(\sum_{j=1}^{N}\frac{1}{3}|\psi_{N,j}|^2 + \sum_{j=0}^{N}\frac{1}{6}|\psi_{N,j} + \psi_{N,j+1}|^2\right) + \frac{h}{2}\sum_{j=0}^{N}\left|\frac{\psi_{N,j+1} - \psi_{N,j}}{h}\right|^2 \\ &= h\sum_{j=0}^{N}\left|\frac{\psi_{N,j+1} - \psi_{N,j}}{h}\right|^2 \end{aligned} \tag{135}$$

By Lemma 7, the discretized observation of (109) is

$$\begin{aligned} \int_0^T \left|\frac{\psi_{N,N}}{h}\right|^2 dt = T\left|\frac{\psi_{N,N}}{h}\right|^2 &= \frac{Th}{L}\frac{12 - \lambda_N h^2}{6 + \lambda_N h^2}\sum_{j=0}^{N}\left|\frac{\psi_{N,j+1} - \psi_{N,j}}{h}\right|^2 \\ &= \frac{T}{L}\frac{12 - \lambda_N h^2}{6 + \lambda_N h^2}E_h(0). \end{aligned} \tag{136}$$

So,

$$\frac{E_h(0)}{\int_0^T \left|\frac{\psi_{N,N}}{h}\right|^2 dt} = \frac{L}{T}\frac{6 + \lambda_N h^2}{12 - \lambda_N h^2}. \tag{137}$$

So by (117),

$$\lim_{h\to 0} \frac{E_h(0)}{\int_0^T \left|\frac{\psi_{N,N}}{h}\right|^2 dt} = \lim_{h\to 0} \frac{L}{T}\frac{6+\lambda_N h^2}{12-\lambda_N h^2} = \lim_{h\to 0}\frac{18L}{0^+} \to +\infty. \quad \square \tag{138}$$

4.3 Filtering the numerical scheme and numerical observability

Lemma 10 *For all* $0 < \epsilon < 1$*. the gap between the roots of consecutive eigenvalues associated to indexes such that* $(j+1)h \le \epsilon L$ *satisfies*

$$\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} \ge \frac{\pi}{L}\left(\frac{3\,(1+\cos(\epsilon\pi))}{2\,(2+\cos(\epsilon\pi))}\right)^{1/2}. \tag{139}$$

Proof: From (23) and (120),

$$\begin{aligned}
\lambda_n(M^{-1}A_h) &= \frac{\lambda_k(A_h)}{1-\frac{h^2}{6}\lambda_k(A_h)} = \frac{\frac{4}{h^2}\sin^2\frac{\pi kh}{2L}}{1-\frac{h^2}{6}\left(\frac{4}{h^2}\sin^2\frac{\pi kh}{2L}\right)} \\
&= \frac{6}{4}\left(\frac{\frac{4}{h^2}\sin^2\frac{\pi kh}{2L}}{1-\frac{4}{6}\sin^2\frac{\pi kh}{2L}}\right) = \frac{\frac{6}{h^2}\sin^2\frac{\pi kh}{2L}}{\frac{3}{2}-\sin^2\frac{\pi kh}{2L}} = \frac{6}{h^2}\left[\frac{\sin^2\frac{\pi kh}{2L}}{\frac{3}{2}-\sin^2\frac{\pi kh}{2L}}\right] \\
&= \frac{6}{h^2}\left[\frac{\frac{1-\cos\frac{\pi kh}{L}}{2}}{\frac{3}{2}-\frac{1-\cos\frac{\pi kh}{L}}{2}}\right] = \frac{6}{h^2}\left[\frac{1-\cos\frac{\pi kh}{L}}{2+\cos\frac{\pi kh}{L}}\right].
\end{aligned} \tag{140}$$

And from (122), we have $\lambda_N(h)h^2 \to 12$ as $h \to 0$. For j fixed, we can use L'Hospital's rule twice to find

$$\begin{aligned}
&\lim_{h\to 0}\lambda_j(h) = \lim_{h\to 0}\frac{6}{h^2}\left[\frac{1-\cos\frac{\pi kh}{L}}{2+\cos\frac{\pi kh}{L}}\right] = \lim_{h\to 0}\frac{\frac{6j\pi}{L}\sin\frac{\pi jh}{L}}{4h+2h\cos\frac{\pi jh}{L}-\frac{h^2 j\pi}{L}\sin\frac{\pi jh}{L}} \\
&\lim_{h\to 0}\frac{6\left(\frac{j\pi}{L}\right)^2\cos\frac{\pi jh}{L}}{4+2\cos\frac{\pi jh}{L}-2h\frac{j\pi}{L}\sin\frac{\pi jh}{L}-2h\frac{j\pi}{L}\sin\frac{\pi jh}{L}-h^2\left(\frac{j\pi}{L}\right)^2\cos\frac{\pi jh}{L}} \\
&= \lim_{h\to 0}\frac{6\left(\frac{j\pi}{L}\right)^2}{4+2} = \left(\frac{j\pi}{L}\right)^2
\end{aligned} \tag{141}$$

Looking at the gap between the roots of the eigenvalues, we have $|i\sqrt{\lambda_{j+1}(h)} - i\sqrt{\lambda_j}|$, but because this is an increasing function and the modulo of i is 1, we have, for some $\xi \in [j\pi h/L, (j+1\pi h/L)]$,

$$\begin{aligned}
\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} &= \frac{\sqrt{6}}{h}\left[\left(\frac{1-\cos\frac{(j+1)\pi h}{L}}{2+\cos\frac{(j+1)\pi h}{L}}\right)^{1/2} - \left(\frac{1-\cos\frac{j\pi h}{L}}{2+\cos\frac{j\pi h}{L}}\right)^{1/2}\right] \\
&= \frac{\sqrt{6}}{h}\left[\frac{1}{2}\left(\frac{1-\cos\xi}{2+\cos\xi}\right)^{-1/2}\left(\frac{(2+\cos\xi)\sin\xi-(1-\cos\xi)(-\sin\xi)}{(2+\cos\xi)^2}\right)\right. \\
&\qquad \left. * \left(\frac{(j+1)\pi h}{L} - \frac{j\pi h}{L}\right)\right] \\
&= \frac{\sqrt{6}\pi}{L}\left[\frac{1}{2}\left(\frac{2+\cos\xi}{1-\cos\xi}\right)^{1/2}\left(\frac{2\sin\xi+\cos\xi\sin\xi+\sin\xi-\cos\xi\sin\xi}{(2+\cos\xi)^2}\right)\right] \\
&= \frac{\sqrt{6}\pi}{L}\left[\frac{1}{2}\left(\frac{2+\cos\xi}{1-\cos\xi}\right)^{1/2}\left(\frac{3\sin\xi}{(2+\cos\xi)^2}\right)\right] \\
&= \frac{\sqrt{6}\pi}{L}\left[\frac{1}{2}\left(\frac{(2+\cos\xi)^{1/2}}{(1-\cos\xi)^{1/2}}\right)\left(\frac{3(1-\cos^2\xi)^{1/2}}{(2+\cos\xi)^2}\right)\right] \\
&= \frac{\sqrt{6}\pi}{L}\left[\frac{1}{2}\left(\frac{(2+\cos\xi)^{1/2}}{(1-\cos\xi)^{1/2}}\right)\left(\frac{3(1-\cos\xi)^{1/2}(1+\cos\xi)^{1/2}}{(2+\cos\xi)^2}\right)\right] \\
&= \frac{\sqrt{6}\pi}{L}\left[\frac{1}{2}\left(\frac{3(1+\cos\xi)^{1/2}}{(2+\cos\xi)^{3/2}}\right)\right] \\
&= 3\sqrt{\frac{3}{2}}\frac{\pi(1+\cos\xi)^{1/2}}{L(2+\cos\xi)^{3/2}} = 3\sqrt{\frac{3}{2}}\frac{\pi(1+\cos\xi)^{1/2}}{L(2+\cos\xi)(2+\cos\xi)^{1/2}} \qquad (142) \\
&\geq \sqrt{\frac{3}{2}}\frac{\pi}{L}\left(\frac{1+\cos\xi}{2+\cos\xi}\right)^{1/2},
\end{aligned}$$

where we used the Mean Value Theorem and the fact that the maximum of $(2+\cos\xi)$ is 3.

