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Deterministic and Stochastic Bellman's Optimality Principles on Isolated Time Domains and Their Applications in Finance

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DETERMINISTIC AND STOCHASTIC BELLMAN'S OPTIMALITY PRINCIPLES ON ISOLATED TIME DOMAINS AND THEIR APPLICATIONS IN FINANCE

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TABLE OF CONTENTS

DETERMINISTIC AND STOCHASTIC BELLMAN'S OPTIMALITY PRINCIPLES ON ISOLATED TIME DOMAINS AND THEIR APPLICATIONS IN FINANCE

Nezihe Turhan May 2011 72 Pages

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The concept of dynamic programming was originally used in late 1949, mostly during the 1950s, by Richard Bellman to describe decision making problems. By 1952, he refined this to the modern meaning, referring specifically to nesting smaller decision problems inside larger decisions. Also, the Bellman equation, one of the basic concepts in dynamic programming, is named after him. Dynamic programming has become an important argument which was used in various fields; such as, economics, finance, bioinformatics, aerospace, information theory, etc. Since Richard Bellman's invention of dynamic programming, economists and mathematicians have formulated and solved a huge variety of sequential decision making problems both in deterministic and stochastic cases; either finite or infinite time horizon. This thesis is comprised of five chapters where the major objective is to study both deterministic and stochastic dynamic programming models in finance.

In the first chapter, we give a brief history of dynamic programming and we introduce the essentials of theory. Unlike economists, who have analyzed the dynamic programming on discrete, that is, periodic and continuous time domains, we claim that trading is not a reasonably periodic or continuous act. Therefore, it is more accurate to demonstrate the dynamic programming on non-periodic time domains. In the second chapter we introduce time scales calculus. Moreover, since it is more realistic to analyze

a decision maker's behavior without risk aversion, we give basics of Stochastic Calculus in this chapter. After we introduce the necessary background, in the third chapter we construct the deterministic dynamic sequence problem on isolated time scales. Then we derive the corresponding Bellman equation for the sequence problem. We analyze the relation between solutions of the sequence problem and the Bellman equation through the principle of optimality. We give an example of the deterministic model in finance with all details of calculations by using guessing method, and we prove uniqueness and existence of the solution by using the Contraction Mapping Theorem. In the fourth chapter, we define the stochastic dynamic sequence problem on isolated time scales. Then we derive the corresponding stochastic Bellman equation. As in the deterministic case, we give an example in finance with the distributions of solutions.

CHAPTER 1

INTRODUCTION

Dynamic optimization is a mathematical theory used to solve optimization problems in a variety of fields; such as, economics, finance, industry, information theory, bioinformatics, aerospace, etc. The theory consists of many theories; for instance, theory of calculus of variations, optimal control theory, which is an extension of calculus of variations, and lastly dynamic programming. Dynamic programming (DP) is a recursive method for solving *sequential (or multistage) decision problems*. Theory has enabled economists and mathematicians to construct and solve a huge variety of sequential decision making problems both in deterministic and stochastic cases; the former is decision making under certainty, and the latter is under uncertainty.

The concept of dynamic programming has become an important principle in decision making processes with studies and contributions of a mathematician named Richard Bellman. Bellman started working for RAND Corporation in the summer of 1949, and was encouraged to work on multistage decision processes by Ed Paxson, who was an employee of the corporation and friend of Richard Bellman. The story that lies behind the choice of the name "dynamic programming" was told by Richard Bellman in his autobiography and retold by Dreyfus in [**8**] . The RAND Corporation was employed by the Air Force, and the person in charge was a gentleman named Wilson. Wilson had a fear for the word *research* in the literal meaning. Therefore, not to offend this gentleman during his visit to the corporation, Bellman used the word *programming* instead of *mathematical.* Since the process is multistage, i.e., time-varying, he decided on the word *dynamic*. Since then, the theory has been called dynamic programming. Bellman published his first paper [**4**] on the subject in 1952, and his first book [**5**] in 1957. He published many papers and books on the theory and its applications in both deterministic and stochastic cases. Also, *the Bellman equation,* which is a necessary condition for optimal solution, is named after him.

Since Richard Bellman's discovery of the theory, economists have been studying both discrete, i.e., periodic, and continuous time. Stokey and Lucas did groundwork in [**18**] that involves both cases with deterministic and stochastic versions. Romer [**15**] studied different economics models, such as, the Solow Growth model, the Ramsey-Cass-Koopmans model, and the Diamond model on continuous time. Conversely, Hassler analyzed the discrete time growth model and solved the problem explicitly in [**10**] . However, we intend to attract attention to a different view of the theory, which is called dynamic programming on non-periodic time domains.

In this thesis we claim that, *trading* is not a reasonably periodic and continuous act. Consider a stock market example. A stockholder purchases stocks of a particular company on varying time intervals. Assume that on the first day of the month, a stockholder purchased stocks in every hour, and in the following two days, the company did a good profit, which made the stockholder purchase several times in *an hour*. Unluckily, the company could not manage well the rest of the month, so the stockholder bought only one stock on the last day of the month. Since there is not a strict rule on the behavior of the stockholder, we cannot tell if purchases had been made on a regular and continuous base. On the contrary, depending on the management of the profit and goodwill of the company, the stockholder did purchase on non-periodic times. This argument leads our research to time scales calculus, exclusively isolated time scales, and we formulate both deterministic and stochastic growth models on any isolated time scales.

Dynamic programming can be studied on both infinite or finite horizons. First we give a brief description of both cases on discrete time deterministic and stochastic models.

The Finite Horizon Case Consider the finite horizon control problem on discrete time, for $t = 0, 1, 2, ..., T$ and $T < \infty$,

$$(SP) \max_{\{x(t),c(t)\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t F\left(x\left(t\right), c\left(t\right)\right)$$

$$s.t.\ x\left(t+1\right) \in \Gamma\left(x\left(t\right), c\left(t\right)\right)$$

$$x\left(0\right) \in X \ given,$$

where $x\left(t\right)$ is the state and $c(t)$ is the control variable, and $0 < \beta < 1$ is the discount factor. F is the concave and bounded objective function, and $\Gamma\left(x\left(t\right), c\left(t\right)\right)$ is the set of constraints that determines the feasible choices of $c\left(t\right)$ given $x\left(t\right)$. Deterministic dynamic programming is also called decision making under certainty. The consumer makes decisions in each stage with risk aversion; thus, there occurs a sequence of decisions $\left\{k\left(t\right), c(t)\right\}_{t=0}^{T}$. Our goal is to find the sequence $\left\{k\left(t\right), c(t)\right\}_{t=0}^{T}$ that gives maximum discounted utility in (SP). To do so, it is worthwhile to introduce the concept of the value function, $V\left(t, x\left(t\right)\right)$, which is the solution of the corresponding *Bellman equation* that is given by

$$V\left(t, x\left(t\right)\right) = F\left(x\left(t\right), c\left(t\right)\right) + \beta V\left(t+1, x\left(t+1\right)\right)$$

$$s.t.\ x\left(t+1\right) \in \Gamma\left(x\left(t\right), c\left(t\right)\right).$$

A value function describes the maximum present discounted value of the objective function from a specific point in time as a function of the state variables at that date. Thus, we achieve our goal by obtaining the value function $V\left(t, x\left(t\right)\right)$. Since there is a *terminal condition* in finite horizon problems, we can solve the Bellman equation, either by *guessing* or *iteration* of the value function.

The Infinite Horizon Case This time we let the time variable t be infinite, i.e., $t = 0, 1, 2, \ldots$. Then the sequence problem has the form

$$(SP) \max_{\{x(t),c(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x(t), c(t))$$

$$s.t.\ x(t+1) \in \Gamma(x(t), c(t))$$

$$x(0) \in X \ given.$$

In infinite horizon, there is no terminal condition. Instead of this, we have a *transversality condition*.

Since time is infinite, it is not practical to use *iteration* of the value function on the infinite horizon. Moreover, since

$$\max \lim_{T\to\infty} \sum_{s=0}^{T} \beta^t F(x(s), c(s)) \neq \lim_{T\to\infty} \max \sum_{s=0}^{T} \beta^t F(x(s), c(s)),$$

we cannot just find the solution on the infinite horizon by taking the limit of the finite horizon solution as $T \to \infty$. Therefore, we use *Richard Bellman's Principle of Optimality* to solve the infinite horizon sequence problem.

Bellman's Principle of Optimality *An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the initial decision.*

By *guessing method,* we identify the value function $V(t, x(t))$, and then analyze the link between optimal solution for (SP) and $V(t, x(t))$ via *Bellman's Principle of Optimality.*

We also formulate the stochastic dynamic programming in infinite horizon in chapter 4 because unlike the deterministic model, a stochastic model is more realistic. It reflects the behavior of the decision maker without risk aversion. In the stochastic model, the sequence problem above has the form

$$(SP) \max_{\{x(t),c(t)\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t F(x(t), c(t))$$

$$s.t.\ x(t+1) \in \Gamma(x(t), c(t), z(t)),$$

where $z(t)$ is the *random state variable.* That means future consumption is uncertain. Then, the corresponding *stochastic Bellman equation* has the form

$$V(t, x(t)) = F(x(t), c(t)) + \beta \mathbb{E}[V(t+1, x(t+1))]$$

$$s.t.\ x(t+1) \in \Gamma(x(t), c(t), z(t)).$$

Eventually, the solution of the Bellman equation involves the random variable $z(t)$, which makes the explicit solution impossible. Therefore, we characterize the distributions of solutions. We refer to Karatzas [**12**] and Bergin [**6**] for more detailed discussion.

Consequently, we underline the fact that dynamic programming is a better alternative to calculus of variations and optimal control theory. Unlike optimal control or calculus of variations, dynamic programming has been applied to problems in both discrete and continuous time. Dynamic programming can also deal with problems that are formulated on infinite horizon, where the other two methods remain inadequate to deal with the time inconsistency. Moreover, we have more flexibility on constraints in dynamic programming. Another benefit of using dynamic programming in decision making problems is that, we can extend the theory to the stochastic case, in which there is uncertainty. We refer to Kamien and Schwartz [**11**] for further reading on calculus of variations and optimal control in economics.

Recently, some papers were published about dynamic programming on time scales. Zhan, Wei, and Xu analyzed the Hamilton-Jacobi-Bellman equations on time scales in [**19**]. Similarly, Seiffert, Sanyal and Wunsch studied the Hamilton-Jacobi-Bellman equations and approximate dynamic programming on time scales in [**17**]. Also Atici, Biles and Lebedinsky studied applications of time scales to economics and analyzed a utility maximization problem on multiple time scales in [**2**] and [**3**], respectively.

CHAPTER 2

PRELIMINARIES

As mentioned in the previous chapter, since trading is not a reasonably periodic and continuous act, we formulate both deterministic and stochastic models on isolated time domains to imply the scattered way of trading over time. Because time scale calculus is a mathematical theory that explains the structure of such domains, it is beneficial to comprehend this theory. Also, we introduce stochastic calculus briefly to understand the behavior of solution under uncertainty.

2.1 Time Scale Calculus

In this section, for the reader's convenience, we shall mention some basic definitions on time scale calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Examples of such sets are the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, *Cantor set*, the harmonic numbers $\{H_n\}_{n=0}^{\infty}$ where $H_0 = 0$ and $H_n = \Sigma_{k=1}^{n} \frac{1}{k}$, and $q^{\mathbb{N}} = \{q^n \mid n \in \mathbb{N}\}$ where $q > 1$. On the other hand, the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R} \backslash \mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval $(0, 1)$ are not time scales.

A time scale $\mathbb{T}$ may or may not be connected. Therefore we should define *the forward jump operator* $\sigma(t)$ and *the backward jump operator* $\rho(t)$. For $t \in \mathbb{T}$ we define $\sigma, \rho \colon \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} \colon s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} \colon s < t\}.$$

For the empty set $\emptyset$, the above definition includes $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called *right-scattered* if $\sigma(t) > t$, *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$, and *left-dense* if $\rho(t) = t$. If $\rho(t) < t < \sigma(t)$ then t is called an *isolated point,* and if $\rho(t) = t = \sigma(t)$, then t is called a *dense point.* If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is *left-scattered,* we let $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{\sup \mathbb{T}\}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$.

A time scale $\mathbb{T}$ is called *an isolated time scale* if every $t \in \mathbb{T}$ is an isolated point. For instance, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, and $q^{\mathbb{N}} = \{q^n \mid n \in \mathbb{N}\}$ where $q > 1$ are all isolated time scales. For our future purposes, we will assume that $\mathbb{T}$ is an isolated time scale with $\sup \mathbb{T} = \infty$.

The *graininess function* $\mu \colon \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Note that for any isolated time scale, the *graininess function* is strictly positive.

Definition 1 *If $f \colon \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^{\kappa}$, we define the delta derivative of f at point t to be the number $f^{\Delta}(t)$ (provided it exists) with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t) [\sigma(t) - s] \right| \leq \varepsilon \left| [\sigma(t) - s] \right|,$$

for all $s \in \mathcal{U}_{\mathbb{T}} = U \cap \mathbb{T}$.

Moreover, if the number $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$, then we say that it is *delta differentiable* on $\mathbb{T}^{\kappa}$. The delta derivative is given by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(s)}{\mu(t)} \tag{2.1}$$

if it is continuous (provided f is *differentiable*) at t and t is *right-scattered.* If t is *right-dense,* then the derivative is given by

$$f^{\Delta}(t) = \lim_{s \to t^{+}} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^{+}} \frac{f(t) - f(s)}{t - s}$$

provided the limit exists. As an example, if $\mathbb{T} = \mathbb{R}$ we have $f^{\Delta}(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$ we have $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$.

For the function $f : \mathbb{T} \to \mathbb{R}$ we define $f^{\sigma} : \mathbb{T}^{\kappa} \to \mathbb{R}$ by

$$f^{\sigma}(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

If f is differentiable at $t \in \mathbb{T}^{\kappa}$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t),$$

and for any differentiable function g, the *product rule* on time scale $\mathbb{T}$ is given by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)\, g(t) + f(\sigma(t))\, g^{\Delta}(t) = f(t)\, g^{\Delta}(t) + f^{\Delta}(t)\, g(\sigma(t)). \quad (2.2)$$

Definition 2 *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted by C_{rd}.*

Definition 3 *A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Then we define the Cauchy integral of $F^{\Delta}(t) = f(t)$ from a to t by*

$$\int_a^t f(s)\, \Delta s = F(t) - F(a).$$

If $\mathbb{T}$ is an isolated time scale, then the Cauchy integral has the form

$$\int_a^b f(t)\, \Delta t = \sum_{t \in [a,b) \cap \mathbb{T}} f(t)\, \mu(t) \;,$$

where $a, b \in \mathbb{T}$ with $a < b$. In the case where $\mathbb{T} = \mathbb{R}$, we have

$$\int_a^b f(t)\, \Delta t = \int_a^b f(t)\, dt,$$

and if $\mathbb{T}=\mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t=\sum_{t=a}^{b-1} f(t)\ \ , a<b.$$

As an example, if $\mathbb{T}=\mathbb{Z}$, and $f(t)=r^t$ where $r\neq 1$ constant, then we have

$$\int_a^b r^t\,\Delta t=\sum_{t=a}^{b-1} r^t=\left.\frac{r^t}{r-1}\right]_a^b=\frac{r^b-r^a}{r-1}.$$

Theorem 4 *If $f\in C_{rd}$ and $t\in\mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau=\mu(t)\,f(t)\,.$$

If $f,g\in C_{rd}$ and $a,b\in\mathbb{T}$, then the *integration by parts formula* on time scales is given by

$$\int_a^b f(\sigma(t))\,g^\Delta(t)\,\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^\Delta(t)\,g(t)\,\Delta t.$$

Now in order to denote the generalized exponential function $e_p(t,s)$, we need to give some definitions.

First, we say that a function $p:\mathbb{T}\to\mathbb{R}$ is *regressive* if

$$1+\mu(t)\,p(t)\neq 0\qquad\text{for all } t\in\mathbb{T}^\kappa.$$

The set of all regressive functions on time scale $\mathbb{T}$ forms an Abelian group under the "circle plus" addition $\oplus$ defined by

$$(p\oplus q)(t)=p(t)+q(t)+\mu(t)\,p(t)\,q(t)\quad\textit{for all}\quad t\in\mathbb{T}^\kappa,$$

and the inverse $\ominus p$ of the function p is defined by

$$(\ominus p)(t)=\frac{-\,p(t)}{1+\mu(t)\,p(t)}\quad\textit{for all}\quad t\in\mathbb{T}^\kappa.$$

The set $\mathcal{R}$ is given with

$$\mathcal{R} = \{f : \mathbb{T} \to \mathbb{R} \colon f \quad is\ rd - continuous\ and\ regressive\}.$$

Now, for $p \in \mathcal{R}$ we define the *exponential function* $e_p(., t_0)$, to be the unique solution of the initial value problem

$$y^{\Delta} = p(t)\, y, \qquad y(t_0) = 1.$$

Next we will list the properties of the exponential function $e_p(., t_0)$ that are very useful for our goals.

