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Cagan Type Rational Expectations Model on Time Scales with Their Applications to Economics

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CAGAN TYPE RATIONAL EXPECTATIONS MODEL ON TIME SCALES WITH THEIR APPLICATIONS TO ECONOMICS

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CAGAN TYPE RATIONAL EXPECTATIONS MODEL ON TIME SCALES WITH THEIR APPLICATIONS TO ECONOMICS

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Rational expectations provide people or economic agents making future decision with available information and past experiences. The first approach to the idea of rational expectations was given approximately fifty years ago by John F. Muth. Many models in economics have been studied using the rational expectations idea. The most familiar one among them is the rational expectations version of the Cagans hyperinflation model where the expectation for tomorrow is formed using all the information available today. This model was reinterpreted by Thomas J. Sargent and Neil Wallace in 1973. After that time, many solution techniques were suggested to solve the Cagan type rational expectations (CTRE) model. Some economists such as Muth [13], Taylor [26] and Shiller [27] consider the solutions admitting an infinite moving-average representation. Blanchard and Kahn [28] find solutions by using a recursive procedure. A general characterization of the solution was obtained using the martingale approach by Broze, Gourieroux and Szafarz in [22], [23]. We choose to study martingale solution of CTRE model. This thesis is comprised of five chapters where the main aim is to study the CTRE model on isolated time scales.

Most of the models studied in economics are continuous or discrete. Discrete models are more preferable by economists since they give more meaningful and accurate results. Discrete models only contain uniform time domains. Time scale calculus enables us to study on m-periodic time domains as well as non periodic time domains. In the first chapter, we give basics of time scales calculus and stochastic calculus. The second chapter is the brief introduction to rational expectations and the CTRE model. Moreover, many other solution techniques are examined in this chapter. After we introduce the necessary background, in the third chapter we construct the CTRE Model on isolated time scales. Then we give the general solution of this model in terms of martingales. We continue our work with defining the linear system and

higher order CTRE on isolated time scales. We use Putzer Algorithm to solve the system of the CTRE Model. Then, we examine the existence and uniqueness of the solution of the CTRE model. In the fourth chapter, we apply our solution algorithm developed in the previous chapter to models in Finance and stochastic growth models in Economics.

CHAPTER 1
PRELIMINARIES

Many discrete and continuous models have been studied extensively in economics. Discrete models are more preferable by economists since they give more meaningful and accurate results. Discrete models only contain uniform time domains such as day, month, year. Time scale calculus enables us to study on m-periodic time domains as well as non periodic time domains. Thus it is beneficial to comprehend the time scale calculus. Also, we introduce stochastic calculus to understand the martingales and conditional expectations.

1.1 Time Scale Calculus

In this section, we will give some basic definitions and theorems on time scales. Many of these definitions, theorems and their proofs can be found in the book by Bohner and Peterson [1].

Definition 1.1. *A time scale* $\mathbb{T}$ *is any nonempty closed subset of the real numbers* $\mathbb{R}$.

The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the Cantor set, and $[2,3] \cup \mathbb{N}$ are examples of time scales. On the other hand, the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval $(1,2)$ are not time scales.

Definition 1.2. *The* forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, *and the* backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *are defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

The above definition for the empty set $\emptyset$ will be $\inf\ \emptyset = \sup\ \mathbb{T}$ and $\sup\ \emptyset = \inf\ \mathbb{T}$. For a point $t \in \mathbb{T}$ if $\sigma(t) > t$, we say that t is *right-scattered*, and if $\rho(t) < t$ we say that t is *left-scattered*. Also if $\sigma(t) = t$, t is called *right-dense*, and if $\rho(t) = t$, t is called *left-dense*. Points that are right-scattered and left scattered at the same time are called *isolated* i.e., $\rho(t) < t < \sigma(t)$. Points that are right-dense and left-dense at the same time are called *dense* i.e., $\rho(t) = t = \sigma(t)$. If $\sup\ \mathbb{T} < \infty$ and $\sup\ \mathbb{T}$ is left-scattered, we let $\mathbb{T}^\kappa = \mathbb{T} \setminus \sup\ \mathbb{T}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1.3. *The* graininess function $\mu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\mu(t) = \sigma(t) - t.$$

A time scale $\mathbb{T}$ is called an *isolated time scale* if every $t \in \mathbb{T}$ is an isolated point. For example, if $\mathbb{T} = \mathbb{Z}$, then for any $t \in \mathbb{Z}$ $\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = t + 1$ and similarly $\rho(t) = \sup\{s \in \mathbb{Z} : s < t\} = t - 1$. Thus every point $t \in \mathbb{Z}$ is isolated. Hence the graininess function is $\mu(t) = 1$. The natural numbers $\mathbb{N}$, and $q^{\mathbb{N}} = \{q^n \mid n \in \mathbb{N}\}$ where $q > 1$ are other examples of isolated time scales.

Definition 1.4. *For* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^\kappa$, *we define the* delta derivative *of* $f(t)$, $f^\Delta(t)$, *to be the number (provided it exists) with the property that, for any* $\epsilon > 0$, *there exists a neighborhood* U *of* t *such that*

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \epsilon\, |\sigma(t) - s| \text{ for all } s \in U.$$

Moreover if f is delta differentiable for every $t \in \mathbb{T}^\kappa$, then we say that it is *delta differentiable* on $\mathbb{T}^\kappa$. If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{1.1.1}$$

If f is differentiable at $t \in \mathbb{T}$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t),$$

where $f(\sigma(t)) = f^\sigma(t)$ for all $t \in \mathbb{T}$.

Note that when $\mathbb{T} = \mathbb{R}$, f^Δ is precisely f' and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta = \Delta f = f(t+1) - f(t)$ is the forward difference operator.

Assume f, g are differentiable at $t \in \mathbb{T}^\kappa$. Then the product rule on time scale $\mathbb{T}$ is given by

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)) \tag{1.1.2}$$

and the quotient rule is given by

$$(\frac{f}{g})^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. \tag{1.1.3}$$

Definition 1.5. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* rd-continuous *provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (i.e. finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 1.6. *The* Cauchy integral *is defined by*

$$\int_a^b f(t)\Delta t = F(b) - F(a) \textit{ for all } a, b \in \mathbb{T}.$$

where a function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$.

Theorem 1.1. *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t)f(t) \qquad \textit{if } a < b.$$

(ii) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

(iii) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \qquad \textit{if } a < b.$$

If $f, g \in C_{rd}$ and $a, b \in \mathbb{T}$, then the integration by parts formula on the time scales is given by

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t. \tag{1.1.4}$$

Definition 1.7. *The function $p : \mathbb{T} \to \mathbb{R}$ is* regressive *if*

$$1 + \mu(t)p(t) \neq 0 \qquad \textit{for all } t \in \mathbb{T}^\kappa.$$

Definition 1.8. *The operation "circle minus" $\ominus$ defined by*

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \qquad \textit{for all } t \in \mathbb{T}^\kappa.$$

The generalized *exponential function* on time scales is given as $e_p(., t_0)$ where $p \in \mathcal{R}$ and $\mathcal{R}$ is the set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$.

Definition 1.9. *The exponential function $e_p(., t_0)$ is the unique solution of the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1. \tag{1.1.5}$$

Now we will list some basic but important properties of the exponential function $e_p(., t_0)$.

Lemma 1.1. *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t, s) \equiv 1$

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$

(iii) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$

(iv) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$

(v) $\left(\dfrac{1}{e_p(., s)}\right)^\Delta = -\dfrac{p(t)}{e_p^\sigma(., s)}.$

Definition 1.10. *If $p \in \mathcal{R}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then the dynamic equation*

$$y^{\Delta}(t) = p(t)y(t) + f(t) \tag{1.1.6}$$

is called regressive*.*

Theorem 1.2. *(Variation of Constants) Suppose (1.1.6) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, then the unique solution to the first order dynamic equation on $\mathbb{T}$*

$$y^{\Delta}(t) = p(t)y(t) + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(t)) f(\tau) \Delta\tau,$$

Next, we will list some definitions for the linear system of dynamic equations,

$$y^{\Delta} = A(t)y \tag{1.1.7}$$

where A is an $n \times n$ matrix -valued function.

Definition 1.11. *An $n \times n$ matrix -valued function A on a time scale $\mathbb{T}$ is called* regressive *provided*

$$I + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T}^{\kappa},$$

and the class of all such regressive and rd-continuous functions is denoted by,

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}).$$

We say that the system (1.1.7) is regressive *provided $A \in \mathcal{R}$.*

Definition 1.12. *Let $t_0 \in \mathbb{T}$ and assume that $A(t) \in \mathcal{R}$ is an $n \times n$ matrix-valued function. The unique matrix-valued solution of IVP*

$$Y^{\Delta} = A(t)Y \qquad Y(t_0) = I,$$

where I denotes as usual the $n \times n$ identity matrix, is called the matrix exponential function *at t_0, and it is denoted by $e_A(., t_0)$.*

Lemma 1.2. *If $A \in \mathcal{R}$ are matrix-valued functions on $\mathbb{T}$, then*

(i) $e_0(t,s) \equiv I$

(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t,s)$

(iii) $\dfrac{1}{e_A(t,s)} = e_{\ominus A}(t,s)$

(iv) $e_A(t,s) = \dfrac{1}{e_A(s,t)} = e_{\ominus A}(s,t)$

(v) $[e_A(t,.)]^{\Delta} = -[e_A(t,.)]^{\sigma} A.$

The following two results can be found in a paper [4] by C. Peterson and his students.

Theorem 1.3. *(Variation of Constants for First Order Recurrence Relations)*

Assume $p(t) \neq 0$, for every $t \in \mathbb{T}^{\kappa}$. Then the unique solution to the IVP

$$y^{\sigma} - p(t)y = r(t),\ y(t_0) = y_0$$

is given by

$$y(t) = e_{\frac{p-1}{\mu}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{p-1}{\mu}}(t,\sigma(s))\frac{r(s)}{\mu(s)}\Delta s.$$

Lemma 1.3. *The exponential function $e_{\frac{p-1}{\mu}}(t,t_0)$ is given by*

$$\begin{aligned} e_{\frac{p-1}{\mu}}(t,t_0) &= \textstyle\prod_{\tau\in[t_0,t)} p(\tau) \qquad && \text{if } t \geq t_0 \\ e_{\frac{p-1}{\mu}}(t,t_0) &= \textstyle\prod_{\tau\in[t,t_0)} \frac{1}{p(\tau)} \qquad && \text{if } t < t_0. \end{aligned}$$

Let $0 < p < 1$ be a constant number, and for $t > t_0$, let $t = t_n$ on time scale $\mathbb{T} = \{t_0, t_1, ..., t_n, t_{n+1}, ...\}$. Also, let n_t be a function of t that counts the number of isolated points on the interval $[t_0, t) \cap \mathbb{T}$. Then by Lemma 1.3, the exponential function becomes

$$e_{\frac{p-1}{\mu}}(t,t_0) = \prod_{\tau\in[t_0,t)} p(\tau) = p^{n_t}, \tag{1.1.8}$$

where the counting function n_t on isolated time scales, is given as

$$n_t(t,s) := \int_s^t \frac{\Delta(\tau)}{\mu(\tau)}. \tag{1.1.9}$$

Next we refer the Putzer Algorithm given by W. G. Kelley and A. C. Peterson in [6] to calculate A^t for $t \in \mathbb{Z}$, where A is $n \times n$ matrix.

Theorem 1.4. *(Putzer Algorithm) Let A be a $n \times n$-matrix. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the* eigenvalues *of A, then*

$$A^t = \sum_{i=0}^{n-1} r_{i+1}(t) P_i,$$

where $r_i(t), (i = 1, 2, ..., n)$ are chosen to satisfy the system:

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \\ . \\ . \\ . \\ r_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 1 & \lambda_2 & 0 & ... & 0 \\ 0 & 1 & \lambda_3 & ... & 0 \\ . & & & . & \\ . & . & & & \\ . & . & & & \\ 0 & ... & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ . \\ . \\ . \\ r_n(t) \end{bmatrix}, \begin{bmatrix} r_1(0) \\ r_2(0) \\ . \\ . \\ . \\ r_n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

and the P_i are defined by

$$P_0 = I$$
$$P_i = (A - \lambda_i I) P_{i-1}, \qquad (1 \leq i \leq n).$$

Next, we continue with properties of the nabla dynamic equation whose definition and some related theorems can be found in the text by Bohner and Peterson [7].