Assume we consider eigenvalues corresponding to the indexes $(j+1)h \leq \epsilon L$ with $0 < \epsilon < 1$. Then $\xi \leq \frac{(j+1)\pi h}{L} \leq \frac{\epsilon\pi L}{L} = \epsilon\pi$. So $0 \leq \xi \leq \pi$. Thus, because cos is a decreasing function between 0 and π, $\cos\xi \geq \cos(\epsilon\pi)$. We can combine this with (142) to find

$$\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} \geq \frac{\pi}{L}\left(\frac{3(1+\cos\xi)}{2(2+\cos\xi)}\right)^{1/2} \geq \frac{\pi}{L}\left(\frac{3(1+\cos(\epsilon\pi))}{2(2+\cos(\epsilon\pi))}\right)^{1/2}, \qquad (143)$$

as needed. □

Lemma 11 *For any $h > 0$ and any solution u of (109), the following identity* *holds:*

$$
\begin{aligned}
&TE_h(0) - \frac{h}{12}\sum_{j=0}^{N}\int_0^T |u_j' - u_{j+1}'|^2 dt + X_h(t)|_0^T \\
&= \frac{L}{2}\int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right] dt, \qquad (144)
\end{aligned}
$$

with

$$
X_h(t) = h\sum_{j=1}^{N} j(u_{j+1} - u_{j-1})(\frac{1}{3}u_j' + \frac{1}{12}u_{j+1}' + \frac{1}{12}u_{j-1}'). \qquad (145)
$$

Proof: We multiply in (109) by $j\frac{(u_{j+1}-u_{j-1})}{2}$ to obtain

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_0^T \left[\frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}''\right] j\frac{(u_{j+1} - u_{j-1})}{2}dt \\
&\quad -\sum_{j=1}^{N}\int_0^T \frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} j\frac{(u_{j+1} - u_{j-1})}{2}dt = I_1 - I_2 = 0. \qquad (146)
\end{aligned}
$$

The second term, I_2, can be treated as in the proof of Lemma 4. So we have

$$
I_2 = -\frac{1}{2}\sum_{j-0}^{N}\int_0^T \left|\frac{u_j - u_{j+1}}{h}\right|^2 dt + \frac{(N+1)}{2}\int_0^T \left|\frac{u_N}{h}\right|^2 dt. \qquad (147)
$$

For the first term, in view of (49), we have

$$I_1 = \sum_{j=1}^{N}\int_0^T \left[\frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}''\right] j\frac{(u_{j+1}-u_{j-1})}{2}dt = \frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt$$
$$+\frac{2}{3}\sum_{j=1}^{N}u_j'j\frac{(u_{j+1}-u_{j-1})}{2}\Bigg|_0^T + \frac{1}{12}\sum_{j=1}^{N}\int_0^T (u_{j+1}''+u_{j-2}'')j(u_{j+1}-u_{j-1})dt$$
$$= \frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt + \frac{1}{3}\sum_{j=1}^{N}u_j'j(u_{j+1}-u_{j-1})\Bigg|_0^T$$
$$+\frac{1}{12}\sum_{j=1}^{N}\left[(u_{j+1}'+u_{j-1}')j(u_{j+1}-u_{j-1})\Big|_0^T\right.$$
$$\left.-\int_0^T (u_{j+1}'+u_{j-1}')j(u_{j+1}'-u_{j-1}')\right] = \frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt$$
$$+\frac{1}{3}\sum_{j=1}^{N}u_j'j(u_{j+1}-u_{j-1})\Bigg|_0^T - \frac{1}{12}\sum_{j=1}^{N}\int_0^T j(|u_{j+1}'|^2-|u_{j-1}'|^2)dt$$
$$+\frac{1}{12}\sum_{j=1}^{N}(u_{j+1}'+u_{j-1}'j(u_{j+1}-u_{j-1})\Bigg|_0^T$$
$$= \frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt + \sum_{j=1}^{N}j(u_{j+1}-u_{j-1})\left[\frac{1}{3}u_j'\frac{1}{12}u_{j+1}'+\frac{1}{12}u_{j-1}'\right]\Bigg|_0^T$$
$$-\frac{1}{12}\int_0^T [(|u_2'|^2-|u_0'|^2)+2(|u_3'|^2-u_1'|^)+3(|u_4'|^2-|u_2'|^2)+...$$
$$+(N-1)(|u_N'|^2-|u_{N-2}'|^2)+N(|u_{N+1}'|^2-|u_{N-1}'|^2)]dt$$
$$= \frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt + \sum_{j=1}^{N}j(u_{j+1}-u_{j-1})\left[\frac{1}{3}u_j'\frac{1}{12}u_{j+1}'+\frac{1}{12}u_{j-1}'\right]\Bigg|_0^T$$
$$-\frac{1}{12}\int_0^T [-2|u_1'|^2-2|u_2'|^2-...-2|u_{N-1}'|^2+(N-1)|u_N'|^2]dt$$
$$= \frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt + \sum_{j=1}^{N}j(u_{j+1}-u_{j-1})\left[\frac{1}{3}u_j'\frac{1}{12}u_{j+1}'+\frac{1}{12}u_{j-1}'\right]\Bigg|_0^T$$
$$+\frac{1}{6}\sum_{j=1}^{N}\int_0^T |u_j'|^2dt - \frac{N+1}{12}\int_0^T |u_N'|^2dt. \tag{148}$$

Combining (146-148), we have

$$\frac{1}{3}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'dt + \frac{1}{6}\sum_{j=1}^{N}\int_0^T |u_j'|^2dt - \frac{N+1}{12}\int_0^T |u_N'(t)|^2dt + X_h(t)|_0^T$$
$$-\left(-\frac{1}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{u_j-u_{j+1}}{h}\right|^2 dt + \frac{N+1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt\right) = 0. \tag{149}$$

Using $h = \frac{L}{N+1}$, we can find

$$-\frac{N+1}{12}\int_0^T |u_N'|^2 dt + \frac{N+1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = -\frac{L}{12h}\int_0^T |u_N'(t)|^2 dt$$
$$+\frac{L}{2h}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = \frac{L}{2h}\int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right] dt. \tag{150}$$

Combining (149) and (150), we find

$$\frac{L}{2}\int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right] dt = \frac{h}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{u_j - u_{j+1}}{h}\right|^2 dt$$
$$+\frac{h}{3}\sum_{j=1}^{N}\int_0^T u_j' u_{j+1}' dt + \frac{h}{6}\sum_{j=1}^{N}\int_0^T |u_j'|^2 dt + X_h(t)|_0^T$$
$$= \int_0^T \left[E_h(0) - \frac{h}{6}\sum_{j=1}^{N}|u_j'|^2 - \frac{h}{12}\sum_{j=0}^{N}|u_j' + u_{j+1}'|^2\right] dt$$
$$+\frac{h}{3}\sum_{j=1}^{N}\int_0^T u_j' u_{j+1}' dt + \frac{h}{6}\sum_{j=1}^{N}\int_0^T |u_j'|^2 dt + X_h(t)|_0^T$$
$$= TE_h(0) + \frac{h}{3}\sum_{j=0}^{N}\int_0^T u_j' u_{j+1}' dt - \frac{h}{12}\sum_{j=0}^{N}\int_0^T |u_j' + u_{j+1}|^2 dt + X_h(t)|_0^T$$
$$= TE_h(0) + \frac{h}{3}\sum_{j=0}^{N}\int_0^T \left[u_j' u_{j+1}' - \frac{|u_j' + u_{j+1}'|^2}{4}\right] dt + X_h(t)|_0^T. \tag{151}$$

Next, we find that

$$\sum_{j=0}^{N}\int_0^T \left[u_j' u_{j+1}' - \frac{|u_j' + u_{j+1}'|^2}{4}\right] dt = \sum_{j=0}^{N}\int_0^T \left[u_j' u_{j+1}' - \frac{|u_j'|^2}{4} - \frac{|u_{j+1}'|^2}{4}\right.$$
$$\left. - \frac{u_j' u_{j+1}'}{2}\right] dt = -\sum_{j=0}^{N}\int_0^T \left[\frac{|u_j'|^2}{4} + \frac{|u_{j+1}'|^2}{4} + \frac{u_j' u_{j+1}'}{2}\right] dt$$
$$= -\sum_{j=0}^{N}\int_0^T \frac{|u_j' - u_{j+1}'|^2}{4} dt. \tag{152}$$

Combining (151) and (152), we find

$$TE_h(0) - \frac{h}{12}\sum_{j=0}^{N}\int_0^T |u_j' - u_{j+1}'|^2 dt + X_h(t)|_0^T$$
$$= \frac{L}{2}\int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right] dt, \tag{153}$$

as needed. □

Lemma 12 *For any $h > 0$ and any solution u of (109), the following identity holds:*

$$h\int_0^T \sum_{j=0}^{N} \left|\frac{u_{j+1}-u_j}{h}\right|^2 dt = h\int_0^T \sum_{j=0}^{N} \left[\frac{1}{3}|u_j'|^2 + \frac{1}{6}|u_j' + u_{j+1}'|^2\right] dt - Y_h(t)|_0^T, \quad (154)$$

with

$$Y_h(t) = h\sum_{j=1}^{N} \left(\frac{2}{3}u_j' + \frac{1}{6}u_{j+1}' + \frac{1}{6}u_{j-1}'\right) u_j. \quad (155)$$

Proof: We multiply in (109) by u_j and find

$$h\sum_{j=1}^{N}\int_0^T \left(\frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}''\right) u_j dt = h\sum_{j=1}^{N}\int_0^T \left[\frac{u_{j+1}+u_{j-1}-2u_j}{h^2}\right] u_j dt. (156)$$

For the left-hand side, we integrate by parts to find

$$\begin{aligned}
&h\sum_{j=1}^{N}\int_0^T \left(\frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}''\right) u_j dt = h\sum_{j=1}^{N} \left(\frac{2}{3}u_j' + \frac{1}{6}u_{j+1}' + \frac{1}{6}u_{j-1}'\right) u_j \Big|_0^T \\
&\quad -h\sum_{j=1}^{N}\int_0^T \left(\frac{2}{3}u_j' + \frac{1}{6}u_{j+1}' + \frac{1}{6}u_{j-1}'\right) u_j' dt \\
&= Y_h(t)|_0^T - h\sum_{j=1}^{N}\int_0^T \left(\frac{2}{3}|u_j'|^2 + \frac{1}{3}u_j'u_{j+1}'\right) dt \\
&= Y_h(t)|_0^T - h\int_0^T \left[\left(\frac{2}{3}|u_1'|^2 + \frac{1}{3}u_1'u_2'\right) + \left(\frac{2}{3}|u_2'|^2 + \frac{1}{3}u_2'u_3'\right) + ...\right. \\
&\quad \left. + \left(\frac{2}{3}|u_{N-1}'|^2 + \frac{1}{3}u_{N-1}'u_N'\right) + \left(\frac{2}{3}|u_N'|^2 + \frac{1}{3}u_N'u_{N+1}'\right)\right] dt \\
&= Y_h(t)|_0^T - h\int_0^T \left[\frac{1}{3}|u_0'|^2 + \frac{1}{3}|u_1'|^2 + ... + \frac{1}{3}|u_N'|^2 + \left(\frac{1}{6}|u_0'|^2 + \frac{1}{6}|u_1'|^2\right.\right. \\
&\quad \left. +\frac{1}{3}u_0'u_1'\right) + \left(\frac{1}{6}|u_1'|^2 + \frac{1}{6}|u_2'|^2 + \frac{1}{3}u_1'u_2'\right) + ... + \left(\frac{1}{6}|u_{N-1}'|^2 + \frac{1}{6}|u_N'|^2\right. \\
&\quad \left.\left. +\frac{1}{3}u_{N-1}'u_N'\right) + \left(\frac{1}{6}|u_N'|^2 + \frac{1}{6}|u_{N+1}'|^2 + \frac{1}{3}u_N'u_{N+1}'\right)\right] dt \\
&= Y_h(t)|_0^T - h\sum_{j=0}^{N}\int_0^T \left(\frac{1}{3}|u_j'|^2 + \frac{1}{6}|u_j' + u_{j+1}'|^2\right) dt. \quad (157)
\end{aligned}$$

