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Floquet Theory on Banach Space

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FLOQUET THEORY ON BANACH SPACE

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In this thesis we study Floquet theory on a Banach space. We are concerned about the linear differential equation of the the form:

$$y'(t) = A(t)y(t),$$

where $t \in R$, $y(t)$ is a function with values in a Banach space X, and $A(t)$ are linear, bounded operators on X. If the system is periodic, meaning $A(t + \omega) = A(t)$ for some period ω, then it is called a Floquet system. We will investigate the existence and uniqueness of the periodic solution, as well as the stability of a Floquet system.

This thesis will be presented in five main chapters. In the first chapter, we review the system of linear differential equations on R^n:

$$y' = A(t)y(t) + f(t),$$

where $A(t)$ is an $n \times n$ matrix-valued function, $y(t)$ are vectors, and $f(t)$ are functions with values in R^n. We establish the general form of the all solutions by using the fundamental matrix, consisting of n independent solutions. Also, we can find the stability of solutions directly by using the eigenvalues of A.

In the second chapter, we study the Floquet system on R^n, where $A(t + \omega) = A(t)$. We establish the Floquet theorem, in which we show that the Floquet system is closely related to a linear system with constant coefficients, so the properties of those systems can be applied. In particular, we can answer the questions about the stability of the Floquet system.

Then we generalize the Floquet theory to a linear system on Banach spaces. So, we introduce to the readers Banach spaces in chapter three and the linear operators on Banach spaces in chapter four. In the fifth chapter we study the asymptotic properties of solutions of the system:

$$y'(t) = A(t)y(t),$$

where $y(t)$ is a function with values in a Banach space X and $A(t)$ are linear, bounded operators on X with $A(t+\omega) = A(t)$. For that system, we can show that the evolution family $U(t,s)$ representing the solutions is periodic, i.e. $U(t+\omega, s+\omega) = U(t,s)$. Next we study the monodromy of the system $V := U(\omega, 0)$. We point out that the spectrum set of V actually determines the stability of the Floquet system. Moreover, we show that the existence and uniqueness of the periodic solution of the nonhomogeneous equation in a Floquet system is equivalent to the fact that 1 belongs to the resolvent set of V.

Linear Differential Equations On R^n

In mathematics, linear differential equations are differential equations that have solutions which can be added together to form other solutions. The solutions to linear equation form a vector space. Linear differential equation are of the form $L(y) = 0$ which we call a homogeneous differential equation or of the form $L(y) = f(t)$ which we call an inhomogeneous differential equation. If $L(y)$ is linear then $L(y) = 0$ is a linear differential equation. For example, since $L(y) = y' + 2y$ is linear then $L(y) = y' + 2y = 0$ is linear differential equation.

Linear Systems of Differential Equations

Consider the system of n linear differential equations of n variables:

$$y_1' = a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t)$$

$$y_2' = a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t)$$

$$\cdots$$

$$y_n' = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t)$$

This system can be written as an equivalent vector differential equation in R^n,

$$y' = A(t)y + f(t)$$

where $y := \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $y' = \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix}$, $A(t) := \begin{bmatrix} a_{11}(t) \ldots a_{1n}(t) \\ \vdots \ddots \vdots \\ a_{n1}(t) \ldots a_{nn}(t) \end{bmatrix}$, and

$$f(t) := \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

If the initial condition is given by $y(s) = y_0$, then we have an initial value problem. First we would like to give an existence and uniqueness result for the initial value problem. The reader can see the proof of this theorem for general case in Theorem V.50.

Theorem I.1. *Assume that the $n \times n$ matrix-valued function $A(t)$ and the vector-valued function $f(t)$ are continuous for $t \geq 0$, then the initial value problem:*

$$(IVP) \begin{cases} y'(t) = A(t)y(t) + f(t) \\ y(0) = y_0 \end{cases} \tag{I.1}$$

has a unique solution.

We first consider the homogeneous equation on R^n:

$$y'(t) = A(t)y(t). \tag{I.2}$$

Before giving the general form of the solution, we would like to give some definitions.

Definition I.2. *We say that k vectors $v_1, v_2, \cdots, v_k$ are linearly independent vectors in R^n if*

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

implies that

$$c_1 = c_2 = \cdots = c_k = 0.$$

Definition I.3. *We say that k vector-valued functions $f_1(t), f_2(t), \cdots, f_k(t)$ are linearly dependent on an interval I in R^n if there are constants c_1, c_2, ..., c_k, not all* *zero such that*

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_k(t) = 0$$

for all $t \in I$. Otherwise we say that these k functions are linearly independent.

We will start working on the homogeneous differential equation (I.2) with the following theorem:

Theorem I.4. (See [4], Theorem 2.11)

The linear differential equation (I.2) has (at most) n linearly independent solutions, and if ϕ_1, ϕ_2, ..., ϕ_n are n linearly independent solutions, then

$$y(t) = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t),$$

where c_1, c_2, ..., c_n are constants, is a general solution

We first solve the autonomous differential equation (I.2), when $A(t) = A$ is independent of t.

Theorem I.5. *If (λ_0, x_0) is an eigen pairs for the matrix A, then*

$$y(t) = e^{\lambda_0 t}x_0$$

defines a solution of

$$y'(t) = Ay(t) \tag{I.3}$$

on R.