Definition 1.13. *If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$: otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. The backward graininess $\nu : \mathbb{T}_\kappa \to \mathbb{R}_0^+$ is defined by*

$$\nu(t) = t - \rho(t).$$

If f is nabla differentiable for every $t \in \mathbb{T}_\kappa$ then f is continuous at t and if f is continuous at a left-scattered point t, then f is nabla differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}. \tag{1.1.10}$$

If $a, b \in \mathbb{T}$ and $f, g : \mathbb{T} \to \mathbb{R}$ are ld-continuous; then the integration by parts formula for nabla integration is given by

$$\int_a^b h(t) g^\nabla(t) \nabla t = (hg)(b) - (hg)(a) - \int_a^b h^\nabla(t) g(\rho(t)) \nabla t. \tag{1.1.11}$$

Definition 1.14. *The exponential function $\hat{e}(., t_0)$ is the unique solution of the initial value problem*

$$y^\nabla = p(t) y, \qquad y(t_0) = 1.$$

Now we will list some basic properties of the exponential function $\hat{e}(., t_0)$, which is known as the nabla exponential function.

Lemma 1.4. *If $p \in \mathcal{R}_\nu$ and $s, t, r \in \mathbb{T}$. Then*

(*i*) $\hat{e}_p(\rho(t), s) = (1 - \nu(t) p(t)) \hat{e}_p(t, s)$

(*i*) $\dfrac{1}{\hat{e}_p(t, s)} = \hat{e}_{\ominus p}(t, s)$

(*iii*) $(\dfrac{1}{\hat{e}_p(t, s)})^\nabla = -\dfrac{p(t)}{\hat{e}_p(t, s)}.$

Theorem 1.5. *(Equivalence of Delta and Nabla Exponential Functions) If p is continuous and regressive, then*

$$e_p(t, t_0) = \hat{e}_{\frac{p^\rho}{1 + p^\rho \nu}}(t, t_0) = \hat{e}_{\ominus \nu(-p^\rho)}(t, t_0)$$

If q is continuous and ν- regressive, then

$$\hat{e}_q(t, t_0) = e_{\frac{q^\sigma}{1 - q^\sigma \mu}}(t, t_0) = e_{\ominus(-q^\sigma)}(t, t_0).$$

Next, we state the relationship between the delta derivative and the nabla derivative.

Theorem 1.6. (i) *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^\kappa$. *Then* f *is nabla differentiable at* t *and*

$$f^\nabla(t) = f^\Delta(\rho(t))$$

for $t \in \mathbb{T}_\kappa$ *such that* $\sigma(\rho(t)) = t$.

(ii) *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is nabla differentiable on* $\mathbb{T}_\kappa$. *Then* f *is delta differentiable at* t *and*

$$f^\Delta(t) = f^\nabla(\sigma(t))$$

for $t \in \mathbb{T}^\kappa$ *such that* $\rho(\sigma(t)) = t$.

Throughout this study, we assume that $\mathbb{T}$ is an isolated scale.

1.2 Stochastic Calculus

In this section, we give some definitions and properties from stochastic calculus so that the reader can follow our work easily. Many of these definitions and properties can be found in the books by Mikosch [2] and Klebaner [3]. We start with the definition of a *random variable*. The outcome of an experiment or game is random. Consider a coin tossing; the possible outcomes "head" or "tail". We can write "1" for "head" and "0" for "tail". Hence we get a *random variable* $X = X(w) \in \{0, 1\}$ where w belongs to the outcome space $\Omega = \{head, tail\}$. In mathematical language $X = X\{w\}$ is a real-valued function defined on Ω, such that

$$X : \Omega \to \mathbb{R}.$$

If we consider x_t, a random variable on isolated time scales as

$$X : \Omega \times \mathbb{T} \to \mathbb{R}.$$

This means $x_t = x(w, t)$ where $w \in \Omega$ and $t \in \mathbb{T}$.

A *partition* of Ω is a collection of exhaustive and mutually exclusive subsets,

$$\{D_1, ..., D_k\} \text{ such that } D_i \cap D_j = \emptyset \text{ and } \bigcup_i D_i = \Omega.$$

The field generated by the partition is the collection of all finite unions of D_j's and their complements.

Definition 1.15. *Let X take values $x_1, ..., x_p$ and $A_1 = \{X = x_1\}, ..., A_p = \{X = x_p\}$. Let the field $\mathcal{F}$ be generated by a partition $\{D_1, D_2, ..., D_k\}$ of Ω. Then the* conditional expectation *of X given $\mathcal{F}$ is defined by*

$$E(X|\mathcal{F}) = \sum_{t=1}^{p} x_i P(A_i|\mathcal{F}),$$

where $P(A_i|\mathcal{F})$ is the conditional probability of A given $\mathcal{F}$.

Basic Properties of Conditional Expectation

(i) The conditional expectation is *linear*: For random variables X_1, X_2 and constants c_1, c_2,

$$E([c_1 X_1 + c_2 X_2]|\mathcal{F}) = c_1 E(X_1|\mathcal{F})) + c_2 E(X_2|\mathcal{F}).$$

(ii) The expectation law: The expectation of X and the expectation of $E(X|\mathcal{F})$ are the same, i.e.

$$EX = E[E(X|\mathcal{F})].$$

(iii) Positivity: If $X \geq 0$, then $E(X|\mathcal{F}) \geq 0$.

(iv) Independence law: If X is independent of $\mathcal{F}$, then $E(X|\mathcal{F}) = E(X)$.

(iv) Tower Property: If $\mathcal{F}$ and $\mathcal{F}'$ are two field with $\mathcal{F} \subset \mathcal{F}'$, then

$$E(E(X|\mathcal{F}')|\mathcal{F}) = E(X|\mathcal{F})$$

or

$$E(E(X|\mathcal{F})|\mathcal{F}') = E(X|\mathcal{F}).$$

(v) Stability: If X is $\mathcal{F}$ measurable, then $E(XZ|\mathcal{F}) = XE(Z|\mathcal{F})$.

(vi) Constants: For any scalar a, $E(a|\mathcal{F}) = a$.

Before giving the definition and properties of martingales, we define the $\sigma - field$ and filtration.

Definition 1.16. *A set of subsets of* Ω*, denote it by* $\mathcal{F}$*, is called a* $\sigma - field$ *if*

1. $\Omega \in \mathcal{F}$

2. *If* $A \in \mathcal{F}$*, then* $A^c \in \mathcal{F}$

3. *If* $A_1, A_2, ...$ *is a sequence of elements of* $\mathcal{F}$*, then*

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}.$$

Example 1.(Examples of fields)

It is easy to verify that any of the following is a field of sets.

1. $\{\Omega, \emptyset\}$ is called the trivial field $\mathcal{F}_0$.

2. $\{\Omega, \emptyset, A, A^c\}$ is called the field generated by set A, and denoted by $\mathcal{F}_A$.

3. $\{A : A \subseteq \Omega\}$ the field of all the subsets of Ω. It is denoted by 2^{Ω}.

Assume that $(\mathcal{F}_t, t \geq 0)$ is a collection of $\sigma - fields$ on the same space Ω and that all $\mathcal{F}_t s$ are subsets of a larger $\sigma - field$ $\mathcal{F}$ on Ω.

The collection $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ of $\sigma - fields$ on Ω is called a *filtration* if

$$\mathcal{F}_s \subset \mathcal{F}_t \text{ for all } 0 \leq s \leq t.$$

Thus one can think a filtration, an increasing stream of information.

Example 2. $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_A, 2^{\Omega}\}$ is an example of filtration.

Definition 1.17. *(Field Generated by a Random Variable)*

Let $(\Omega, 2^{\Omega})$ *be a sample space with the field of all events, and* X *be a random variable with values* x_i*,* $i = 1, 2, ..., k$*. Consider sets*

$$A_i = \{w : X(w) = x_i\} \subseteq \Omega.$$

These sets form a partition of Ω*, and the field generated by this partition is called the field generated by* X*. It is the smallest field that contains all the sets of the form* $A_i = \{X = x_i\}$ *and it is denoted by* $\sigma(X)$*.*

The discrete-time process $Y = (Y_t, t = 0, 1, ...)$ is said to be *adapted to the filtration* $(\mathcal{F}_t, t = 0, 1, ...)$ if

$$\sigma(Y_t) \subset \mathcal{F}_t \qquad \text{for all} \qquad t = 0, 1, 2, ...$$

where $\sigma(Y_t)$ is the field generated by random variable Y_t.

Definition 1.18. *The stochastic process* $X = (X_n, n = 0, 1, ...)$ *is called a* discrete-time martingale *with respect to the filtration* $(\mathcal{F}_n, n = 0, 1, ...)$, *we write* $(X, (\mathcal{F}_n))$, *if*

(*i*) $E|X_n| < \infty$ *for all* $n = 0, 1, ...$

(*ii*) X *is adapted to* $(\mathcal{F}_n)$.

(*iii*) $E(X_{n+1}|\mathcal{F}_n) = X_n$ *for all* $n = 0, 1, ...,$

i.e. X_n *is the best prediction of* X_{n+1} *given* $\mathcal{F}_n$.

The continuous-time process $Y = (Y_t, t \geq 0)$ is said to be *adapted to the filtration* $(\mathcal{F}_t, t \geq 0)$ if

$$\sigma(Y_t) \subset \mathcal{F}_t \qquad \text{for all} \qquad t \geq 0$$

where $\sigma(Y_t)$ is the field generated by random variable Y_t.

Definition 1.19. *The stochastic process* $X = (X_t, t \geq 0)$ *is called a* continuous-time martingale *with respect to the filtration* $(\mathcal{F}_t, t \geq 0)$, *we write* $(X, (\mathcal{F}_t))$, *if*

(*i*) $E|X_t| < \infty$ *for all* $t \geq 0$

(*ii*) X *is adapted to* $(\mathcal{F}_t)$.

(*iii*) $E(X_t|\mathcal{F}_s) = X_s$ *for all* $0 \leq s < t$,

i.e. X_s *is the best prediction of* X_t *given* $\mathcal{F}_s$.

Example 3. Let $X_1, X_2, ...$ be independent random variables with $E[X_n] = 1$ for all n. Let $Z_n = \prod_{i=1}^{n} X_i$, $n \geq 1$. Then $E|Z_n| = \prod_{i=1}^{n} E|X_i| < \infty$ for every n, and

$$\begin{aligned} E[Z_{n+1}|Z_1, ...Z_n] &= E[(X_{n+1}Z_n|Z_1, ...Z_n)] \\ &= Z_n E[X_{n+1}|Z_1, ...Z_n] \\ &= Z_n, \end{aligned}$$

so that $\{Z_n\}$ is a martingale.

Before giving the definition of lag and forward operators on isolated time scales, we will give the invariance property on the conditional expectations. The following result can be found in the paper [9] by Broze, Gourieroux and Szafarz. A conditional expectation is written by

$$E[y_{t+h-k}|I_{t-k}] = E_{t-k}[y_{t+h-k}]$$

where the information I_t is increasing with t and composed of the current and past observations. The condition $k \geq 0$ implies that one cannot make predictions using future observations of the variables. There is at most, for $k = 0$, a simultaneity between the dates at which y_t and the expectations are determined. This assumptions implies that y_t is a function of the variables appearing in I_t. Thus we have

$$E_{t-k}[y_{t+h-k}] = y_{t+h-k} \qquad \text{if } h \leq 0.$$

This invariance property allows us to consider, in the case $h = 0$, we have

$$E_{t-k}[y_{t-k}] = y_{t-k}.$$

Definition 1.20. *Let L and F denote respectively, a lag operator and a forward operator. L and F are defined as*

$$Ly_t \equiv F^{-1}y_t \equiv y_t^{\rho}, \qquad L^n y_t \equiv F^{-n}y_t \equiv y_t^{\rho^n},$$

$$L^{-1}y_t \equiv Fy_t \equiv y_t^{\sigma}, \quad L^{-n}y_t \equiv F^n y_t \equiv y_t^{\sigma^n}. \tag{1.2.1}$$

For example, the equation $y_t + by_t^{\rho^2} = ax_t^{\rho}$ can be written as $y_t + bL^2y_t = aLx_t$. Then, we can divide throughout by L^2 and use the fact that $L^{-2} = F^2$ to get $F^2y_t + by_t = aFx_t$, or $y_t^{\sigma^2} + by_t = ax_t^{\sigma}$.

Next, we state a lemma which provides us a significant result for our future work.

Lemma 1.5. *Let y_t be a random variable so that $y_t^{\Delta} = x_t$ where Δ derivative is with respect to t, then*

$$\int x_t \Delta t = y_t + M^*(t)$$

where $M^*(t)$ is any arbitrary martingale.