For the right-hand side, we have

$$h\sum_{j=1}^{N}\int_0^T \left[\frac{u_{j+1}+u_{j-1}-2u_j}{h^2}\right]u_j dt = \frac{h}{h^2}\sum_{j=1}^{N}\int_0^T \left[u_ju_{j+1}+u_ju_{j-1}-2|u_j|^2\right]dt$$
$$= \frac{h}{h^2}\int_0^T \Big[(u_1u_2+u_1u_0-2|u_1|^2)+(u_2u_3+u_2u_1-2|u_2|^2)+...+(u_{N-1}$$
$$u_N+u_{N-1}u_{N-2}-2|u_{N-1}|^2)+(u_Nu_{N+1}+u_Nu_{N-1}-2|u_N|^2)\Big]dt$$
$$= \frac{h}{h^2}\int_0^T \Big[(2u_0u_1-|u_0|^2-|u_1|^2)+(2u_1u_2-|u_1|^2-|u_2|^2)+...$$
$$+(2u_{N-1}u_N-|u_{N-1}|^2-|u_N|^2)+(2u_Nu_{N+1}-|u_N|^2-|u_{N+1}|^2)\Big]dt$$
$$= h\sum_{j=0}^{N}\int_0^T \left|\frac{u_j-u_{j+1}}{h}\right|^2 dt. \tag{158}$$

By substituting the results from (157) and (158) into (156), we find (154) immediately. □

Lemma 13 *For any $h>0$ and any solution u of (109), it follows that*

$$|Z_h(t)| \le \sqrt{L^2-\frac{\Lambda h^4}{48}+\frac{3\Lambda h^2}{16\lambda_1}}E_h(0), \quad \forall 0<t<T, \tag{159}$$

with

$$Z_h(t) = X_h(t)-\frac{\Lambda h^2}{24}Y_h(t) \tag{160}$$

and Λ being the largest eigenvalue entering the Fourier develpoment of the solution u of (109).

Proof: We start by putting $Z_h(t)$ into the tensor notation.

$$Z_h(t) = X_h(t)-\frac{\Lambda h^2}{24}Y_h(t)$$
$$= h\sum_{j=1}^{N}\left(j\frac{(u_{j+1}-u_j}{2}-\frac{\Lambda h^2}{24}u_j\right)\left(\frac{2}{3}u_j'+\frac{1}{6}u_{j+1}'+\frac{1}{6}u_{j-1}\right)$$
$$= h\begin{bmatrix}a_1 & a_2 & ... & a_N\end{bmatrix}M\begin{bmatrix}b_1 & b_2 & ... & b_N\end{bmatrix}^T = h\vec{a}^TM\vec{b}$$
$$= h\sum_{i,j=1}^{N}m_{ij}a_ib_j, \tag{161}$$

where

$$a_i = i\frac{(u_{i+1} - u_{i-1})}{2} - \frac{\Lambda h^2}{24}u_i; \quad b_i = u_i', \tag{162}$$

M is the matrix (111), and m_{ij} are the entries of M. Then, because M is a positive definite and symmetric matrix, we have

$$Z_h(t) = h\sum_{i,j=1}^{N} m_{ij}a_ib_j = h\vec{a}^T M\vec{b} = h(\vec{a}^T M^{1/2})(M^{1/2}\vec{b}).$$

Using the Cauchy-Schwartz Inequality (223), we find

$$\begin{aligned} |Z_h| &\le h(\vec{a}^T M^{1/2}\cdot \vec{a}^T M^{1/2})(M^{1/2}\vec{b}\cdot M^{1/2}\vec{b}) \\ &= h(\vec{a}^T M^{1/2}\cdot M^{1/2}\vec{a}^T)(\vec{b}^T M^{1/2}\cdot M^{1/2}\vec{b}) \\ &= h(\vec{a}^T M\vec{a})(\vec{b}^T M\vec{b}) = h\left(\sum_{i,j=1}^{N} m_{ij}a_ia_j\right)\left(\sum_{i,j=1}^{N} m_{ij}b_ib_j\right) \end{aligned} \tag{163}$$

On the right-hand side, we have

$$\begin{aligned} h\sum_{i,j=1}^{N} m_{ij}b_ib_j &= h\sum_{j=1}^{N}\left[\frac{2}{3}|u_j'|^2 + \frac{1}{6}(u_{j+1}'u_{j-1}')u_j'\right] = h\left[\left(\frac{2}{3}|u_1'|^2 + \frac{1}{6}(u_2' + u_0')u_1'\right)\right. \\ &+ \left(\frac{2}{3}|u_2'|^2 + \frac{1}{6}(u_3' + u_1')u_2'\right) + ... + \left(\frac{2}{3}|u_{N-1}'|^2 + \frac{1}{6}(u_N' + u_{N-2}')u_{N-1}'\right) \\ &+ \left.\left(\frac{2}{3}|u_N'|^2 + \frac{1}{6}(u_{N+1}' + u_{N-1}')u_N'\right)\right] = h\left[\frac{1}{3}|u_0'|^2 + \frac{1}{3}|u_1'|^2 + ... + \frac{1}{3}|u_N'|^2\right. \\ &\left(\frac{2}{6}u_0'u_1' + \frac{1}{6}|u_0'|^2 + \frac{1}{6}|u_1'|^2\right) + \left(\frac{2}{6}u_1'u_2' + \frac{1}{6}|u_1'|^2 + \frac{1}{6}|u_2'|^2\right) + ... \\ &+ \left.\left(\frac{2}{6}u_N'u_{N+1}' \; + \frac{1}{6}|u_N'|^2 + \frac{1}{6}|u_{N+1}'|^2\right)\right] = h\sum_{j=0}^{N}\left[\frac{1}{3}|u_j'|^2 + \frac{1}{6}|u_j' + u_{j+1}'|^2\right] \end{aligned} \tag{164}$$

By using (123) and the fact that $\lambda_1 < \lambda_2 < ... < \lambda_N$, we can find that

$$\frac{h}{\lambda_1}\sum_{j=0}^{N}\left|\frac{\psi_{j+1} - \psi_j}{h}\right|^2 \ge \sum_{j=1}^{N}\left[\frac{1}{3}|\psi_j|^2 + \frac{1}{6}|\psi_j + \psi_{j+1}|^2\right] \ge \sum_{j=1}^{N}\frac{h}{3}|u_j|^2. \tag{165}$$

Now, letting $\eta = \frac{\Lambda h^2}{24}$ and using (165) and the fact that $\eta \leq \frac{1}{2}$, we also find

$$
\begin{aligned}
h\sum_{i,j=1}^{N} m_{ij}a_i a_j &= h\sum_{j=1}^{N}(m_{j,j}a_j^2 + m_{j-1,j}a_{j-1}a_j + m_{j+1,j}a_{j+1}a_j) \\
&= h[(m_{1,1}a_1^2 + m_{0,1}a_0 a_1 + m_{2,1}a_2 a_1) + (m_{2,2}a_2^2 + m_{1,2}a_1 a_2 + m_{3,2}a_3 a_2)] \\
&+ ... + (m_{N-1,N-1}a_{N-1}^2 + m_{N-2,N-1}a_{N-2}a_{N-1} + m_{N,N-1}a_N aN - 1) \\
&+(m_{N,N}a_N^2 + m_{N-1,N}a_{N-1}a_N + m_{N+1,N}a_{N+1}a_N).
\end{aligned}
$$

We can use the inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$ and the values of the elements of M to show

$$
\begin{aligned}
&h\sum_{i,j=1}^{N} m_{ij}a_i a_j \leq h\left[m_{11}a_1^2 + ... + m_{NN}a_N^2 + \frac{m_{21}}{2}(a_2^2 + a_1^2) + \frac{m_{12}}{2}(a_1^2 + a_2^2) + ...\right. \\
&\qquad \left. + \frac{m_{(N+1)N}}{2}(a_{N+1}^2 + a_N^2)\right] \\
&= h[(m11 + \frac{m_{12}}{2} + \frac{m_{21}}{2})a_1^2 + ... + (m_{NN} + \frac{m_{N(N+1)}}{2} + \frac{m_{(N+1)N}}{2})a_N^2] \\
&= h[(2/3 + 1/6 + 1/6)a_1^2 + ... + (2/3 + 1/6 + 1/6)a_N^2] = h[a_1^2 + ... + a_N^2] \\
&= h\sum_{j=1}^{N}|a_j|^2.
\end{aligned}
$$