Proof. Let

$$y(t) = e^{\lambda_0 t}x_0$$

then

$$
\begin{aligned}
y'(t) &= \lambda_0 e^{\lambda_0 t} x_0 \\
&= e^{\lambda_0 t} \lambda_0 x_0 \\
&= e^{\lambda_0 t} A x_0 \\
&= A e^{\lambda_0 t} x_0 \\
&= A y(t),
\end{aligned}
$$

for $t \in R$. □

Example 1. Solve the differential equation

$$y' = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} y$$

The eigen pairs of matrix

$$A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$$

are

$$\text{-1}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By theorem (I.5) the vector function ϕ_1, ϕ_2 defined by:

$$\phi_1(t) = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \phi_2 = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

are solutions on R. Since the vector functions ϕ_1, ϕ_2 are linearly independent on R, a general solution y is given by

$$y(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

To solve the nonautonomous differential equation (I.2) (where $A(t)$ is varying), we define the matrix differential equation:

$$X'(t) = A(t)X(t), \tag{I.4}$$

where

$$X = \begin{bmatrix} x_{11}(t) \dots x_{1n}(t) \\ \vdots \ddots \vdots \\ x_{n1}(t) \dots x_{nn}(t) \end{bmatrix},$$

The following theorem gives a relationship between the vector differential equation (I.2) and the matrix differential equation (I.4).

Theorem I.6. *Assume that $A(t)$ is a continuous $n \times n$ matrix function on an interval I and assume that ϕ is defined by*

$$\phi(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{bmatrix}$$

is an $n \times n$ matrix function. Then $\phi(t)$ is a solution of the matrix differential equation (I.4)on I iff each column ϕ_i is a solution of the vector differential equation (I.2) on I. Moreover, if ϕ is a solution of the matrix differential equation (I.4) then for any constant $n \times 1$ vector c

$$y(t) = \phi(t)c$$

is a solution of the vector differential equation (I.2).

Proof. Define

$$\phi(t) := \begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{bmatrix}$$

and assume that $\phi_1, \phi_2, \cdots, \phi_n$ are the solution of (I.2). Then ϕ is a continuously differentiable matrix function on I and

$$\begin{aligned}
\phi'(t) &= \left[\phi_1'(t), \phi_2'(t) \cdots \phi_n'(t) \right] \\
&= \left[A(t)\phi_1(t), A(t)\phi_2(t) \cdots A(t)\phi_n(t) \right] \\
&= A(t) \left[\phi_1(t), \phi_2(t) \cdots \phi_n(t) \right] \\
&= A(t)\phi(t),
\end{aligned}$$

for $t \in I$. Thus, ϕ is a solution of the matrix differential equation (I.4) on I.

Now, let

$$y(t) := \phi(t)c, \quad for \;\; t \in I,$$

where c is a constant $n \times 1$ vector and ϕ is a solution of the matrix differential equation (I.4). Then

$$\begin{aligned}
y'(t) &= \phi'(t)c \\
&= A(t)\phi(t)c \\
&= A(t)y(t)
\end{aligned}$$

for $t \in I$. □

From the above theorem, we have a direct result:

Corollary I.7. *Let $A(t)$ be a continuous matrix function on an interval I. Then the initial value problem*

$$(IVP) \begin{cases} X' = A(t)X, \\ X(t_0) = X_0, \end{cases}$$

where $t_0 \in I$ and X_0 is an $n \times n$ constant matrix, has a unique solution X that is a solution on the whole interval I.

The next theorem presents the relationship between the determinants and solution of a matrix differential equation.

Theorem I.8. (Liouville's Theorem [4], Theorem 2.23)

Assume that there are n solutions $\phi_1, \phi_2, \cdots, \phi_n$ of Equation (I.2) on I and ϕ is the matrix function with columns $\phi_1, \phi_2, \cdots, \phi_n$. Then if $t_0 \in I$,

$$\det \phi(t) = e^{\int_{t_0}^{t} tr[A(s)]ds} \det \phi(t_0)$$

where

$$tr[A(t)] = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)$$

and $t \in I$.

From the above theorem we see that $\det \phi(t) = 0$ for all $t \in I$ if and only if $\det \phi(t_0) = 0$. Hence, we can conclude the status of the determinants of a solution of the matrix equation as follows:

Corollary I.9. *Suppose that ϕ is the matrix function with the columns $\phi_1, \phi_2, \cdots, \phi_n$ and assume that $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of Equation (I.2). Then*

(a) The solution $\phi_1, \phi_2, \cdots, \phi_n$ are linearly dependent on I iff

$$\det \phi(t) = 0 \quad for \quad all \quad t \in I$$

(b) The solution $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I iff

$$\det \phi(t) \neq 0 \quad for \quad all \quad t \in I$$

Since each linear differential equation on R^n has n independent solutions, we would like to call them "fundamental" as in the following definition.

Definition I.10. (Fundamental matrix) *If an $n \times n$ matrix function ϕ is a solution of the matrix equation (I.4) on I and $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ are n linearly* *independent on I so that* $\det \phi(t) \neq 0$, *then*

$$\phi(t) := \begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{bmatrix}$$

is called fundamental matrix.