Lemma 5 *For $p \in \mathcal{R}$ we have*

i) $e_0(t, s) \equiv 1$

ii) $e_p(t, t) \equiv 1$

iii) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$

iv) $e_p(t, s)\ e_p(s, r) = e_p(t, r)$

v) $e_p(t, s)\ e_q(t, s) = e_{p \oplus q}(t, s)$

where $r, s, t \in \mathbb{T}$.

We define the set of positive regressive functions by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)\, p(t) > 0 \ for\ all\ t \in \mathbb{T}\}.$$

If p is positive regressive, then the exponential function $e_p(t, t_0)$ is positive for *all* $t \in \mathbb{T}$ [*Theorem* 2.48 (*i*) in [**7**]].

Here are some examples of $e_p(t,t_0)$ on some various time scales. If $\mathbb{T}=\mathbb{R}$, then $e_\alpha(t,t_0)=e^{\alpha(t-t_0)}$, if $\mathbb{T}=h\mathbb{Z}$ then, $e_\alpha(t,t_0)=(1+\alpha h)^{\frac{t-t_0}{h}}$. This implies that $e_\alpha(t,t_0)=(1+\alpha)^{t-t_0}$ is the exponential function for $\mathbb{T}=\mathbb{Z}$.

Now consider the initial value problem

$$y^\sigma - p(t)\,y = r(t), \qquad y(t_0)=y_0 \tag{2.3}$$

on isolated time scales, where $p(t)\neq 0$ for all $t\in\mathbb{T}^\kappa$. Then the unique solution to the initial value problem is given by

$$y(t)=e_{\frac{p-1}{\mu}}(t,t_0)\,y_0+\int_{t_0}^{t} e_{\frac{p-1}{\mu}}(t,\sigma(s))\,\frac{r(s)}{\mu(s)}\Delta s.$$

This problem was extensively studied in the paper by A. C. Peterson and his students [**14**], and it brings us significant outcomes that are beneficial for our work.

Remark 6 *The exponential function $e_{\frac{p-1}{\mu}}(t,t_0)$ is given by*

$$e_{\frac{p-1}{\mu}}(t,t_0)=\prod_{\tau\in[t_0,t)} p(\tau)$$

for $t\geq t_0$; otherwise we have

$$e_{\frac{p-1}{\mu}}(t,t_0)=\prod_{\tau\in[t,t_0)} \frac{1}{p(\tau)}.$$

This remark deduces two important conclusions. First, $e_{\frac{p-1}{\mu}}(t,t_0)$ is the unique solution of the homogeneous equation $y^\sigma - p(t)\,y=0$ with the given initial value $y(t_0)=y_0$. Thus we have the recurrence relation

$$e^\sigma_{\frac{p-1}{\mu}}(t,t_0)=p(t)\,e_{\frac{p-1}{\mu}}(t,t_0) \tag{2.4}$$

for $t\geq t_0$. Second, let $0<p<1$ be a constant number, and for $t\geq t_0$, let $t=t_n$ on time scale $\mathbb{T}=\{t_0,t_1,...,t_n,...\}$. Also, let $g(t)$ be an function of t that counts the number of isolated points on the interval $[t_0, t) \cap \mathbb{T}$. Then the exponential function becomes

$$e_{\frac{p-1}{\mu}}(t, t_0) = \prod_{\tau \in [t_0, t)} p(\tau) = p^{g(t)}.$$

Thus, we have

$$\lim_{t \to \infty} e_{\frac{p-1}{\mu}}(t, t_0) = 0. \tag{2.5}$$

For further reading and for the proofs, we refer the reader to the book by A. C. Peterson and M. Bohner [**7**].

2.2 Stochastic Calculus

Since we formulate our models on isolated time domains, in this section, we define basics of a discrete probability model for our purposes. We start by defining a sample set Ω. Consider we have a single stock with price S_t at time $t = 1, 2, ..., T$. Then, the set of all possible values, Ω, is given by

$$\Omega = \{\omega : \omega = (S_1, S_2, ..., S_T)\}.$$

We have to list *all* possible future prices, i.e., *all* possible states of the world, to model uncertainty about the price in the future. The unknown future is just one of many possible outcomes, called the true state of the world.

Now say $\mathcal{F}_t$ is the information available to investors at time t, which consists of stock prices before and at time t. Then, for a finite sample space Ω, we define *the random variable* X as a real-valued function given by

$$X : (\Omega, \mathcal{F}_t) \to \mathbb{R}.$$

Definition 7 *A stochastic process is a collection of random variables* $\{X(t)\}$.

Next we introduce expectation, variance, and covariance of random variables. Prior to this, we shall give some basic definitions. Define the probability function of an outcome $\omega \in \Omega$, where Ω is finite, as $P(\omega)$. $P(\omega)$ is a real number which describes the likelihood of ω occurring. It satisfies the following properties:

i) $P(\omega) \geq 0$,

ii) $\sum\limits_{\omega\in\Omega} P(\omega) = P(\Omega) = 1$.

Notice that, for a random variable $X : \Omega \to \mathbb{R}$, where Ω is finite, $X(\Omega)$ also denotes a finite set given by

$$X(\Omega) = \{x_i\}_{i=1}^{k}.$$

We define the collection of probabilities p_i of sets

$$\{X = x_i\} = \{\omega : X(\omega) = x_i\}$$

as the *probability distribution of* X, provided

$$p_i = P(X = x_i) = \sum_{\omega : X(\omega) = x_i} P(\omega).$$

Definition 8 *For the random variable* X *given on* $(\Omega, \mathcal{F})$, *and probability* P, *expectation of* X *with respect to* P *is given by*

$$\mathbb{E}(X) \equiv \mathbb{E}_P X = \sum_{all\ \omega\in\Omega} X(\omega) P(\omega).$$

Next, we give properties of an expectation, $\mathbb{E}(X)$, for any random variable X.

Remark 9 *Assume* X *and* Y *are random variables, and* α, β *are constants. Then, we say* $\mathbb{E}$ *is linear, provided*

$$\mathbb{E}(\alpha X + \beta Y) = \alpha\mathbb{E}(X) + \beta\mathbb{E}(Y)$$

or

$$\mathbb{E}\left(\alpha X+\beta\right)=\alpha\mathbb{E}\left(X\right)+\beta.$$

The covariance of two random variables X and Y is defined by

$$Cov\left(X,Y\right)=\mathbb{E}\left[\left(X-\mathbb{E}\left(X\right)\right)\left(Y-\mathbb{E}\left(Y\right)\right)\right]=\mathbb{E}\left(XY\right)-\mathbb{E}\left(X\right)\mathbb{E}\left(Y\right).$$

If two random variables X and Y are uncorrelated, then

$$Cov\left(X,Y\right)=0.$$

Definition 10 *Variance is a measure that describes how far a set of numbers are spread out from each other. In a probability distribution, it describes how far the numbers lie from the expectation (or mean).*

The variance of X is the covariance of X with itself, i.e.,

$$Var\left(X\right)=Cov\left(X,X\right).$$

Then, we say

$$\begin{aligned} Var\left(X\right) &= Cov\left(X,X\right) \\ &= \mathbb{E}\left[\left(X-\mathbb{E}X\right)\left(X-\mathbb{E}X\right)\right] \\ &= \mathbb{E}\left[\left(X-\mathbb{E}X\right)^2\right] \\ &= \mathbb{E}\left[X^2\right]-\left(\mathbb{E}\left[X\right]\right)^2. \end{aligned}$$

For further reading, we refer the reader to the book by F. C. Klebaner [**13**].

CHAPTER 3

DETERMINISTIC DYNAMIC PROGRAMMING

In this chapter, we study the concept of dynamic programming when the decision maker gives decisions with risk aversion, i.e., decision making under certainty. For our purposes, we assume $\mathbb{T}$ is an isolated time scale with $\sup \mathbb{T} = \infty$. In Section 3.1, we construct the infinite deterministic dynamic sequence problem on $\mathbb{T}$ and introduce the concept of utility function provided with examples. We continue our work by deriving the corresponding Bellman equation in Section 3.2. In Section 3.3, the relation between solutions of the sequence problem and the Bellman equation is identified through the Bellman's Optimality Principle. Afterwards, in Section 3.4, we solve an example of optimal growth model with a given logarithmic utility function in economical finance, and in Section 3.5, we examine the existence and uniqueness of the solution of the Bellman equation via Contraction Mapping Theorem.

3.1 Deterministic Dynamic Sequence Problem

Let $\sup \mathbb{T} = \infty$ and $\mathbb{T} \cap [0, \infty) = [0, \infty)_{\mathbb{T}}$. Then, we define the deterministic sequence problem on isolated time scale $\mathbb{T}$ as

$$(SP) \quad \sup_{\{x(t),c(t)\}_{t=0}^{\infty}} \int_{s\in[0,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\, F(x(s), c(s))\, \Delta s \tag{3.1}$$

$$s.t.\ x^{\Delta}(t) \in \Gamma(t, x(t), c(t))$$

$$x(0) \in X given,$$

where

- $x(t)$ is the state variable,

- $c(t)$ is the optimal control or choice variable,
- $F(.,.)$ is a strictly concave, continuous and differentiable real-valued objective function,
- X is the space of sequences $\{x(t)\}_{t=0}^{\infty}$ that maximizes the sequence problem,
- $\Gamma : X \to X$ is the correspondence describing the feasibility constraints where $\Gamma(t, x(t), c(t))$ is the set of constraints that determine the feasible choices of t and $c(t)$ given $x(t)$.

Also, we shall talk about the budget constraint. A budget constraint is an accounting identity that describes the consumption options available to a decision maker with a limited income (or wealth) to allocate among various goods. It is important to understand that the budget constraint is an accounting identity, not a behavioral relationship. A decision maker may well determine the next decision step in order to find the best path by considering the budget constraint. Thus, the budget constraint is a given element of the problem that a decision maker faces. In this thesis, we define the budget constraint on any isolated time scales $\mathbb{T}$ as

$$x^{\Delta}(t) \in \Gamma(t, x(t), c(t)),$$

which is also known as the first order dynamic inclusion. That means $x^{\Delta}(t)$ can be chosen as any of the feasible plans that belongs to $\Gamma(t, x(t), c(t))$. For instance, we can choose the budget constraint as one of the following:

i) $x^{\Delta}(t) < g(t, x(t), c(t))$, or

ii) $x^{\Delta}(t) = g(t, x(t), c(t))$, or

iii)$x^{\Delta}(t) \quad \Re \quad g(t, x(t), c(t))$, etc., where $\Re$ defines the "relation" between functions.

We refer to the paper by F. M. Atici and D. C. Biles [**1**] for more detailed discussion about the existence results on first order dynamic inclusions on time scales.

The sequence problem given with (3.1) is a multi-stage problem, and as we mentioned in the first chapter it is efficient to use dynamic programming to solve the problem. The basic idea in dynamic programming is to take the infinite time maximizing problem and somehow break it down into a reasonable number of subproblems. Then we use the optimal solutions to the smaller subproblems to find the optimal solutions to the larger ones. These subproblems can overlap, so long as there are not too many of them.

3.2 Deterministic Bellman Equation

We define the deterministic Bellman equation on any isolated time scale $\mathbb{T}$ as

$$\left(e_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t))\right)^{\Delta}+e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t))=0 \tag{3.2}$$

where $V(t,x(t))$ is the value function.

Remark 11 *Equation* (3.2) *is equivalent to*

$$V(t,x(t))=\mu(t)\,F(x(t),c(t))+\beta V(\sigma(t),x(\sigma(t))) \tag{3.3}$$

$$s.t.\ x^{\Delta}(t)\in\Gamma(t,x(t),c(t)).$$

Indeed, if we apply the product rule (2.2) on the LHS of the equation (3.2), then we have

$$0 = \left(e_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t))\right)^{\Delta} + e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t))$$

$$\begin{aligned} 0 &= e^{\Delta}_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t)) + e^{\sigma}_{\frac{\beta-1}{\mu}}(t,0)\,V^{\Delta}(t,x(t)) \\ &\quad + e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t)). \end{aligned}$$

Then by applying the definition of delta derivative (2.1), we have

$$\begin{aligned} 0 &= e^{\Delta}_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t)) + e^{\sigma}_{\frac{\beta-1}{\mu}}(t,0)\,V^{\Delta}(t,x(t)) \\ &\quad + e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t)) \end{aligned}$$

$$\begin{aligned} 0 &= \frac{\beta-1}{\mu(t)}e_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t)) + \beta e_{\frac{\beta-1}{\mu}}(t,0)\,\frac{V^{\sigma}(t,x(t)) - V(t,x(t))}{\mu(t)} \\ &\quad + e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t)) \end{aligned}$$

$$\begin{aligned} 0 &= -\frac{1}{\mu(t)}e_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t)) + \frac{\beta}{\mu(t)}e_{\frac{\beta-1}{\mu}}(t,0)\,V^{\sigma}(t,x(t)) \\ &\quad + e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t)) \end{aligned}$$

$$0 = e_{\frac{\beta-1}{\mu}}(t,0)\left[-V(t,x(t)) + \beta V^{\sigma}(t,x(t)) + \mu(t)\,F(x(t),c(t))\right].$$

Since $e_{\frac{\beta-1}{\mu}}(t,0) \neq 0$, the last equation above implies

$$0 = -V(t,x(t)) + \beta V^{\sigma}(t,x(t)) + \mu(t)\,F(x(t),c(t)).$$

If we rearrange it, we have

$$V(t,x(t)) = \mu(t)\,F(x(t),c(t)) + \beta V^{\sigma}(t,x(t)).$$

∎

3.3 Bellman's Optimality Principle

In this section, we prove the main results of this thesis that examine the relation between the optimum of the sequence problem and the solution of the Bellman equation. Once we obtain the link between these solutions, we can say a solution to the Bellman equation is also a solution for the sequence problem. Thus, instead of solving the sequence problem, we solve the Bellman equation.

The following theorem relates the solutions to the Bellman equation to those of (SP).

Theorem 12 *Assume (SP) attains its supremum at $\{x(t), c(t)\}_{t=0}^{\infty}$. Then*

$$V(t, x(t)) = \int_{s\in[t,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s)) \Delta s \tag{3.4}$$

$$s.t.\ x^{\Delta}(s) \in \Gamma(s, x(s), c(s))$$

is the solution to the Bellman equation with initial condition at t=0.