We can then plug in (162) to obtain

$$
\begin{aligned}
&h\sum_{i,j=1}^{N} m_{ij}a_i a_j \leq \frac{h}{4}\sum_{j=1}^{N}\left[j(u_{j+1} - u_{j-1}) - \frac{\Lambda h^2}{12}u_j\right]^2 \\
&\leq \frac{h}{4}\sum_{j=1}^{N}\left[2j^2|u_{j+1} - u_j|^2 + 2j^2|u_j - u_{j-1}|^2 + \frac{\Lambda^2 h^4}{144}|u_j|^2\right. \\
&\qquad \left. - \frac{\Lambda h^2}{6}ju_j(u_{j+1} - u_{j-1})\right],
\end{aligned}
$$

then use $j \leq \frac{L}{h}$ to find

$$\begin{aligned}
h\sum_{i,j=1}^{N} m_{ij}a_ia_j &\leq L^2 h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2 + \frac{\Lambda^2 h^5}{576}\sum_{j=1}^{N}|u_j|^2 + \frac{\Lambda h^3}{24}\sum_{j=1}^{N}u_ju_{j+1} \\
&\leq L^2 h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2 + \frac{\nu h^2}{2}h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2 + (\nu^2+\nu)h\sum_{j=1}^{N}|u_j|^2 \\
&\leq (L^2 - \frac{\nu h^2}{2} - \frac{3(\nu^2+\nu)}{\lambda_1})h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2 \\
&\leq (L^2 - \frac{\nu h^2}{2} - \frac{9\nu}{2\lambda_1})h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2 \qquad (166)
\end{aligned}$$

Combining (163), (164), and (166), we find

$$\begin{aligned}
|Z_h| &\leq \left(h\sum_{i,j=1}^{N} m_{ij}a_ia_j\right)^{1/2}\left(h\sum_{i,j=1}^{N} m_{ij}b_ib_j\right)^{1/2} \\
&\leq \left[h\sum_{j=1}^{N}\left(\frac{1}{3}|u_j'|^2 + \frac{1}{6}|u_j'+u_{j+1}'|^2\right)\right]^{1/2}\left[\left(L^2 - \frac{\eta h^2}{2} + \frac{9\eta}{2\lambda_1}\right)h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2\right]^{1/2} \\
&\leq [E_h(0)]^{1/2}\left[\left(L^2 - \frac{\eta h^2}{2} + \frac{9\eta}{2\lambda_1}\right)E_h(0)\right]^{1/2} = \sqrt{L^2 - \frac{\eta h^2}{2} + \frac{9\eta}{2\lambda_1}}E_h(0). \qquad (167)
\end{aligned}$$

Theorem 7 *For any $0 < \gamma < 12$, there exists $T(\gamma) > 0$ such that, for any $T > T(\gamma)$, there exists a positive constant $C(T,\gamma)$ such that*

$$E_h(0) \leq C(T,\gamma)\int_0^T\left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right]dt \qquad (168)$$

for any solution u of (109) in the class $\mathcal{C}_\langle(\gamma)$. Moreover,

(a) $T(\gamma) \nearrow \infty$ as $\gamma \nearrow 12$ and $T(\gamma) \searrow 2L$ as $\gamma \searrow 0$,

(b) $C(T,\gamma) \to \frac{L}{2(T-2L)}$ as $\gamma \searrow 0$.

Proof: Let Λ be the largest eigenvalue entering the Fourier development of the solution u of (109). That is,

$$u = \sum_{|\mu_k|\leq\sqrt{\Lambda}} a_k e^{i\mu_k t}\psi^k, \qquad (169)$$

with $\mu_k = \sqrt{\lambda_k}$ for $k > 0$ and $\mu_k = -\mu_{-k}$ when $k < 0$. Therefore,

$$u' = i \sum_{|\mu_k| \le \sqrt{\Lambda}} a_k \mu_k e^{i\mu_k t} \psi^k. \tag{170}$$

So we can use Lemmas 1 and 8 to find an estimate for the term $\sum_{j=0}^{N} \int_0^T |u'_j - u'_{j+1}|^2 dt$ in (144).

So similarly to (84), we have

$$\begin{aligned}
&\sum_{j=0}^{N} |u'_j - u'_{j+1}|^2 = \sum_{j=0}^{N} \left| \sum_{|\mu_k| \le \sqrt{\Lambda}} a_k \mu_k e^{i\mu_k t} (\psi_{k,j} - \psi_{k,j+1}) \right|^2 \\
&= \sum_{j=0}^{N} \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \mu_k^2 |\psi_{k,j} - \psi_{k,j+1}|^2 \\
&\quad + \sum_{j=0}^{N} \sum_{\substack{|\mu_k| \le \sqrt{\Lambda} \\ |\mu_l| \le \sqrt{\Lambda} \\ \mu_k \ne \mu_l}} \mu_k \mu_l a_k \bar{a}_l e^{i(\mu_k - \mu_l)t} (\psi_{k,j} - \psi_{k,j+1})(\psi_{l,j} - \psi_{l,j+1}) \\
&= \sum_{j=0}^{N} \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \lambda_k |\psi_{k,j} - \psi_{k,j+1}|^2 \\
&= h^2 \sum_{j=1}^{N} \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \lambda_k^2 \left(\frac{1}{3} |\psi_{k,j}|^2 + \frac{1}{6} |\psi_{k,j} + \psi_{k,j+1}|^2 \right) \\
&= \mu_k^2 \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \lambda_k h^2 \sum_{j=1}^{N} \left(\frac{1}{3} |\psi_{k,j}|^2 + \frac{1}{6} |\psi_{k,j} + \psi_{k,j+1}|^2 \right) \\
&\le \Lambda \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \lambda_k h^2 \sum_{j=1}^{N} \left(\frac{1}{3} |\psi_{k,j}|^2 + \frac{1}{6} |\psi_{k,j} + \psi_{k,j+1}|^2 \right) \\
&= \Lambda h^2 \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \mu_k^2 h^2 \sum_{j=1}^{N} \left(\frac{1}{3} |\psi_{k,j}|^2 + \frac{1}{6} |\psi_{k,j} + \psi_{k,j+1}|^2 \right) \\
&= \Lambda h^2 \sum_{j=1}^{N} \left[\frac{1}{3} \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \mu_k^2 |\psi_{k,j}|^2 + \frac{1}{6} \sum_{|\mu_k| \le \sqrt{\Lambda}} |a_k|^2 \mu_k^2 |\psi_{k,j} + \psi_{k,j+1}|^2 \right] \\
&= \Lambda h^2 \sum_{j=1}^{N} \left[\frac{1}{3} \left| \sum_{|\mu_k| \le \sqrt{\Lambda}} i a_k \mu_k e^{i\mu_k t} \psi_{k,j} \right|^2 + \frac{1}{6} \left| \sum_{|\mu_k| \le \sqrt{\Lambda}} i a_k \mu_k e^{i\mu_k t} (\psi_{k,j} + \psi_{k,j+1}) \right|^2 \right] \\
&= \Lambda h^2 \sum_{j=0}^{N} \left[\frac{1}{3} |u'_j|^2 + \frac{1}{6} |u'_j + u'_{j+1}|^2 \right].
\end{aligned} \tag{171}$$

Now, using Lemmas 9 and 12, the conservation of energy identity and equipartition of energy, respectively, we find

$$
\begin{aligned}
&h\sum_{j=0}^{N}\int_0^T\left(\frac{1}{3}|u_j'|^2+\frac{1}{6}|u_j'+u_{j+1}|^2\right)dt = h\sum_{j=0}^{N}\int_0^T\left|\frac{u_{j+1}-u_j}{h}\right|^2 dt + Y_h(t)|_0^T\\
&= 2\left[\int_0^T E_h(0)-\frac{h}{6}\sum_{j=1}^{N}\int_0^T|u_j'|^2dt+\frac{h}{12}\sum_{j=0}^{N}\int_0^T|u_j'+u_{j+1}'|^2dt\right]+Y_h(t)|_0^T\\
&= TE_h(0)+\frac{1}{2}Y_h(t)|_0^T. \qquad (172)
\end{aligned}
$$

Combining (151), (152), (171), and (172), we find

$$
\begin{aligned}
&\frac{L}{2}\int_0^T\left[\left|\frac{u_N(t)}{h}\right|^2+\frac{|u_N'(t)|^2}{6}\right]dt\\
&= TE_h(0)+\frac{h}{3}\sum_{j=0}^{N}\int_0^T\left[u_j'u_{j+1}' - \frac{|u_j'+u_{j+1}'|^2}{4}\right]dt + X_h(t)|_0^T\\
&= TE_h(0)+\frac{h}{3}\left[-\sum_{j=0}^{N}\int_0^T\frac{|u_j'-u_{j+1}'|^2}{4}dt\right]+X_h(t)|_0^T\\
&\geq TE_h(0)-\frac{h}{12}\int_0^T\left[h^2\Lambda\sum_{j=0}^{N}\left[\frac{1}{3}|u_j'|^2 \;+\frac{1}{6}|u_j'+u_{j+1}'|^2\right]\right]dt+X_h(t)|_0^T\\
&= TE_h(0)-\frac{h^2\Lambda}{12}\left(TE_h(0)+\left.\frac{Y_h(t)}{2}\right|_0^T\right)+X_h(t)|_0^T\\
&= \left(1-\frac{h^2\Lambda}{12}\right)TE_h(0)-\frac{h^2\Lambda}{24}Y_h(t)|_0^T+X_h(t)|_0^T \qquad (173)\\
&= \left(1-\frac{h^2\Lambda}{12}\right)TE_h(0)+Z_h(t)|_0^T,
\end{aligned}
$$

where

$$
Z_h(t) = X_h(t)-\frac{h^2\Lambda}{24}Y_h(t). \qquad (174)
$$

In Lemma 13, we estimated $|Z_h(t)|$ as

$$
|Z_h| \leq \sqrt{L^2-\frac{h^4\Lambda}{48}+\frac{3h^2\Lambda}{16\lambda_1}E_h(0)}. \qquad (175)
$$