Example 2. Find a fundamental matrix ϕ for

$$y' = \begin{bmatrix} -2 & 3 \\ 2 & 3 \end{bmatrix} y. \tag{I.5}$$

The characteristic equation is

$$\lambda^2 - \lambda - 12 = 0$$

and the eigen pairs are

$$-3, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

and

$$4, \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence the vector functions ϕ_1, ϕ_2 defined by

$$\phi_1(t) = e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \textit{and} \quad \phi_2(t) = e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

for $t \in R$ are solution of (I.5). It follows from Theorem (I.6) that the matrix function ϕ defined by

$$\phi(t) = \begin{bmatrix} \phi_1(t), \phi_2(t) \end{bmatrix} = \begin{bmatrix} 3e^{-3t} & e^{4t} \\ -e^{-3t} & 2e^{4t} \end{bmatrix},$$

for $t \in R$ is a matrix solution of the matrix equation corresponding to (I.4). Since

$$\det \phi(t) = \begin{bmatrix} 3e^{-3t} & e^{4t} \\ -e^{-3t} & 2e^{4t} \end{bmatrix} = 7e^{t} \neq 0,$$

for all $t \in R$, ϕ is a fundamental matrix of (I.5) on R. By Theorem (I.6), the general solution y of (I.5) is given by

$$y(t) = \phi(t)c = \begin{bmatrix} 3e^{-3t} & e^{4t} \\ -e^{-3t} & 2e^{4t} \end{bmatrix} c,$$

for $t \in R$, where c is an arbitrary 2×1 constant vector.

Theorem I.11. *If ϕ is a fundamental matrix for (I.4), then $\psi = \phi \cdot C$, where C is an arbitrary nonsingular constant matrix, is a general fundamental matrix for (I.4).*

Proof. Assume that ϕ is a fundamental matrix for (I.4) and let

$$\psi = \phi \cdot C$$

where C is an arbitrary nonsingular constant matrix. Then we have

$$\begin{aligned} \psi' &= \phi'(t) \cdot C \\ &= A(t)\phi(t)C \\ &= A(t)\psi(t). \end{aligned}$$

Thus, $\psi = \phi \cdot C$ is a solution of matrix for (I.4). Moreover,

$$\begin{aligned} \det[\psi(t)] &= \det[\phi(t)C] \\ &= \det[\phi(t)] \det[C] \\ &\neq 0, \end{aligned}$$

for $t \in I$. Therefore, ψ is a fundamental matrix.

Furthermore, to show that any fundamental matrix ψ is of the form $\psi(t) = \phi(t)C_0$, assume that ψ is an arbitrary but fixed fundamental matrix of (I.4). Let $t_0 \in I$ and let

$$C_0 := \phi^{-1}(t_0)\psi(t_0).$$

Then C_0 is a nonsingular constant matrix and

$$\psi(t_0) = \phi(t_0)C_0.$$

Hence, both $\psi(t)$ and $\phi(t) \cdot C_0$ are solutions of the IVP

$$(IVP)\begin{cases} X' = A(t)X, \\ X(t_0) = \psi(t_0). \end{cases}$$

By Corollary I.7, we have

$$\psi(t) = \phi(t)C_0,$$

for $t \in I$. □

Now we try to find the fundamental matrix for the differential equation (I.4). First we consider the autonomous case, where $A(t) = A$.

Definition I.12. *Suppose that A is an $n \times n$ constant matrix. Then the matrix exponential function defined by e^{At} is the solution of*

$$(IVP)\begin{cases} X' = AX \\ X(0) = I, \end{cases} \tag{I.6}$$

where I is the $n \times n$ identity matrix.

By Liouville's Theorem, $\det[e^{At}] = e^{\int_0^t tr[A(s)]ds} \det[e^{A \cdot 0}] = e^{tr[A]t} \neq 0$. Hence e^{At} is a fundamental matrix. In the following theorem we give some properties of matrix exponential function

Theorem I.13. **([4], Theorem 2.39)**

Let A and B be $n \times n$ constant matrices. Then

(i) $\frac{d}{dt}e^{At} = Ae^{At}$, *for* $t \in R$;

(ii) $\det[e^{At}] \neq 0$ *for* $t \in R$ *and* e^{At} *is a fundamental matrix for* (I.3);

(iii) $e^{At}e^{As} = e^{A(t+s)}$, *for* $t, s \in R$;

(iv) $(e^{At})^{-1} = e^{-At}$, *for* $t \in R$ *and, in particular,*

$$(e^A)^{-1} = e^{-A},$$

(v) *if* $AB = BA$, *then* $e^{At}B = Be^{At}$ *for* $t \in R$ *and, in particular,*

$$e^A B = Be^A;$$

(vi) *if* $AB = BA$, *then* $e^{At}e^{Bt} = e^{(A+B)t}$ *for* $t \in R$ *and, in particular,*

$$e^A e^B = e^{(A+B)}$$

(vii) $e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots$, *for* $t \in R$;

(viii) *if* P *is a nonsingular matrix, then* $e^{PBP^{-1}} = Pe^BP^{-1}$.

We now are able to explicitly present the solution of an initial value problem.

Theorem I.14. *The following statements hold.*

(**i**) *The initial value problem*

$$(IVP)\begin{cases} y'(t) = Ay(t) \\ y(0) = y_0 \end{cases} \tag{I.7}$$

has the solution:

$$y'(t) = e^{At}y_0.$$

(ii) *The inhomogeneous IVP*

$$(IVP)\begin{cases} y'(t) = Ay(t) + f(t) \\ y(0) = y_0 \end{cases} \tag{I.8}$$

has solution

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}(s)f(s)ds$$

(iii) *If ϕ is a fundamental matrix for* (I.4), *then the solution of the IVP*

$$(IVP)\begin{cases} y'(t) = A(t)y(t) + f(t) \\ y(t_0) = y_0 \end{cases} \tag{I.9}$$

is given by

$$y(t) = \phi(t)\phi^{-1}(t_0)y_0 + \phi(t)\int_{t_0}^t \phi^{-1}(s)f(s)ds$$

First we state the following fact, which will be used in the proof.