Proof. It is sufficient to show that the value function (3.4) satisfies the Bellman equation (3.3). Then by plugging the value function, $V(t, x(t))$, in the RHS of equation (3.3), we have

$$\begin{aligned} & \mu(t) F(x(t), c(t)) + \beta V(\sigma(t), x(\sigma(t))) \\ = \ & \mu(t) F(x(t), c(t)) + \beta \int_{s\in[\sigma(t),\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s, \sigma(t)) F(x(s), c(s)) \Delta s. \end{aligned}$$

By properties of the exponential function we have

$$= \mu(t) F(x(t), c(t)) + \beta \int_{s\in[\sigma(t),\infty)_{\mathbb{T}}} \frac{e_{\frac{\beta-1}{\mu}}(s,0)}{e_{\frac{\beta-1}{\mu}}(\sigma(t),0)} F(x(s), c(s)) \Delta s.$$

Then by the recurrence relation (2.4) and the property of exponential function, that is, $e_p(t,s)\, e_p(s,r) = e_p(t,r)$, we have

$$= \mu(t) F(x(t), c(t)) + \frac{\beta}{\beta} \int\limits_{s\in[\sigma(t),\infty)_{\mathbb{T}}} \frac{e_{\frac{\beta-1}{\mu}}(s,0)}{e_{\frac{\beta-1}{\mu}}(t,0)} F(x(s), c(s))\, \Delta s$$

$$= \mu(t) F(x(t), c(t)) + \int\limits_{s\in[\sigma(t),\infty)_{\mathbb{T}}} \frac{e_{\frac{\beta-1}{\mu}}(s,0)}{e_{\frac{\beta-1}{\mu}}(t,0)} F(x(s), c(s))\, \Delta s$$

$$= \mu(t) F(x(t), c(t)) + \int\limits_{s\in[\sigma(t),\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s.$$

Next in the last equation above, we add and subtract the integral

$$\int_t^{\sigma(t)} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s,$$

which gives us

$$\begin{aligned} = \;& \mu(t) F(x(t), c(t)) + \int\limits_{s\in[\sigma(t),\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s \\ & + \int_t^{\sigma(t)} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta - \int_t^{\sigma(t)} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s \end{aligned}$$

$$\begin{aligned} = \;& \mu(t) F(x(t), c(t)) - \int_t^{\sigma(t)} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s \\ & + \int\limits_{s\in[t,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s. \end{aligned}$$

Since we are on isolated time scale $\mathbb{T}$, by Theorem 4 and since $e_{\frac{\beta-1}{\mu}}(t,t) = 1$, we have

$$\begin{aligned} \int_t^{\sigma(t)} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s))\, \Delta s &= \mu(t)\, e_{\frac{\beta-1}{\mu}}(t,t) F(x(t), c(t)) \\ &= \mu(t) F(x(t), c(t)). \end{aligned}$$

Then this implies

$$
\begin{aligned}
&= \mu(t) F(x(t), c(t)) - \int_{t}^{\sigma(t)} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s)) \Delta s \\
&\qquad + \int_{s\in[t,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s)) \Delta s \\
&= \mu(t) F(x(t), c(t)) - \mu(t) F(x(t), c(t)) \\
&\qquad + \int_{s\in[t,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s)) \Delta s \\
&= \int_{s\in[t,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,t) F(x(s), c(s)) \Delta s \\
&= V(t, x(t)).
\end{aligned}
$$

■

Before we continue, we shall introduce the *Euler equations* and the *transversality condition.*

Euler Equations An Euler equation is an intertemporal version of a first-order condition characterizing an optimal choice. In many economic problems, whatever method you use, such as calculus of variations, optimal control theory or dynamic programming, part of the solution is typically a solution of the Euler equation. We define the Euler equations for our problem as follows.

Let $V(t, x(t))$ be the solution of the Bellman equation, and assume (SP) attains its supremum at $\{x^*(t), c^*(t)\}_{t=0}^{\infty}$. Then, define the supremum of all solutions of the Bellman equation as

$$\begin{aligned} V^*(t, x(t)) &= \sup_{\{x(s)\}_{s=t}^{\infty}} V(s, x(s)) \\ &= \sup_{\{x(s)\}_{s=t}^{\infty}} \left[\mu(s) F(x(s), c(s)) + \beta V(\sigma(s), x(\sigma(s)))\right]. \end{aligned}$$

Assume $x(\sigma(t)) = g(t, x(t), c(t))$ suitable to the budget constraint that is given with the Bellman equation. Then by differentiating both sides of the above equation with respect to the control and state variables, respectively $c(t)$ and $x(t)$, we obtain the *Euler equations* of the Bellman equation as

$$\begin{aligned} 0 &= \beta V_x(\sigma(s), g(s, x^*(s), c^*(s)))\, g_c(s, x^*(s), c^*(s)) \\ &\quad + \mu(s) F_c(x^*(s), c^*(s)), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} V_x(t, x^*(t)) &= \beta V_x(\sigma(s), g(s, x^*(s), c^*(s)))\, g_x(s, x^*(s), c^*(s)) \\ &\quad + \mu(s) F_x(x^*(s), c^*(s)). \end{aligned} \tag{3.6}$$

The Transversality Condition In order to solve the Euler equations for the optimal solution of the problem, we need boundary conditions. These boundary conditions are obtained from the transversality condition. The transversality condition states that

$$\lim_{T\to\infty} V_x(T, x^*(T)) = 0 \tag{3.7}$$

if

$$V_x(T, x^*(T)) = \int_{s\in[0,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s, 0)\, F_x(x^*(s), c^*(s))\, \Delta s.$$

Under the existence of the transversality condition given with (3.7), the following theorem shows that any sequence satisfying this condition is an optimal solution for the maximizing sequence problem.

Theorem 13 *Assume* $V(t, x(t))$ *is the solution of the Bellman equation. Let the supremum of all solutions of the Bellman equation be*

$$V^*(t, x(t)) = \sup_{\{x(s)\}_{s=t}^{\infty}} V(s, x(s)). \tag{3.8}$$

Then $V^*(t, x(t))$ *characterizes the sequence problem* (SP) *at initial state* $x(0)$.

Proof. Assume $\{x^*(t), c^*(t)\}_{t=0}^{\infty}$ attain the supremum in (SP). It is sufficient to show that the difference between the objective functions in (SP) evaluated at $\{x^*(t), c^*(t)\}$ and at $\{x(t), c(t)\}$ is nonnegative. Therefore, we start by setting the difference as

$$(SP)^* - (SP) = \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s, 0)\left[F(x^*(s), c^*(s)) - F(x(s), c(s))\right]\Delta s.$$

Since F is concave, continuous and differentiable, we have

$$\begin{aligned}(SP)^* - (SP) \ \geq\ & \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s, 0)\left[F_x(x^*(s), c^*(s))(x^*(s) - x(s))\right. \\ & \left. + F_c(x^*(s), c^*(s))(c^*(s) - c(s))\right]\Delta s. \end{aligned}\tag{3.9}$$

If we solve the Euler equations given with (3.5) and (3.6), respectively, for F_c and F_x, we obtain

$$F_c(x^*(s), c^*(s)) = -\frac{\beta}{\mu(s)} V_x(\sigma(s), g(s, x^*(s), c^*(s)))\, g_c(s, x^*(s), c^*(s)),$$

and

$$\begin{aligned} F_x(x^*(s), c^*(s)) \ =\ & -\frac{\beta}{\mu(s)} V_x(\sigma(s), g(s, x^*(s), c^*(s)))\, g_x(s, x^*(s), c^*(s)) \\ & + \frac{1}{\mu(s)} V_x(t, x^*(t)). \end{aligned}$$

Now we substitute F_c and F_x in (3.9). Then we have,

$$\begin{aligned} =\ & \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s, 0)\, \frac{1}{\mu(s)} \left\{V_x(s, x^*(s))(x^*(s) - x(s))\right. \\ & -\beta V_x(\sigma(s), g(s, x^*(s), c^*(s)))\left[g_x(s, x^*(s), c^*(s))(x^*(s) - x(s))\right. \\ & \left.\left. + g_c(s, x^*(s), c^*(s))(c^*(s) - c(s))\right]\right\}\Delta s \end{aligned}$$

$$\geq \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\frac{1}{\mu(s)}\{V_x(s,x^*(s))(x^*(s)-x(s))$$
$$-\beta V_x(\sigma(s),g(s,x^*(s),c^*(s)))[g(s,x^*(s),c^*(s))-g(s,x(s),c(s))]\}\Delta s.$$

Since $x(\sigma(t)) = g(t,x(t),c(t))$, we have

$$= \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\frac{1}{\mu(s)}\{V_x(s,x^*(s))(x^*(s)-x(s))$$
$$-\beta V_x(\sigma(s),x^*(\sigma(s)))[x^*(\sigma(s))-x(\sigma(s))]\}\Delta s.$$

By plugging

$$x(\sigma(s)) = x(s)+\mu(s)x^{\Delta}(s)$$

in the last expression above, we have

$$\mathcal{I} = \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\frac{1}{\mu(s)}\{V_x(s,x^*(s))(x^*(s)-x(s))$$
$$-\beta V_x(\sigma(s),x^*(\sigma(s)))\left[x^*(s)+\mu(s)x^{*^{\Delta}}(s)-x(s)-\mu(s)x^{\Delta}(s)\right]\Big\}\Delta s$$

$$= \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\frac{1}{\mu(s)}\{V_x(s,x^*(s))(x^*(s)-x(s))$$
$$-\beta V_x(\sigma(s),x^*(\sigma(s)))\left[(x^*(s)-x(s))+\mu(s)(x^{*^{\Delta}}(s)-x^{\Delta}(s))\right]\Big\}\Delta s$$

$$= \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} \Big\{-e_{\frac{\beta-1}{\mu}}(s,0)\frac{1}{\mu(s)}\beta V_x(\sigma(s),x^*(\sigma(s)))\mu(s)(x^{*^{\Delta}}(s)-x^{\Delta}(s))$$
$$+e_{\frac{\beta-1}{\mu}}(s,0)\frac{1}{\mu(s)}[V_x(s,x^*(s))-\beta V_x(\sigma(s),x^*(\sigma(s)))](x^*(s)-x(s))\Big\}\Delta s.$$

By using the recurrence relation of the exponential function, i.e., $e^{\sigma}_{\frac{\beta-1}{\mu}}(t,0) = \beta e_{\frac{\beta-1}{\mu}}(t,0)$, we rewrite the above equation. Then, we have

$$\mathcal{I} = \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} \Big\{-e_{\frac{\beta-1}{\mu}}(\sigma(s),0)V_x(\sigma(s),x^*(\sigma(s)))(x^{*^{\Delta}}(s)-x^{\Delta}(s))$$
$$-\left[\frac{e^{\sigma}_{\frac{\beta-1}{\mu}}(s,0)V_x^{\sigma}(s,x^*(s))-e_{\frac{\beta-1}{\mu}}(s,0)V_x(s,x^*(s))}{\mu(s)}\right](x^*(s)-x(s))\Big\}\Delta s$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} - e_{\frac{\beta-1}{\mu}}(\sigma(s),0)\, V_x(\sigma(s), x^*(\sigma(s)))\,(x^{*^{\Delta}}(s) - x^{\Delta}(s))\Delta s \\
&\quad - \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} \left[e_{\frac{\beta-1}{\mu}}(s,0)\, V_x(s, x^*(s))\right]^{\Delta} (x^*(s) - x(s))\,\Delta s.
\end{aligned}
$$

In the first integral above, we use the integration by parts formula on time scales, i.e.,

$$
\int f(\sigma(t))\, g^{\Delta}(t)\,\Delta t = (fg)(t) - \int f^{\Delta}(t)\, g(t)\,\Delta t.
$$

Now let

$$
\begin{aligned}
f(\sigma(t)) &= -e_{\frac{\beta-1}{\mu}}(\sigma(s),0)\, V_x(\sigma(s), x^*(\sigma(s))) \quad \text{and} \\
g^{\Delta}(t) &= \left(x^{*^{\Delta}}(s) - x^{\Delta}(s)\right),
\end{aligned}
$$

and apply the integration by parts formula. Then we have

$$
\begin{aligned}
\mathcal{I} &= \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} - e_{\frac{\beta-1}{\mu}}(\sigma(s),0)\, V_x(\sigma(s), x^*(\sigma(s)))\,(x^{*^{\Delta}}(s) - x^{\Delta}(s))\Delta s \\
&\quad - \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} \left[e_{\frac{\beta-1}{\mu}}(s,0)\, V_x(s, x^*(s))\right]^{\Delta} (x^*(s) - x(s))\,\Delta s \\
&= -\lim_{T\to\infty} e_{\frac{\beta-1}{\mu}}(t,0)\, V_x(t, x^*(t))\,(x^*(t) - x(t))\Big]_0^T \\
&\quad + \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} \left[e_{\frac{\beta-1}{\mu}}(s,0)\, V_x(s, x^*(s))\right]^{\Delta} (x^*(s) - x(s))\,\Delta s \\
&\quad + \lim_{T\to\infty} \int\limits_{s\in[0,T)_{\mathbb{T}}} - \left[e_{\frac{\beta-1}{\mu}}(s,0)\, V_x(s, x^*(s))\right]^{\Delta} (x^*(s) - x(s))\,\Delta s \\
&= -\lim_{T\to\infty} \Big(e_{\frac{\beta-1}{\mu}}(T,0)\, V_x(T, x^*(T))\,(x^*(T) - x(T)) \\
&\quad - e_{\frac{\beta-1}{\mu}}(0,0)\, V_x(0, x^*(0))\,(x^*(0) - x(0))\Big).
\end{aligned}
$$

Because $x^*(0) - x(0) = 0$, we have

$$
\mathcal{I} = -\lim_{T\to\infty} e_{\frac{\beta-1}{\mu}}(T,0)\, V_x(T, x^*(T))\,(x^*(T) - x(T)).
$$

It then follows from the transversality condition (3.7) that $(SP)^* - (SP) \geq 0$, which demonstrates the desired result. ■

3.4 An Example in Financial Economics: Optimal Growth Model

In this section, we solve *the social planner's problem* on isolated time scale $\mathbb{T}$ by *the guessing method.* In the social planner's problem the objective function F is assumed to be as U, and it is called utility function. First, we shall give the definition and the properties of the utility function U.

A utility is a numerical rating assigned to every possible outcome a decision maker may be faced with. It is often modeled to be affected by consumption of various goods and services, possession of wealth, and spending of leisure time. Next we give the mathematical description of the utility function. Let $U : (0, \infty) \to \mathbb{R}$ be a strictly increasing, concave and continuously differentiable function, with

1. $U(0)$ means $\lim_{c \to 0^+} U(c) \geq -\infty$
2. $U'(0)$ means $\lim_{c \to 0^+} U(c) \leq \infty$
3. $U'(\infty)$ means $\lim_{c \to \infty} U'(c) = 0.$

A function with these properties is called a *utility function.* Here we give some examples of the best known utility functions. The first one is the Cobb-Douglas utility function

$$U(x, y) = x^a y^b, \ a, b > 0.$$

Another utility function is Constant Relative Risk Aversion (CRRA), or power utility function, given by

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & for\ \gamma > 0,\ \gamma \neq 1 \\ \ln C & for\ ,\ \gamma = 1 \end{cases}$$

where $\frac{1}{\gamma}$ is the intertemporal substitution elasticity between consumption in any two periods; i.e., it measures the willingness to substitute consumption between different periods. If γ is getting smaller, then the household is more willing to substitute consumption over time. Another utility function is Constant Absolute Risk Aversion (CARA) which is

$$U(C) = -\frac{1}{\alpha}e^{(-\alpha C)}, \quad \alpha > 0.$$

Lastly, we have the Quadratic utility function, which is given by

$$U(C) = C - \frac{a}{2}C^2, \quad a > 0.$$

For further reading, we refer the reader to the paper by I. Karatzas [**12**], and the lecture notes by D. Romer [**15**].

3.4.1 Solving the Bellman Equation by Guessing Method

As we mentioned in the introduction, *trading* is not a reasonably periodic and continuous act. A stock or a merchandise will be purchased in a period of time sometimes daily or hourly, and sometimes in every minute or second. Thus, there is not a rule that tells us the frequency of a purchase is periodic or continuous. Each purchase may be scattered randomly over a period of time. Therefore, we formulate the problem on isolated time domains.

First, we define the deterministic utility maximization problem. Assume $\sup \mathbb{T} = \infty$ for *all* $t \in \mathbb{T}$. Then the infinite horizon dynamic sequence problem on isolated time scale $\mathbb{T}$ has the form

$$\sup_{\{k(\sigma(t))\}_{t=0}^{\infty}} \int_{s\in[0,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\,U(k(s),c(s))\,\Delta s \tag{3.10}$$

$$s.t. \;\; k^{\Delta}(t) \in \Gamma(t,k(t),c(t))$$

$$k(0) \in X \; given,$$

where $k(t)$ is the state variable and $c(t)$ is the control variable. In the social planner's problem, we use CRRA utility function, i.e., $U(k(t),c(t)) = \ln(c(t))$. Then, we define the corresponding Bellman equation as

$$V(t,k(t)) = \mu(t)\,U(k(t),c(t)) + \beta V(\sigma(t),k(\sigma(t))) \tag{3.11}$$

$$s.t. \;\; k^{\Delta}(t) \in \Gamma(t,k(t),c(t)).$$

Our goal is to find the sequence $\{k(\sigma(t))\}_{t=0}^{\infty}$ which maximizes the given utility function. The budget constraint for the social planner's problem is given as

$$k(\sigma(t)) = k^{\alpha}(t) - c(t), \tag{3.12}$$

where $0 < \alpha < 1$. First, we need to show the budget constraint is appropriately chosen. Indeed, we get the desired result by doing elementary algebra and by using the definition of the Δ-*derivative* on isolated time scales

$$k(\sigma(t)) = k^{\alpha}(t) - c(t)$$

$$\frac{k(\sigma(t)) - k(t)}{\mu(t)} = \frac{k^{\alpha}(t) - k(t) - c(t)}{\mu(t)}$$

$$k^{\Delta}(t) \in \Gamma^{*}\left(\frac{k^{\alpha}(t) - k(t) - c(t)}{\mu(t)}\right) \equiv \Gamma(t,k(t),c(t)).$$

Now substitute the utility function $U(k(t),c(t)) = \ln(c(t))$ and the budget constraint $k(\sigma(t)) = k^{\alpha}(t) - c(t)$ in the Bellman equation. Then, we have the Bellman equation as

$$V(t,k(t)) = \mu(t)\ln\left(k^{\alpha}(t) - k(\sigma(t))\right) + \beta V(\sigma(t), k(\sigma(t))). \tag{3.13}$$

By guessing, we say the solution of this Bellman equation, $V(t,k(t))$, has the form

$$V(t,k(t)) = A(t)\ln(k(t)) + B(t). \tag{3.14}$$

Theorem 14 *The value function $V(t,k(t))$ is equal to*

$$\begin{aligned} V(t,k(t)) \;&=\; \alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right]\ln(k(t)) \\ &\quad + e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)\left\{B(0) - \int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right.\right. \\ &\quad \left.\left. + \frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)\right]\Delta s\right\}. \end{aligned} \tag{3.15}$$

for the sequence $\{k(\sigma(t))\}_{t=0}^{\infty}$, which maximizes (SP).