Proof.

$$\int x_t \Delta t = y_t + M^*(t)$$

means that

$$(y_t + M^*(t))^\Delta = x_t.$$

Using the invariance property of conditional expectation we rewrite $M^*(t)$ as

$$(y_t + E_t[M^*(t)])^\Delta = x_t$$
$$\text{where } E[M^*(t)|I_t] = E_t[M^*(t)].$$

$$y_t^\Delta + (E_t M^*(t))^\Delta = x_t.$$

By the assumption we know that $y_t^\Delta = x_t$, then this implies that

$$(E_t[M^*(t)])^\Delta = 0.$$

Indeed, by the definition of Δ-derivative and the property of martingale on the LHS of the above equation we obtain

$$\frac{E_t[(M^*)^\sigma(t)] - E_t[M^*(t)]}{\mu(t)} = \frac{M^*(t) - M^*(t)}{\mu(t)} = 0.$$

$\square$

Lemma 1.6. *Let y_t be a random variable so that $y_t^\nabla = x_t$, then*

$$\int x_t \nabla t = y_t + (M^{**} \circ \sigma)(t)$$

where $(M^{**} \circ \sigma)(t)$ is any arbitrary martingale.

Proof.

$$\int x_t \nabla t = y_t + (M^{**} \circ \sigma)(t)$$

means that

$$(y_t + (M^{**} \circ \sigma)(t))^{\nabla} = x_t$$

Using the invariance property of conditional expectation we rewrite $(M^{**} \circ \sigma)(t)$ as

$$(y_t + E_t[(M^{**} \circ \sigma)(t)])^{\nabla} = x_t$$

$$y_t^{\nabla} + (E_t[(M^{**} \circ \sigma)(t)])^{\nabla} = x_t$$

By the assumption we know that $y_t^{\nabla} = x_t$, then this implies that

$$(E_t[(M^{**} \circ \sigma)(t)])^{\nabla} = 0$$

Indeed, by the definition of ∇-derivative and the property of martingale on the LHS of the above equation we obtain

$$\frac{E_t[(M^{**} \circ \sigma)(t)] - E_t[M^{**}(t)]}{\nu(t)} = \frac{M^{**}(t) - M^{**}(t)}{\nu(t)} = 0.$$

$\square$

The treatment of the notations $M^*(t)$ and $M^{**}(t)$ are similar to the notation of arbitrary constant in deterministic setting.

CHAPTER 2
A BRIEF INTRODUCTION TO RATIONAL EXPECTATIONS

In economics, *expectations* are defined as the prediction of future economic events or economists' opinions about the future prices, incomes, taxes or other important variables. According to modern economic theory, there is an important difference between economics and natural sciences which is the forward-looking decisions made by economic agents. Expectations are included in many areas of economics such as wage bargaining in the labor market, cost benefit analysis, exchange rates, financial market investment, etc.

In this thesis we are considering the rational expectations which is commonly used in literature. Rational expectations is an economic theory which provides the people or economic agents making future decision with available information and past experiences. The first approach of the rational expectations was begun approximately fifty years ago. At that time, rational expectations and forecast of future development were not clear nor perfect. Even though, proven forecasts are not exactly rational, still it has ring of truth. The purpose of the rational expectations is to give the optimal forecast of the future which means that rational expectations should have the best guess of the future with all information available such as weather conditions, market conditions, supply demand curves, etc. The result of the expectations depends on other available information, thus it changes as external factors change and affect the situation. For instance, assume that part of a crop was destroyed due to bad weather condition, so that typical price of the crop rises above normal. Depending on this situation, if the farmer expects that this high price will prevail, he will plant more than usual. Eventually, more planting will cause the price of the harvested crops to fall below normal. Referring to this example, forecasting the future will not be clear for rational expectations. In economics, agents form expectations that are accepted rationally because they are based on past experiences. These expectations must be adjusted when external influences change the situation, as in our example. Equilibrium of a dynamic model can be described by a probability distribution over order of data. Also, data for every agent is consistent with this equilibrium probability distribution, so that there is relation between outcomes which are generated by the model and expectations. On field rational expectations, there have been three influential economists: John F. Muth, Robert E. Lucas, and Thomas J. Sargent.

Rational expectations was proposed by John F. Muth in the early 1960s. He is an American Economist and is known as the father of the rational expectations idea. He gave the idea of rational expectations in his linear microeconomic model. He published the first paper [13] in this area. John F Muth got his Ph.D. degree in Mathematical Economics from the Carnegie Mellon University and was the first recipient of the Alexander Henderson Award. Eventually, rational expectations was established by Muth, and it has become the way of other economists. In the 1970s, Robert E. Lucas, another American Economist, began working on rational expectations equilibrium for a model whose agents have a different approach for determining the rational expectations [see [17]]. He obtained his Ph.D. degree in Economics in 1964 from the University of Chicago. Lucas received the Nobel Prize in 1995 for developing and applying the theory of rational expectations to an econometric hypothesis. Rational expectations have been transformed from micro-economics to macro-economics by Robert E. Lucas. Although, the first rational expectations hypothesis was introduced by John Muth, the process of the rational expectations did not gain too much attention until Lucas extended this approach. Another well-known economist is Thomas J. Sargent who focuses on the field of macroeconomics. Additionally, he specializes in the area of rational expectations and developed the rational expectations revolution. He also argued that decision makers cannot systematically manipulate the economy through predictable variables. According to Sargent's article, which was published at the Library Economics Liberty, "The concept of rational expectations asserts that outcomes do not differ systematically (i.e., regularly or predictably) from what people expected them to be. The concept is motivated by the same thinking that led Abraham Lincoln to assert, 'You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time.' From the viewpoint of the rational expectations doctrine, Lincoln's statement gets things right. It does not deny that people often make forecasting errors, but it does suggest that errors will not persistently occur on one side or the other". Also rational expectations has been updated by economists over the last three decades through articles and books by Sargent in [14], Lucas and Sargent in [15], and Hansen and Sargent [16]. Rational expectations is not only used in one specific economic field, but it has also been extended to many other fields of economics such as finance, labor economics, and industrial organization. Therefore, all influential economists who have studied rational expectations, have had a different approach and focus related to rational expectations. Cagan's hyperinflation model is an example of such an approach. Phillip D. Cagan is an American scholar, author and economist. He got his MA degree in 1951 and his Ph.D. in Economics in 1954 from the University of Chicago. Cagan's work focuses on controlling the inflation model. In 1956, he wrote a book [18] about the demand for money during hyperinflation. The demand for cash balance is a future inflation expectations, for which Cagan suggested the adaptive expectations. Cagan's model was a catalyst for a significant body of work in microeconomics and leading economists extended this idea and used it for their model. Sargent and Wallace transformed Cagan's model into a rational expectations model in 1973 [20], by adding three assumptions which are 1. conditions are such that adaptive expectations of inflation are rational, 2. the money demand disturbance is econometrically exogenous with respect to money growth and inflation, and 3. the money demand disturbance follows a random walk [see [19]]. Then, the Cagan's Hyperinflation model has the following form

$$y_t = aE[y_{t+1}|I_t] + cx_t, \tag{2.0.1}$$

where y_t is endogenous variable which is known as the independent variable generated within a model and z_t is exogenous variable which is known as the dependent variable generated within a model, $E[y_{t+1}|I_t] \equiv E_t[y_{t+1}]$ is the conditional expectation.

There are several techniques for solving the equation (2.0.1). We examine three techniques: repeated substitution, undetermined coefficients and Sargent's factorization method which are taken from Thompson's lecture notes [25]. Even though they were used extensively in literature, we detect some weaknesses in these three solution techniques.

METHOD 1. Repeated Substitution

Rewrite the equation (2.0.1) for $t+1$:

$$y_{t+1} = aE[y_{t+2}|I_{t+1}] + cx_{t+1},$$

and take conditional expectation on I_t:

$$E[y_{t+1}|I_t] = aE[y_{t+2}|I_t] + cE[x_{t+1}|I_t],$$

where, in the first term on the right hand side, applied the tower property of conditional expectation. Now substitute this expression for $E[y_{t+1}|I_t]$ into (2.0.1):

$$y_t = a^2E[y_{t+2}|I_t] + acE[x_{t+1}|I_t] + cx_t.$$

Repeat this substitution up to time $t+T$:

$$y_t = c\sum_{i=0}^{T} a^i E[x_{t+i}|I_t] + a^{T+1}E[y_{t+T+1}|I_t].$$

Now if y_t is bounded, then as $|a| < 1$ we have,

$$\lim_{T\to\infty} a^{T+1}E[y_{t+T+1}|I_t] = 0 \tag{2.0.2}$$

and so,

$$y_t = c\sum_{i=0}^{\infty} a^i E[x_{t+i}|I_t] \tag{2.0.3}$$

which is the solution of the problem under assumption (2.0.2). At this point, one might ask " what if $|a| \geq 1$ or y_t is unbounded? ". Thus, we cannot use this method for any coefficients.

METHOD 2. Undetermined Coefficients

As in the deterministic case, we guess a functional form for the solution and then verify it. Let's guess a form for the solution

$$y_t = \sum_{i=0}^{\infty} \lambda_i E[x_{t+i}|I_t] \tag{2.0.4}$$

where $\lambda_i = 1, 2, 3, ...$ are coefficients to be determined. If the guess is <u>correct</u>, then imposing rational expectations give us

$$E[y_{t+1}|I_t] = \sum_{i=0}^{\infty} \lambda_i E[x_{t+i+1}|I_t]. \tag{2.0.5}$$

If we substitute guesses (2.0.4) and (2.0.5) into the original equation (2.0.1) to obtain

$$\sum_{i=0}^{\infty} \lambda_i E[x_{t+i}|I_t] = a \sum_{i=0}^{\infty} \lambda_i E[x_{t+i+1}|I_t] + cx_t.$$

This equation should hold for any realizations of the sequences x_{t+i} for $i = 1, 2, 3, ...$, and the only way this can happen is if for every i, the coefficient on x_{t+i} on the LHS of the equation is identical to the coefficient on x_{t+i} on the RHS. Matching up the coefficients, we get

$$\lambda_0 = c \text{ and } \lambda_{i+1} = a\lambda_i = a^i c$$

and this again yields (2.0.4).

We, mathematicians, do not prefer to use guessing method if there is an accurate method to solve the equation.

METHOD 3. Sargent's Factorization Method

1) Lag and forward operators were used in solving stochastic rational expectations model by Sargent in (1975).

2) With the introduction of expectations operators, it is important to note that the lag and forward operators work on the time-subscript of the variable and not on the time subscript of the information set. That is,

$$LE[p_{t+i}|I_t] \equiv E[p_{t+i-1}|I_t]$$
$$LE[p_{t+1}|I_t] \equiv E[p_t|I_t] \equiv p_t.$$

Series expansions of forward operator is given as

$$(1-\alpha F)^{-1} = \sum_{i=0}^{\infty} \alpha^i F^i, forall \quad |\alpha| < 1, \tag{2.0.6}$$

$$(1-\alpha F)^{-1} = -\sum_{i=0}^{\infty} \alpha^{-i} F^{-i}, \text{ for all } |\alpha| > 1,$$

Same expansions are also valid for lag operator.

Sargent's factorization method first involves taking expectations on both sides of the equations conditional on the oldest information set that appears anywhere in the equation. In the equation (2.0.1), there is only one information set,I_t, so we take expectations over the entire equation based on I_t

$$E[y_t|I_t] = aE[y_{t+1}|I_t] + cE[x_t|I_t]. \tag{2.0.7}$$

The second step in Sargent's method is to write (2.0.7) in terms of the lag and forward operators:

$$E[y_t|I_t] = aFE[y_t|I_t] + cE[x_t|I_t]$$

which implies

$$(1-aF)E[y_t|I_t] = cE[x_t|I_t],$$

or

$$E[y_t|I_t] = c(1-aF)^{-1}E[x_t|I_t].$$

Using forward operator expansion (2.0.6), we obtain

$$E[y_t|I_t] = c\sum_{i=0}^{\infty} a^i F^i E[x_t|I_t].$$

Sargent's method is the most powerful one, particularly for problems with multiple solutions. On the other hand, it is often to be the most conceptually challenging since one needs to make a decision to use either forward shift operator or backward shift (lag) operator.

For the model (2.0.1), it was also put forward that the general solution may be expressed in terms of martingales (see Broze, Jansen, and Szafarz [21], Broze and Szafarz [22], Gourieroux, Laffont, Monfort [23], Pesaran [24]). The method we develop in this study is based on the calculus of time scales and martingales solution of the model.

CHAPTER 3
GENERALIZATION OF CAGAN TYPE RATIONAL EXPECTATIONS MODEL

In this chapter, we study the Cagan Type Rational Expectations (CTRE) model. For our purposes, we assume $\mathbb{T}$ is an isolated time scale. In section 3.1, we introduce the Cagan's hyperinflation model and explain how to convert it to the rational expectations model. In Section 3.2, we construct the (CTRE) model on $\mathbb{T}$. We continue our work with defining the linear system and higher order (CTRE) on $\mathbb{T}$ in Section 3.3. Afterwards, in Section 3.4, we consider a second order (CTRE) model. Then we use the discrete Putzer Algorithm to find the general solution for the second order (CTRE) model. In Section 3.5, we examine the existence and uniqueness of the solution of the (CTRE) model.