Lemma I.15. ([4], Theorem 5.21)

Assume the vector-valued function $f(t,s)$ and the partial derivative $f_t(t,s)$ are continuous on an interval $I \times I$ and $a \in I$. Then

$$\frac{d}{dt}\int_a^t f(t,s)ds = \int_a^t f_t(t,s)ds + f(t,t).$$

Proof. (of Theorem I.8)

(i) Let $y(t) = e^{At}y_0$. Then

$$\begin{aligned} y'(t) &= Ae^{At}y_0 \\ &= Ay(t). \end{aligned}$$

Also,

$$\begin{aligned} y(0) &= e^{A(0)}y_0 \\ &= y_0. \end{aligned}$$

Thus $y(t) = e^{At}y_0$ is the solution of the initial value problem (I.7).

(ii) Let

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}(s)f(s)ds.$$

Then

$$\begin{aligned} y'(t) &= Ae^{At}y_0 + \int_0^t Ae^{A(t-s)}(s)f(s)d(s) + e^{A(t-t)}f(t) \\ &= Ae^{At}y_0 + \int_0^t Ae^{A(t-s)}(s)f(s)d(s) + f(t) \\ &= A[e^{At}y_0 + \int_0^t e^{A(t-s)}(s)f(s)d(s)] + f(t) \\ &= Ay(t) + f(t). \end{aligned}$$

Here, we used Lemma I.15 with $f(t,s) = e^{A(t-s)}f(s)$. Also, it is not hard too see that $y(0) = y_0$. Hence, $y(t)$ is the solution of (I.8).

(iii) Let ϕ be a fundamental matrix and let

$$y(t) = \phi(t)\phi^{-1}(t_0)y_0 + \phi(t)\int_{t_0}^t \phi^{-1}(s)f(s)ds$$

Then

$$
\begin{aligned}
y'(t) &= \phi'(t)\phi^{-1}(t_0)y_0 + \phi'(t)\int_{t_0}^{t}\phi^{-1}(s)f(s)ds + \phi(t)\phi^{-1}(t)f(t) \\
&= A(t)\phi(t)\phi^{-1}(t_0)y_0 + A(t)\phi(t)\int_{t_0}^{t}\phi^{-1}(s)f(s)ds + f(t) \\
&= A(t)[\phi(t)\phi^{-1}(t_0)y_0 + \phi(t)\int_{t_0}^{t}\phi^{-1}(s)f(s)ds] + f(t) \\
&= A(t)y(t) + f(t).
\end{aligned}
$$

Also,

$$
\begin{aligned}
y(t_0) &= \phi(t_0)\phi^{-1}(t_0)y_0 \\
&= y_0.
\end{aligned}
$$

Hence, $y(t)$ is the solution of (I.9).

□

In the general nonautonomous case, where $A(t)$ varies, it is not easy to find a fundamental matrix. The next theorem shows how a fundamental matrix can be found when the matrices $A(t)$ are commutative.

Theorem I.16. ([4], Theorem 2.42)

Assume that $A(t)$ is a continuous $n \times n$ matrix function on an interval I. If

$$A(t)A(s) = A(s)A(t)$$

for all $t, s \in I$, then

$$\phi(t) = e^{\int_{t_0}^{t} A(s)ds}$$

Stability of Solutions of Linear Differential Equations on R^n

We now study the asymptotic behavior of the solutions of autonomous, homogeneous differential equations in R^n as $t \to \infty$. First, we want to equip with some definitions of norms.

Definition I.17. (Norms on R^n)

A norm on R^n is a function $\|\cdot\| : R^n \mapsto R$ having the following properties:

(i) $\|x\| \geq 0$ *for all* $x \in R^n$*;*

(ii) $\|x\| = 0$ *if and only if* $x = 0$*;*

(iii) $\|cx\| = |c|\|x\|$ *for all* $c \in R$ *and* $x \in R^n$*;*

(iv) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in R^n$*.*

Example 3. For each $1 \leq p \leq \infty$ we define the functional $\|\cdot\|_p : R^n \to R$ by

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}.$$

In particular

1) $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$ (the traffic norm);

2) $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ (the Euclidean norm);

3) $\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$ (the maximum norm).

By standard arguments, we can show that $\|\cdot\|_p$ is a norm for all $1 \leq p \leq \infty$.

Definition I.18. (Induced Matrix Norm) *Assume that $\|\cdot\|$ is a norm on R^n. Let M_n denote the set of all $n \times n$ matrices. We define the matrix norm on M_n induced by the vector norm by*

$$\|A\| := \sup_{\|x\|=1} \|Ax\|.$$

We can list some basic properties of matrix norms without proving them

Theorem I.19. *The following statements hold:*

(i) For each matrix A, we have

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|};$$

(ii) For each matrix $A \in M_n$ and any vector $x \in R^n$ we have

$$\|Ax\| \leq \|A\| \cdot \|x\|;$$

(iii) For every matrices A and B in M_n we have

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|.$$

We now are able to state some results on the stability of solutions of autonomous, homogeneous differential equations, in which the equivalence of part a. and part e. is the famous Lyapunov's Theorem.

Theorem I.20. *Consider the system*

$$y'(t) = Ay(t)$$

Then the following statements are equivalent:

(a) The system is asymptotically stable, i.e.

$$\lim_{t \to \infty} e^{At} y_0 = 0 \quad \text{for each vector } y_0 \in R^n;$$

.