So,

$$
\begin{aligned}
&Z_h(t)|_0^T = |Z_h(T) - Z_h(0)| \leq |Z_h(T)| + |Z_h(0)| \\
&\leq 2\sqrt{L^2 - \frac{h^4\Lambda}{48} + \frac{3h^2\Lambda}{16\lambda_1}} E_h(0).
\end{aligned} \tag{176}
$$

Therefore,

$$
Z_h(T) - Z_h(0) \geq -2\sqrt{L^2 - \frac{h^4\Lambda}{48} + \frac{3h^2\Lambda}{16\lambda_1}} E_h(0). \tag{177}
$$

Now, let $\Lambda = \frac{\gamma}{h^2}$. Combining (173) and (177), we have

$$
\begin{aligned}
&\frac{L}{2}\int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right] dt \\
&\geq \left[\left(1 - \frac{\gamma}{12}\right) T - 2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}\right] E_h(0).
\end{aligned} \tag{178}
$$

So from (178), we know

$$
E_h(0) \leq \frac{L}{2\left[\left(1 - \frac{\gamma}{12}\right) T - 2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}\right]} \int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right] dt
$$

provided that

$$
T > \frac{2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}}{\left(1 - \frac{\gamma}{12}\right)}.
$$

So we have

$$
T(\gamma) = \frac{2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}}{\left(1 - \frac{\gamma}{12}\right)} \tag{179}
$$

and

$$
C(T, \gamma) = \frac{L}{2\left[\left(1 - \frac{\gamma}{12}\right) T - 2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}\right]}, \tag{180}
$$

where

$$\lambda_1 = \frac{6}{h^2}\left(\frac{1-\cos\frac{\pi h}{L}}{2+\cos\frac{\pi h}{L}}\right). \tag{181}$$

So as $\gamma \searrow 0$,

$$\lim_{\gamma\to 0} T(\gamma) = \lim_{\gamma\to 0} \frac{2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}}{\left(1-\frac{\gamma}{12}\right)} = \lim_{\gamma\to 0} 2\sqrt{L^2} = 2L \tag{182}$$

and

$$\lim_{\gamma\to 0} C(T,\gamma) = \lim_{\gamma\to 0} \frac{L}{2\left[\left(1-\frac{\gamma}{12}\right)T - 2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}}\right]} = \lim_{\gamma\to 0} \frac{L}{2(T-2L)}. \tag{183}$$

As $\gamma \searrow 0$, $T(\gamma) \searrow 2L$ and $C(T,\gamma) \searrow \frac{L}{2(T-2L)}$. And, as $\gamma \nearrow 12$,

$$\lim_{\gamma\to 12} T(\gamma) = \lim_{\gamma\to 12} \frac{2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\left(\frac{6}{h^2}\left(\frac{1-\cos\frac{\pi h}{L}}{2+\cos\frac{\pi h}{L}}\right)\right)}}}{\left(1-\frac{\gamma}{12}\right)} \tag{184}$$

and as $\gamma \to 12$, $\left(1-\frac{\gamma}{12}\right) \to 0$, so $T(\gamma) \nearrow \infty$. Finally, as $\gamma \nearrow 12$,

$$\begin{aligned}\lim_{\gamma\to 12} C(T,\gamma) &= \lim_{\gamma\to 12} \frac{L}{2\left[\left(1-\frac{\gamma}{12}\right)T - 2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\left(\frac{6}{h^2}\left(\frac{1-\cos\frac{\pi h}{L}}{2+\cos\frac{\pi h}{L}}\right)\right)}}\right]} \\ &= \lim_{\gamma\to 12} -\frac{L}{4\sqrt{L^2 - \frac{h^2}{12} + \frac{3h^2\left(2+\cos\frac{\pi h}{L}\right)}{8\left(1-\cos\frac{\pi h}{L}\right)}}}.\end{aligned} \tag{185}$$

$C(T,\gamma)$ must be positive, so as $\gamma \nearrow 12$, $C(T,\gamma)$ does not exist. So the statements of Theorem 7 hold. □

5 Stabilization result for the Finite Difference semi-discretized wave equation

The model for the 1-dimensional damped wave equation with clamped-free boundary conditions is

$$\begin{cases} y_{tt} - y_{xx} = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+ \\ y(0,t) = 0, \quad y_x(L,t) + \alpha y_t(L,t) = 0, \quad t \in \mathbb{R}^+ \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), x \in (0,L) \end{cases} \tag{186}$$

such that $(y_0, y_1) \in H_0^1(0,L) \times L^2(0,L)$, $L = 1$, and α is positive.

The energy for the system is

$$E(t) = \frac{1}{2} \int_0^1 (|y_t(x,t)|^2 + |y_x(x,t)|^2) dx, \quad \forall t \geq 0. \tag{187}$$

By the dissipation law, $\frac{dE(t)}{dt} = -\alpha |y_t(1,t)|^2$. So, for all $M > 0, \omega > 0$ independent of solution, we have

$$E(t) \leq M e^{-\omega t} E(0), \quad \forall t \geq 0.$$

Set $y = u + z$ with $u_j^0 = y_j^0$, $u_j^1 = y_j^1$, where $y_0 \in \mathscr{C}_h(\gamma)$ and $y_1 \in \mathscr{C}_h(\gamma)$. Then, discretizing using the Finite Difference Method, our model becomes

$$\begin{cases} y_j''(t) - \frac{y_{j+1}(t) + y_{j-1}(t) - 2y_j(t)}{h^2} = 0, \quad 0 < t < T, j = 1, ..., N \\ y_0(t) = 0, \frac{y_{N+1}(t) - y_N(t)}{h} + \alpha y_{n+1}' = 0, \quad 0 < t < T \\ y_j(0) = y_j^0, y_j'(0) = y_j^1, \quad j = 0, ..., N+1 \end{cases} \tag{188}$$

The energy of the solutions of the system is given by

$$E_h(y,t) = \frac{h}{2} \sum_{j=0}^{N} \left[|y_j'(t)|^2 + \left| \frac{y_{j+1}(t) - y_j(t)}{h} \right|^2 \right]. \tag{189}$$

Therefore,

$$E_h'(y,t) = -\alpha |y_{n+1}|.$$

The unfiltered system is given by

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0, \quad 0 < t < T, j = 1, ..., N \\ u_0(t) = 0, u_{N+1}(t) = u_N(t), \quad 0 < t < T \\ u_j(0) = u_j^0, u_j'(0) = u_j^1, \quad j = 0, ..., N+1 \end{cases} \tag{190}$$

Let z solve

$$\begin{cases} z_j''(t) - \frac{z_{j+1}(t)+z_{j-1}(t)-2z_j(t)}{h^2} = 0, \quad 0 < t < T, j = 1, ..., N \\ z_0(t) = 0, \frac{z_{N+1}(t)-z_N(t)}{h} + \alpha y_{n+1}' = 0, \quad 0 < t < T \\ z_j(0) = 0, z_j'(0) = 0, \quad j = 0, ..., N+1 \end{cases} \tag{191}$$

The observability result for the clamped-free case can be found in [13], and the proof mimics the proof of Theorem 4.

Theorem 8 *Assume $\gamma < 4$. Then there exists $T_1(\gamma) > 2$ such that for all $T > T_1(\gamma)$, the following inequality holds:*

$$\left[T(1 - \frac{\gamma}{4}) - 2\sqrt{1 + \frac{3\gamma}{16\lambda_0}}\right] E_h(u, 0) \leq \frac{1}{2} \int_0^T |u_{N+1}'|^2 dt. \tag{192}$$

Lemma 14 *Let $T > 0$. There exists $C > 0$ and $K > 0$ such that, for all $0 < h < 1$, we have*

$$\int_0^T |z_{N+1}|^2 dt \leq C \int_0^T |y_{N+1}'|^2 dt + K E_h(y, 0). \tag{193}$$

Proof: The energy of the solutions of system (191) is given by

$$E_h(z, t) = \frac{h}{2} \sum_{j=0}^{N} \left[|z_j'(t)|^2 + \left| \frac{z_{j+1}(t) - z_j(t)}{h} \right|^2 \right], \tag{194}$$

so its derivative is

$$E_h'(z, t) = -\alpha z_{N+1}' y_{N+1}. \tag{195}$$

So

$$|E_h(z,t)| = \left|\int_0^t -\alpha z'_{N+1} y'_{N+1} dt\right| = \left|\int_0^t \alpha y'_{N+1} z'_{N+1} dt\right|.$$

Applying Young's inequality, i.e. (226) for $\epsilon > 0$, we find

$$E_h(z,t) \le \int_0^t \left(\frac{|\alpha y'_{N+1}|^2}{2\epsilon} + \frac{\epsilon |z'_{N+1}|^2}{2}\right) dt. \tag{196}$$

Let $\epsilon \to 2\epsilon$. Then (196) becomes

$$\begin{aligned} E_h(z,t) &\le \int_0^T \left(\frac{|\alpha y'_{N+1}|^2}{4\epsilon} + \epsilon |z'_{N+1}|^2\right) dt \\ &\le \frac{\alpha^2}{4\epsilon} \int_0^t |y'_{N+1}|^2 dt + \epsilon \int_0^T |z'_{N+1}|^2 dt. \end{aligned} \tag{197}$$

