1. Introduction

Much recent significant work on the arithmetic and analytic properties of Stieltjes constants has been done in a series of papers such as [1], [2], and [3]. Here we briefly review some of these results and relate them to results of this thesis. Mark Coffey is thanked for bringing our attention to this work. The omission of these citations is regretted.

The Stieltjes constants have been well-studied from a variety of viewpoints. In particular, their analytic and asymptotic properties have been investigated in [1] and various arithmetic properties investigated in [2], and [3]. As presented in [2], the Stieltjes constants can be defined as

$$\gamma_k(a) = \lim_{n \to \infty} \left( \sum_{j=0}^{n} \frac{\ln^k(j + a)}{j + a} - \frac{\ln^{k+1}(n + a)}{k + 1} \right),$$

and they occur as coefficients in the Laurent expansion about $s = 1$ of the Hurwitz zeta function

$$\zeta(s, a) = \frac{1}{s - 1} + \sum_{n=0}^\infty \frac{(-1)^n \gamma_n(a)}{n!} (s - 1)^n, \quad s \neq 1.$$
One of the consequences of the methods of the thesis is that by looking at derivatives of $\zeta(s, a)$, certain relations concerning the Stieltjes constants can be found. These relations can essentially be found in [2]. There, several very general results are given which have numerous relations of Stieltjes constants as straightforward consequences. For example, in (3.27) of [2], a relation of the Stieltjes constants to series involving logarithms is found. In particular, it is shown that

$$\gamma_k(a) - \gamma_k(b) = \sum_{n=0}^{\infty} \left( \frac{\ln^k(n + a)}{n + a} - \frac{\ln^k(n + b)}{n + b} \right).$$

Further, various general summation relations for Stieltjes constants are shown, such as the following.

**Proposition 1** (Proposition 1, [2]). If $\Re(a) > 0$ and $\Re(b) > 0$ then

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\gamma_{n+1}(a) - \gamma_{n+1}(b)) = \ln \left( \frac{\Gamma(b)}{\Gamma(a)} \right).$$

In the next section of [2], various relations to derivatives of certain Dirichlet $L$-series are obtained. A very general result encompassing many relations of the numbers $\gamma_1(a)$ is shown. More precisely, the following result (for odd Dirichlet characters) is given.
Proposition 2 (Proposition 3, [2]). Suppose that $\chi_k$ is a nonprincipal Dirichlet character of conductor $k$ so that $\chi_k(-1) = -1$. Then

$$- \sum_{m=1}^{k} \chi_k(m) \gamma_1 \left( \frac{m}{k} \right) = kL'_{-k}(1) - \ln(k) \sum_{m=1}^{k} \chi_k(m) \psi \left( \frac{m}{k} \right)$$

$$= k \int_{0}^{\infty} \frac{\ln(u) + \gamma}{1 - e^{-ku}} \left( \sum_{m=1}^{k} \chi_k(m) e^{-mu} \right) du - \ln(k) \sum_{m=1}^{k} \chi_k(m) \psi \left( \frac{m}{k} \right)$$

$$= -\frac{\pi}{k^{1/2}} (2\ln(2\pi) + \gamma) \sum_{m=1}^{k} m\chi_k(m) - \pi k^{1/2} \ln(\prod_{m=1}^{k} \Gamma^{\chi_k(m)} \left( \frac{m}{k} \right)) - \ln(k) \sum_{m=1}^{k} \chi_k(m) \psi \left( \frac{m}{k} \right).$$

Here $L_k(s)$ is the Dirichlet $L$-function $L(s, \chi_k) = \sum_{n=1}^{\infty} \chi_k(n)n^{-s}$. The analogous result for even Dirichlet characters is also given.