(b) $\lim_{t \to \infty} \|e^{At}\| = 0$;

(c) There exist positive numbers M and ω such that

$$\|e^{At}\| \leq Me^{-\omega t}$$

for all $t > 0$.

(d) There exists a number $t_0 > 0$ *such that* $\|e^{At_0}\| < 1$;

(e) $Re\lambda < 0$ *for each eigenvalue* λ *of* A.

Proof. The implication (c.$\rightarrow$ b.),(b.$\rightarrow$ d.) and (b.$\rightarrow$ a.) are obvious. We now prove (a.$\rightarrow$ b.), (d.$\rightarrow$ c.) and (a.$\leftrightarrow$ e.) one at a time.

(a.$\rightarrow$ b.) Assume that $(e^{At})_i$ be the i^{th} column of e^{At}. Now, take $y_0 = [1, 0, \cdots, 0]^T$ then, from part a, the solution of the system $y'(t) = Ay(t)$ is

$$y(t) = e^{At}y_0 = (e^{At})_1$$

Since $\lim_{t\to\infty}(e^{At})_1 = 0$, we have $\lim_{t\to\infty}\|(e^{At})_1\| = 0$. With the same reasoning, we can prove that

$$\lim_{t\to\infty}\|(e^{At})_i\| = 0 \quad for \quad i = 1, 2, \cdots, n.$$

Hence,

$$\lim_{t\to\infty}\|e^{At}\| = \lim_{t\to\infty}\sum_{i=1}^{n}\|(e^{At})_i\| = 0.$$

(d.$\rightarrow$ c.) Let t_0 be the number with $\|e^{At_0}\| = r_0 < 1$. Let $t = mt_0 + s$, where t is any number such that $t > t_0$, m is a natural number, and $0 \leq s \leq t_0$. Since the function e^{At} is continuous, the maximum $M = \max_{0\leq t\leq t_0}\|e^{At}\|$ exists. So we have that

$$
\begin{aligned}
\|e^{At}\| &= \|e^{A(mt_0+s)}\| \\
&= \|e^{Amt_0}e^{As}\| \\
&\leq \|e^{Amt_0}\|\|e^{As}\| \\
&\leq M\|e^{Amt_0}\| \\
&= M\|(e^{At_0})^m\| \\
&\leq M\|e^{At_0}\|^m \\
&= Mr_0^m = Me^{m\ln r_0} \\
&= Me^{(t-s)/t_0 \ln r_0} \quad (\text{ since } \quad m = (t-s)/t_0) \\
&= Me^{-s/t_0 \ln r_0} e^{(lnr_0/t_0)t}.
\end{aligned}
$$

Let $\omega = -(\ln r_0)/t_0$. Since $r_0 < 1$, hence $\ln r_0 < 0$ and $\omega > 0$. Thus we have

$$\|e^{At}\| \leq Me^{-s/t_0 \ln r_0} e^{-\omega t} \leq M_1 e^{-\omega t},$$

where $M_1 = M \max_{0 \leq s \leq t_0} e^{(-s/t_0) \ln r_0}$.

(a.→e.) We need to prove if $y(t) = e^{At}y_0 \to 0$ as $t \to \infty$ for all the initial values y_0, then $Re\ \lambda < 0$ for each eigenvalue λ of A. On the contrary, suppose that there exists an eigenvalue λ, where $Re\ \lambda \geq 0$. Let x_0 be the corresponding eigenvector to λ, then

$$y(t) = Re[e^{\lambda t}x_0] = e^{Re\lambda t} cos(Im\lambda t)x_0$$

is a solution of the system, which doesn't approach 0 as $t \to \infty$, since $x \neq 0$ and $Re\ \lambda \geq 0$. This is a contradiction to the assumptions. Thus, $Re\ \lambda < 0$.

(e.→a.) Assume that $Re\lambda < 0$ for each eigenvalue λ of A. We will prove part

(a) by three cases

(i) If λ is a single real eigenvalue and x_0 is corresponding eigenvector, then clearly the solution

$$y(t) = e^{\lambda t} x_0 \to 0 \quad as \quad t \to \infty$$

.

(ii) If λ is a real eigenvalue multiplicity k then

$$y(t) = t^k e^{\lambda t} \to 0 \quad as \quad t \to \infty$$

(iii) If λ is a complex eigenvalue such that $\lambda = \alpha + i\beta$ then

$$y(t) = e^{\alpha t} \cos(\beta t) x_0 \to 0 \quad as \quad t \to \infty$$

and

$$y(t) = e^{\alpha t} \sin(\beta t) x_0 \to 0 \quad as \quad t \to \infty$$

Thus, the general solution

$$y(t) = a_1 y_1 + a_2 y_2 + \cdots a_n y_n \to 0$$

□

With the same manner as the proof of Theorem I.20 we can also have prove the following result:

Theorem I.21. *Consider the system*

$$y'(t) = Ay(t).$$

(i) If the real parts of all eigenvalues of A are non-positive, and each eigenvalue with zero real part is simple, then all solutions are stable (i.e. they are

bounded)

(ii) If A has an eigenvalue with positive real part, then the solution is unstable (i.e. there is an unbounded solution).

Floquet Theory on R^n

Differential equations involving periodic functions play an important roll in many applications. We consider the linear system:

$$y'(t) = A(t)y(t) + f(t),$$

where $A(t)$ is a matrix-valued, continuous periodic function with period ω (i.e. $A(t+\omega) = A(t)$) and $f(t)$ is also ω-periodic. Such systems are called Floquet systems. A natural question is: Does a Floquet system have a periodic solution with period ω? Unfortunately, the answer is NO in general case, as Theorem II.26 states. However, the Floquet system turns out to be closely related to a linear system with constant coefficients, so that the properties which have been obtained in the previous chapters, can be used.