We use multiplier $j\frac{z_{j+1}-z_{j-1}}{2}$ and integrate by parts to find

$$\begin{aligned} &h\sum_{j=1}^{N}\int_0^T \left(z''_j - \frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right) j\frac{z_{j+1}-z_{j-1}}{2} dt = h\sum_{j=1}^{N}\int_0^T jz''_j \frac{z_{j+1}-z_{j-1}}{2} dt \\ &-h\sum_{j=1}^{N}\int_0^T j\left(\frac{z_{j+1}-zj-1}{2}\right)\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right) dt \\ &= h\sum_{j=1}^{N} jz'_j \left(\frac{z_{j+1}-z_{j-1}}{2}\right)\Bigg|_0^T - h\sum_{j=1}^{N}\int_0^T jz'_j \left(\frac{z'_{j+1}-z'_{j-1}}{2}\right) dt \\ &-h\sum_{j=1}^{N}\int_0^T j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right) dt = 0. \end{aligned} \tag{198}$$

By rewriting the following summation, we find

$$\begin{aligned} &h\sum_{j=1}^{N} jz'_j \frac{z_{j+1}-z_{j-1}}{2}\Bigg|_0^T = h\sum_{j=1}^{N} jz'_j \left(\frac{z_{j+1}-z_j}{2} + \frac{z_j - z_{j-1}}{2}\right)\Bigg|_0^T \\ &= h\sum_{j=1}^{N} jz'_j \frac{z_{j+1}-z_j}{2}\Bigg|_0^T + h\sum_{j=1}^{N} jz'_j \frac{z_j-z_{j-1}}{2}\Bigg|_0^T. \end{aligned} \tag{199}$$

Then, since

$$
\begin{aligned}
h\sum_{j=1}^{N} jz'_j \frac{z_j - z_{j-1}}{2} &= h\left[z'_1\frac{z_1 - z_0}{2} + 2z'_2\frac{z_2 - z_1}{2} + \dots\right. \\
&\quad \left. +(N-1)z'_{N-1}\frac{z_{N-1} - z_{N-2}}{2} + Nz'_N\frac{z_N - z_{N-1}}{2}\right] \\
&= h\sum_{j=0}^{N-1}(j+1)z'_{j+1}\left(\frac{z_{j+1} - z_j}{2}\right),
\end{aligned}
$$

(199) becomes

$$
h\sum_{j=1}^{N} jz'_j \frac{z_{j+1} - z_{j-1}}{2}\Bigg|_0^T = h^2\sum_{j=0}^{N} jz'_j\frac{z_{j+1} - z_j}{2h}\Bigg|_0^T + h^2\sum_{j=0}^{N-1}(j+1)z'_{j+1}\frac{z_{j+1} - z_j}{2h}\Bigg|_0^T \tag{200}
$$

We can also rewrite the following summation in order to find

$$
\begin{aligned}
-h\sum_{j=1}^{N}\int_0^T jz'_j\frac{z'_{j+1} - z'_{j-1}}{2}dt &= -\frac{h}{2}\int_0^T (z'_1z'_2 - z'_1z'_0 + 2z'_2z'_3 - 2z'_2z'_1 \\
&\quad +3z'_3z'_4 - 3z'_3z'_2 + \dots + (N-1)z'_{N-1}z'_N - (N-1)z'_{N-1}z'_{N-2} + Nz'_Nz'_{N+1} \\
&\quad -Nz'_Nz'_{N-1})dt = \frac{h}{2}\int_0^T\sum_{j=0}^{N} z'_jz'_{j-1}dt - \frac{Nh}{2}\int_0^T z'_Nz'_{N+1}dt \\
&= \frac{h}{2}\int_0^T\sum_{j=0}^{N} z'_jz'_{j-1}dt + \frac{h}{2}\int_0^T z'_{N+1}z'_Ndt - \frac{h}{2}\int_0^T z'_{N+1}z'_Ndt - \frac{Nh}{2}\int_0^T z'_Nz'_{N+1}dt \\
&= \frac{h}{4}\sum_{j=0}^{N}\int_0^T |z'_j|^2dt + \frac{h}{4}\int_0^T |z'_{N+1}|^2dt - \frac{h}{4}\sum_{j=0}^{N}\int_0^T (|z'_{j+1}|^2 + |z'_j|^2 - 2|z'_{j+1}z'_j|)dt \\
&\quad -\frac{(N+1)h}{2}\int_0^T z'_Nz'_{N+1}dt = \frac{h}{2}\sum_{j=0}^{N}\int_0^T |z'_j|^2dt + \frac{h}{4}\int_0^T |z'_{N+1}|^2dt \\
&\quad -\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1} - z'_j}{h}\right|^2 dt - \frac{1}{2}\int_0^T |z'_{n+1} - z'_N|dt \\
&= \frac{h}{2}\sum_{j=0}^{N}\int_0^T |z'_j|^2dt + \frac{h}{4}\int_0^T |z'_{N+1}|^2dt - \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1} - z'_j}{h}\right|^2 dt \\
&\quad +\frac{1}{4}\int_0^T (|z'_{N+1}|^2 + |z'_N|^2 - 2|z'_{N+1}z'_N|)dt - \frac{1}{4}\int_0^T (|Z'_{N+1}|^2 + |z'_N|^2)dt \\
&= \frac{h}{2}\sum_{j=0}^{N}\int_0^T |z'_j|^2dt + \frac{h}{4}\int_0^T |z'_{N+1}|^2dt - \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1} - z'_j}{h}\right|^2 dt \\
&\quad +\frac{\alpha^2h^2}{4}\int_0^T |y''_{N+1}|^2dt - \frac{1}{4}\int_0^T (|Z'_{N+1}|^2 + |z'_N|^2)dt.
\end{aligned}
$$

Next, we rewrite the following summation and use the boundary conditions to find

$$-h\sum_{j=1}^{N}\int_0^T j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right)dt$$
$$=-h\sum_{j=1}^{N}\int_0^T \frac{j}{2h^2}\left(|z_{j+1}|^2-2z_jz_{j+1}\ +2z_jz_{j-1}-|z_{j-1}|^2\right)dt$$

$$\begin{aligned}
&=\frac{-h}{2h^2}\int_0^T[(|z_2|^2-2z_1z_2+2z_1z_0-|z_0|^2)+2(|z_3|^2-2z_2z_3+2z_2z_1-|z_1|^2)\\
&\quad+3(|z_4|^2-2z_3z_4+2z_3z_2-|z_2|^2)+...+(N-1)(|z_N|^2-2z_{N-1}z_N\\
&\quad+2z_{N-1}z_{N-2}-|z_{N-2}|^2)+N(|z_{N+1}|^2-2z_Nz_{N+1}+2z_Nz_{N-1}-|z_{N-1}|^2)]dt\\
&=-\frac{h}{2h^2}\int_0^T[-2|z_1|^2-2|z_2|^2-...-|z_{N-1}|^2+(N-1)|z_N|^2+N|z_{N+1}|^2\\
&\quad+2z_1z_0+2z_1z_2+2z_2z_3+...+2z_{N-1}z_{N-2}+2z_Nz_{N-1}-2Nz_Nz_{N+1}]dt\\
&=-\frac{h}{2h^2}\int_0^T[-2|z_1|^2-...-2|z_{N-1}|^2+(N-1)|z_N|^2+N|z_{N+1}|^2+(|z_0|^2+|z_1|^2\\
&\quad-|z_0-z_1|^2)+(|z_1|^2+|z_2|^2-|z_1-z_2|^2)+(|z_2|^2+|z_3|^2-|z_2-z_3|^2)+...\\
&\quad+(|z_N|^2+|z_{N-1}|^2-|z_N-z_{N-1}|^2)-N(|z_N|^2+|z_{N+1}|^2-|z_N-z_{N+1}|^2)]dt\\
&=-\frac{-h}{2h^2}\int_0^T[-|z_0-z_1|^2-|z_1-z_2|^2-|z_2-z_3|^2-...-|z_N-z_{N-1}|^2\\
&\quad+N|z_N-z_{N+1}|^2]dt=\frac{h}{2}\int_0^T\sum_{j=0}^{N}\left|\frac{z_j-z_{j+1}}{h}\right|^2dt-\frac{h(N-1)}{2h^2}\int_0^T|z_N-z_{N+1}|^2dt\\
&=\frac{h}{2}\int_0^T\sum_{j=0}^{N}\left|\frac{z_{j+1}-z_j}{h}\right|^2dt-\frac{(\frac{1}{h}+1-1)h}{2}\int_0^T\left|\frac{z_N-z_{N+1}}{h}\right|^2dt\\
&=\frac{h}{2}\int_0^T\sum_{j=0}^{N}\left|\frac{z_{j+1}-z_j}{h}\right|^2dt-\frac{\alpha^2}{2}\int_0^T\left|y'_{N+1}\right|^2dt \qquad (201)
\end{aligned}$$

Plugging (200-201) into (198), we find

$$
\begin{aligned}
& h\sum_{j=1}^{N} j z_j'\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\Bigg|_0^T - h\sum_{j=1}^{N}\int_0^T j z_j'\left(\frac{z_{j+1}'-z_{j-1}'}{2}\right)dt \\
& \quad -h\sum_{j=1}^{N}\int_0^T j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right)dt \\
& = h^2\sum_{j=0}^{N} j z_j'\left(\frac{z_{j+1}-z_j}{2h}\right)\Bigg|_0^T + h^2\sum_{j=0}^{N-1}(j+1)z_{j+1}'\left(\frac{z_{j+1}-z_j}{2h}\right)\Bigg|_0^T \\
& \quad +\frac{h}{2}\sum_{j=0}^{N}\int_0^T |z_j'|^2 dt + \frac{h}{4}\int_0^T |z_{N+1}'|^2 dt - \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z_{j+1}'-z_j'}{h}\right|^2 dt \\
& \quad +\frac{\alpha^2 h^2}{4}\int_0^T |y_{N+1}''|^2 dt - \frac{1}{4}\int_0^T (|z_{N+1}'|^2 + |z_N'|^2)dt + \frac{h}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{z_{j+1}-z_j}{h}\right|^2 dt \\
& \quad -\frac{\alpha^2}{2}\int_0^T |y_{N+1}'|^2 dt = 0
\end{aligned}
$$