Proof. Because the value function $V(t,k(t))$, given with (3.14), is solution of the Bellman equation (3.3), it satisfies equation (3.13). Therefore, we have

$$\begin{aligned} A(t)\ln(k(t)) + B(t) = \mu(t)\ln\left(k^{\alpha}(t) - k(\sigma(t))\right) \\ + \beta\left[A(\sigma(t))\ln(k(\sigma(t))) + B(\sigma(t))\right]. \end{aligned} \tag{3.16}$$

Then by differentiating both sides of equation (3.16) with respect to $k(\sigma(t))$, we have

$$\begin{aligned} \frac{\partial}{\partial k(\sigma(t))}\{A(t)\ln(k(t)) + B(t)\} \;&=\; \frac{\partial}{\partial k(\sigma(t))}\{\mu(t)\ln\left(k^{\alpha}(t) - k(\sigma(t))\right) \\ &\quad + \beta\left[A(\sigma(t))\ln(k(\sigma(t))) + B(\sigma(t))\right]\} \end{aligned}$$

$$0 = \mu(t)\left(\frac{-1}{k^{\alpha}(t) - k(\sigma(t))}\right) + \beta A(\sigma(t))\frac{1}{k(\sigma(t))}$$

We substitute the budget constraint in (3.12) back, and then we find the *control variable* $c(t)$ as

$$\begin{aligned}
\frac{\mu(t)}{c(t)} &= \frac{\beta A(\sigma(t))}{k^{\alpha}(t) - c(t)} \\
\mu(t)\, k^{\alpha}(t) &= [\mu(t) + \beta A(\sigma(t))]\, c(t) \\
c(t) &= \frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))} k^{\alpha}(t).
\end{aligned}$$

Hence, we find the *state variable* $k(\sigma(t))$ as

$$\begin{aligned}
k(\sigma(t)) &= k^{\alpha}(t) - c(t) \\
&= k^{\alpha}(t) - \frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))} k^{\alpha}(t) \\
&= \frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))} k^{\alpha}(t).
\end{aligned}$$

Now we plug these values in equation (3.16). Then we have

$$\begin{aligned}
A(t)\ln(k(t)) + B(t) &= \mu(t)\ln\left(\frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))} k^{\alpha}(t)\right) \\
&\quad + \beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))} k^{\alpha}(t)\right) \\
&\quad + \beta B(\sigma(t)) \\
&= \alpha\mu(t)\ln(k(t)) + \mu(t)\ln\left(\frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))}\right) \\
&\quad + \alpha\beta A(\sigma(t))\ln(k(t)) \\
&\quad + \beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))}\right) \\
&\quad + \beta B(\sigma(t)). \qquad (3.17)
\end{aligned}$$

Next we have to find the coefficients, $A(t)$ and $B(t)$, in the value function. From equivalence of RHS and LHS of equation (3.17), we find $A(t)$ and $B(t)$ as follows.

For the coefficient $A(t)$, we have

$$A(t) = \alpha\mu(t) + \alpha\beta A(\sigma(t)).$$

By doing elementary algebra, we obtain

$$\frac{1}{\alpha\beta}A(t) = \frac{1}{\beta}\mu(t) + A(\sigma(t)),$$

$$A(\sigma(t)) - \frac{1}{\alpha\beta}A(t) = -\frac{1}{\beta}\mu(t),$$

$$\frac{A(\sigma(t))}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)} - \frac{\frac{1}{\alpha\beta}A(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)} = \frac{-\frac{1}{\beta}\mu(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)}. \tag{3.18}$$

Then, by the recurrence relation (2.4) of the exponential function, we have

$$e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0) = \frac{1}{\alpha\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0).$$

When we plug this equality in equation (3.18), we obtain

$$\frac{A(\sigma(t))}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)} - \frac{A(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)} = \frac{-\alpha\mu(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)}.$$

We divide the entire equation by $\mu(t)$, and then we obtain the first order linear dynamic equation on time scales as

$$\frac{\frac{A(\sigma(t))}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)} - \frac{A(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)}}{\mu(t)} = \frac{-\alpha}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)},$$

which is equivalent to

$$\left(\frac{A(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)}\right)^{\Delta} = \frac{-\alpha}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)} = -\alpha e_{\ominus\left(\frac{\frac{1}{\alpha\beta}-1}{\mu}\right)}(t,0).$$

By the definition of "circle minus" in time scales, we have

$$e_{\ominus\left(\frac{\frac{1}{\alpha\beta}-1}{\mu}\right)}(t,0) = e_{\frac{\alpha\beta-1}{\mu}}(t,0).$$

This implies

$$\left(\frac{A(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)}\right)^{\Delta} = -\alpha e_{\frac{\alpha\beta-1}{\mu}}(t,0). \tag{3.19}$$

Equation (3.19) is a first order linear dynamic equation on isolated time scale $\mathbb{T}$. Then by integrating both sides of equation (3.19) on the domain

$$\mathbb{T}\cap[0,t) = [0,t)_{\mathbb{T}},$$

we have

$$\int\limits_{s\in[0,t)_{\mathbb{T}}} \left(\frac{A(s)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(s,0)}\right)^{\Delta} \Delta s = -\alpha \int\limits_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\Delta s.$$

We can rewrite the Cauchy integral in terms of the sum operator on isolated time scales, and then for $e_p(t,t)=1$, $\forall t\in\mathbb{T}$, we have

$$\frac{A(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)} - \frac{A(0)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(0,0)} = -\alpha \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)$$

$$A(t) = e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\,A(0) - \alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0) \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s).$$

Notice that for $\mathbb{T}=\mathbb{Z}$, we have $\mu(t)=1$. This implies

$$A(t) = e_{\frac{1}{\alpha\beta}-1}(t,0)\,A(0) - \alpha e_{\frac{1}{\alpha\beta}-1}(t,0) \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0).$$

Because the exponential function is equivalent to

$$e_{\frac{1}{\alpha\beta}-1}(t,0) = \left(1+\frac{1}{\alpha\beta}-1\right)^{t-0} = \left(\frac{1}{\alpha\beta}\right)^{t}$$

for $\mathbb{T}=\mathbb{Z}$, we have

$$A(t) = \left(\frac{1}{\alpha\beta}\right)^{t} A(0) - \alpha\left(\frac{1}{\alpha\beta}\right)^{t} \sum_{s=0}^{t-1} (\alpha\beta)^{s}$$

where $\alpha\beta$ is constant and $0<\alpha\beta<1$. Then the sum of this product in the last equation above is

$$A(t) = \frac{1}{(\alpha\beta)^{t}} A(0) - \alpha\frac{1}{(\alpha\beta)^{t}} \left(\frac{(\alpha\beta)^{s}}{\alpha\beta-1}\right)\Bigg]_{0}^{t}$$

$$
\begin{aligned}
&= \frac{1}{(\alpha\beta)^t} A(0) - \alpha \frac{1}{(\alpha\beta)^t} \left[\frac{(\alpha\beta)^t}{\alpha\beta - 1} - \frac{1}{\alpha\beta - 1} \right] \\
&= \frac{1}{(\alpha\beta)^t} A(0) - \frac{\alpha}{\alpha\beta - 1} + \frac{\alpha}{(\alpha\beta)^t (\alpha\beta - 1)}.
\end{aligned}
$$

J. Hassler [**10**] formulated $A(t)$ as constant in discrete time as

$$A = \frac{\alpha}{1 - \alpha\beta}.$$

As a result of this, we assume the initial value $A(0)$ as

$$A(0) = \frac{\alpha}{1 - \alpha\beta}.$$

Hence, we find $A(t)$ as

$$A(t) = \frac{\alpha}{1 - \alpha\beta} e_{\frac{\frac{1}{\alpha\beta} - 1}{\mu}}(t, 0) - \alpha e_{\frac{\frac{1}{\alpha\beta} - 1}{\mu}}(t, 0) \sum_{s \in [0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta - 1}{\mu}}(s, 0) \mu(s),$$

which is

$$A(t) = \alpha e_{\frac{\frac{1}{\alpha\beta} - 1}{\mu}}(t, 0) \left[\frac{1}{1 - \alpha\beta} - \sum_{s \in [0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta - 1}{\mu}}(s, 0) \mu(s) \right] \tag{3.20}$$

for any isolated time scale $\mathbb{T}$.

Next we solve $B(t)$ in the same manner we solved $A(t)$. For the coefficient $B(t)$, we have

$$B(t) = \mu(t) \ln\left(\frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))} \right) + \beta A(\sigma(t)) \ln\left(\frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))} \right) + \beta B(\sigma(t)).$$

By rewriting the equation, we have

$$
\begin{aligned}
\frac{1}{\beta} B(t) &= \frac{1}{\beta} \left[\mu(t) \ln\left(\frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))} \right) \right. \\
&\quad \left. + \beta A(\sigma(t)) \ln\left(\frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))} \right) \right] + B(\sigma(t))
\end{aligned}
$$

$$
\begin{aligned}
B(\sigma(t)) - \frac{1}{\beta} B(t) &= -\frac{\mu(t)}{\beta} \ln\left(\frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))} \right) \\
&\quad - A(\sigma(t)) \ln\left(\frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))} \right).
\end{aligned}
$$

By dividing both sides of the equation with integrating factor $e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)$, we have

$$\frac{B\left(\sigma\left(t\right)\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)}-\frac{\frac{1}{\beta}B\left(t\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)} = -\frac{\frac{\mu(t)}{\beta}\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)} -\frac{A\left(\sigma\left(t\right)\right)\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)}.$$

From recurrence relation of exponential function, i.e.,

$$e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)=\frac{1}{\beta}e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(t,0\right),$$

we have

$$\frac{B\left(\sigma\left(t\right)\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(\sigma\left(t\right),0\right)}-\frac{B\left(t\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(t,0\right)} = -\frac{\mu\left(t\right)\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(t,0\right)} -\frac{\beta A\left(\sigma\left(t\right)\right)\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(t,0\right)}.$$

To constitute the Δ-*derivative* of $B\left(t\right)$ on the LHS, we divide the entire equation by $\mu\left(t\right)$. Thus, we have

$$\frac{\frac{B(\sigma(t))}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)}-\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}}{\mu\left(t\right)} = -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}\left(t,0\right)\ln\left(\frac{\mu\left(t\right)}{\mu\left(t\right)+\beta A\left(\sigma\left(t\right)\right)}\right) -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}\left(t,0\right)\frac{\beta A\left(\sigma\left(t\right)\right)}{\mu\left(t\right)}\ln\left(\frac{\beta A\left(\sigma\left(t\right)\right)}{\mu\left(t\right)+\beta A\left(\sigma\left(t\right)\right)}\right),$$

which is equivalent to

$$\left(\frac{B\left(t\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}\left(t,0\right)}\right)^{\Delta} = -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}\left(t,0\right)\left[\ln\left(\frac{\mu\left(t\right)}{\mu\left(t\right)+\beta A\left(\sigma\left(t\right)\right)}\right) + \frac{\beta A\left(\sigma\left(t\right)\right)}{\mu\left(t\right)}\ln\left(\frac{\beta A\left(\sigma\left(t\right)\right)}{\mu\left(t\right)+\beta A\left(\sigma\left(t\right)\right)}\right)\right].$$

By the definition of "circle minus" in time scales, we have

$$e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}(t,0)=e_{\frac{\beta-1}{\mu}}(t,0).$$

This implies

$$\left(\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}\right)^{\Delta}=-e_{\frac{\beta-1}{\mu}}(t,0)\left[\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)\right.$$
$$\left.+\frac{\beta A(\sigma(t))}{\mu(t)}\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)\right]. \tag{3.21}$$

Equation (3.21) is a first order linear dynamic equation on isolated time scale $\mathbb{T}$, and by integrating both sides on the domain $[0,t)$, we have

$$\int_{s\in[0,t)_{\mathbb{T}}}\left(\frac{B(s)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(s,0)}\right)^{\Delta}\Delta s=-\int_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right.$$
$$\left.+\frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)\right]\Delta s$$

$$\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}-\frac{B(0)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(0,0)}=-\int_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right.$$
$$\left.+\frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)\right]\Delta s.$$

Hence, we find $B(t)$ as

$$B(t)=e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)\left\{B(0)-\int_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right.\right.$$
$$\left.\left.+\frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)\right]\Delta s\right\},$$

where

$$A(\sigma(t))=-\frac{1}{\beta}\mu(t)+\frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{(1-\alpha\beta)}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s)\right].$$

Notice that for $\mathbb{T}=\mathbb{Z}$, we have $\mu(t)=1$. Since $A(t)$ is constant for $\mathbb{T}=\mathbb{Z}$, we have $A(\sigma(t))=A(t)=\frac{\alpha}{1-\alpha\beta}$. This implies

$$B(t)=e_{\frac{1}{\beta}-1}(t,0)\left\{B(0)-\int_{s\in[0,t)_{\mathbb{T}}}e_{\beta-1}(s,0)\left[\ln\left(\frac{1}{1+\beta\frac{\alpha}{1-\alpha\beta}}\right)\right.\right.$$
$$\left.\left.+\beta\frac{\alpha}{1-\alpha\beta}\ln\left(\frac{\beta\frac{\alpha}{1-\alpha\beta}}{1+\beta\frac{\alpha}{1-\alpha\beta}}\right)\right]\Delta s\right\}.$$

Because the exponential function is equivalent to

$$e_{\frac{1}{\beta}-1}(t,0)=\left(1+\frac{1}{\beta}-1\right)^{t-0}=\left(\frac{1}{\beta}\right)^{t}$$

and

$$e_{\beta-1}(t,0)=(1+\beta-1)^{t-0}=\beta^{t}$$

for $\mathbb{T}=\mathbb{Z}$, we have from the definition of sum operator on time scales

$$\begin{aligned}
B(t) &= \left(\frac{1}{\beta}\right)^{t}\left\{B(0)-\int\limits_{s\in[0,t)_{\mathbb{T}}}\beta^{s}\left[\ln(1-\alpha\beta)+\beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta)\right]\Delta s\right\}\\
&= \left(\frac{1}{\beta}\right)^{t}B(0)-\left(\frac{1}{\beta}\right)^{t}\sum_{s=0}^{t-1}\beta^{s}\left[\ln(1-\alpha\beta)+\beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta)\right]\\
&= \left(\frac{1}{\beta}\right)^{t}B(0)-\left(\frac{1}{\beta}\right)^{t}\left[\ln(1-\alpha\beta)+\beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta)\right]\sum_{s=0}^{t-1}\beta^{s},
\end{aligned}$$

where β is constant and $0<\beta<1$. By calculating the sum in the last equation above, we have

$$\begin{aligned}
B(t) &= \left(\frac{1}{\beta}\right)^{t}B(0)-\left(\frac{1}{\beta}\right)^{t}\left[\ln(1-\alpha\beta)+\beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta)\right]\left(\frac{\beta^{s}}{\beta-1}\right)\Bigg]_{0}^{t}\\
&= \left(\frac{1}{\beta}\right)^{t}B(0)-\left(\frac{1}{\beta}\right)^{t}\left[\ln(1-\alpha\beta)+\beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta)\right]\left[\frac{\beta^{t}}{\beta-1}-\frac{1}{\beta-1}\right]\\
&= \left(\frac{1}{\beta}\right)^{t}B(0)-\left(\frac{1}{\beta}\right)^{t}\left(1-\beta^{t}\right)\left[\frac{\ln(1-\alpha\beta)}{1-\beta}+\frac{\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}\right].
\end{aligned}$$

J. Hassler [**10**] formulated $B(t)$ as constant in discrete time as

$$B = \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}.$$

As a result of this, we assume the initial value $B(0)$ to be

$$B(0) = 2\left[\frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}\right].$$