First we have a result about the existence of the logarithm of a matrix.

Theorem II.22. (Log of a Matrix)

If C is $n \times n$ nonsingular matrix then there is a matrix B such that

$$e^B = C.$$

Proof. We will prove this by 2×2 matrices. The general case can be proved by the same manner. Let λ_1, λ_2 be the two eigenvalues of C. Since C is nonsingular then $\lambda_1, \lambda_2 \neq 0$. We have 3 cases to prove the result:

Case 1. Assume that

$$C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Then, it is not hard to see that $C = e^B$, where

$$B = \begin{bmatrix} \ln \lambda_1 & 0 \\ 0 & \ln \lambda_2 \end{bmatrix}.$$

Case 2. Assume

$$C = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

In this case, we are looking for a matrix B of the form

$$B = \begin{bmatrix} b_1 & b_2 \\ 0 & b_1 \end{bmatrix},$$

so that $e^B = C$. For such matrix B, we can calculate that

$$e^{Bt} = \begin{bmatrix} e^{b_1 t} & b_2 t e^{b_1 t} \\ 0 & e^{b_1 t} \end{bmatrix}$$

and hence,

$$e^B = \begin{bmatrix} e^{b_1} & b_2 e^{b_1} \\ 0 & e^{b_1} \end{bmatrix}.$$

We now solve for b_1 and b_2 such that $e^B = C$, we have $b_1 = \ln \lambda_1$ and $b_2 = \frac{1}{\lambda_1}$.

So we can just take

$$B = \begin{bmatrix} \ln \lambda_1 & \frac{1}{\lambda_1} \\ 0 & \ln \lambda_1 \end{bmatrix}.$$

Case 3. C is an arbitrary 2×2 nonsingular constant matrix. By the Jordan canonical there is a nonsingular matrix P such that $C = PJP^{-1}$, where

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad or \quad J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

By the previous two cases there is a matrix B_1 so that

$$e^{B_1} = J.$$

Let

$$B := PB_1P^{-1};$$

then by theorem (I.13) part (viii)

$$e^B = e^{PB_1P^{-1}} = Pe^{B_1}P^{-1} = C.$$

□

Example 4. Find a log of the matrix

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

The characteristic equation for C is

$$\lambda^2 - 6\lambda + 5 = 0$$

The eigen pairs of C are

$$1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$5, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The Jordan canonical form of C is

$$J := \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

From the proof of theorem (II.22), if we let

$$P := \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix},$$

then

$$PB_1P^{-1}$$

is a log of C provided B_1 is a log of J. Note that

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & \ln 5 \end{bmatrix},$$

is a log of J. Hence a log of C is given by

$$\begin{aligned} B &= PB_1P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \ln 5 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4}\ln 5 & \frac{1}{4}\ln 5 \\ \frac{3}{4}\ln 5 & \frac{3}{4}\ln 5 \end{bmatrix}. \end{aligned}$$

We now have the main result of this chapter, which is called Floquet's

Theorem, for the Floquet system

$$y'(t) = A(t)y(t), \tag{II.10}$$

where $A(t+\omega) = A(t)$ for all t.

Theorem II.23. *If ϕ is a fundamental matrix for the Floquet system* (II.10) *where the matrix $A(t)$ is continuous on R and has period ω, then*

1) *The function ψ defined by*

$$\psi(t) = \phi(t+\omega)$$

is also a fundamental matrix for the Floquet system.

2) *There is a matrix B and a nonsingular $n \times n$ matrix function $\rho(t)$, which is periodic with period ω so that*

$$\phi(t) = \rho(t)e^{Bt}$$

for all $t \in R$.

Proof. 1) Assume that ϕ is a fundamental matrix for (II.10) and define

$$\psi(t) = \phi(t+\omega).$$

We want to show that ψ is also a fundamental matrix. So

$$\begin{aligned}
\psi'(t) &= \phi'(t+\omega) \\
&= A(t+\omega)\phi(t+\omega) \\
&= A(t+\omega)\psi(t) \\
&= A(t)\psi(t).
\end{aligned}$$

Because we define $\psi(t) = \phi(t+\omega)$ and ϕ is a fundamental matrix, then

$\det \psi = \det \phi(t+\omega) \neq 0$ for all t. Thus, ψ is also a fundamental matrix of (II.10).

2) Since ϕ and ψ are fundamental matrices for (II.10), then by theorem (I.11) there is constant matrix C such that

$$\phi(t+\omega) = \phi(t)C$$

And by theorem (II.22) there is a matrix B such that $e^{B\omega} = C$.