3.1 The Cagan's Hyperinflation Model

A simple model which contains a future expectation of the endogenous variable (the independent variable generated within a model) is called the Cagan's (1956) hyper-inflation model. This is a model on the money market that ascribes an important role on the expected inflation. The real demand for money is given by

$$m_t^d - y_t = \alpha(y_{t+1}^* - y_t), \tag{3.1.1}$$

where m_t^d is the logarithm of the nominal money demand at the date t, y_t is the logarithm of the price level at date t and y_{t+1}^* is the price level expected by agent at time $t+1$ given all information available at time t. The nominal money demand is defined by stochastic process;

$$m_t^d = \widetilde{z}_t.$$

Thus the demand on the money market yields;

$$\alpha(y_{t+1}^* - y_t) = \widetilde{z}_t - y_t$$

$$\alpha y^*_{t+1} - \alpha y_t = \widetilde{z}_t - y_t$$

$$(1-\alpha)y_t = \widetilde{z}_t - \alpha y^*_{t+1}$$

$$y_t = \frac{\widetilde{z}_t}{1-\alpha} - \frac{\alpha}{1-\alpha} y^*_{t+1}$$

$$y_t = \frac{\alpha}{\alpha - 1} y^*_{t+1} - \frac{\widetilde{z}_t}{\alpha - 1}$$

let $a = \dfrac{\alpha}{\alpha - 1}$ and $f(t, z_t) = -\dfrac{\widetilde{z}_t}{\alpha - 1}$.

Hence the Cagan's model can be written as

$$y_t = aE[y_{t+1}|I_t] + f(t, z_t), \tag{3.1.2}$$

with y_t is endogenous variable which is known as the independent variable generated within a model and z_t is exogenous variable which is known as the dependent variable generated within a model, $E[y_{t+1}|I_t] \equiv E_t[y_{t+1}]$ is the subjective expectation formed by the only one economic agent. Following the rational expectation (RE) hypothesis, it is assumed that this expectation is identical to the conditional mathematical expectation of y_{t+1} with respect to all the information available at time t and included in I_t. The information set contains the observations on y_t, z_t and their past values, i.e. $I_t = (z_t, z_{t-1}, ...)$. Consequently, it represents a increasing set with $I_t \supset I_{t-1} \supset I_{t-2} \supset$ ..... This implies that the economic agent has infinite memory. Furthermore, under the RE hypothesis, the agent "knows the model, namely its formal structure and the true values of the parameters"(L. Broze and A. Szafarz [10]).

For further reading we refer the reader to the book by M. P. Tucci [8].

3.2 First Order CTRE Model on Isolated Time Scales

Let y_t be an endogenous variable and z_t be an exogenous variable, a be the parameter associated with the future expectation and $E[y_t^\sigma|I_t] \equiv E_t[y_t^\sigma]$ is the conditional expectation at time $\sigma(t)$ given all information available at time t. The first order

CTRE model is given as

$$y_t = aE_t[y_t^\sigma] + f(t, z_t), \tag{3.2.1}$$

where $t \in \mathbb{T}$.

Before giving the solution of the equation (3.2.1), we state a useful lemma.

Lemma 3.1. *Let y_t be a random variable, then*

$$\int E_t[y_t^\sigma]\Delta t = E_t[\int y_t^\sigma \Delta t] + M^*(t)$$

where $M^*(t)$ is an arbitrary martingale.

Proof. To prove the lemma we show that

$$(E_t[\int y_t^\sigma \Delta t] + M^*(t))^\Delta = E_t[y_t^\sigma].$$

By the invariance property of conditional expectation we write $M^*(t) = E_t[M^*(t)]$ on the LHS of the above equation, then we get

$$(E_t[\int y_t^\sigma \Delta t] + E_t[M^*(t)])^\Delta$$
$$(E_t[\int y_t^\sigma \Delta t])^\Delta + (E_t[M^*(t)])^\Delta.$$

By Lemma 1.5 we have $(E_t[M^*(t)])^\Delta = 0$, thus we obtain

$$(E_t[\int y_t^\sigma \Delta t])^\Delta.$$

We define $\int y_t^\sigma \Delta t = G_t$ and using the definition of Δ derivative we get

$$(E_t[G_t])^\Delta = \frac{FE_t[G_t] - E_t[G_t]}{\mu(t)}$$

By the properties of forward operator (1.2.1), we obtain

$$(E_t[G_t])^\Delta = \frac{E_t[FG_t] - E_t[G_t]}{\mu(t)}.$$

$$(E_t[G_t])^\Delta = \frac{E_t[G_t^\sigma] - E_t[G_t]}{\mu(t)}.$$

The linearity property of conditional expectation on the RHS of the above equation gives

$$E_t[\frac{G_t^\sigma - G_t}{\mu(t)}] = E_t[G_t^\Delta],$$

writing back the $G_t = \displaystyle\int y_t^\sigma \Delta t$ on the above equation, we obtain

$$E_t[G_t^\Delta] = E_t[(\int y_t^\sigma \Delta t)^\Delta] = E_t[y_t^\sigma].$$

Thus we conclude that $\displaystyle\int E_t[y_t^\sigma]\Delta t = E_t[\int y_t^\sigma \Delta t] + M^*(t)$. This completes the proof. □

Theorem 3.1. *Let $\mathbb{T}$ be an isolated time scale. Then the solution of the first order CTRE model (3.2.1) is given by*

$$y_t = e_{\frac{1-a}{a\mu}}(t,0)M(t) - e_{\frac{1-a}{a\mu}}(t,0)\int e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}f(t,z_t)\Delta t$$

where $t \in \mathbb{T}$ and the $M(t)$ is an arbitrary martingale, i.e. satisfies the martingale property
$E_t[M^\sigma(t)] = M(t)$.

Proof. If we rewrite y_t using the invariance property of conditional expectation with the information set I_t, that is, $E_t[y_t] = y_t$, on the equation (3.2.1), we have

$$E_t[y_t] = aE_t[y_t^\sigma] + f(t,z_t)$$

$$E_t[y_t^\sigma] - \frac{1}{a}E_t[y_t] = -\frac{1}{a}f(t,z_t).$$

Then, dividing both side by $e^\sigma_{\frac{1-a}{a\mu}}(t,0)\mu(t)$ we get

$$\frac{E_t[y_t^\sigma] - \frac{1}{a}E_t[y_t]}{e^\sigma_{\frac{1-a}{a\mu}}(t,0)\mu(t)} = \frac{\frac{-1}{a}f(t,z_t)}{e^\sigma_{\frac{1-a}{a\mu}}(t,0)\mu(t)},$$

by Lemma 1.1(ii) we obtain that $e^{\sigma}_{\frac{1-a}{a\mu}}(t,0) = \dfrac{1}{a} e_{\frac{1-a}{a\mu}}(t,0)$. Thus, it follows that

$$\frac{e^{\sigma}_{\ominus\frac{1-a}{a\mu}}(t,0)E_t[y_t^{\sigma}] - e_{\ominus\frac{1-a}{a\mu}}(t,0)E_t[y_t]}{\mu(t)} = -e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}E_t[f(t,z_t)].$$

By the invariance property of the conditional expectation we rewrite the LHS of the above equation as

$$\frac{E_t[e^{\sigma}_{\ominus\frac{1-a}{a\mu}}(t,0)y_t^{\sigma}] - E_t[e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t]}{\mu(t)} = -e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}E_t[f(t,z_t)].$$

Then the linearity property of the conditional expectation on the LHS of the equation gives

$$\frac{E_t[e^{\sigma}_{\ominus\frac{1-a}{a\mu}}(t,0)y_t^{\sigma} - e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t]}{\mu(t)} = -e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}E_t[f(t,z_t)].$$

LHS of the above equation is Δ-derivative of $e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t$, that is

$$E_t[(e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t)^{\Delta}] = -e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}f(t,z_t).$$

Taking the integral both of side of the above equation and by Lemma 1.5 we obtain

$$\int E_t[(e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t)^{\Delta}]\Delta t = M^{**}(t) - \int e_{\ominus\frac{1-a}{a\mu}(t,0)}\frac{1}{\mu(t)}f(t,z_t)\Delta t$$

using Lemma 3.1 on the LHS of the above equation we have

$$E_t[\int (e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t)^{\Delta}\Delta t] + M^{*}(t) = M^{**}(t) - \int e_{\ominus\frac{1-a}{a\mu}(t,0)}\frac{1}{\mu(t)}f(t,z_t)\Delta t$$
$$E_t[e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t] = (M^{**}(t) - M^{*}(t)) - \int e_{\ominus\frac{1-a}{a\mu}(t,0)}\frac{1}{\mu(t)}f(t,z_t)\Delta t$$

where $M^{**}(t) - M^{*}(t) = M(t)$ is a martingale. Then we have

$$e_{\ominus\frac{1-a}{a\mu}}(t,0)y_t = M(t) - \int e_{\ominus\frac{1-a}{a\mu}(t,0)}\frac{1}{\mu(t)}f(t,z_t)\Delta t,$$

dividing both side of the above equality by $e_{\ominus\frac{1-a}{a\mu}}(t,0)$ we obtain

$$y_t = e_{\frac{1-a}{a\mu}}(t,0)M(t) - e_{\frac{1-a}{a\mu}}(t,0)\int e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}f(t,z_t)\Delta t.$$

$\square$

Next, we derive the nabla solution of the equation (3.2.1).

Theorem 3.2. *Let $\mathbb{T}$ be an isolated time scale. Then the nabla solution of the equation (3.2.1) is given by*

$$y_t = \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(\rho(t),0)f(\rho(t),z_t^{\rho})\nabla t$$

where $t \in \mathbb{T}$ and the $\hat{M}(t)$ is an arbitrary martingale, i.e. satisfies the martingale property
$E_t[M^{\sigma}(t)] = M(t)$.

Proof. We rewrite y_t using the invariance property of conditional expectation on I_t, on the equation (3.2.1), then we have

$$E_t[y_t] = aE_t[y_t^{\sigma}] + f(t,z_t)$$
$$E_t[y_t^{\sigma}] - \frac{1}{a}E_t[y_t] = -\frac{1}{a}f(t,z_t).$$

To obtain the nabla derivative on the RHS of the above equation we multiply both sides by $\dfrac{1}{\nu(t)}\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)$, we get

$$\frac{\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)E_t[y_t^{\sigma}] - \frac{1}{a}\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)E_t[y_t]}{\nu(t)} = -\frac{1}{a\nu(t)}\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)$$

$$\frac{E_t[\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y_t^{\sigma}] - E_t[\frac{1}{a}\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y_t]}{\nu(t)} = -\frac{1}{a\nu(t)}\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t). \tag{3.2.2}$$

.

By Theorem 1.5 we have

$$\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0) = e^{\sigma}_{\frac{a-1}{\mu}}(t,0),$$

by Lemma 1.1(ii) we have

$$e^{\sigma}_{\frac{a-1}{\mu}}(t,0) = ae_{\frac{a-1}{\mu}}(t,0),$$

and using Theorem 1.5 we get an useful equality

$$\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0) = a\hat{e}_{\frac{a-1}{a\nu}}(t,0).$$

Then it follows that the equation (3.2.2) becomes

$$\frac{E_t[\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y^{\sigma}_t] - E_t[\frac{1}{a}a\hat{e}_{\frac{a-1}{a\nu}}(t,0)y_t]}{\nu(t)} = -\frac{1}{a\nu(t)}a\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t).$$

By the linearity property of conditional expectation we have

$$\frac{E_t[\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y^{\sigma}_t - \frac{1}{a}a\hat{e}_{\frac{a-1}{a\nu}}(t,0)y_t]}{\nu(t)} = -\frac{1}{a\nu(t)}a\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)$$

which is equivalent to

$$E_t[(\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y^{\sigma}_t)^{\nabla}] = -\frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t). \tag{3.2.3}$$

Integrating both sides of the equation (3.2.3) and using Lemma 1.6, we obtain

$$\int E_t[(\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y^{\sigma}_t)^{\nabla}]\nabla t = (M^{**}\circ\sigma)(t) - \int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t,$$

by Lemma 3.1 on the LHS of the above equation we get

$$E_t[\int(\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y^{\sigma}_t)^{\nabla}\nabla t] + M^{*}(t) = (M^{**}\circ\sigma)(t) - \int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t$$

$$E_t[\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)y^{\sigma}_t] = ((M^{**}\circ\sigma)(t) - M^{*}(t)) - \int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t$$

$$\hat{e}^{\sigma}_{\frac{a-1}{a\nu}}(t,0)E_t[y^{\sigma}_t] = \hat{M}(t) - \int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t,$$

where $\hat{M}(t) = (M^{**} \circ \sigma)(t) - M^*(t)$ is a martingale.