Hence, we find $B(t)$ is

$$\begin{aligned} B(t) = e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)\Bigg\{B(0) - \int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right. \\ \left. + \frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)\right]\Delta s\Bigg\}, \end{aligned} \tag{3.22}$$

where

$$A(\sigma(t)) = -\frac{1}{\beta}\mu(t) + \frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{(1-\alpha\beta)} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right],$$

and

$$B(0) = 2\left[\frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}\right].$$

Therefore, the value function $V(t)$ given with equation (3.14), is found as

$$\begin{aligned} V(t,k(t)) &= \alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right]\ln(k(t)) \\ &\quad + e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)\Bigg\{B(0) - \int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right. \\ &\quad \left. + \frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)\right]\Delta s\Bigg\}. \end{aligned}$$

■

Lemma 15 *If $0 < \mu(t) \le 1$, then $A(t)$ is a positive valued function of t over $[0,\infty)_{\mathbb{T}}$.*

Proof. In equation (3.20), let D be the difference given in the brackets. We know $0<\alpha<1$, and for $p=\frac{\frac{1}{\alpha\beta}-1}{\mu}\in\mathcal{R}^{+}$, the sign of the exponential function $e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)$ is positive (*see* [**7**], *Theorem 2.48(i)*). In order to show $A(t)$ is positive, we need to show

$$D=\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)$$

is positive. By rewriting the second term in the above difference, we have

$$\begin{aligned}
D &= \frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\\
&= \frac{1}{1-\alpha\beta}-\Big[\mu(0)+e_{\frac{\alpha\beta-1}{\mu}}(\sigma(s),0)\,\mu(\sigma(s))+e_{\frac{\alpha\beta-1}{\mu}}\left(\sigma^{2}(s),0\right)\mu\left(\sigma^{2}(s)\right)\\
&\qquad +\cdots+e_{\frac{\alpha\beta-1}{\mu}}\left(\sigma^{\rho(t)}(s),0\right)\mu\left(\sigma^{\rho(t)}(s)\right)\Big]\\
&= \frac{1}{1-\alpha\beta}-\Big[\mu(0)+(\alpha\beta)\,\mu(\sigma(s))+(\alpha\beta)^{2}\mu\left(\sigma^{2}(s)\right)\\
&\qquad +\cdots+(\alpha\beta)^{\rho(t)}\mu\left(\sigma^{\rho(t)}(s)\right)\Big].
\end{aligned}$$

Then by using the geometric series, we have

$$\begin{aligned}
D &= \frac{1}{1-\alpha\beta}-\Big[\mu(0)+(\alpha\beta)\,\mu(\sigma(s))+(\alpha\beta)^{2}\mu\left(\sigma^{2}(s)\right)\\
&\qquad +\cdots+(\alpha\beta)^{\rho(t)}\mu\left(\sigma^{\rho(t)}(s)\right)\Big]\\
&= \sum_{i=0}^{\infty}(\alpha\beta)^{i}-\left[\sum_{i=0}^{\rho(t)}(\alpha\beta)^{i}\mu\left(\sigma^{i}(0)\right)\right].
\end{aligned}$$

For $0<\mu(t)\leq 1$, we know

$$\sum_{i=0}^{\infty}(\alpha\beta)^{i}>\sum_{i=0}^{\rho(t)}(\alpha\beta)^{i}\geq\sum_{i=0}^{\rho(t)}(\alpha\beta)^{i}\mu\left(\sigma^{i}(0)\right).$$

This implies

$$D=\sum_{i=0}^{\infty}(\alpha\beta)^{i}-\left[\sum_{i=0}^{\rho(t)}(\alpha\beta)^{i}\mu\left(\sigma^{i}(0)\right)\right]>0.$$

Thus, for isolated time scale $\mathbb{T}$ with $0<\mu(t)\leq 1$, $A(t)$ is positive over $[0,\infty)_{\mathbb{T}}$. ■

From (3.20) we find $A(\sigma(t))$ as

$$-\frac{1}{\beta}\mu(t)+\frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{(1-\alpha\beta)}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s)\right].$$

Indeed, plugging in $\sigma(t)$ for each t in equation (3.20) gives us

$$\begin{aligned}A(\sigma(t)) &= \frac{\alpha}{1-\alpha\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)\\ &\quad -\alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(\sigma(t),0)\sum_{s\in[0,\sigma(t))_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s).\end{aligned}$$

By recurrence relation of the exponential function, we have

$$\begin{aligned}A(\sigma(t)) &= \frac{\alpha}{1-\alpha\beta}\left(\frac{1}{\alpha\beta}\right)e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\\ &\quad -\alpha\left(\frac{1}{\alpha\beta}\right)e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s)\\ &\quad -\alpha\left(\frac{1}{\alpha\beta}\right)e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\sum_{s\in[t,\sigma(t))_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s).\end{aligned}$$

This implies that

$$\begin{aligned}A(\sigma(t)) &= \frac{1}{\beta(1-\alpha\beta)}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\\ &\quad -\frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s)\\ &\quad -\frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\,e_{\frac{\alpha\beta-1}{\mu}}(t,0)\mu(t).\end{aligned}$$

Since

$$e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\,e_{\frac{\alpha\beta-1}{\mu}}(t,0)=e_{\left(\frac{\frac{1}{\alpha\beta}-1}{\mu}\right)\oplus\left(\frac{\alpha\beta-1}{\mu}\right)}(t,0)=e_0(t,0)=1,$$

we have

$$\begin{aligned}A(\sigma(t)) &= \frac{1}{\beta(1-\alpha\beta)}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)-\frac{1}{\beta}\mu(t)\\ &\quad -\frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s),\end{aligned}$$

and that is

$$A(\sigma(t)) = -\frac{1}{\beta}\mu(t) + \frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right].$$

By having the solution of the Bellman equation, we can find the *maximizing sequence* $\{k(\sigma(t))\}_{t=0}^{\infty}$ by obtaining the *state variable* $k(t)$. First, let $f(t)$ be equal to

$$f(t) = \frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))}.$$

Then, we have

$$k(\sigma(t)) = \frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))}k^{\alpha}(t)$$

$$k(\sigma(t)) = f(t)\,k^{\alpha}(t). \tag{3.23}$$

Now, by iterating equation (3.23) for the initial point $t = 0$, we have

$$k(\sigma(0)) = f(0)\,k^{\alpha}(0)$$

$$\begin{aligned} k\left(\sigma^2(t)\right) &= f(\sigma(0))\,k^{\alpha}(\sigma(0)) \\ &= f(\sigma(0))\,[f(0)\,k^{\alpha}(0)]^{\alpha} \\ &= f(\sigma(0))\,f^{\alpha}(0)\,k^{\alpha^2}(0) \end{aligned}$$

$$\begin{aligned} k\left(\sigma^3(0)\right) &= f\left(\sigma^2(0)\right)k^{\alpha}\left(\sigma^2(0)\right) \\ &= f\left(\sigma^2(0)\right)\left[f(\sigma(0))\,f^{\alpha}(0)\,k^{\alpha^2}(0)\right]^{\alpha} \\ &= f\left(\sigma^2(0)\right)f^{\alpha}(\sigma(0))\,f^{\alpha^2}(0)\,k^{\alpha^3}(0), \end{aligned}$$

$$\vdots$$

and for $t = \sigma^n(0) \in [0,\infty)_{\mathbb{T}}$,

$$k(t) = k(\sigma^n(0)) = \left(\prod_{j=0}^{n-1} f\left(\sigma^j(0)\right)^{\alpha^{(n-1)-j}}\right)k^{\alpha^n}(0). \tag{3.24}$$

Lastly, we find the *optimal control variable* $c(t)$, for the maximizing sequence $\{k(\sigma(t))\}_{t=0}^{\infty}$. By plugging $A(\sigma(t))$ in the equation of $c(t)$, we have

$$\begin{aligned} c(t) &= \frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))} k^{\alpha}(t) \\ &= \frac{\mu(t) k^{\alpha}(t)}{\mu(t)+\beta\left(-\frac{1}{\beta}\mu(t)+\frac{1}{\beta} e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{(1-\alpha\beta)}-\sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s)\right]\right)}, \end{aligned}$$

which is

$$c(t) = \frac{\mu(t)}{e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{(1-\alpha\beta)}-\sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\mu(s)\right]} k^{\alpha}(t) \tag{3.25}$$

for the *state variable* $k(t)$ in (3.24).

3.5 Existence of Solution of the Bellman Equation

We can solve the Bellman equation either by *guessing* or *iteration* of the value function. Since we formulate the utility maximization problem on infinite horizon, it is not practical to use *iteration* of the value function method. Therefore, in the previous section, we solved the utility maximization problem by *guessing* method. We guessed the value function, $V(t,k(t))$, has the form $A(t)\ln(k(t))+B(t)$, and we said it satisfies the Bellman equation

$$V(t,k(t)) = F(t,U)+\beta V(\sigma(t),k(\sigma(t))), \quad t\in\mathbb{T}$$

where $F(t,U)=\mu(t)U(k(t),c(t))$. Therefore, we need to show the existence and uniqueness of the solution of the Bellman equation. Before we show the existence and uniqueness of this solution, we shall give some definitions about metric spaces, norms and contraction mappings.

First, assume that $\mathcal{U}$ is the metric space of continuous, real valued, and bounded functions with *metric* ρ defined by

$$\rho(f, g) = \| f - g \| \qquad (f, g \in \mathcal{U}).$$

We associate with each $f \in \mathcal{U}$ its *supremum norm*

$$\| f \| = \sup_{f \in \mathcal{U}} |f(x)|.$$

Completeness of $\mathcal{U}$ is proved by W. Rudin ([**16**], *Thm. 7.15*). Now we can define a contraction mapping, T, on the complete metric space $\mathcal{U}$.

Next, we define the utility function U by

$$U : (0, \infty) \to \mathbb{R}.$$

Note that the utility function is assumed to be strictly increasing, strictly concave and continuously differentiable. For $t \in \mathbb{T}$, we define the state variable $k(t)$, and the control variable $c(t)$ by

$$k, c : \mathbb{T} \to \mathbb{R}.$$

We assume that $k(t)$ and $c(t)$ are bounded. Then, we define the value function by

$$V : \underset{(t, k(t))}{\mathbb{T} \times \mathbb{R}} \underset{\longmapsto}{\longrightarrow} \underset{V(t, k(t))}{\mathcal{U}}.$$

Now we need to show that the value function we obtained in the previous section is the unique solution of the Bellman equation. To achieve our goal, we use the Contraction Mapping Theorem, which deals with operators that map a given metric space into itself. Prior to that, we must show necessary assumptions of the contraction mapping theorem hold. First we define the mapping T as

$$TV = F(., U) + \beta V^{\sigma}. \tag{3.26}$$

Then, by the following lemma we prove that T is invariant.

Lemma 16 *T is a mapping on the complete metric space of bounded and continuous functions. Assume that for all V, $t \in \mathbb{T}$, we have $V \in \mathcal{U}$. Then we say $TV \in \mathcal{U}$, i.e., T is invariant.*

Proof. By the definition of utility function, we know U is a continuous function. Also, we know the control and state variables, respectively $c(t)$ and $k(t)$, are continuous and bounded on $\mathbb{T}$. The image of bounded functions under a continuous function is also bounded. Hence, we say $U(k(t), c(t))$ is bounded. Additionally, both terms on the RHS of equation (3.26) are bounded. Therefore we say TV is bounded. The continuity of vV follows from continuity of utility function. Thus, $TV \in \mathcal{U}$; i.e., the mapping T is invariant $(T(V) \subset \mathcal{U})$. ■

Next, we need to show that T is a contraction mapping. First we give the basics of a contraction.

Definition 17 *Suppose f is a real-valued function on $(-\infty, \infty)$. If $f(x) = x$, then we say x is the fixed point of f.*

Definition 18 *For a metric space $\mathcal{U}$, let $T : \mathcal{U} \longrightarrow \mathcal{U}$ be a mapping. T is said to be a contraction on $\mathcal{U}$ if for each $x, y \in \mathcal{U}$*

$$\rho(Tx, Ty) \leq \alpha\rho(x, y), \qquad 0 \leq \alpha < 1.$$

We refer to the books by W. Rudin [**16**] and R. R. Goldberg [**9**] for further reading on the subject. Now, we state the contraction mapping theorem.

Theorem 19 (Contraction Mapping Theorem) *A contraction mapping T, defined on a complete metric space, has a unique fixed point; that is, there is a unique element, $x \in \mathcal{U}$, such that $Tx = x$.*

Proof. First of all we need to show $T : \mathcal{U} \longrightarrow \mathcal{U}$ is a contraction. Therefore, choose any V_1, $V_2 \in \mathcal{U}$. Then, we have

$$\begin{aligned}
\rho\left(TV_1, TV_2\right) &= \parallel TV_1 - TV_2 \parallel \\
&= \sup_t \sup_k \left|TV_1\left(k\left(t\right)\right) - TV_2\left(k\left(t\right)\right)\right| \\
&= \sup_t \sup_k \left|\mu\left(t\right) U\left(k\left(t\right), c\left(t\right)\right) + \beta V_1\left(\sigma\left(t\right), k\left(\sigma\left(t\right)\right)\right)\right. \\
&\qquad \left. -\mu\left(t\right) U\left(k\left(t\right), c\left(t\right)\right) - \beta V_2\left(\sigma\left(t\right), k\left(\sigma\left(t\right)\right)\right)\right| \\
&= \beta \sup_{t,k} \left|V_1\left(\sigma\left(t\right), k\left(\sigma\left(t\right)\right)\right) - V_2\left(\sigma\left(t\right), k\left(\sigma\left(t\right)\right)\right)\right|.
\end{aligned}$$

Because we find the supremum of the difference in the absolute value for *all* t and k's, we can write

$$\begin{aligned}
\rho\left(TV_1, TV_2\right) &= \beta \sup_{t,k} \left|V_1\left(t, k\left(t\right)\right) - V_2\left(t, k\left(t\right)\right)\right| \\
&= \beta \parallel V_1 - V_2 \parallel,
\end{aligned}$$

where $0 < \beta < 1$. Hence, we say $T : \mathcal{U} \longrightarrow \mathcal{U}$ is a contraction on the complete metric space $\mathcal{U}$, and by the Contraction Mapping Theorem, V is the unique fixed point such that

$$V = TV = F\left(., U\right) + \beta V^{\sigma}.$$

■

CHAPTER 4

STOCHASTIC DYNAMIC PROGRAMMING

In this chapter, we formulate the *stochastic* sequence problem and the corresponding *stochastic* Bellman equation. Stochastic dynamic programming reflects the behavior of the decision maker without risk aversion; i.e., decision making under uncertainty. Uncertainty is a state of having limited knowledge where it is impossible to exactly describe the existing state or future outcome. There may be more than one possible outcome. Therefore, for the measurement of uncertainty, we have a set of possible states or outcomes where probabilities are assigned to each possible state or outcome. Moreover, risk is a state of uncertainty where some possible outcomes have an undesired effect or significant loss. Deterministic problems are formulated with known parameters, yet real-world problems almost invariably include some unknown parameters. As a conclusion, we say the stochastic case is more realistic, and it gives more accurate results. For our purposes, we assume $\mathbb{T}$ is an isolated time scale with $\sup \mathbb{T} = \infty$. In Section 4.1, we formulate the infinite stochastic dynamic sequence problem on $\mathbb{T}$. Then we define the corresponding stochastic Bellman equation in Section 4.2. In Section 4.3, we solve the social planner's problem, but this time it is given with a budget constraint that includes a random variable. Thus, the random variable also appears in the solution. Since it is not possible to find a solution explicitly that involves a random variable, we examine its distribution in Section 4.4.

4.1 Stochastic Dynamic Sequence Problem

Let $\sup \mathbb{T} = \infty$ and $\mathbb{T} \cap [0, \infty) = [0, \infty)_{\mathbb{T}}$. Then, we define the stochastic sequence problem on isolated time scale $\mathbb{T}$ as

$$(SP) \quad \sup_{\{x(t),z(t),c(t)\}_{t=0}^{\infty}} \mathbb{E} \int_{s \in [0,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}} (s, 0) F (x (s), c (s)) \Delta s \tag{4.1}$$

$$s.t. \; x^{\Delta} (t) \in \Gamma (t, x (t), z (t), c (t))$$

$$x (0) \in X \; given,$$

where $\{z (t)\}_{t=0}^{\infty}$ is a sequence of independently and identically distributed random state variables. Deviations of the random variable $z (t)$ are uncorrelated in different time periods; i.e., $Cov (z (t_i), z (t_j)) = 0$, for all $i \neq j$. $\mathbb{E}$ denotes the mathematical expectation of the objective function F. As we defined in the deterministic model, $x (t)$ is the state variable, and $c(t)$ is the control variable. Our goal is to find the optimal sequence $\{x (t), z (t), c(t)\}_{t=0}^{\infty}$ that maximizes the expected utility in the sequence problem.