Define the matrix function ρ by

$$\rho(t) = \phi(t)e^{-Bt}$$

Now, we want to prove that $\rho(t)$ is periodic with period ω. We have

$$\begin{aligned}
\rho(t+\omega) &= \phi(t+\omega)e^{-B(\omega+t)} \\
&= \phi(t+\omega)e^{-B\omega-Bt} \\
&= \phi(t)\cdot C \cdot e^{-B\omega}\cdot e^{-Bt} \\
&= \phi(t)\cdot e^{B\omega}\cdot e^{-B\omega}\cdot e^{-Bt} \text{ (because } e^{B\omega} = C \text{ and } e^{-B\omega} = C^{-1}) \\
&= \phi(t)e^{-Bt} \\
&= \rho(t).
\end{aligned}$$

Hence, $\rho(t+\omega) = \rho(t)$ and $\rho(t)$ is a periodic matrix function. Therefore, $\rho(t) = \phi(t)e^{-Bt}$ implies that $\phi(t) = \rho(t)e^{Bt}$. □

We know that if $\phi(t)$ is a fundamental matrix for the Floquet system (II.10) then $\phi(t+\omega)$ is also fundamental matrix for (II.10). So, $\phi(t+\omega) = \phi(t)C$ where C is a constant matrix. Now if we want to find C, let $t = 0$ then we have

$$\phi(\omega) = \phi(0)C$$

then

$$C = \phi^{-1}(0)\phi(\omega).$$

Definition II.24. *Let ϕ be a fundamental matrix for the Floquet system* (II.10) *and $C = \phi^{-1}(0)\phi(\omega)$. The eigenvalues of C are called* <u>*Floquet multipliers*</u>*.*

Note that if we have another fundamental matrix ψ for the Floquet system (II.10) and

$$D = \psi^{-1}(0)\psi(\omega),$$

then all the eigenvalue of C must be the same as the eigenvalue of D.

To see that, by Theorem (I.11), there is an arbitrary $n \times n$ nonsingular constant matrix, let's call it M, such that

$$\psi(t) = \phi(t)M \quad \text{for all} \quad t \in R$$

It follows that

$$\begin{aligned} D &= \psi^{-1}(0)\psi(\omega) \\ &= (\phi(0)M)^{-1} \cdot \phi(\omega)M \\ &= M^{-1}\phi^{-1}(0) \cdot \phi(\omega)M \\ &= M^{-1}CM. \end{aligned}$$

Therefore, C and D are similar matrices and hence have the same eigenvalues. Hence, Floquet multipliers are well defined.

Example 5. Find the Floquet multiplier for the Floquet system

$$y' = \begin{bmatrix} 1 & 1 \\ 0 & \frac{(\cos t + sint)}{(2+\sin t - \cos t)} \end{bmatrix} y.$$

First we solve this equation for $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. From the above equation we have linear differential equations for y_2 and y_1:

$$y_1' = y_1(t) + y_2(t).$$

and

$$y_2' = \frac{(\cos t + \sin t)}{(2 + \sin t - \cos t)} y_2(t)$$

Solving these equations first for y_2 and then for y_1, we obtain the general solutions:

$$\begin{aligned} y_1 &= \alpha e^t - \beta(\cos t + \sin t), \\ y_2 &= \beta(2 + \sin t - \cos t) \end{aligned}$$

for $t \in R$. Take $\alpha = 0$, $\beta = 1$, and then $\alpha = 1$ and $\beta = 0$, we have two independent solutions:

$$y(t) = \begin{bmatrix} -2 - \sin t \\ 2 + \cos t + \sin t \end{bmatrix}$$

and

$$z(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}.$$

It follows that a fundamental matrix is

$$\phi(t) = \begin{bmatrix} -2 - \sin t & e^t \\ 2 + \sin t - \cos t & 0 \end{bmatrix}.$$

Since

$$C = \phi^{-1}(0)\phi(2\pi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi} \end{bmatrix}, \tag{II.11}$$

the Floquet multipliers are $\mu_1 = 1$ and $\mu_2 = e^{2\pi}$.

The following theorem shows the importance of the Floquet multipliers.

Theorem II.25. *Assume that $\mu_1, \mu_2, \cdots, \mu_n$ are the Floquet multipliers of the Floquet system $y' = A(t)y$. Then the solutions of the Floquet system are*

i. asymptotically stable on $[0, \infty)$ (i.e. $\lim_{t\to\infty} y(t) = 0$ for all solutions $y(t)$) iff $|\mu_i| < 1$, $1 \le i \le n$;

ii. stable on $[0, \infty)$ (i.e. all solutions are bounded) provided $|\mu_i| < 1$, $1 \le i \le n$ and whenever $|\mu_i| = 1$, μ_i is a simple eigenvalue;

iii. unstable on $[0, \infty)$ (i.e. there is an unbounded solution) provided there is an i_0, $1 \le i_0 \le n$, such that $|\mu_{i_0}| > 1$.

Proof. We will prove this theorem for just two-dimensional cases. Assume that $y(t) = \phi(t)y_0 = \rho(t)e^{Bt}y_0$ is a solution of the Floquet system and let C be as in Floquet's theorem, i.e.

$$C = e^{B\omega}.$$

By the Jordan canonical form theorem, there are matrices M and J so that

$$B = MJM^{-1},$$

where either

$$J = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}, \quad or \quad J = \begin{bmatrix} \rho_1 & 1 \\ 0 & \rho_1 \end{bmatrix},$$

where ρ_1, ρ_2 are the eigenvalues of B. It follows that

$$\begin{aligned} C &= e^{B\omega} \\ &= e^{MJM^{-1}\omega} \\ &= Me^{J\omega}M^{-1} \quad \textit{By theorem (I.13) part (viii)} \\ &= MKM^{-1}, \end{aligned}$$

where either

$$K = \begin{bmatrix} e^{\rho_1 \omega} & 0 \\ 0 & e^{\rho_2 \omega} \end{bmatrix}, \quad or \quad K = \begin{bmatrix} e^{\rho_1 \omega} & \omega e^{\rho_1 \omega} \\ 0 & e^{\rho_1 \omega} \end{bmatrix}.$$