Hence we find $E_t[y_t^\sigma]$ as

$$E_t[y_t^\sigma] = \hat{e}^\sigma_{\ominus \frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}^\sigma_{\ominus \frac{a-1}{a\nu}}(t,0) \int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t$$

Since our aim is to find y_t, we solve the equation (3.2.1) for $E_t[y_t^\sigma]$ as $E_t[y_t^\sigma] = \dfrac{y_t - f(t,z_t)}{a}$, then we substitute this in the above equation and obtain

$$\frac{y_t - f(t,z_t)}{a} = \hat{e}^\sigma_{\ominus \frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}^\sigma_{\ominus \frac{a-1}{a\nu}}(t,0) \int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t.$$

By the property of exponential function we write the equality $\hat{e}^\sigma_{\ominus \frac{a-1}{a\nu}}(t,0) = \dfrac{1}{a}\hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0)$ which gives us

$$\begin{aligned}
\frac{y_t - f(t,z_t)}{a} &= \frac{1}{a}\hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \frac{1}{a}\hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0) \int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t \\
y_t - f(t,z_t) &= \hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0) \int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t \\
y_t &= \hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}_{\ominus \frac{a-1}{a\nu}}(t,0) \int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t + f(t,z_t).
\end{aligned}$$

Now, we use the integration by part formula in $\displaystyle\int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t$ by the equation (1.1.11)
$g^\nabla(t) = \dfrac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)$ and $h(t) = f(t,z_t)$, hence we get $g(t) = \dfrac{a}{a-1}\hat{e}_{\frac{a-1}{a\nu}}(t,0)$ and by the definition of nabla $(f(t,z_t))^\nabla = \dfrac{f(t,z_t) - f(\rho(t), z_t^\rho)}{\nu(t)}$, then we have

$$\begin{aligned}
\int \frac{1}{\nu(t)} \hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \nabla t = \frac{a}{a-1}&\hat{e}_{\frac{a-1}{a\nu}}(t,0) f(t,z_t) \\
&- \frac{a}{a-1} \int \hat{e}^\rho_{\frac{a-1}{a\nu}}(t,0) \frac{f(t,z_t) - f(\rho(t), z_t^\rho)}{\nu(t)} \nabla t
\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t = {} & \frac{a}{a-1}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\\ & -\frac{a}{a-1}\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\frac{1}{\nu(t)}\nabla t\\ & +\frac{a}{a-1}\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z^{\rho}_t)\frac{1}{\nu(t)}\nabla t.\end{aligned}$$

By the property of the nabla exponential function $\hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0) = \frac{1}{a}\hat{e}_{\frac{a-1}{a\nu}}(t,0)$, which implies

$$\begin{aligned}\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t = {} & \frac{a}{a-1}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\\ & -\frac{a}{a(a-1)}\int \hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\frac{1}{\nu(t)}\nabla t\\ & +\frac{a}{a-1}\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z^{\rho}_t)\frac{1}{\nu(t)}\nabla t.\end{aligned}$$

Then it follows from algebraic steps

$$\begin{aligned}(1+\frac{1}{a-1})\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t = {} & \frac{a}{a-1}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\\ & +\frac{a}{a-1}\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z^{\rho}_t)\frac{1}{\nu(t)}\nabla t.\end{aligned}$$

By doing elementary algebra we have,

$$\begin{aligned}\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t = {} & \hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\\ & +\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z^{\rho}_t)\frac{1}{\nu(t)}\nabla t.\end{aligned}$$

Multiplying both side of the above equation by $-\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)$ we obtain

$$\begin{aligned}-\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t = {} & -\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\\ & -\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z^{\rho}_t)\frac{1}{\nu(t)}\nabla t\end{aligned}$$

Using the property of exponential function, $\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\hat{e}_{\frac{a-1}{a\nu}}(t,0) = 1$, we get

$$\begin{aligned}-\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t + f(t,z_t) = {} & \\ & -\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z^{\rho}_t)\frac{1}{\nu(t)}\nabla t.\end{aligned}$$

Hence we can conclude that

$$y_t = \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \frac{1}{\nu(t)}\hat{e}_{\frac{a-1}{a\nu}}(t,0)f(t,z_t)\nabla t + f(t,z_t)$$

which equals to

$$y_t = \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z_t^{\rho})\frac{1}{\nu(t)}\nabla t. \qquad \square$$

Lemma 3.2. *Delta and nabla solution of the equation (3.2.1) are the same.*

Proof. We obtained the Δ solution of the equation (3.2.1) as

$$y_t = e_{\frac{1-a}{a\mu}}(t,0)M(t) - e_{\frac{1-a}{a\mu}}(t,0)\int e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}f(t,z_t)\Delta t \tag{3.2.4}$$

and ∇ solution as

$$y_t = \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\hat{M}(t) - \hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0)\int \hat{e}^{\rho}_{\frac{a-1}{a\nu}}(t,0)f(\rho(t),z_t^{\rho})\frac{1}{\nu(t)}\nabla t. \tag{3.2.5}$$

By the property of the exponential function and Theorem 1.5 we have

$$\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0) = \frac{1}{\hat{e}_{\frac{a-1}{a\nu}}(t,0)} = \frac{1}{e_{\frac{a-1}{\mu}}(t,0)} = e_{\ominus\frac{a-1}{\mu}}(t,0).$$

Then by the definition of the circle minus $\ominus\frac{a-1}{\mu} = \frac{1-a}{a\mu}$, which shows that the exponential functions on the equation (3.2.4) and (3.2.5) are equals

$$\hat{e}_{\ominus\frac{a-1}{a\nu}}(t,0) = e_{\ominus\frac{a-1}{\mu}}(t,0) = e_{\frac{1-a}{a\mu}}(t,0).$$

It remains to show equivalence of the integrals and the martingales on the equation (3.2.4) and (3.2.5). To show that, we give a lemma.

Lemma 3.3. *Let $f : \mathbb{T} \to \mathbb{R}$ be given function. Then*

$$\int f(t)\Delta t = \int f(\rho(t))\nabla t.$$

Proof. Assume that

$$\int f(\rho(t))\nabla t = F(t) + C$$

If we take the nabla derivative of both side we have

$$(\int f(\rho(t))\nabla(t))^{\nabla} = (F(t) + C)^{\nabla}$$
$$f(\rho(t)) = F^{\nabla}(t).$$

If we apply forward shift operator to each side of the above equation, we have

$$f(\sigma(\rho(t))) = F^{\nabla}(\sigma(t))$$
$$f(t) = F^{\nabla}(\sigma(t)).$$

By Theorem 1.6 (ii) $F^{\Delta}(t) = F^{\nabla}(\sigma(t))$, then we have

$$f(t) = (F(t))^{\Delta},$$

taking integral of the both sides with respect Δ operator we obtain

$$\int f(t)\Delta(t) = \int F^{\Delta}(t)\Delta(t)$$
$$\int f(t)\Delta(t) = F(t) + D.$$

Then we have

$$\int f(t)\Delta(t) = \int f(\rho(t))\nabla(t) + D - C.$$

Notice that $\hat{M}(t) = M(t) - D + C$. Thus we conclude that the equivalence of the equation (3.2.4) and (3.2.5). $\square$

Notice that for $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$. Thus the equation (3.2.1) becomes as

$$y_t = aE_t[y_{t+1}] + f(t, z_t). \tag{3.2.6}$$

Corollary 3.1. *$y_t = a^{-t}M(t) - a^{-t}\sum a^t f(t, z_t)$ is the solution of the equation (3.2.6).*

Corollary 3.2. *$y_t = a^{-t}M(t) - a^{-t}\sum a^{t-1} f(t-1, z_{t-1})$ is the backward difference solution of the equation (3.2.6).*

3.3 Linear Systems and Higher Order CTRE Model on Isolated Time Scales

As mentioned in the text [8], it may happen that several future expectations appear as explanatory variables on the right-hand side of the Cagan type rational expectation model (REM) (3.1.2). Thus the general form of the CTRE Model (3.1.2) is given by

$$y_t = a_n E_t[y_{t+n}] + a_{n-1}E_t[y_{t+n-1}] + ... + a_1 E_t[y_{t+1}] + f(t, z_t)$$

where y_t, z_t are endogenous and exogenous variables, respectively, $E_t[y_{t+n}]$ is the conditional expectation and $a_n, a_{n-1}, ...$ are constants. The presence of more than future expectations means that economic agent suffers the consequences of the rational prediction errors.

We define the general CTRE Model on isolated time scales $\mathbb{T}$ as

$$y_t = a_n E_t[y_t^{\sigma^n}] + a_{n-1}E_t[y_t^{\sigma^{n-1}}] + ... + a_1 E_t[y_t^{\sigma}] + f(t, z_t). \tag{3.3.1}$$

The next goal is to find the solution of this new formulation of CTRE Model (3.3.1). At this point, we can consider the equation (3.3.1) as a nonlinear stochastic equation. The idea is similar to solution techniques of differential equations.

The n^{th} order nonhomogenous CTRE and homogenous CTRE models are given by the following, respectively,

$$y_t = a_n E_t[y_t^{\sigma^n}] + a_{n-1} E_t[y_t^{\sigma^{n-1}}] + ... + a_1 E_t[y_t^{\sigma}] + f(t, z_t)$$

$$y_t = a_n E_t[y_t^{\sigma^n}] + a_{n-1} E_t[y_t^{\sigma^{n-1}}] + ... + a_1 E_t[y_t^{\sigma}]. \tag{3.3.2}$$

We characterize the general solution of the equation (3.3.1) through a sequence of theorems. Without loss of generality we consider a second order equation, i.e.

$$y_t = a_2 E_t[y_t^{\sigma^2}] + a_1 E_t[y_t^{\sigma}].$$

Theorem 3.3.

(i) If $u_1(t), u_2(t)$ are solutions of the homogenous equation $y_t = a_2 y_t^{\sigma^2} + a_1 y_t^{\sigma}$. Then $u(t) = M_{1t} u_1(t) + M_{2t} u_2(t)$,where the $M_{it}, i = 1, 2$ are arbitrary martingales, is a solution for $y_t = a_2 E_t[y_t^{\sigma^2}] + a_1 E_t[y_t^{\sigma}]$

(ii) If $w(t)$ solves the equation

$$y_t = a_2 E_t[y_t^{\sigma^2}] + a_1 E_t[y_t^{\sigma}] \tag{3.3.3}$$

and $v(t)$ solves the equation

$$y_t = a_2 E_t[y_t^{\sigma^2}] + a_1 E_t[y_t^{\sigma}] + f(t, z_t) \tag{3.3.4}$$

then $w(t) + v(t)$ solves the equation (3.3.4).

(iii) If $y_1(t)$ and $y_2(t)$ solve the equation (3.3.4), then $y_1(t) - y_2(t)$ solves the equation (3.3.3).

Proof.

(i) Let $u_1(t)$ and $u_2(t)$ be solutions of

$$y_t = a_2 y_t^{\sigma^2} + a_1 y_t^{\sigma}.$$

Thus $u_1(t)$ and $u_2(t)$ satisfy the above equation, then we get

$$u_1(t) = a_2 u_1^{\sigma^2}(t) + a_1 u_1^{\sigma}(t)$$
$$u_2(t) = a_2 u_2^{\sigma^2}(t) + a_1 u_2^{\sigma}(t).$$

If we define $u(t) = M_{1t}u_1(t) + M_{2t}u_2(t)$, then

$M_{1t}u_1(t) + M_{2t}u_2(t) = a_2 E_t[M_{1t}u_1^{\sigma^2}(t) + M_{2t}u_2^{\sigma^2}(t)] + a_1 E_t[M_{1t}u_1^{\sigma}(t) + M_{2t}u_2^{\sigma}(t)],$

where $M_1(t)$ and $M_2(t)$ are martingales.

The LHS of the above equation can be written with conditional expectation as

$E_t[M_{1t}u_1(t)] + E_t[M_{2t}u_2(t)] = a_2 E_t[M_{1t}u_1^{\sigma^2}(t) + M_{2t}u_2^{\sigma^2}(t)] + a_1 E_t[M_{1t}u_1^{\sigma}(t) + M_{2t}u_2^{\sigma}(t)].$

By the linearity property of conditional expectation we have

$$E_t[M_{1t}\underbrace{(u_1(t) - a_2 u_1^{\sigma^2}(t) - a_1 u_1^{\sigma}(t))}_{\text{zero}}] + E_t[M_{2t}\underbrace{(u_2(t) - a_2 u_2^{\sigma^2}(t) - a_1 u_2^{\sigma}(t))}_{\text{zero}}]$$
$$= 0$$

(ii) If $w(t) + v(t)$ solves the equation (3.3.4) then,

$$w(t) + v(t) = a_2 E_t[w^{\sigma^2}(t) + v^{\sigma^2}(t)] + a_1 E_t[w^{\sigma}(t) + v^{\sigma}(t)] + f(t, z_t)$$

using the linearity property of conditional expectation, above equation can be written as

$$\underbrace{w(t) - a_2 E_t[w^{\sigma^2}(t)] - a_1 E_t[w^{\sigma}(t)]} + \underbrace{v(t) - a_2 E_t[v^{\sigma^2}(t)] - a_1 E_t[v^{\sigma}(t)] - f(t, z_t)}.$$

Since $w(t)$ solves the equation (3.3.3) we have

$$w(t) - a_2 E_t[w^{\sigma^2}(t)] - a_1 E_t[w^{\sigma}(t)] = 0$$

and $v(t)$ solves the equation (3.3.4) we have

$$v(t) - a_2 E_t[v^{\sigma^2}(t)] - a_1 E_t[v^{\sigma}(t)] - f(t, z_t) = 0$$

this shows $w(t) + v(t)$ solves the equation (3.3.4).