We move terms over and combine them to find

$$
\begin{aligned}
& \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z_{j+1}'-z_j'}{h}\right|^2 dt + \frac{1-h}{4}\int_0^T |z_{N+1}'|^2 dt + \frac{1}{4}\int_0^T |z_N'|^2 dt \\
& = h^2\sum_{j=0}^{N} j z_j'\left(\frac{z_{j+1}-z_j}{2h}\right)\Bigg|_0^T + h^2\sum_{j=0}^{N-1}(j+1)z_{j+1}'\left(\frac{z_{j+1}-z_j}{2h}\right)\Bigg|_0^T + \int_0^T E_h(z,t)dt \\
& \quad -\frac{\alpha^2}{2}\int_0^T |y_{N+1}'|^2 dt + \frac{\alpha^2 h^2}{4}\int_0^T |y_{N+1}''|^2 dt
\end{aligned} \tag{202}
$$

From the left-hand side, we use the boundary conditions and the inequality $(a+b)^2 \geq \frac{a^2}{2} - b^2$ to find

$$
\begin{aligned}
& \frac{1-h}{4}\int_0^T |z_{N+1}'|^2 dt + \frac{1}{4}\int_0^T |z_N'|^2 dt = \frac{1-h}{4}\int_0^T |z_{N+1}'|^2 dt + \frac{1}{4}\int_0^T |\alpha h y_{N+1}'' + z_{N+1}'|^2 \\
& \geq \left(\frac{1}{4}-\frac{h}{4}\right)\int_0^T |z_{N+1}'|^2 dt + \frac{1}{8}\int_0^T |z_{N+1}'^2 dt - \frac{\alpha^2 h^2}{4}\int_0^T |y_{N+1}''|^2 dt \\
& = \frac{3-2h}{8}\int_0^T |z_{N+1}'|^2 dt - \frac{\alpha^2 h^2}{4}\int_0^T |y_{N+1}''|^2 dt.
\end{aligned} \tag{203}
$$

From the right-hand side, we can find

$$\int_0^T E_h(z,t)dt \leq \left|\int_0^T E_h(z,t)dt\right| \leq \int_0^T |E_h(z,t)|dt \leq TE_h(z,T)$$
$$\leq T\left(\frac{\alpha^2}{4\epsilon}\int_0^T |y'_{N+1}|^2 dt + \epsilon\int_0^T |z'_{N+1}|^2 dt\right) \tag{204}$$

and

$$h^2\sum_{j=0}^{N} jz'_j\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T + h^2\sum_{j=0}^{N-1}(j+1)z'_{j+1}\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T$$
$$\leq \left|h^2\sum_{j=0}^{N} j\frac{z'_j}{\sqrt{2}}\left(\frac{z_{j+1}-z_j}{\sqrt{2}h}\right)\right|_0^T + \left|h^2\sum_{j=0}^{N-1}(j+1)\frac{z'_{j+1}}{\sqrt{2}}\left(\frac{z_{j+1}-z_j}{\sqrt{2}h}\right)\right|_0^T \tag{205}$$

Using the fact that $j \leq N$ and Young's Inequality, (205) becomes

$$h^2\sum_{j=0}^{N} jz'_j\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T + h^2\sum_{j=0}^{N-1}(j+1)z'_{j+1}\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T$$
$$\leq h^2N\sum_{j=0}^{N}\left(\frac{|z'_j|^2}{2}+\frac{|z_{j+1}-z_j|^2}{2h^2}\right)\Bigg|_0^T + h^2N\sum_{j=0}^{N-1}\left(\frac{|z'_{j+1}|^2}{2}+\frac{|z_{j+1}-z_j|^2}{2h^2}\right)\Bigg|_0^T$$
$$= \frac{h^2N}{2}\sum_{j=0}^{N}\left(|z'_j|^2+\left|\frac{z_{j+1}-z_j}{h}\right|^2\right)\Bigg|_0^T + \frac{h^2N}{2}\sum_{j=0}^{N-1}\left(|z_{j+1}|^2+\left|\frac{z_{j+1}-z_j}{h}\right|^2\right)\Bigg|_0^T$$
$$= hNE_h(z,t)|_0^T + hN\left[E_h(z,t) - \frac{|z_{N+1}-z_N|^2}{h^2}\right]_0^T \leq 2h(\frac{1}{h}-1)E_h(z,t)|_0^T$$
$$= 2(1-h)E_h(z,t)|_0^T \leq E_h(z,T) \tag{206}$$

Substituting (203),(204), and (206) into (202), we find

$$
\begin{aligned}
&\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1}-z'_j}{h}\right|^2 dt + \frac{3-2h}{8}\int_0^T |z'_{N+1}|^2 dt \\
&\leq E_h(z,T) + T\left(\frac{\alpha^2}{4\epsilon}\int_0^T |y'_{N+1}|^2 dt + \epsilon\int_0^T |z'_{N+1}|^2 dt\right) \\
&\quad -\frac{\alpha^2}{2}\int_0^T |y'_{N+1}|^2 dt + \frac{\alpha^2 h^2}{2}\int_0^T |y''_{N+1}|^2 dt \\
&\leq \frac{\alpha^2}{4\epsilon}\int_0^T |y'_{N+1}|^2 dt + \epsilon\int_0^T |z'_{N+1}|^2 dt + T\left(\frac{\alpha^2}{4\epsilon}\int_0^T |y'_{N+1}|^2 dt + \epsilon\int_0^T |z'_{N+1}|^2 dt\right) \\
&\quad +\frac{\alpha^2 h^2}{2}\int_0^T |y''_{N+1}|^2 dt \\
&= \frac{\alpha^2}{4\epsilon}(1+T)\int_0^T |y'_{N+1}|^2 + \epsilon(1+T)\int_0^T |z'_{N+1}|^2 dt + \frac{\alpha^2 h^2}{2}\int_0^T |y''_{N+1}|^2 dt. \quad (207)
\end{aligned}
$$

We choose $\epsilon = \frac{3-2h}{16(1+T)}$ so that

$$
\begin{aligned}
&\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1}-z'_j}{h}\right|^2 dt + \frac{3-2h}{8}\int_0^T |z'_{N+1}|^2 dt \\
&\leq \frac{16\alpha^2(1+T)}{4(3-2h)}(1+T)\int_0^T |y'_{N+1}|^2 dt + \frac{(3-2h)}{16(1+T)}(1+T)\int_0^T |z'_{N+1}|^2 dt \\
&\quad +\frac{\alpha^2 h^2}{2}\int_0^T |y''_{N+1}|^2 dt
\end{aligned}
$$

becomes

$$
\begin{aligned}
4h^3\sum_{j=0}^{N}\int_0^T \left|\frac{z'_{j+1}-z'_j}{h}\right|^2 dt + (3-2h)\int_0^T |z'_{N+1}|^2 dt \quad (208)\\
\leq \frac{64\alpha^2(1+T)^2}{(3-2h)}\int_0^T |y'_{N+1}|^2 dt + 8\alpha^2 h^2\int_0^T |y''_{N+1}|^2 dt. \quad (209)
\end{aligned}
$$

We can also check that

$$
8\alpha^2 h^2\int_0^T |y''_{N+1}|^2 dt \leq K E_h(y,0). \quad (210)
$$

If $\vec{y}_h$ is a solution to (186), then so is $\vec{y}'_h$ since (186) is linear. By substituting $(\vec{y}_h)'$

for $\vec{y}_h$ in $E_h(y,t)$, we can obtain the energy F_h given by

$$F_h = \frac{h}{2}\sum_{j=0}^{N}\left[\left|y_j''(t)\right|^2 + \left|\frac{y_{j+1}'(t) - y_j'(t)}{h}\right|^2\right].$$

F_h is a non-increasing function of the time variable t with

$$F_h'(t) = -\alpha|y_{N+1}''|^2.$$

Consequently,

$$\alpha^2h^2\int_0^T |y_{N+1}''|^2dt \leq \alpha h^2[F_h(0) - F_h(T)] \leq \alpha h^2 F_h(0).$$

Now all that is left is to check that there exists $K > 0$ independent of h, the initial data, and T such that $h^2F_h(0) \leq KE_h(y,0)$. If we take $K = \lambda^2h^2$, we obtain $h^2F_h(0) \leq \lambda h^2E_h(y,0)$, which satisfies the inequality.