In the stochastic model, we define the budget constraint on any isolated time scale $\mathbb{T}$, as

$$x^{\Delta} (t) \in \Gamma (t, x (t), z (t), c (t)),$$

which is also called the first order dynamic inclusion. That means future consumption is uncertain. As we explained in the deterministic model, $x^{\Delta} (t)$ can be chosen as any of the feasible plans that belong to $\Gamma (t, x (t), z (t), c (t))$.

4.2 Stochastic Bellman Equation

In section 3.2, we defined the deterministic Bellman equation as

$$\left(e_{\frac{\beta-1}{\mu}}(t,0)\,V(t,x(t))\right)^{\Delta}+e_{\frac{\beta-1}{\mu}}(t,0)\,F(x(t),c(t))=0,$$

where $V(t,x(t))$ is the solution of the Bellman equation and showed that it is equivalent to the equation

$$V(t,x(t))=\mu(t)\,F(x(t),c(t))+\beta V(\sigma(t),x(\sigma(t))),$$

which is given in (3.3). In the stochastic model, the solution of the Bellman equation involves the random variable $z(t)$, i.e., $V(t,x(t),z(t))$. Then we define the stochastic Bellman equation as

$$V(t,x(t),z(t))=\mu(t)\,F(x(t),c(t))+\beta\mathbb{E}\left[V(\sigma(t),x(\sigma(t)),z(\sigma(t)))\right] \quad (4.2)$$

$$x^{\Delta}(t)\in\Gamma(t,x(t),z(t),c(t)).$$

4.3 Stochastic Optimal Growth Model

In this section, we solve *the stochastic social planner's problem* by guessing method. For the same reasons we stated in chapter 2, we formulate the stochastic model on any isolated time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$. As we did in the deterministic case, we assume the objective function F is the utility function U. In this case, the optimal control variable $c(t)$ and state variable $x(t)$ are all random variables because both of them are now functions of $z(t)$, which is a random variable. Therefore, after we obtain the optimal sequence $\{x(t),z(t),c(t)\}_{t=0}^{\infty}$, we describe its distributions.

4.3.1 Solving the Stochastic Bellman Equation by Guessing Method

Assume $\mathbb{T}$ is an isolated time scale with $\sup \mathbb{T} = \infty$, for *all* $t \in \mathbb{T}$. Then the *stochastic* social planner's problem has the form

$$\sup_{\{k(\sigma(t))\}_{t=0}^{\infty}} \mathbb{E} \int_{s \in [0,\infty)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}} (s, 0) \, U (k (s), c (s)) \, \Delta s \tag{4.3}$$

$$s.t. \; k^{\Delta} (t) \in \Gamma (t, k (t), z (t), c (t))$$

$$k (0) \in X \; given,$$

where $k(t)$ is the state variable, $z(t)$ is the *random* state variable, and $c(t)$ is the control variable. We assume the random variables are *lognormally* distributed such that

$$\ln z (t) \sim N \left(\mu^*, \sigma^2\right).$$

In other words, $\ln z(t)$ is *normally* distributed with mean μ^* and variance σ^2. Therefore, $\mathbb{E} [\ln z (t_i)] = \mu^*$ for *every* $t_i \in [0, \infty)_{\mathbb{T}}$. Additionally, because $k(t)$ and $c(t)$ are functions of the random variable $z(t)$, they also have the *log normal distribution.* We define the corresponding *stochastic* Bellman equation as

$$V (t, k (t), z (t)) = \mu (t) U (k (t), c (t)) + \beta \mathbb{E} [V (\sigma (t), k (\sigma (t)), z (\sigma (t)))] \tag{4.4}$$

$$k^{\Delta} (t) \in \Gamma (t, k (t), z (t), c (t)).$$

As in the deterministic case, we use CRRA utility function, i.e., $U (k (t), c (t)) = \ln (c (t))$. Our goal is to find the sequence $\{k (\sigma (t))\}_{t=0}^{\infty}$ and the distribution of the state variable $k(t)$ that will maximize (SP). The budget constraint for the stochastic case is given as

$$k (\sigma (t)) = z (t) k^{\alpha} (t) - c (t),$$

where $0 < \alpha < 1$. From the definition of $\Delta - derivative$ on isolated time scales, we obtain

$$\begin{aligned} k(\sigma(t)) &= z(t)k^{\alpha}(t) - c(t) \\ \frac{k(\sigma(t)) - k(t)}{\mu(t)} &= \frac{z(t)k^{\alpha}(t) - k(t) - c(t)}{\mu(t)} \\ k^{\Delta}(t) \in \Gamma^{*}\left(\frac{z(t)k^{\alpha}(t) - k(t) - c(t)}{\mu(t)}\right) &\equiv \Gamma(t, k(t), z(t), c(t)), \end{aligned}$$

which shows that the budget constraint is appropriately chosen.

By substituting the utility function $U(k(t), c(t)) = \ln(c(t))$ and the budget constraint $k(\sigma(t)) = z(t)k^{\alpha}(t) - c(t)$ in the stochastic Bellman equation given in (4.4), we have

$$\begin{aligned} V(t, k(t), z(t)) &= \mu(t)\ln(z(t)k^{\alpha}(t) - k(\sigma(t))) \\ &\quad + \beta\mathbb{E}[V(\sigma(t), k(\sigma(t)), z(\sigma(t)))]. \end{aligned} \tag{4.5}$$

By guessing, we say the solution of the stochastic Bellman equation, $V(t, k(t), z(t))$, has the form

$$V(t, k(t), z(t)) = A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t). \tag{4.6}$$

Theorem 20 *The value function* $V(t, k(t))$ *is equal to*

$$\begin{aligned} V(t, k(t), z(t)) &= \alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t, 0)\left[\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s, 0)\,\mu(s)\right]\ln(k(t)) \\ &\quad + e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t, 0)\left[\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s, 0)\,\mu(s)\right]\ln(z(t)) \\ &\quad + e_{\frac{\frac{1}{\beta}-1}{\mu}}(t, 0)\left\{B(0) - \int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s, 0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right.\right. \\ &\quad \left.\left. + \frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right) + \frac{\beta\mu^{*}R(\sigma(s))}{\mu(s)}\right]\Delta s\right\} \end{aligned} \tag{4.7}$$

for the sequence $\{k(\sigma(t))\}_{t=0}^{\infty}$, *which maximizes* (SP).

Proof. Because this $V(t, k(t), z(t))$, given with (4.6), is the solution of the stochastic Bellman equation (4.4), it satisfies equation (4.5). Therefore, we have

$$\begin{aligned} A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t) &= \mu(t)\ln(z(t)k^{\alpha}(t) - k(\sigma(t))) \\ &\quad +\beta\mathbb{E}\{A(\sigma(t))\ln(k(\sigma(t))) \\ &\quad +R(\sigma(t))\ln(z(\sigma(t))) + B(\sigma(t))\}. \end{aligned}$$

Because $\mathbb{E}$ is linear we have

$$\begin{aligned} A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t) &= \mu(t)\ln(z(t)k^{\alpha}(t) - k(\sigma(t))) \\ &\quad +\beta\mathbb{E}[A(\sigma(t))\ln(k(\sigma(t)))] \\ &\quad +\beta\mathbb{E}[R(\sigma(t))\ln(z(\sigma(t)))] \\ &\quad +\beta\mathbb{E}[B(\sigma(t))], \end{aligned} \tag{4.8}$$

and since all other terms except $\ln(z(\sigma(t)))$ are nonrandom in the RHS of (4.8), we have

$$\begin{aligned} A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t) &= \mu(t)\ln(z(t)k^{\alpha}(t) - k(\sigma(t))) \\ &\quad +\beta A(\sigma(t))\ln(k(\sigma(t))) \\ &\quad +\beta R(\sigma(t))\,\mathbb{E}[\ln(z(\sigma(t)))] \\ &\quad +\beta B(\sigma(t)). \end{aligned} \tag{4.9}$$

Since $\mathbb{E}[\ln z(t_i)] = \mu^*$ for *every* $t_i \in [0,\infty)_{\mathbb{T}}$, we rewrite (4.9) as

$$\begin{aligned} A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t) &= \mu(t)\ln(z(t)k^{\alpha}(t) - k(\sigma(t))) \\ &\quad +\beta A(\sigma(t))\ln(k(\sigma(t))) \\ &\quad +\beta\mu^* R(\sigma(t)) + \beta B(\sigma(t)). \end{aligned} \tag{4.10}$$

Now to find the first order conditions we differentiate both sides of equation (4.10) with respect to $k(\sigma(t))$. Hence, we have

$$\begin{aligned} & \frac{\partial}{\partial k(\sigma(t))}\{A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t)\} \\ = & \frac{\partial}{\partial k(\sigma(t))}\{\mu(t)\ln(z(t)k^{\alpha}(t) - k(\sigma(t))) + \beta A(\sigma(t))\ln(k(\sigma(t))) \\ & \qquad + \beta\mu^{*}R(\sigma(t)) + \beta B(\sigma(t))\} \end{aligned}$$

$$0 = \mu(t)\frac{-1}{z(t)k^{\alpha}(t) - k(\sigma(t))} + \beta A(\sigma(t))\frac{1}{k(\sigma(t))}.$$

We substitute the budget constraint back, and then we find the *control variable* $c(t)$ as

$$\begin{aligned} \frac{\mu(t)}{c(t)} &= \frac{\beta A(\sigma(t))}{z(t)k^{\alpha}(t) - c(t)} \\ \mu(t)z(t)k^{\alpha}(t) &= [\mu(t) + \beta A(\sigma(t))]c(t) \\ c(t) &= \frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))}z(t)k^{\alpha}(t). \end{aligned}$$

By plugging $c(t)$ in the budget constraint, we obtain the *state variable* $k(t)$ as

$$\begin{aligned} k(\sigma(t)) &= z(t)k^{\alpha}(t) - c(t) \\ &= z(t)k^{\alpha}(t) - \frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))}z(t)k^{\alpha}(t) \\ &= \frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))}z(t)k^{\alpha}(t). \end{aligned}$$

When we plug these values in equation (4.10), we have

$$\begin{aligned} A(t)\ln(k(t)) + R(t)\ln(z(t)) + B(t) &= \mu(t)\ln\left(\frac{\mu(t)}{\mu(t) + \beta A(\sigma(t))}z(t)k^{\alpha}(t)\right) \\ &\quad + \beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))}z(t)k^{\alpha}(t)\right) \\ &\quad + \beta\mu^{*}R(\sigma(t)) + \beta B(\sigma(t)) \end{aligned}$$

$$
\begin{aligned}
A(t)\ln(k(t))+R(t)\ln(z(t))+B(t) &= \alpha\mu(t)\ln(k(t))+\mu(t)\ln(z(t)) \\
&\quad +\mu(t)\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right) \\
&\quad +\alpha\beta A(\sigma(t))\ln(k(t))+\beta A(\sigma(t))\ln(z(t)) \\
&\quad +\beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right) \\
&\quad +\beta\mu^{*}R(\sigma(t))+\beta B(\sigma(t)). \qquad (4.11)
\end{aligned}
$$

Next we have to find the coefficients, $A(t)$, $R(t)$ and $B(t)$, in the value function. From equivalence of RHS and LHS of equation (4.11), we find $A(t)$, $R(t)$ and $B(t)$ as follows. For the coefficient $A(t)$, we have

$$A(t)=\alpha\mu(t)+\alpha\beta A(\sigma(t)).$$

In chapter 3, we solved this equation for $A(t)$, and now similarly find $A(t)$ as in the equation (3.20), which is

$$A(t)=\alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right] \qquad (4.12)$$

for any isolated time scale $\mathbb{T}$. Next we find $R(t)$ from

$$R(t)=\mu(t)+\beta A(\sigma(t)).$$

Since

$$R(t)=\mu(t)+\beta A(\sigma(t))=\frac{A(t)}{\alpha},$$

we find $R(t)$ as

$$R(t)=e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right] \qquad (4.13)$$

by dividing both sides of equation (4.12) by α. Lastly, we find $B(t)$ by solving

$$
\begin{aligned}
B(t) &= \mu(t)\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)+\beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right) \\
&\quad +\beta B(\sigma(t))+\beta\mu^{*}R(\sigma(t)),
\end{aligned}
$$

from equation (4.11). By rewriting the equation, we have

$$\begin{aligned} \frac{1}{\beta}B(t) &= \frac{1}{\beta}\left[\mu(t)\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)\right. \\ &\quad \left.+\beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)+\beta\mu^{*}R(\sigma(t))\right]+B(\sigma(t)), \end{aligned}$$

$$\begin{aligned} B(\sigma(t))-\frac{1}{\beta}B(t) &= -\frac{\mu(t)}{\beta}\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right) \\ &\quad -A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)-\mu^{*}R(\sigma(t)). \end{aligned}$$

By dividing both sides of the equation with integrating factor $e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)$, we have

$$\begin{aligned} \frac{B(\sigma(t))}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)}-\frac{\frac{1}{\beta}B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)} &= -\frac{\frac{\mu(t)}{\beta}\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)} \\ &\quad -\frac{A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)}-\frac{\mu^{*}R(\sigma(t))}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)}. \end{aligned}$$

From recurrence relation of exponential function, i.e.,

$$e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)=\frac{1}{\beta}e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0),$$

we have

$$\begin{aligned} \frac{B(\sigma(t))}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)}-\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)} &= -\frac{\mu(t)\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)} \\ &\quad -\frac{\beta A(\sigma(t))\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}-\frac{\beta\mu^{*}R(\sigma(t))}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}. \end{aligned}$$

To constitute the Δ-*derivative* of $B(t)$ on the LHS, we divide the whole equation by $\mu(t)$. Thus, we have

$$\begin{aligned}\frac{\frac{B(\sigma(t))}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(\sigma(t),0)}-\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}}{\mu(t)} &= -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}(t,0)\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)\\ &\quad -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}(t,0)\frac{\beta A(\sigma(t))}{\mu(t)}\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)\\ &\quad -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}(t,0)\frac{\beta\mu^{*}R(\sigma(t))}{\mu(t)},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\left(\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}\right)^{\Delta} &= -e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}(t,0)\left[\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)\right.\\ &\quad\left.+\frac{\beta A(\sigma(t))}{\mu(t)}\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)+\frac{\beta\mu^{*}R(\sigma(t))}{\mu(t)}\right].\end{aligned}$$

By the definition of "circle minus" on time scales, we have

$$e_{\ominus\left(\frac{\frac{1}{\beta}-1}{\mu}\right)}(t,0)=e_{\frac{\beta-1}{\mu}}(t,0).$$

This implies

$$\begin{aligned}\left(\frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)}\right)^{\Delta} &= -e_{\frac{\beta-1}{\mu}}(t,0)\left[\ln\left(\frac{\mu(t)}{\mu(t)+\beta A(\sigma(t))}\right)\right.\\ &\quad\left.+\frac{\beta A(\sigma(t))}{\mu(t)}\ln\left(\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))}\right)+\frac{\beta\mu^{*}R(\sigma(t))}{\mu(t)}\right].\end{aligned}\tag{4.14}$$

Equation (4.14) is a first order linear dynamic equation on isolated time scale $\mathbb{T}$, and by integrating both sides on the domain $[0,t)_{\mathbb{T}}$, we have

$$\int_{s\in[0,t)_{\mathbb{T}}} \left(\frac{B(s)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(s,0)} \right)^{\Delta} \Delta s = -\int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0) \left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))} \right) \right.$$

$$\left. + \frac{\beta A(\sigma(s))}{\mu(s)} \ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))} \right) + \frac{\beta\mu^* R(\sigma(s))}{\mu(s)} \right] \Delta s,$$

$$\begin{aligned} \frac{B(t)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)} - \frac{B(0)}{e_{\frac{\frac{1}{\beta}-1}{\mu}}(0,0)} &= -\int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0) \left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))} \right) \right. \\ &\quad \left. + \frac{\beta A(\sigma(s))}{\mu(s)} \ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))} \right) + \frac{\beta\mu^* R(\sigma(s))}{\mu(s)} \right] \Delta s. \end{aligned}$$

Hence, we find $B(t)$ as

$$\begin{aligned} B(t) &= e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0) \left\{ B(0) - \int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0) \left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))} \right) \right. \right. \\ &\quad \left. \left. + \frac{\beta A(\sigma(s))}{\mu(s)} \ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))} \right) + \frac{\beta\mu^* R(\sigma(s))}{\mu(s)} \right] \Delta s \right\}, \end{aligned} \tag{4.15}$$

where

$$A(\sigma(t)) = -\frac{1}{\beta}\mu(t) + \frac{1}{\beta} e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0) \left[\frac{1}{(1-\alpha\beta)} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s) \right],$$

and

$$\begin{aligned} R(\sigma(t)) &= \frac{A(\sigma(t))}{\alpha} \\ &= -\frac{1}{\alpha\beta}\mu(t) + \frac{1}{\alpha\beta} e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0) \left[\frac{1}{(1-\alpha\beta)} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s) \right]. \end{aligned}$$

Therefore, the value function $V(t)$, given with equation (4.6), is found as

$$
\begin{aligned}
V(t,k(t),z(t)) &= \alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right]\ln(k(t)) \\
&\quad + e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}(t,0)\left[\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)\right]\ln(z(t)) \\
&\quad + e_{\frac{\frac{1}{\beta}-1}{\mu}}(t,0)\left\{B(0)-\int_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(s,0)\left[\ln\left(\frac{\mu(s)}{\mu(s)+\beta A(\sigma(s))}\right)\right.\right. \\
&\quad \left.\left. + \frac{\beta A(\sigma(s))}{\mu(s)}\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)+\frac{\beta\mu^{*}R(\sigma(s))}{\mu(s)}\right]\Delta s\right\}
\end{aligned}
$$

for the sequence $\{k(\sigma(t))\}_{t=0}^{\infty}$, which maximizes (SP). ■

4.3.2 Distributions of the Optimal Solution

We are searching for the optimal state variable $k(t)$ which is a function of the random state variables $z(t)$. Because we cannot obtain these optimal solutions explicitly, we shall characterize their distributions. We assume $\ln z(t) \sim N(\mu^{*},\sigma^{2})$, and deviations of the random variable $z(t)$ are uncorrelated in different time periods; i.e., $Cov(z(t_i),z(t_j))=0$, for all $i\neq j$. Now we need to find the mean and variance of $\ln(k(t))$.