Proposition 3 (Proposition 4, [2]). Suppose that $\chi_k$ is a nonprincipal Dirichlet character of conductor $k$ so that $\chi_k(-1) = 1$. Then

$$- \sum_{m=1}^{k} \chi_k(m) \gamma_1 \left( \frac{m}{k} \right) = kL'_{+k}(1) - \ln(k) \sum_{m=1}^{k} \chi_k(m) \psi \left( \frac{m}{k} \right)$$

$$= k \int_{0}^{\infty} \frac{\ln(u) + \gamma}{1 - e^{-ku}} \left( \sum_{m=1}^{k} \chi_k(m) e^{-mu} \right) du - \ln(k) \sum_{m=1}^{k} \chi_k(m) \psi \left( \frac{m}{k} \right)$$

$$= k^{1/2} \left( 2\ln(2\pi) + \gamma \right) \ln(\prod_{m=1}^{k} \Gamma^{\chi_k(m)} \left( \frac{m}{k} \right)) - \sum_{m=1}^{k} \chi_k(m) \zeta'' \left( 0, \frac{m}{k} \right)$$

$$- \ln(k) \sum_{m=1}^{k} \chi_k(m) \psi \left( \frac{m}{k} \right).$$

These are every general results, and they encompass the Stieltjes constant identities found in the thesis. In particular, several explicit identities involving the first Stieltjes constant can be shown from this formula. For example, taking $k = 3$ (see (3.20) of [2])
we get
\[
\gamma_1 \left( \frac{1}{3} \right) - \gamma_1 \left( \frac{2}{3} \right) = -\frac{\pi}{\sqrt{3}} \left( \ln(2\pi) + \gamma - 3 \ln \left( \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right) + \ln(3) \right)
\]
and (3.21) of [2] is
\[
\gamma_1 \left( \frac{1}{4} \right) - \gamma_1 \left( \frac{3}{4} \right) = -\pi \left( \ln(8\pi) + \gamma - 2 \ln \left( \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \right) \right).
\]

Taking \( \gamma_m = \gamma_m(1) \), from (3.28) of [2] (and also Proposition 3 of [1] and Proposition 5.1 of [3] and Corollary 3 of [2] for the \( k = 1 \) case) we have
\[
\sum_{r=1}^{q-1} \gamma_k \left( \frac{r}{q} \right) = -\gamma_k + (-1)^k \frac{\ln^{k+1}(q)}{k+1} + q \sum_{r=0}^{k} (-1)^r \binom{k}{r} \ln^r(q) \gamma_{k-r}.
\]

Further, expressions for \( \gamma_k(a) \) for \( k \geq 2 \) are also obtained, such as (1.21) and (1.22) in [2]. These expressions give \( \gamma_2(a) \) and, more generally, \( \gamma_n(a) \) in terms of certain values of derivatives of Hurwitz zeta functions. For example, Proposition 8 in [2] is, for \( \text{Re}(a) > 0 \),
\[
\gamma_2(a) = -\frac{1}{3} \ln^3(a+1) + \frac{\ln^2(a)}{a} - \sum_{k=1}^{\infty} \frac{1}{k+1} \left( (-1)^k \zeta''(k+1, a+1) \right)
\]
\[
- \frac{2}{k!} s(k+1, 2) \zeta'(k+1, a+1) + \frac{2}{k!} s(k+1, 3) \zeta(k+1, a+1)
\]
where \( s(n, m) \) are the Stirling numbers of the first kind.
There also is a general expression for such Stieltjes constants given in Proposition 6.1 of [3] as follows,

\[
\gamma_m = -m! \sum_{\ell=1}^{m} \frac{B_{m-\ell+1}}{(m-\ell+1)!} \ln^{m-\ell}(2) \sum_{n=1}^{\infty} \frac{1}{2n+1} \sum_{k=1}^{n} (-1)^{k} \binom{n}{k} \frac{\ln^{k}(k+1)}{k+1}
\]

\[
- \frac{1}{\ln(2)} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sum_{k=1}^{n} (-1)^{k} \binom{n}{k} \frac{\ln^{n+1}(k+1)}{k+1} - \frac{B_{m+1}}{m+1} \ln^{m+1}(2)
\]

where the \( B_m \) are the Bernoulli numbers.

In this thesis, properties of the Hurwitz zeta function were investigated and higher derivatives of the Hurwitz zeta function at \( s = 0 \) were evaluated. Several relations among the Stieltjes constants were then shown and these relations can be encompassed in the general results of [2].

**References**