(iii) If $y_1(t) - y_2(t)$ solves (3.3.3), then

$$y_1(t) - y_2(t) = a_2 E_t[y_1^{\sigma^2}(t) - y_2^{\sigma^2}(t)] + a_1 E_t[y_1^{\sigma}(t) - y_2^{\sigma}(t)]$$

$$\underbrace{y_1(t) - a_2 E_t[y_1^{\sigma^2}(t)] - a_1 E_t[y_1^{\sigma}(t)]} - \underbrace{y_2(t) - a_2 E_t[y_2^{\sigma^2}(t)] - a_1 E_t[y_2^{\sigma}(t)]}$$

both parts of the above equation are solutions of equation (3.3.4) so they are equal to $f(t, z_t)$, then this provide that $y_1(t) - y_2(t)$ is a solution of the equation (3.3.3).

Up to this point we have focused on the single CTRE on isolated time scales. However, many CTRE models frequently involve several unknown quantities with an equal number of equations. We consider a system of the form

$$\begin{aligned}
y_1(t) &= a_{11} E_t[y^{\sigma}(t)] + ... + a_{1n} E_t[y^{\sigma^n}(t))] + f_1(t, z_t) \\
y_2(t) &= a_{21} E_t[y^{\sigma}(t)] + ... + a_{2n} E_t[y^{\sigma^n}(t)] + f_2(t, z_t) \\
&. \\
&. \\
&. \\
y_n(t) &= a_{n1} E_t[y^{\sigma}(t)] + ... + a_{nn}(t) E_t[y^{\sigma^n}(t)] + f_n(t, z_t)
\end{aligned}$$

This system can be written as an equivalent vector equation,

$$Y_t = A E_t[Y_t^{\sigma}] + F(t, Z_t) \tag{3.3.5}$$

where A is invertible $n \times n$ matrix and

$$Y_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ . \\ . \\ . \\ y_n(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & ... & a_{nn} \end{bmatrix}, \qquad F(t, Z_t) = \begin{bmatrix} f_1(t, z_t) \\ f_2(t, z_t) \\ . \\ . \\ . \\ f_n(t, z_t) \end{bmatrix}.$$

Theorem 3.4. *The solution of (3.3.5) is given as*

$$Y_t = e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M(t) - e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0) \int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t \tag{3.3.6}$$

where $t \in \mathbb{T}$ and I is the $n \times n$ identity matrix.

Proof. We prove using substitution method. Then, Y_t solves the equation (3.3.5) and we have

$$e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M(t) - e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0) \int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t \tag{3.3.7}$$

$$= AE_t[e^{\sigma}_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M^{\sigma}(t) - e^{\sigma}_{(I-A)A^{-1}\frac{1}{\mu}}(t,0) \int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(\sigma(t))}F(\sigma(t), Z^{\sigma}_t)\sigma^{\Delta}(t)\Delta t] + F(t, Z_t).$$

We rearrange the RHS of the equation (3.3.7) then we get,

$$\begin{aligned} &= AE_t[e^{\sigma}_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M^{\sigma}(t)] \\ &- AE_t[e^{\sigma}_{(I-A)A^{-1}\frac{1}{\mu}}(t,0) \int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(\sigma(t))}F(\sigma(t), Z^{\sigma}_t)\sigma^{\Delta}(t)\Delta t] \\ &+ e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)E_t[e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t)]. \end{aligned}$$

By Lemma 1.2(ii) we obtain that $e^{\sigma}_{(I-A)A^{-1}\frac{1}{\mu}}(t,0) = A^{-1}e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)$ and by the martingale property $E_t[M^{\sigma}(t)] = M(t)$ we have,

$= AA^{-1}e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M(t)$

$$-AA^{-1}e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)E_t[\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(\sigma(t))}F(\sigma(t),Z^{\sigma}_t)\sigma^{\Delta}(t)\Delta t]$$
$$+e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)E_t[e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t)]$$
$$= e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M(t)$$
$$-e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)E_t[\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(\sigma(t))}F(\sigma(t),Z^{\sigma}_t)\sigma^{\Delta}(t)\Delta t+e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t)].$$

To show the equality of RHS and LHS of the equation (3.3.7) it remains to show

$$\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(\sigma(t))}F(\sigma(t),Z^{\sigma}_t)\sigma^{\Delta}(t)\Delta t + e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t)$$
$$= \int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t.$$

By Lemma 1.2(ii) and (v) we have

$$P_t = \int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta(t) = A^{-1}\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t \tag{3.3.8}$$
$$= A^{-1}\int(e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))^{\Delta}(-(I-A)^{-1})A\mu(t)\frac{1}{\mu(t)}F(t,Z_t)\Delta t$$
$$= -(I-A)^{-1}\int(e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))^{\Delta}F(t,Z_t)\Delta t.$$

After applying the integration by parts for the above last delta integral, we obtain

$$P_t = -(I-A)^{-1}[e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t) - \int(e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F^{\Delta}(t,Z_t)\Delta t]$$
$$= -(I-A)^{-1}e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t)+(I-A)^{-1}\int(e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F^{\Delta}(t,Z_t)\Delta t].$$

By the definition of $\Delta - \mathit{derivative}$, we have $F^{\Delta}(t,Z_t) = \dfrac{F(\sigma(t),Z^{\sigma}_t)-F(t,Z_t)}{\mu(t)}$ and by the equation (3.3.8)
$P_t = A^{-1}\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t$, hence we get

$$A^{-1}\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t = -(I-A)^{-1}e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t)$$
$$+(I-A)^{-1}\int(e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(\sigma(t),Z_{\sigma(t)})\frac{1}{\mu(t)}\Delta t$$
$$-(I-A)^{-1}\int(e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(t,Z_t)\frac{1}{\mu(t)}\Delta(t).$$

Multiplying both side by $(I-A)$ we get

$$
\begin{aligned}
(A^{-1}-I)\int e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t = & -e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t) \\
& + \int (e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(\sigma(t),Z_{\sigma(t)})\frac{1}{\mu(t)}\Delta t \\
& - \int (e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(t,Z_t)\frac{1}{\mu(t)}\Delta t.
\end{aligned}
$$

After rearranging the above equation we have

$$
\begin{aligned}
A^{-1}\int (e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(t,Z_t)\frac{1}{\mu(t)}\Delta t = & -e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t) \\
& + \int (e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(\sigma(t),Z_{\sigma(t)})\frac{1}{\mu(t)}\Delta t.
\end{aligned}
$$

Here note that $\frac{1}{\mu(t)} = \frac{\sigma^{\Delta}(t)}{\mu(\sigma(t))}$. Then, we have

$$
\begin{aligned}
\int (e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(t,Z_t)\frac{1}{\mu(t)}\Delta t = & -e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)F(t,Z_t) \\
& + \int (e^{\sigma}_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0))F(\sigma(t),Z_{\sigma(t)})\frac{\sigma^{\Delta}(t)}{\mu(\sigma(t))}\Delta t.
\end{aligned}
$$

This last expression is what we need to see to finish the proof. This indicates the RHS of the equation (3.3.7) can be given as

$$
e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M(t) - e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)\int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t
$$

This completes the proof. □

For $\mathbb{T}=\mathbb{Z}$, the equation (3.3.5) will be

$$
Y_t = AE_t[Y_{t+1}] + F(t,Z_t). \tag{3.3.9}
$$

Corollary 3.3. *$Y_t = A^{-t}M(t) - A^{-t}\sum A^t F(t,Z_t)$ is the general solution of equation (3.3.9).*

3.4 Second Order Linear CTRE Model

In this section we consider the second order CTRE with constant coefficients

$$
a_2E_t(y_t^{\sigma^2}) + a_1E_t(y_t^{\sigma}) + a_0y_t = 0 \tag{3.4.1}
$$

with $a_2, a_1, a_0 \in \mathbb{R}$ on a isolated time scale $\mathbb{T}$.

The characteristic equation of the (3.4.1) is given as

$$a_2\lambda^2 + a_1\lambda + a_0 = 0$$

Now without loss of generality we write the equation (3.4.1) as

$$E_t(y_t^{\sigma^2}) - (\lambda_1 + \lambda_2)E_t(y_t^{\sigma}) + (\lambda_1\lambda_2) = 0 \tag{3.4.2}$$

where λ_1 and λ_2 are roots of the characteristic equation.

Next, we convert the equation (3.4.2) to the system, using the reduction of order, that is

$$\begin{array}{lll} y_1(t) = y_t & y_1^{\sigma}(t) = y_t^{\sigma} & E_t(y_1^{\sigma}(t)) = E_t(y_t^{\sigma}) \\ y_2(t) = E_t(y_t^{\sigma}) & y_2^{\sigma}(t) = E_t(y_t^{\sigma^2}) & E_t(y_2^{\sigma}(t)) = E_t(y_t^{\sigma^2}). \end{array}$$

Thus in terms of $y_1(t)$ and $y_2(t)$, the system of the equation (3.4.2) is given as

$$E_t(y_1^{\sigma}(t)) = y_2(t)$$

$$E_t(y_2^{\sigma}) = (\lambda_1 + \lambda_2)y_2 - \lambda_1\lambda_2 y_1$$

$$E_t \begin{bmatrix} y_1^{\sigma}(t) \\ y_2^{\sigma}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

where $A^{-1} = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$. we have already stated that solution of the equation (3.3.6). Then the solution of the equation (3.4.2) can be given as

$$Y_t = e_{(A^{-1}-I)\frac{1}{\mu}}(t, 0)M(t) \tag{3.4.3}$$

where $M(t)$ is an arbitrary martingale.

To write the solution explicitly we need to calculate $e_{(A^{-1}-I)\frac{1}{\mu}}(t, 0)$. We refer the paper by Merrell, Ruger and Severs [4]. According to the paper the exponential function $e_{(A^{-1}-I)\frac{1}{\mu}}(t, 0)$ can be given as

$$e_{(A^{-1}-I)\frac{1}{\mu}}(t, 0) = \prod_{s\in[0,t)} A^{-1} = (A^{-1})^{n_t}$$

where $n_t(t,0) := \displaystyle\int_0^t \frac{\Delta(\tau)}{\mu(\tau)}$ is a counting function for any isolated time scale $\mathbb{T}$. Next, we calculate $(A^{-1})^t$ by using the Putzer algorithm by Theorem (1.4) for $\mathbb{T} = \mathbb{Z}$. First, we find the characteristic roots of A^{-1}.

$$det \begin{vmatrix} -\lambda & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0.$$

Hence, λ_1 and λ_2 are the characteristic roots.

CASE I. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$

$$P_0 = I$$

$$P_1 = (A^{-1} - \lambda_1 I)I = \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_1\lambda_2 & \lambda_2 \end{bmatrix}$$

.

and

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The initial value problem

$$r_1(t+1) = \lambda_1 r_1(t)\ ,\ r_1(0) = 1$$

has the solution $r_1(t) = \lambda_1^t$, and

$$r_2(t+1) = \lambda_1^t + \lambda_2 r_2(t),\ r_2(0) = 0$$

has the solution $r_2(t) = \dfrac{\lambda_1^t - \lambda_2^t}{\lambda_1 - \lambda_2}$. Then

$$(A^{-1})^t = P_0 r_1(t) + P_1(t) r_2(t)$$

$$= \begin{bmatrix} \dfrac{-\lambda_2\lambda_1^t + \lambda_2^t\lambda_1}{\lambda_1 - \lambda_2} & \dfrac{\lambda_1^t - \lambda_2^t}{\lambda_1 - \lambda_2} \\ \\ \dfrac{-\lambda_2\lambda_1^{t+1} + \lambda_1\lambda_2^{t+1}}{\lambda_1 - \lambda_2} & \dfrac{\lambda_1^{t+1} - \lambda_2^{t+1}}{\lambda_1 - \lambda_2} \end{bmatrix}$$

Since n_t is an positive integer, we can write

$$(A^{-1})^{n_t} = \begin{bmatrix} \frac{-\lambda_2\lambda_1^{n_t} + \lambda_2^{n_t}\lambda_1}{\lambda_1 - \lambda_2} & \frac{\lambda_1^{n_t} - \lambda_2^{n_t}}{\lambda_1 - \lambda_2} \\ \\ \frac{-\lambda_2\lambda_1\lambda_1^{n_t} + \lambda_1\lambda_2\lambda_2^{n_t}}{\lambda_1 - \lambda_2} & \frac{\lambda_1\lambda_1^{n_t} - \lambda_2\lambda_2^{n_t}}{\lambda_1 - \lambda_2} \end{bmatrix}$$

Finally we obtain the solution of the equation (3.4.2) as,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-\lambda_2\lambda_1^{n_t} + \lambda_2^{n_t}\lambda_1}{\lambda_1 - \lambda_2} & \frac{\lambda_1^{n_t} - \lambda_2^{n_t}}{\lambda_1 - \lambda_2} \\ \\ \frac{-\lambda_2\lambda_1\lambda_1^{n_t} + \lambda_1\lambda_2\lambda_2^{n_t}}{\lambda_1 - \lambda_2} & \frac{\lambda_1\lambda_1^{n_t} - \lambda_2\lambda_2^{n_t}}{\lambda_1 - \lambda_2} \end{bmatrix} \begin{bmatrix} M_1(t) \\ \\ M_2(t) \end{bmatrix}$$

where $M(t) = \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}$ is an arbitrary bivariate martingale.