Plugging (210) in, we find

$$\int_0^T |z_{N+1}'|^2dt \leq 64\alpha^2(1+T)^2\int_0^T |y_{N+1}'|^2dt + KE_h(y,0), \tag{211}$$

which means we must have $C = 64\alpha(1+T)^2$. □

Theorem 9 *The exponential decay of E_h to zero is uniform with respect to the mesh size in the range $\mathscr{C}_h(\gamma)$, i.e., there exists positive constants M_1 and ω_1 independent of h such that, for every y^0 and y^1 in the class $\mathscr{C}_h(\gamma)$,*

$$e_h(y,t) \leq M_1e^{-\omega t}E_h(y,0), \quad t \geq 0, \quad 0 < h < 1. \tag{212}$$

Proof: From the observability result Theorem 8, and the fact that $y = u + z$,

$$\begin{aligned}&\left[T(1-\frac{\gamma}{4}) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}}\right] E_h(u,0) \le \frac{1}{2}\int_0^T |u'_{N+1}|^2 dt \\ &\le \int_0^T |z'_{N+1}|^2 dt + \int_0^T |y'_{N+1}|^2 dt. \qquad (213)\end{aligned}$$

Because we have $E_h(y,0) = E_h(u,0)$, which implies that $E_h'(y,t) < 0$, and therefore E_h is strictly decreasing, we can use (193) from Lemma 14 to find

$$\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}} - K\right] E_h(u,0) \le (C+1)\int_0^T |y'_{N+1}|^2 dt. \qquad (214)$$

So, for $T > \frac{2\sqrt{1+\frac{3\gamma}{16\lambda_0}}+K}{1-\frac{\gamma}{4}}$, we have

$$E_h(y,0) \le (C+1)\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}} - K\right]^{-1} \int_0^T |y'_{N+1}|^2 dt. \qquad (215)$$

Since $E_h'(y,t) = -\alpha |y'_{N+1}|^2$ and $T > 0$, we have

$$\begin{aligned}&E_h(y,T) \le E_h(y,0) \\ &\le \frac{C+1}{\alpha}\left[T\left(1-\frac{\gamma}{4}\right) -2\sqrt{1+\frac{3\gamma}{16\lambda_0}} - K\right]^{-1} \int_0^T -E_h'(y,t) dt. \qquad (216)\end{aligned}$$

Letting $C' = \frac{C+1}{\alpha}\left[T\left(1-\frac{\gamma}{4}\right) - 2\sqrt{1+\frac{3\gamma}{16\lambda_0}} - K\right]^{-1}$, we have

$$E_h(y,T) \le C' \int_0^T -E_h'(y,t) = C'(E_h(y,0) - E_h(y,T)),$$

which becomes

$$(1+C')E_h(y,T) \le C' E_h(y,0).$$

So we find

$$E_h(y,T) \le \frac{C'}{1+C'} E_h(y,0).$$

As the system is invariant by translation, we can deduce that for all $n \in \mathbb{N}$,

$$E_h(y, (n+1)T \leq \frac{C'}{C'+1} E_h(y, nT).$$

By iteration, we find

$$E_h(y, (n+1)T) \leq \left(\frac{C'}{C'+1} \right)^{n+1} E_h(y, 0)$$

Therefore, with $\omega_1 = \frac{1}{T} \ln \frac{C'+1}{C'}$, we have

$$E_h(y, (n+1)T) \leq e^{-\omega_1 (n+1)T} E_h(y, 0). \tag{217}$$

For $t > 0$, there exists $n \in \mathbb{N}$ such that $nT \leq t \leq (n+1)T$. Using the equation $E'_h(y, t) = -\alpha |y'_{N+1}|^2$, which implies that E_h is strictly decreasing, we find

$$E_h(y, t) \leq E_h(y, nT), \tag{218}$$

which implies that

$$E_h(y, t) \leq e^{-\omega_1 nT} E_h(y, 0). \tag{219}$$

Hence,

$$E_h(y, t) \leq \frac{C'+1}{C'} e^{-\omega_1 (n+1)T} E_h(y, 0).$$

Using the inequality $t < (n+1)T$, we find

$$E_h(y, t) \leq \frac{C'+1}{C'} e^{-\omega_1 t} E_h(y, 0). \tag{220}$$

Thus, we have $M_1 = \frac{C'+1}{C'}$. $\square$

6 Conclusion

This paper first proves the exact observability of the continuous model of the 1-D wave equation, then the lack of exact observability for the FDM (Finite Differences) and FEM (Finite Elements) semi-discretized models. The latter is due to the disappearance of the uniform gap condition for the eigenvalues, which an observer cannot differentiate between different high frequency eigenvalues. As a remedy, a direct filtering technique is applied the FDM and FEM semi-discretized models to successfully prove the exact observability of the discrete models, mimicking the continuous system. Finally the exponential stabilization of the FDM semi-discretized model is proved for the clamped-free wave equation.

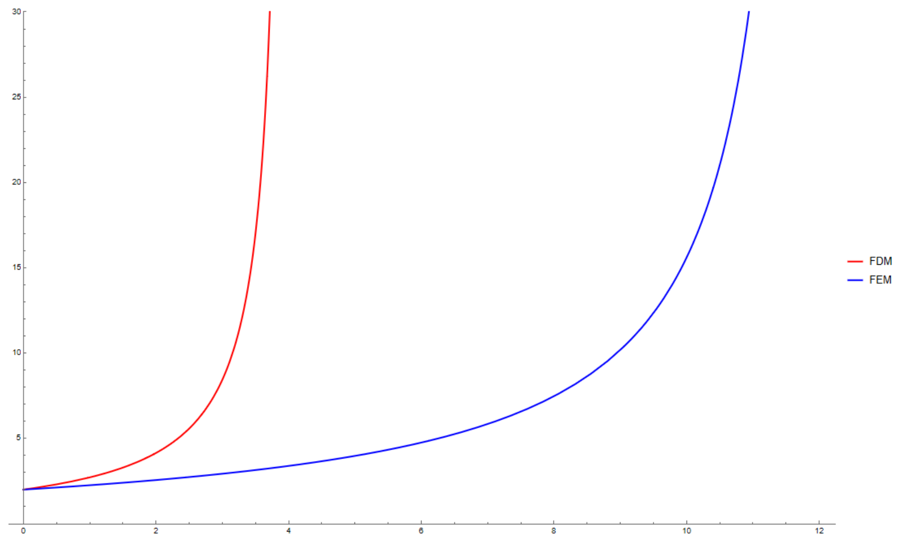

Figure 7: For $N = 20$, the observation time $T(\gamma)$ as γ approaches its maximum value blows up faster for FDM than for FEM, so FEM produces better observation times for a given γ. (Note that for FDM $0 \leq \gamma < 4$ and for FEM $0 \leq \gamma < 12$).

The results shown in this paper allow a comparison between these two discretization techniques. For FDM and FEM the filtering parameter γ comes with different ranges , $0 \leq \gamma < 4$ and $0 \leq \gamma < 12$, respectively, due to the different spectrums (eigenvalues) for each discretization, see also (5). The optimal γ to retain $m < N$ eigen-solutions after filtering can be calculated by taking the maximum of $h\sqrt{\lambda_k}$ for set L and N and taking $k = 1, ..., m$. As an example, let $m = 10$, $N = 20$, and $L = 1$. Then, we can find filtering parameters ro retain 10 eigen-solutions as $\gamma_{FDM} = 1.85054$ and $\gamma_{FEM} = 2.66026$. Plugging that into $T(\gamma)$ for each discretization, we have the minimal observation times $T_{FMD} = 6.26476$ and $T_{FEM} = 2.51077$. Notice that the minimal observation time found in Theorem 1 is $2L = 2$ seconds. Comparing the minimum observation times found in (94) and (179) and plotted in Figure 7, it is evident that FEM results in faster observation times for a given γ. Therefore, the FEM results are more reliable and more realistic in comparison to the FDM results.

The methodology and techniques in the proofs in this projects are needed for future applications, i.e., the following 1-D, strongly coupled PDE model used by a former graduate student of Dr. Özer describing the longitudinal vibrations of a piezoelectric beam of length L with the addition of magnetic effects [14]:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} - \gamma\beta p_{xx} = 0, & \\ \mu p_{tt} - \beta p_{xx} - \gamma\beta v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ v(0,t) = p(0,t) = 0, & \\ \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \\ p_x(L,t) - \gamma v_x(L,t) = 0, & t \in \mathbb{R}^+ \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), & \\ p(x,0) = p_0(x), p_t(x,0) = p_1(x), & x \in (0,L), \end{cases} \tag{221}$$

where $\rho, \alpha, \gamma, \mu, \beta$ denote material constants. The coupling of the equations makes the discrete energy estimations much more complicated, but the same techniques are still used to prove uniform exact observability of the PDE and its Finite-Difference approximations [15] as the discretization parameter tends to zero. The uniform observability of the Finite Element-based filtered approximated model, or the uniform exact stabilizability of the these models are left as open problems in [15].

Future work will include a master's thesis proving the stabilization of the FEM model from this paper, the stabilization of the FDM semi-discretization of the system (221), and the observability and stabilization of the FEM discretization with linear splines of the system (221), as the exact observability is already proved [15] for the FDM semi-discretization of (221).

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Appendix A: Known Inequalities

Theorem 10 (Triangle Inequality) *Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then*

$$|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|. \tag{222}$$

Theorem 11 (Cauchy-Schwartz Inequality) *Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then,*

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right). \tag{223}$$

Theorem 12 (Discrete Poincar$\acute{e}'$s Inequality) *Let $\vec{u} \in \mathbb{R}^n$, and $h > 0$. Then,*

$$h\sum_{j=1}^{n} u_j^2 \leq \frac{1}{\lambda_1} h \sum_{j=0}^{n} \left|\frac{u_j - u_{j-1}}{h}\right|^2, \tag{224}$$

where λ_1 is the first eigenvalue of the following eigenvalue problem

$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = \lambda u_j, \qquad j = 1, 2, \ldots n. \tag{225}$$

Theorem 13 (Generalized Young's Inequality) *Let f and g be in $L^2(0, L)$. For any $\epsilon > 0$,*

$$\int_0^L f(x)g(x)dx \leq \frac{1}{2\epsilon}\int_0^L |f(x)|^2 dx + \frac{\epsilon}{2}\int_0^L |g(x)|^2 dx. \tag{226}$$