First, let

$$\mathcal{M}_t=\{t_i\colon t_i\in[0,t] \text{ and } t_i \text{ left-scattered}\} \text{ and } N(\mathcal{M})=m.$$

Then, by setting

$$f(t)=\frac{\beta A(\sigma(t))}{\mu(t)+\beta A(\sigma(t))},$$

we rewrite

$$k(\sigma(t)) = \frac{\beta A(\sigma(t))}{\mu(t) + \beta A(\sigma(t))} z(t) k^{\alpha}(t)$$

as

$$k(\sigma(t)) = f(t) z(t) k^{\alpha}(t). \tag{4.16}$$

Now, by applying natural logarithm to both sides of equation (4.16), we have

$$\begin{aligned} \ln(k(\sigma(t))) &= \ln(f(t) z(t) k^{\alpha}(t)) \\ \ln(k(\sigma(t))) &= \ln(f(t) z(t)) + \alpha \ln(k(t)). \end{aligned}$$

Let $\ln(k(t)) = u(t)$. Then we have

$$u(\sigma(t)) = \ln(f(t) z(t)) + \alpha u(t).$$

By doing elementary algebra, we obtain

$$u(\sigma(t)) - \alpha u(t) = \ln(f(t) z(t)),$$

and by dividing both sides of the equation with integrating factor $e_{\frac{\alpha-1}{\mu}}(\sigma(t), 0)$, the last equation above becomes

$$\frac{u(\sigma(t))}{e_{\frac{\alpha-1}{\mu}}(\sigma(t), 0)} - \frac{\alpha u(t)}{e_{\frac{\alpha-1}{\mu}}(\sigma(t), 0)} = \frac{\ln(f(t) z(t))}{e_{\frac{\alpha-1}{\mu}}(\sigma(t), 0)}.$$

Then, by the recurrence relation of exponential function, i.e.,

$$e_{\frac{\alpha-1}{\mu}}(\sigma(t), 0) = \alpha e_{\frac{\alpha-1}{\mu}}(t, 0),$$

we have

$$\frac{u(\sigma(t))}{e_{\frac{\alpha-1}{\mu}}(\sigma(t), 0)} - \frac{u(t)}{e_{\frac{\alpha-1}{\mu}}(t, 0)} = \frac{\ln(f(t) z(t))}{\alpha e_{\frac{\alpha-1}{\mu}}(t, 0)}.$$

We divide the whole equation by $\mu(t)$, and then we obtain the first order linear dynamic equation on time scales as

$$\frac{\frac{u(\sigma(t))}{e_{\frac{\alpha-1}{\mu}}(\sigma(t),0)} - \frac{u(t)}{e_{\frac{\alpha-1}{\mu}}(t,0)}}{\mu(t)} = \frac{\ln(f(t) z(t))}{\alpha \mu(t) e_{\frac{\alpha-1}{\mu}}(t, 0)},$$

which is equivalent to

$$\left(\frac{u(t)}{e_{\frac{\alpha-1}{\mu}}(t,0)}\right)^{\Delta}=\frac{\ln(f(t)z(t))}{\alpha\mu(t)e_{\frac{\alpha-1}{\mu}}(t,0)}. \tag{4.17}$$

Equation (4.17) is a first order linear dynamic equation on isolated time scale $\mathbb{T}$. Then by integrating both sides on the domain $\mathbb{T}\cap[0,t)=[0,t)_{\mathbb{T}}$, we have

$$\int\limits_{s\in[0,t)_{\mathbb{T}}}\left(\frac{u(s)}{e_{\frac{\alpha-1}{\mu}}(s,0)}\right)^{\Delta}\Delta s=\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln(f(s)z(s))}{\alpha\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s$$

$$\int\limits_{s\in[0,t)_{\mathbb{T}}}\left(\frac{u(s)}{e_{\frac{\alpha-1}{\mu}}(s,0)}\right)^{\Delta}\Delta s=\int\limits_{s\in[0,t)_{\mathbb{T}}}\left[\frac{\ln(f(s))}{\alpha\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}+\frac{\ln(z(s))}{\alpha\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\right]\Delta s$$

$$\frac{u(t)}{e_{\frac{\alpha-1}{\mu}}(t,0)}-\frac{u(0)}{e_{\frac{\alpha-1}{\mu}}(0,0)}=\int\limits_{s\in[0,t)_{\mathbb{T}}}\left[\frac{\ln\left(\frac{\beta A(\sigma(s))}{\mu(s)+\beta A(\sigma(s))}\right)}{\alpha\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}+\frac{\ln(z(s))}{\alpha\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\right]\Delta s.$$

Since $A(t)=\alpha\mu(t)+\alpha\beta A(\sigma(t))$, that is, $\frac{A(t)}{\alpha}=\mu(t)+\beta A(\sigma(t))$, we have

$$u(t)=e_{\frac{\alpha-1}{\mu}}(t,0)\left\{u(0)+\frac{1}{\alpha}\int\limits_{s\in[0,t)_{\mathbb{T}}}\left[\frac{\ln\left(\frac{\alpha\beta A(\sigma(s))}{A(s)}\right)}{\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}+\frac{\ln(z(s))}{\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\right]\Delta s\right\}.$$

Hence, we find $u(t)$ as

$$\begin{aligned}u(t) &= e_{\frac{\alpha-1}{\mu}}(t,0)u(0)+\frac{\ln(\alpha\beta)}{\alpha}e_{\frac{\alpha-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s\\ &\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s\\ &\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln(z(s))}{\mu(s)e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s.\end{aligned} \tag{4.18}$$

We can integrate the first integral in equation (4.18). To do so, we use the definition of exponential function on time scales. We know

$$e_p^{\Delta}(t,0)=p(t)e_p(t,0).$$

This implies

$$e^{\Delta}_{\ominus\left(\frac{\alpha-1}{\mu}\right)}(t,0)=\ominus\left(\frac{\alpha-1}{\mu(t)}\right)e_{\ominus\left(\frac{\alpha-1}{\mu}\right)}(t,0)\,. \tag{4.19}$$

Since $\ominus\left(\frac{\alpha-1}{\mu(t)}\right)=\frac{1-\alpha}{\alpha\mu(t)}$ from the definition of "circle minus" and $e_{\ominus\left(\frac{\alpha-1}{\mu}\right)}(t,0)=\frac{1}{e_{\frac{\alpha-1}{\mu}}(t,0)}$ from the properties of the exponential function, we rewrite the equation (4.19) as

$$e^{\Delta}_{\frac{1-\alpha}{\alpha\mu}}(t,0)=\left(\frac{1-\alpha}{\alpha\mu(t)}\right)\frac{1}{e_{\frac{\alpha-1}{\mu}}(t,0)}$$

$$\left(\frac{\alpha}{1-\alpha}\right)e^{\Delta}_{\frac{1-\alpha}{\alpha\mu}}(t,0)=\frac{1}{\mu(t)\,e_{\frac{\alpha-1}{\mu}}(t,0)}. \tag{4.20}$$

By integrating both sides of equation (4.20) on the domain $\mathbb{T}\cap[0,t)=[0,t)_{\mathbb{T}}$, we have

$$\frac{\alpha}{1-\alpha}\int\limits_{s\in[0,t)_{\mathbb{T}}}e^{\Delta}_{\frac{1-\alpha}{\alpha\mu}}(s,0)\,\Delta s=\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu(s)\,e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s$$

$$\frac{\alpha}{1-\alpha}\left(e_{\frac{1-\alpha}{\alpha\mu}}(t,0)-e_{\frac{1-\alpha}{\alpha\mu}}(0,0)\right)=\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu(s)\,e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s,$$

which is equivalent to

$$\frac{\alpha}{1-\alpha}e_{\frac{1-\alpha}{\alpha\mu}}(t,0)-\frac{\alpha}{1-\alpha}=\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu(s)\,e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s. \tag{4.21}$$

Then we interchange equation (4.21) with the first integral in (4.18), and we obtain

$$\begin{aligned}u(t)&=e_{\frac{\alpha-1}{\mu}}(t,0)\,u(0)+\frac{\ln(\alpha\beta)}{\alpha}e_{\frac{\alpha-1}{\mu}}(t,0)\left[\frac{\alpha}{1-\alpha}e_{\frac{1-\alpha}{\alpha\mu}}(t,0)-\frac{\alpha}{1-\alpha}\right]\\&\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu(s)\,e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s\\&\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln(z(s))}{\mu(s)\,e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s\end{aligned}$$

$$
\begin{aligned}
= \; & e_{\frac{\alpha-1}{\mu}}(t,0)\,u(0) + \frac{\ln(\alpha\beta)}{1-\alpha} e_{\frac{\alpha-1}{\mu}}(t,0)\, e_{\frac{1-\alpha}{\alpha\mu}}(t,0) - \frac{\ln(\alpha\beta)}{1-\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \\
& + \frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)} \Delta s \\
& + \frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln(z(s))}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)} \Delta s.
\end{aligned}
$$

Since

$$
e_{\frac{\alpha-1}{\mu}}(t,0)\, e_{\frac{1-\alpha}{\alpha\mu}}(t,0) = e_{\left(\frac{\alpha-1}{\mu}\right)\oplus\left(\frac{1-\alpha}{\alpha\mu}\right)}(t,0) = e_0(t,0) = 1,
$$

we have

$$
\begin{aligned}
u(t) \; = \; & e_{\frac{\alpha-1}{\mu}}(t,0)\,u(0) + \frac{\ln(\alpha\beta)}{1-\alpha} - \frac{\ln(\alpha\beta)}{1-\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \\
& + \frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)} \Delta s \\
& + \frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln(z(s))}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)} \Delta s.
\end{aligned}
$$

Thus, we find $\ln(k(t))$ as

$$
\begin{aligned}
\ln(k(t)) \; = \; & e_{\frac{\alpha-1}{\mu}}(t,0) \left[\ln(k(0)) - \frac{\ln(\alpha\beta)}{1-\alpha}\right] + \frac{\ln(\alpha\beta)}{1-\alpha} \\
& + \frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)} \Delta s \\
& + \frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln(z(s))}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)} \Delta s, \qquad (4.22)
\end{aligned}
$$

for $\ln(k(t)) = u(t)$.

We stated that the state variable $k(t)$ has the *log normal distribution.* Therefore, we need to find the expected value (or mean) and the variance of $k(t)$.

Expected value of the state variable k(t):

We take the expected value of both sides of equation (4.22), i.e.,

$$\begin{aligned}\mathbb{E}\left[\ln\left(k\left(t\right)\right)\right] &= \mathbb{E}\left[e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\ln\left(k\left(0\right)\right)-\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\right]+\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\right.\\ &\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\\ &\quad\left.+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(z\left(s\right)\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right].\end{aligned}$$

Then, by using the properties of expectation and the fact that $\mathbb{E}\left[\ln z\left(t_i\right)\right]=\mu^*$ for *every* $t_i\in[0,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\mathbb{E}\left[\ln\left(k\left(t\right)\right)\right] &= e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\ln\left(k\left(0\right)\right)-\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\right]+\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\\ &\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\\ &\quad+\frac{\mu^*}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s.\end{aligned}\tag{4.23}$$

By plugging equation (4.21) in (4.23), we obtain

$$\begin{aligned}\mathbb{E}\left[\ln\left(k\left(t\right)\right)\right] &= e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\ln\left(k\left(0\right)\right)-\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\right]+\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\\ &\quad+\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\\ &\quad+\frac{\mu^*}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\frac{\alpha}{1-\alpha}e_{\frac{1-\alpha}{\alpha\mu}}\left(t,0\right)-\frac{\alpha}{1-\alpha}\right].\end{aligned}$$

Since

$$e_{\frac{\alpha-1}{\mu}}\left(t,0\right)e_{\frac{1-\alpha}{\alpha\mu}}\left(t,0\right)=e_{\left(\frac{\alpha-1}{\mu}\right)\oplus\left(\frac{1-\alpha}{\alpha\mu}\right)}\left(t,0\right)=e_0\left(t,0\right)=1,$$

we find $\mathbb{E}\left[\ln\left(k\left(t\right)\right)\right]$ as

$$\begin{aligned}
\mathbb{E}\left[\ln\left(k\left(t\right)\right)\right] &= e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\ln\left(k\left(0\right)\right)-\frac{\ln\left(\alpha\beta\right)}{1-\alpha}\right]+\frac{\ln\left(\alpha\beta\right)}{1-\alpha} \\
&\quad +\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s \\
&\quad +\frac{\mu^{*}}{1-\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)e_{\frac{1-\alpha}{\alpha\mu}}\left(t,0\right)-\frac{\mu^{*}}{1-\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right) \\
&= e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\ln\left(k\left(0\right)\right)-\frac{\ln\left(\alpha\beta\right)}{1-\alpha}-\frac{\mu^{*}}{1-\alpha}\right] \\
&\quad +\frac{\ln\left(\alpha\beta\right)}{1-\alpha}+\frac{\mu^{*}}{1-\alpha} \\
&\quad +\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(\frac{A(\sigma(s))}{A(s)}\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s. \qquad (4.24)
\end{aligned}$$

However, we have to find the expected value (or mean) as we move in time. To do so, we have to take the limit of equation (4.24) as $t\to\infty$. Therefore, we have to show the integral in equation (4.24) is a finite number. First we check the proportion $\frac{A(\sigma(t))}{A(t)}$:

$$\begin{aligned}
\frac{A\left(\sigma\left(t\right)\right)}{A\left(t\right)} &= \frac{-\frac{1}{\alpha\beta}\mu\left(t\right)+\frac{1}{\alpha\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}\left(t,0\right)\left[\frac{1}{(1-\alpha\beta)}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}\left(s,0\right)\mu\left(s\right)\right]}{\alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}\left(t,0\right)\left[\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}\left(s,0\right)\mu\left(s\right)\right]} \\
&= \frac{\frac{1}{\beta}e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}\left(t,0\right)\left[\frac{1}{(1-\alpha\beta)}-\mu\left(t\right)e_{\frac{\alpha\beta-1}{\mu}}\left(t,0\right)-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}\left(s,0\right)\mu\left(s\right)\right]}{\alpha e_{\frac{\frac{1}{\alpha\beta}-1}{\mu}}\left(t,0\right)\left[\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}\left(s,0\right)\mu\left(s\right)\right]} \\
&= \frac{1}{\alpha\beta}\left[1-\frac{\mu\left(t\right)e_{\frac{\alpha\beta-1}{\mu}}\left(t,0\right)}{\frac{1}{1-\alpha\beta}-\sum_{s\in[0,t)_{\mathbb{T}}}e_{\frac{\alpha\beta-1}{\mu}}\left(s,0\right)\mu\left(s\right)}\right]. \qquad (4.25)
\end{aligned}$$