Therefore, we can conclude that,

$$y(t) = [\frac{-M_1(t)\lambda_2 + M_2(t)}{\lambda_1 - \lambda_2}]\lambda_1^{n_t} + [\frac{M_1(t)\lambda_1 - M_2(t)}{\lambda_1 - \lambda_2}]\lambda_2^{n_t}$$

is the general solution of the equation (3.4.2).

CASE II. If $\lambda = \lambda_1 = \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$

$$P_0 = I$$

$$P_1 = (A^{-1} - \lambda_1 I)I = \begin{bmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{bmatrix}$$

and

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The initial value problem

$$r_1(t+1) = \lambda r_1(t) \ , \ r_1(0) = 1$$

has the solution $r_1(t) = \lambda^t$, and

$$r_2(t+1) = \lambda r_2(t) + \lambda^t, \ r_2(0) = 0$$

has the solution $r_2(t) = t\lambda^{t-1}$. Then

$$(A^{-1})^t = P_0 r_1(t) + P_1(t) r_2(t)$$

$$= \begin{bmatrix} \lambda^t - t\lambda^t & t\lambda^{t-1} \\ -t\lambda^{t+1} & \lambda^t + t\lambda^t \end{bmatrix}$$

Since n_t is a positive integer, we can write

$$(A^{-1})^{n_t} = \begin{bmatrix} \lambda^{n_t} - n_t\lambda^{n_t} & \frac{n_t}{\lambda}\lambda^{n_t} \\ -n_t\lambda\lambda^{n_t} & \lambda^{n_t} + n_t\lambda^{n_t} \end{bmatrix}$$

Therefore, we can conclude that,

$$y(t) = \lambda^{n_t} M_1(t) + n_t\lambda^{n_t}[\frac{M_2(t)}{\lambda} - M_1(t)]$$

is the general solution of the equation (3.4.2).

CASE III. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ that is $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ or using the polar form

$$\lambda_{1,2} = re^{\pm i\theta} = r(cos\theta \pm isin\theta),$$

where $a^2 + b^2 = r^2$ and $tan\theta = \frac{b}{a}$. Then

$$\lambda_{1,2}^t = r^t e^{\pm i\theta t} = r^t(cos\theta t \pm isin\theta t).$$

$$P_0 = I$$

$$P_1 = (A^{-1} - \lambda_1 I)I = \begin{bmatrix} -re^{i\theta} & 1 \\ -r^2 & re^{-i\theta} \end{bmatrix}$$

.

and

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \end{bmatrix} = \begin{bmatrix} re^{i\theta} & 0 \\ 1 & re^{-i\theta} \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The initial value problem

$$r_1(t+1) = re^{i\theta} r_1(t) \ , \ r_1(0) = 1$$

has the solution $r_1(t) = r^t e^{i\theta t}$, and

$$r_2(t+1) = r^t e^{i\theta t} + re^{-i\theta} r_2(t), \ r_2(0) = 0$$

has the solution $r_2(t) = \dfrac{r^{t-1}}{1 - e^{2i\theta}}[e^{i\theta(1-t)} - e^{i\theta(t+1)}]$. Then

$$(A^{-1})^t = P_0 r_1(t) + P_1(t) r_2(t)$$

$$= \begin{bmatrix} r^t e^{i\theta t} - \dfrac{r^t e^{i\theta}\big[e^{i\theta(1-t)} - e^{i\theta(t+1)}\big]}{1-e^{2i\theta}} & \dfrac{r^{t-1}\big[e^{i\theta(1-t)} - e^{i\theta(t+1)}\big]}{1-e^{2i\theta}} \\ \dfrac{-r^{t+1}\big[e^{i\theta(1-t)} - e^{i\theta(t+1)}\big]}{1-e^{2i\theta}} & r^t e^{i\theta t} + \dfrac{r^t e^{i\theta t}\big[e^{i\theta(1-t)} - e^{i\theta(t+1)}\big]}{1-e^{2i\theta}} \end{bmatrix}$$

Since n_t is a positive integer, we can write

$$(A^{-1})^{n_t} = \begin{bmatrix} r^{n_t} e^{i\theta n_t} - \dfrac{r^{n_t} e^{i\theta}\big[e^{i\theta(1-n_t)} - e^{i\theta(n_t+1)}\big]}{1-e^{2i\theta}} & \dfrac{r^{n_t-1}\big[e^{i\theta(1-n_t)} - e^{i\theta(n_t+1)}\big]}{1-e^{2i\theta}} \\ \dfrac{-r^{n_t+1}\big[e^{i\theta(1-n_t)} - e^{i\theta(n_t+1)}\big]}{1-e^{2i\theta}} & r^{n_t} e^{i\theta n_t} + \dfrac{r^{n_t} e^{i\theta n_t}\big[e^{i\theta(1-n_t)} - e^{i\theta(n_t+1)}\big]}{1-e^{2i\theta}} \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = (A^{-1})^{n_t} \begin{bmatrix} M_1(t) \\ \\ M_2(t) \end{bmatrix}$$

Therefore, we can conclude that,

$$y(t) = r^{n_t} cos(\theta n_t) M_1(t) + r^{n_t} sin(\theta n_t)[\frac{r cos(\theta) M_1(t) - M_2(t)}{r sin(\theta)}]$$

is the general solution of the equation (3.4.2).

If we sum up three cases, we have

1. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then the general solution of the equation (3.4.2) is given as

$$y(t) = M_1^*(t) \lambda_1^{n_t} + M_2^*(t) \lambda_2^{n_t}$$

where $M_1^*(t)$ and $M_2^*(t)$ are arbitrary martingales.

2. If $\lambda = \lambda_1 = \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then the general solution of the equation (3.4.2) is given as

$$y(t) = M_1^*(t)\lambda_1^{n_t} + M_2^*(t)n_t\lambda_2^{n_t}$$

where $M_1^*(t)$ and $M_2^*(t)$ are arbitrary martingales.

3. If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ that is $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$, then the general solution of the equation (3.4.2) is given as

$$y(t) = M_1^*(t)r^{n_t}cos(\theta n_t) + M_2^*(t)r^{n_t}sin(\theta n_t)$$

where $M_1^*(t)$ and $M_2^*(t)$ are arbitrary martingales.

These three cases for $\mathbb{T} = \mathbb{Z}$ were studied on the paper by L. Broze, C. Gourieroux and A. Szafarz [9]. They found characteristic roots, λ_1 and λ_2, as inverse of ours. The authors obtained the similar results as we had here. Despite they claimed the results are in general form, they did not prove them.

3.5 An Observation About the Uniqueness of CTRE Model

At a first glance the CTRE model with an initial condition seems to have an unique solution. This observation forces us to examine the uniqueness of the CTRE model. Thus we add the initial value to the CTRE model. Then the first order IVP of CTRE on isolated time scale is given as

$$Y_t = AE_t[Y_t^\sigma]$$
$$Y_{t_0} = 0.$$

We have already pointed out the solution of

$$Y_t = AE_t[Y_t^\sigma]$$

is

$$Y_t = e_{(A^{-1}-I)\frac{1}{\mu}}(t, t_0)M(t).$$

Next we try to prove $Y_t = 0$ is the only solution for the above IVP. We have

$$Y_{t_0} = e_{(I-A)A^{-1}\frac{1}{\mu}}(t_0, t_0)M(t_0) = 0 \tag{3.5.1}$$

$$M(t_0) = 0.$$

To show $M(t) = 0$ for all $t \in \mathbb{T}$ we begin with a lemma,

Lemma 3.4. *If $A(t) : t \in [0, T]$ is continuous-parameter martingale satisfying*

(i) It is almost surely continuous,

(ii) It is almost surely of bounded variation, and

(iii) $A(0) = 0$,

then $A(t) \equiv 0$.

If we consider the constraints on $M(t)$ such that almost surely continuous and almost surely of bounded variation, then by the lemma we can conclude that $Y_{t_0} = 0$ is the only solution for IVP.

CHAPTER 4
APPLICATIONS

In this section we apply the solution techniques developed in Section 3 to three examples drawn from the literature. In the first example, we apply our solution algorithm to a model in Finance. In the second and third examples we apply our solution method to a model which is known as Stochastic Growth Model in Economics.

4.1 An Example in Finance: Wealth of a Self-Financing Trading Strategy

First, we shall introduce the trading strategy and self-financing trading strategy in Finance.

A *trading strategy* is a predictable process (a process H_t is called predictable if for each t, H_t is $\mathcal{F}_{t-1}$ measurable) with initial investment, $V_0(\theta) = \theta_0 S_0$ and wealth process $V_t(\theta) = \theta_t S_t$. Every trading strategy has an associated gains process defined by

$$G_t(\theta) = \sum_{k=0}^{t-1} \theta_t (S_{k+1} - S_k)$$

where S_k is price of the security.

A trading strategy θ is called *self financing* if the change in wealth is determined solely by capital gains and losses, i.e. if and only if $V_t(\theta) = V_0(\theta) + G_t(\theta)$.

For further reading we refer the book by M. Ammann [5].

In general, trading can be explained as buying and selling securities, commodities, goods or services. Demand of a stock or commodities may change over a time, sometimes monthly, daily or even hourly. Due to trading is not periodic or a continuous action in a certain time, we formulate the self-financing trading strategy formula on isolated time domains.

In the absence of arbitrage, every one-period model has a risk-neutral probability such that

$$R_f = E^Q[R_i], \text{ for all } i$$

where R_i is the return of asset i and the letter Q in E^Q denotes risk-neutral probability.

Consider one generic asset with return R_t and price S_t. Therefore the risky return between t and $\sigma(t)$ is simply

$$R_t^\sigma = \frac{S_t^\sigma}{S_t}.$$

Within the multi-period set-up the one-period pricing equation can be written by using the conditional expectation,

$$R_{ft} = E_t^Q[\frac{S_t^\sigma}{S_t}]. \tag{4.1.1}$$

Example 1. Consider a self-financing strategy with cash value V_t and risky investment θ_t,

$$V_t^\sigma = R_{ft}V_t + \theta_t S_t(\frac{S_t^\sigma}{S_t} - R_{ft}).$$

And if we apply $E_t^Q[.]$ both sides of the above equation we obtain

$$E_t^Q[V_t^\sigma] = E_t^Q[R_{ft}V_t] + E_t^Q[\theta_t S_t(\frac{S_t^\sigma}{S_t} - R_{ft})].$$

By the invariance property of the conditional expectation we acquire

$$E_t^Q[V_t^\sigma] = R_{ft}V_t + \theta_t S_t\{E_t^Q[\frac{S_t^\sigma}{S_t}] - R_{ft}\}.$$

By the equation (4.1.1) we have $E_t^Q[\frac{S_t^\sigma}{S_t}] - R_{ft} = 0$, thus we get

$$E_t^Q[V_t^\sigma] = R_{ft}V_t, \tag{4.1.2}$$

and after dividing both sides of (4.1.2) by R_{ft} we obtain

$$V_t = \frac{1}{R_{ft}} E_t^Q [V_t^\sigma]. \tag{4.1.3}$$

Equation (4.1.3) is the first order homogenous CTRE and its solution by Theorem (3.1) is obtained as

$$V_t = e_{\frac{R_{ft}-1}{\mu}}(t,0) M(t). \tag{4.1.4}$$

As a special case if we consider $\mathbb{T} = \mathbb{Z}$, the equation (4.1.4) is given as

$$V_t = \frac{1}{R_{ft}} M(t).$$

By virtue of the equation (4.1.4), we can conclude that *the wealth of any self-financing strategy* is a martingale under Q.

Next, we continue giving examples for the case $\mathbb{T} = \mathbb{Z}$. The social planner's problem is one of the common ones among the optimization problems in economics. The CTRE model arise from the second constraints of the social planner's problem. In the next example, we solve the second order CTRE on the social planner's problem.

4.2 The Stochastic Growth Models

Example 2. The social planner's problem is given by

$$\begin{aligned} &\sup_{k(t+1)} E \sum_{t=0}^{\infty} \beta^t U(k(t), c(t)) \\ \text{s.t. } \ & k(t+1) \in \Gamma(t, k(t), c(t), A(t)). \end{aligned}$$

where E is the expectation, $\in$ means inclusion.

First using Bellman optimality principle, the value function was obtained. Second, from the first order conditions they got the Euler-Lagrange equation and then log-linearized the Euler-Lagrange equation around the steady state. Linearization of the Euler equation is equivalent to maximizing a quadratic (second order) expansion of the objective function. It was given as

$$a_1 E_t[k_{t+2}] + a_2 E_t[k_{t+1}] + a_3 E_t[k_t] + a_4 A_{t+1} + a_5 A_t = 0. \tag{4.2.1}$$

This is the second order nonhomogenous CTRE model. We rewrite the equation (4.2.1) as

$$a_1 E_t[k_{t+2}] + a_2 E_t[k_{t+1}] + a_3 E_t[k_t] + z_t = 0 \tag{4.2.2}$$

where $z_t = a_4 A_{t+1} + a_5 A_t$.

Next, we convert the equation (4.2.2) to the system, using the reduction of order, that is

$$\begin{aligned} &k_t^1 = k_t, && k_{t+1}^1 = k_{t+1}, && E_t(k_{t+1}^1) = E_t(y_{t+1}), \\ &k_t^2 = E_t(k_{t+1}), && k_{t+1}^2 = E_{t+1}(k_{t+2}), && E_t(k_{t+1}^2) = E_t(y_{t+2}), \end{aligned}$$

$$E_t \begin{bmatrix} k_{t+1}^1 \\ k_{t+1}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_3}{a_1} & -\frac{a_2}{a_1} \end{bmatrix} \begin{bmatrix} k_t^1 \\ k_t^2 \end{bmatrix} \quad \text{and } Z_t = \begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix}.$$

By Corollary 3.3, solution of the equation (4.2.2) can be given as

$$K_t = A^{-t} M(t) - A^{-t} \sum A^t Z_t. \tag{4.2.3}$$

Next, using the Putzer algorithm [by Theorem (1.4)] we calculate A^{-t}, where

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -\frac{a_3}{a_1} & -\frac{a_2}{a_1} \end{bmatrix}.$$

The matrix A^{-1} has the eigenvalues $\lambda_1 = \frac{1}{1.005}$ and $\lambda_2 = -\frac{1}{0.503}$. These values were given in the notes by N. C. Mark [11].

By the Case I which was derived in Chapter 3, $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. We can conclude that

$$A^{-t} = \begin{bmatrix} \dfrac{-\lambda_2\lambda_1^t + \lambda_2^t\lambda_1}{\lambda_1 - \lambda_2} & \dfrac{\lambda_1^t - \lambda_2^t}{\lambda_1 - \lambda_2} \\ \\ \dfrac{-\lambda_2\lambda_1^{t+1} + \lambda_1\lambda_2^{t+1}}{\lambda_1 - \lambda_2} & \dfrac{\lambda_1^{t+1} - \lambda_2^{t+1}}{\lambda_1 - \lambda_2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-(1.98)(1.005)^{-t} - (0.99)(-0.503)^{-t}}{2.98} & \dfrac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} \\ \\ \dfrac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} & \dfrac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} \end{bmatrix}.$$

Thus first part of the solution (4.2.3) is given as

$$A^{-t}M(t)$$

$$= \begin{bmatrix} \dfrac{-(1.98)(1.005)^{-t} - (0.99)(-0.503)^{-t}}{2.98} & \dfrac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} \\ \\ \dfrac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} & \dfrac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} \end{bmatrix} \times$$

$$\begin{bmatrix} M_1(t) \\ \\ M_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-(1.98)(1.005)^{-t} - (0.99)(-0.503)^{-t}}{2.98} M_1(t) + \dfrac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} M_2(t) \\ \\ \dfrac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} M_1(t) + \dfrac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} M_2(t) \end{bmatrix}.$$

In addition to the above part, we calculate second part of the solution (4.2.3), that is

$$-A^{-t}\sum A^t Z_t$$

$$= \begin{bmatrix} \frac{-(1.98)(1.005)^{-t} - (-0.503)^{-t}(0.99)}{2.98} & \frac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} \\ \\ \frac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} & \frac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} \end{bmatrix} \times$$

$$\begin{bmatrix} \sum \frac{(1.98)(1.005)^{t} + (-0.503)^{t}(0.99)}{2.98} z_t^1 + \sum \frac{(1.005)^{t} - (-0.503)^{t}}{2.98} z_t^2 \\ \\ \sum \frac{(1.97)(1.005)^{t} - (1.97)(-0.503)^{t}}{2.98} z_t^1 + \sum \frac{(0.99)(1.005)^{t} + (1.98)(-0.503)^{t}}{2.98} z_t^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-(1.98)(1.005)^{-t} - (-0.503)^{-t}(0.99)}{2.98} f(t) + \frac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} g(t) \\ \\ \frac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} f(t) + \frac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} g(t) \end{bmatrix}$$

where

$$f(t) = \sum \frac{(1.98)(1.005)^{t} + (-0.503)^{t}(0.99)}{2.98} z_t^1 + \sum \frac{(1.005)^{t} - (-0.503)^{t}}{2.98} z_t^2$$

and

$$g(t) = \sum \frac{(1.97)(1.005)^{t} - (1.97)(-0.503)^{t}}{2.98} z_t^1 + \sum \frac{(0.99)(1.005)^{t} + (1.98)(-0.503)^{t}}{2.98} z_t^2$$

Thus, we obtained the explicit solutions as

$$\begin{aligned} k_t^1 = {} & \frac{-(1.98)(1.005)^{-t} - (0.99)(-0.503)^{-t}}{2.98} M_1(t) + \frac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} M_2(t) \\ & + \frac{-(1.98)(1.005)^{-t} - (-0.503)^{-t}(0.99)}{2.98} f(t) + \frac{-(1.005)^{-t} + (-0.503)^{-t}}{2.98} g(t) \end{aligned}$$

and

$$\begin{aligned} k_t^2 = {} & \frac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} M_1(t) + \frac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} M_2(t) \\ & + \frac{-(1.97)(1.005)^{-t} + (1.97)(-0.503)^{-t}}{2.98} f(t) + \frac{-(0.99)(1.005)^{-t} - (1.98)(-0.503)^{-t}}{2.98} g(t). \end{aligned}$$

where $M_1(t)$ and $M_2(t)$ are arbitrary martingales.

Next we consider another stochastic growth model was studied in E. Sims' [12] lecture notes.

Example 3. The non-linear system of difference equation was given as

$$c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma}(\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta))$$
$$k_{t+1} = a_t k_t^{\alpha} - c_t + (1-\delta)k_t$$
$$\ln a_t = \rho \ln a_{t-1} + e_t.$$

After log-linearization around the steady state, the below system was given

$$E_t \begin{bmatrix} c_{t+1} \\ k_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} 1.035 & -0.102 & 0.092 \\ -0.362 & 1.052 & 0.462 \\ 0 & 0 & 0.95 \end{bmatrix} \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}.$$

By Corollary 3.3, the solution of the above system can be given as

$$Y_t = A^{-t} M(t) \tag{4.2.4}$$

where $Y_t = \begin{bmatrix} c_t \\ k_t \\ a_t \end{bmatrix}$ and $M(t) = \begin{bmatrix} M_1(t) \\ M_2(t) \\ M_3(t) \end{bmatrix}$ is a vector valued martingale.

Next we calculate the A^{-t} using the Putzer algorithm by Theorem 1.4, where

$$A^{-1} = \begin{bmatrix} 1.035 & -0.102 & 0.092 \\ -0.362 & 1.052 & 0.462 \\ 0 & 0 & 0.95 \end{bmatrix}.$$

The matrix A^{-1} has the eigenvalues $\lambda_1 = 0.95$, $\lambda_2 = 0.85$ and $\lambda_3 = 1.23$. By Theorem 1.4, we obtain

$$P_0 = I$$

$$P_1 = \begin{bmatrix} 0.085 & -0.102 & 0.092 \\ -0.362 & 0.102 & 0.462 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.052 & -0.029 & -0.03 \\ -0.103 & 0.057 & 0.06 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \\ r_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0 & 0 \\ 1 & 0.85 & 0 \\ 0 & 1 & 1.23 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix}, \quad \begin{bmatrix} r_1(0) \\ r_2(0) \\ r_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The initial value problem

$$r_1(t+1) = (0.95)r_1(t), \quad r_1(0) = 1$$

has the solution $r_1(t) = (0.95)^t$, and

$$r_2(t+1) = (0.95)^t + (0.85)r_2(t), \quad r_2(0) = 0$$

has the solution $r_2(t) = 10(0.95)^t - 10(0.85)^t$, and

$$r_3(t+1) = 10(0.95)^t - (0.85)^t + (1.23)r_3(t), \quad r_3(0) = 0$$

has the solution $r_3(t) = \dfrac{100}{28}((1.23)^t - (0.95)^t) + \dfrac{100}{38}((1.23)^t - (0.85)^t)$.
Thus we obtain A^{-t} as

$$A^{-t} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where

$$a_{11} = (-0.00714)(0.95)^t - (2.70714)(0.85)^t + (3.71429)(1.23)^t$$

$$a_{12} = (0.01571)(0.95)^t + (2.05571)(0.85)^t - (2.07143)(1.23)^t$$

$$a_{13} = (1.99143)(0.95)^t + (0.15143)(0.85)^t - (2.14286)(1.23)^t$$

$$a_{21} = (0.05857)(0.95)^t + (7.29857)(0.85)^t - (7.35714)(1.23)^t$$

$$a_{22} = (-0.01571)(0.95)^t + (3.05571)(0.85)^t - (4.07143)(1.23)^t$$

$$a_{23} = (2.47714)(0.95)^t - (6.76286)(0.85)^t + (4.28571)(1.23)^t$$

$$a_{31} = 0, \quad a_{32} = 0, \quad a_{33} = (0.95)^t$$

Therefore, the solution of the equation (4.2.4) is obtained as,

$$c_t = a_{11}M_1(t) + a_{12}M_2(t) + a_{13}M_3(t)$$

$$k_t = a_{21}M_1(t) + a_{22}M_2(t) + a_{23}M_3(t)$$

$$a_t = a_{31}M_1(t) + a_{32}M_2(t) + a_{33}M_3(t).$$

As a conclusion, we calculated variables c_t, k_t and a_t explicitly. However, in the note [12] the following statement was given as a solution

$$c_t = 0.5557k_t + 0.5728a_t.$$

In addition, our solutions c_t, k_t and a_t satisfy the above relation.

CHAPTER 5
CONCLUSION AND FUTURE WORK

Rational expectations has been studied since 1960 by the many economists. The purpose of the rational expectations is to give the optimal forecast of the future with all information available. The idea of rational expectations has been important for both understanding macroeconomics, financial markets and having essential and remarkable implications to other areas. Despite the rational expectations has impact to develop the macroeconomics, there are still many open questions in this newly developing theory. In this thesis, we developed a new aspect to rational expectations using the time scale calculus. We formulated Cagan type rational expectations model on isolated time scales. Using the martingale approach we proved the theory about the general solution of CTRE model. There are two main findings in our study: 1. Our model unified and generalized the existing model. 2. The solution method we developed works for any given parameters. We also developed the linear system and higher order CTRE model on isolated time scales. We used the Putzer Algorithm to solve the system of CTRE model. Then, we examined the existence and uniqueness of the solution of CTRE model. We applied our solution algorithm to a finance problem and stochastic growth model problems.

For future work, we would like to apply the ideas that we presented for CTRE model to other rational expectations models. For instance,

$$y_t = a_0 E[y_{t+1}|I_t] + a_1 y_{t-1} + a_2 E[y_t|I_{t-1}] + a_3 p_t + e_t$$

or

$$\begin{aligned} D_t &= -\beta p_t && \text{(demand)} \\ S_t &= \gamma E[p_t|I_{t-1}] + \varepsilon_t && \text{(supply)} \end{aligned}$$

$$\begin{aligned} I_t &= \alpha(E[p_{t+1}|I_t] - p_t) && \text{(inventory demand)} \\ S_t &= D_t + (I_t - I_{t-1}). && \text{(market clear)} \end{aligned}$$

Moreover, up to this time martingales are defined on discrete-time and continuous-time, we would like to generalize and unify the martingales on time scales.

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