For the existence of the term $\ln\left(\frac{A(\sigma(t))}{A(t)}\right)$, we should have $\frac{A(\sigma(t))}{A(t)}>0$. We already know $0<\frac{1}{\alpha\beta}<1$. Thus, it is sufficient to check the sign of the expression in the brackets in equation (4.25). We set

$$D = 1 - \frac{\mu(t)\, e_{\frac{\alpha\beta-1}{\mu}}(t,0)}{\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)},$$

and we want it to be positive, i.e.,

$$D = 1 - \frac{\mu(t)\, e_{\frac{\alpha\beta-1}{\mu}}(t,0)}{\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)} > 0.$$

Indeed, by Lemma 15 in chapter 2, we know

$$\sum_{i=0}^{\infty} (\alpha\beta)^i - \left[\sum_{i=0}^{\rho(t)} (\alpha\beta)^i \mu\left(\sigma^i(0)\right)\right] > 0, \tag{4.26}$$

for isolated time scale $\mathbb{T}$ with $0 < \mu(t) \leq 1$. From the property of the exponential function given with Remark 6 in chapter 2, we have

$$\sum_{s\in[t,\sigma(t))_{\mathbb{T}}} (\alpha\beta)^i \mu\left(\sigma^i(0)\right) = (\alpha\beta)^t \mu\left(\sigma^t(0)\right) = e_{\frac{\alpha\beta-1}{\mu}}(t,0)\,\mu(t),$$

which is positive. We add this positive term to both sides of inequality (4.26), and we obtain

$$\sum_{i=0}^{\infty} (\alpha\beta)^i - \sum_{s\in[0,t)_{\mathbb{T}}} (\alpha\beta)^i \mu\left(\sigma^i(0)\right) > e_{\frac{\alpha\beta-1}{\mu}}(t,0)\,\mu(t). \tag{4.27}$$

Then, we divide both sides of inequality (4.27) with

$$\sum_{i=0}^{\infty} (\alpha\beta)^i - \sum_{s\in[0,t)_{\mathbb{T}}} (\alpha\beta)^i \mu\left(\sigma^i(0)\right) > 0;$$

i.e., we have

$$1 > \frac{e_{\frac{\alpha\beta-1}{\mu}}(t,0)\,\mu(t)}{\sum_{i=0}^{\infty} (\alpha\beta)^i - \sum_{s\in[0,t)_{\mathbb{T}}} (\alpha\beta)^i \mu(\sigma^i(0))}$$

$$1 - \frac{e_{\frac{\alpha\beta-1}{\mu}}(t,0)\,\mu(t)}{\sum_{i=0}^{\infty} (\alpha\beta)^i - \sum_{s\in[0,t)_{\mathbb{T}}} (\alpha\beta)^i \mu(\sigma^i(0))} > 0.$$

Hence, we find D as positive; i.e.,

$$\begin{aligned} D &= 1 - \frac{\mu(t)\, e_{\frac{\alpha\beta-1}{\mu}}(t,0)}{\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)} \\ &= 1 - \frac{e_{\frac{\alpha\beta-1}{\mu}}(t,0)\,\mu(t)}{\sum_{i=0}^{\infty} (\alpha\beta)^i - \sum_{s\in[0,t)_{\mathbb{T}}} (\alpha\beta)^i \mu(\sigma^i(0))} > 0. \end{aligned}$$

Finally, this property implies that,

$$1 > \frac{\mu(t)\, e_{\frac{\alpha\beta-1}{\mu}}(t,0)}{\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)},$$

which means

$$\frac{A(\sigma(t))}{A(t)} = \frac{1}{\alpha\beta}\left[1 - \frac{\mu(t)\, e_{\frac{\alpha\beta-1}{\mu}}(t,0)}{\frac{1}{1-\alpha\beta} - \sum_{s\in[0,t)_{\mathbb{T}}} e_{\frac{\alpha\beta-1}{\mu}}(s,0)\,\mu(s)}\right] < \frac{1}{\alpha\beta};$$

i.e., $\ln\left(\frac{A(\sigma(t))}{A(t)}\right)$ exists and is finite. Thus, the limit of (4.24) exists as $t \to \infty$. Also from equation (2.5) , we have

$$\lim_{t\to\infty} e_{\frac{\alpha-1}{\mu}}(t,0) = 0.$$

Hence, as $t \to \infty$, the mean is

$$\mathbb{E}\left[\ln(k(t))\right] = \frac{\ln(\alpha\beta)}{1-\alpha} + \frac{\mu^*}{1-\alpha}.$$

Variance of the state variable k(t):

Next, we shall find the variance of the state variable $k(t)$. From the definition of variance we have

$$\begin{aligned} Var\left[\ln(k(t))\right] &= \mathbb{E}\left[\left(\ln(k(t)) - \mathbb{E}\left[\ln(k(t))\right]\right)^2\right] \\ &= \mathbb{E}\left[\left(\frac{1}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) \int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{\ln(z(s))}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s \right.\right. \\ &\qquad \left.\left. + \frac{\mu^*}{1-\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) - \frac{\mu^*}{1-\alpha}\right)^2\right]. \end{aligned}$$

By multiplying equation (4.21) with nonzero expression $-\frac{\mu^*}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0)$ and using the properties of the exponential function, we have

$$\begin{aligned} -\frac{\mu^*}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0)\left(\int\limits_{s\in[0,t)_{\mathbb{T}}} \frac{1}{\mu(s)\, e_{\frac{\alpha-1}{\mu}}(s,0)}\Delta s\right) &= -\frac{\mu^*}{\alpha} e_{\frac{\alpha-1}{\mu}}(t,0)\, \frac{\alpha}{1-\alpha}\left(e_{\frac{1-\alpha}{\alpha\mu}}(t,0) - 1\right) \\ &= \frac{\mu^*}{1-\alpha} e_{\frac{\alpha-1}{\mu}}(t,0) - \frac{\mu^*}{1-\alpha}. \end{aligned} \tag{4.28}$$

We substitute (4.28) in the variance, and obtain

$$
\begin{aligned}
Var\left[\ln\left(k\left(t\right)\right)\right] &= \mathbb{E}\left[\left(\ln\left(k\left(t\right)\right)-\mathbb{E}\left[\ln\left(k\left(t\right)\right)\right]\right)^{2}\right] \\
&= \mathbb{E}\left[\left(\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\ln\left(z\left(s\right)\right)}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right.\right. \\
&\qquad \left.\left.-\frac{\mu^{*}}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right)^{2}\right].
\end{aligned}
$$

Then since $\mathbb{E}\left[\ln z\left(t_i\right)\right]=\mu^{*}$ for *every* $t_i\in[0,\infty)_{\mathbb{T}}$, we have

$$
Var\left[\ln\left(k\left(t\right)\right)\right]=\mathbb{E}\left[\left(\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\left[\ln\left(z\left(s\right)\right)-\mu^{*}\right]}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right)^{2}\right].
$$

By expanding this and using properties of the expectation and the exponential function, we have

$$
\begin{aligned}
Var\left[\ln\left(k\left(t\right)\right)\right] &= \mathbb{E}\left[\left(\frac{1}{\alpha}e_{\frac{\alpha-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\left[\ln\left(z\left(s\right)\right)-\mu^{*}\right]}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right)^{2}\right] \\
&= \frac{1}{\alpha^{2}}e^{2}_{\frac{\alpha-1}{\mu}}\left(t,0\right)\mathbb{E}\left[\left(\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{\left[\ln\left(z\left(s\right)\right)-\mu^{*}\right]}{\mu\left(s\right)e_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right)^{2}\right] \\
&= \frac{1}{\alpha^{2}}e^{2}_{\frac{\alpha-1}{\mu}}\left(t,0\right)\left[\int\limits_{s=r\in[0,t)_{\mathbb{T}}}\frac{\mathbb{E}\left[\ln\left(z\left(s\right)\right)-\mu^{*}\right]^{2}}{\mu^{2}\left(s\right)e^{2}_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right. \\
&\qquad \left.+\int\limits_{s\neq r}\frac{\mathbb{E}\left[\ln\left(z\left(s\right)\right)-\mu^{*}\right]\mathbb{E}\left[\ln\left(z\left(r\right)\right)-\mu^{*}\right]}{\mu^{2}\left(s\right)e^{2}_{\frac{\alpha-1}{\mu}}\left(s,0\right)}\Delta s\right].
\end{aligned}
$$

where $e^{2}_{\frac{\alpha-1}{\mu}}\left(t,0\right)=e_{\frac{\alpha^{2}-1}{\mu}}\left(t,0\right)$. Since $Var\left[\ln\left(z\left(t_i\right)\right)\right]=\mathbb{E}\left[\ln\left(z\left(s\right)\right)-\mu^{*}\right]^{2}=\sigma^{2}$ for every $t_i\in[0,\infty)_{\mathbb{T}}$, and $Cov\left(z\left(t_i\right),z\left(t_j\right)\right)=0$, for all $i{\neq}j$, we obtain the variance of state variable $k\left(t\right)$ as

$$
Var\left[\ln\left(k\left(t\right)\right)\right]=\frac{\sigma^{2}}{\alpha^{2}}e_{\frac{\alpha^{2}-1}{\mu}}\left(t,0\right)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu^{2}\left(s\right)e_{\frac{\alpha^{2}-1}{\mu}}\left(s,0\right)}\Delta s. \tag{4.29}
$$

Since

$$e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(t,0)=\ominus\left(\frac{\alpha^2-1}{\mu}\right)e_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(t,0),$$

then we have

$$\begin{aligned}
e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(t,0) &= \frac{1-\alpha^2}{\alpha^2\mu(t)}e_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(t,0) \\
\frac{\alpha^2}{1-\alpha^2}e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(t,0) &= \frac{1}{\mu(t)\,e_{\frac{\alpha^2-1}{\mu}}(t,0)}. \qquad (4.30)
\end{aligned}$$

Then we substitute the expression in equation (4.30) in the integral in equation (4.29); that is,

$$\begin{aligned}
Var[\ln(k(t))] &= \frac{\sigma^2}{\alpha^2}e_{\frac{\alpha^2-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu(s)}\frac{\alpha^2}{1-\alpha^2}e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(s,0)\,\Delta s \\
&= \frac{\sigma^2}{1-\alpha^2}e_{\frac{\alpha^2-1}{\mu}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{\mu(s)}e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{\mu}\right)}(s,0)\,\Delta s. \qquad (4.31)
\end{aligned}$$

Thus, we obtain the variance of the optimal state variable $k(t)$ as in equation (4.31).

Example For $\mathbb{T}=h\mathbb{Z}$, we have $\mu(t)=h$. Then the variance given with equation (4.31) becomes

$$Var[\ln(k(t))]=\frac{\sigma^2}{1-\alpha^2}e_{\frac{\alpha^2-1}{h}}(t,0)\int\limits_{s\in[0,t)_{\mathbb{T}}}\frac{1}{h}e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{h}\right)}(s,0)\,\Delta s. \qquad (4.32)$$

By the definition of "circle minus" on time scales, we have

$$e^{\Delta}_{\ominus\left(\frac{\alpha^2-1}{h}\right)}(t,0)=e^{\Delta}_{\frac{1-\alpha^2}{\alpha^2h}}(t,0).$$

We substitute this in equation (4.32), and we obtain

$$Var[\ln(k(t))]=\frac{\sigma^2}{1-\alpha^2}e_{\frac{\alpha^2-1}{h}}(t,0)\frac{1}{h}\int\limits_{s\in[0,t)_{\mathbb{T}}}e^{\Delta}_{\frac{1-\alpha^2}{\alpha^2h}}(s,0)\,\Delta s. \qquad (4.33)$$

Because the exponential function in equation (4.33) is equivalent to

$$e_{\frac{\alpha^2-1}{h}}(t,0)=\left(1+\frac{\alpha^2-1}{h}h\right)^{\frac{t-0}{h}}=\alpha^{\frac{2t}{h}}$$

for $\mathbb{T}=h\mathbb{Z}$, we have

$$\begin{aligned}
Var\left[\ln\left(k\left(t\right)\right)\right] &= \frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\int\limits_{s\in[0,t)_{\mathbb{T}}} e^{\Delta}_{\frac{1-\alpha^2}{\alpha^2 h}}(s,0)\,\Delta s\\
&= \frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\left(e_{\frac{1-\alpha^2}{\alpha^2 h}}(s,0)\right)\Big]_0^t\\
&= \frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\left(e_{\frac{1-\alpha^2}{\alpha^2 h}}(t,0)-e_{\frac{1-\alpha^2}{\alpha^2 h}}(0,0)\right)\\
&= \frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\left(e_{\frac{1-\alpha^2}{\alpha^2 h}}(t,0)-1\right).
\end{aligned}$$

Also for $\mathbb{T}=h\mathbb{Z}$, we have

$$e_{\frac{1-\alpha^2}{\alpha^2 h}}(t,0)=\left(1+\frac{1-\alpha^2}{\alpha^2 h}h\right)^{\frac{t-0}{h}}=\left(\frac{1}{\alpha}\right)^{\frac{2t}{h}}.$$

Thus, we have

$$\begin{aligned}
Var\left[\ln\left(k\left(t\right)\right)\right] &= \frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\left(\left(\frac{1}{\alpha}\right)^{\frac{2t}{h}}-1\right)\\
&= \frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\left(\frac{1}{\alpha}\right)^{\frac{2t}{h}}-\frac{\sigma^2}{1-\alpha^2}\alpha^{\frac{2t}{h}}\frac{1}{h}\\
&= \frac{\sigma^2}{h\left(1-\alpha^2\right)}-\frac{\sigma^2}{h\left(1-\alpha^2\right)}\alpha^{\frac{2t}{h}}.
\end{aligned}$$

Hence, for $0<\alpha<1$, the variance is

$$Var\left[\ln\left(k\left(t\right)\right)\right]=\frac{\sigma^2}{h\left(1-\alpha^2\right)}$$

for $\mathbb{T}=h\mathbb{Z}$, and it is

$$Var\left[\ln\left(k\left(t\right)\right)\right]=\frac{\sigma^2}{1-\alpha^2}$$

for $\mathbb{T}=\mathbb{Z}$, as $t\to\infty$.

CHAPTER 5

CONCLUSION AND FUTURE WORK

Dynamic programming is one of the methods included in the theory of dynamic optimization. The concept of dynamic programming is useful in multistage decision processes where the result of a decision at one stage affects the decision to be made in the succeeding stage. Therefore, if one is interested in a long term effect of the decisions that have been made, rather than an immediate effect, then one has become involved in the concept of dynamic programming. The method has advantages, such as enabling the decision maker to formulate the problem on infinite horizon in both deterministic and stochastic cases with flexibility on the constraints. Therefore, these advantages have made the dynamic programming an effective and useful mathematical principle in optimization theory.

Since Richard Bellman's invention of dynamic programming in the 1950s, economists and mathematicians have formulated and solved a huge variety of sequential decision making problems both in deterministic and stochastic cases; either finite or infinite time horizon. Yet, absence of conditions, which determine the nature of *trading* as periodic or continuous, forces us to reformulate the decision making problem on non-periodic time domains. Thus, we developed the concept of dynamic programming on non-periodic time domains, which is known as isolated time scales. We formulated the sequence problem on isolated time scales for both deterministic and stochastic cases, and then we derived the corresponding Bellman equations, respectively. We developed the Bellman's Principle of Optimality on isolated time scales, and we proved the theory which unifies and generates the existence results. Then we formulated and solved the deterministic optimal growth model, that is, the social planner's problem, by using time scales calculus. This case gave us the optimal solution with risk aversion, which is the optimal solution under certainty. Because we used guessing method to find the solution of the Bellman equation, we showed the existence and uniqueness of this solution by using the Contraction Mapping Theorem.

While deterministic problems have the property that the result of any decision is known to the decision maker before the decision is made, stochastic problems specify the probability of each of the various outcomes of the decision. Thus, the stochastic case gives more accurate results. Therefore, we formulated and stated the optimal solutions of the stochastic social planner's problem. Because these optimal solutions involved random variables which make explicit solutions impossible, we examined the distributions of these solutions.

In this study, we established necessary theory of dynamic programming on isolated time scales. For future work, we will examine an application of this theory in the real world. Then, we will make a comparison between deterministic and stochastic models.

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