Since the eigenvalues of K are the same as the eigenvalue of C, we get that the Floquet multipliers are

$$\mu_i = e^{\rho_i \omega},$$

$i = 1, 2$, where $\rho_1 = \rho_2$ is possible. Since

$$|\mu_i| = e^{Re(\rho_i)\omega},$$

we have that

$$\begin{aligned} |\mu_i| &< 1 \;\; iff \;\; Re(\rho_i) < 0 \\ |\mu_i| &= 1 \;\; iff \;\; Re(\rho_i) = 0 \\ |\mu_i| &> 1 \;\; iff \;\; Re(\rho_i) > 0. \end{aligned}$$

Let

$$Q_1 = \sup_{t \in R} \|\rho(t)\| = \max_{t \in [0,\omega]} \|\rho(t)\|$$

and

$$Q_2 = \sup_{t \in R} \|\rho^{-1}(t)\| = \max_{t \in [0,\omega]} \|\rho^{-1}(t)\|.$$

Then

$$\|y(t)\| = \|\rho(t) e^{Bt} y_0\| \leq \|\rho(t)\| \|e^{Bt} y_0\| \leq Q \|e^{Bt} y_0\|$$

for $t \in R$ and

$$\|e^{Bt} y_0\| = \|\rho^{-1}(t) y(t)\| \leq \|\rho^{-1}(t)\| \|y(t)\| \leq Q_2 \|y(t)\|,$$

for $t \in R$. It shows that the stability of $y(t)$ is equivalent to the stability of $e^{Bt}y_0$. The conclusion of this theorem then follows Theorem I.20 and Theorem I.21. $\square$

The next theorem will show the property of a Floquet multiplier.

Theorem II.26. *A number μ is a Floquet multiplier of the Floquet system* (II.10) *if and only if there is a non trivial solution y such that*

$$y(t+\omega) = \mu y(t).$$

Proof. $\Longrightarrow$ Assume that μ is a Floquet multiplier of (I.1) so that μ is an eigenvalue of $C = \phi^{-1}(0)\phi(\omega)$ and let $y_0 \neq 0$ be an eigenvector corresponding to μ so that $Cy_0 = \mu y_0$. Define the vector function y by

$$y(t) = \phi(t)y_0$$

which is nonzero solution, then we have

$$\begin{aligned} y(t+\omega) &= \phi(t+\omega)y_0 \\ &= \phi(t)Cy_0 \\ &= \phi(t)\mu y_0 \\ &= \mu y(t). \end{aligned}$$

So we have that

$$y(t+\omega) = \mu y(t).$$

$\Longleftarrow$ Assume that there is a nontrivial solution such that $y(t+\omega) = \mu y(t)$ for all

$t \in R$. Let ϕ be a fundamental matrix for (II.10), then

$$y(t) = \phi(t)y_0$$

for some non zero vector y_0. Then

$$\begin{aligned} y(t+\omega) &= \phi(t+\omega)y_0 \\ &= \phi(t)Cy_0. \end{aligned}$$

So, from $y(t+\omega) = \mu y(t) = \mu\phi(t)y_0$ we have that

$$\phi(t)Cy_0 = \mu\phi(t)y_0.$$

Multiply both sides by $\phi^{-1}(t)$, we have

$$\phi^{-1}\phi(t)Cy_0 = \phi^{-1}(t)\mu\phi(t)y_0$$

or

$$\mu y_0 = Cy_0.$$

Thus, μ is eigenvalue of C and it follows that μ is a Floquet multiplier of (II.10). □

From Theorem II.26, we can see immediately that the Floquet system has a nontrivial periodic solution of period ω if and only if $\mu = 1$ is a Floquet multiplier.

Example 6. It is not hard to show that

$$y(t) = \begin{bmatrix} -e^{\frac{t}{2}}\cos t \\ e^{\frac{t}{2}}\sin t \end{bmatrix}$$

is a solution of the Floquet system

$$y'(t) = \begin{bmatrix} -1 + (\frac{3}{2})\cos^2 t & 1 - (\frac{3}{2})\cos t \sin t \\ -1 - (\frac{3}{2})\sin t \cos t & -1 + (\frac{3}{2})\sin^2 t \end{bmatrix} y.$$

Note that

$$\begin{aligned} y(t + 2\pi) &= \begin{bmatrix} -e^{\frac{t+2\pi}{2}} \cos t \\ e^{\frac{t+2\pi}{2}} \sin t \end{bmatrix} \\ &= e^{\pi} y(t). \end{aligned}$$

By Theorem II.26, e^{π} is a Floquet multiplier.

Banach Spaces

In this chapter we are going to introduce Banach spaces and their properties. First, we will review the basic properties of linear vector space.

Definition III.27. (Vector Spaces)

A linear vector space X *over a scalar* F *is a nonempty set* X *with a mapping:* $(x_1, x_2) \to x_1 \oplus x_2$ *from* $X \times X$ *into* X*, which we call addition, and a mapping:* $(c, x) \to cx$ *from* $F \times X$ *into* X *which we call scalar multiplication. These mappings satisfy the conditions:*

1) The commutative property: $x \oplus y = y \oplus x$ *for all* $x, y \in X$*.*

2) The associative property: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *for all* $x, y, z \in X$*.*

3) The existence of the zero element 0*: For each* $x \in X$*, there exists a unique element* 0 *in* X *such that* $x \oplus 0 = 0 \oplus x = x$*.*

4) The existence of an inverse: for each $x \in X$*, there exists a unique element* $-x \in X$ *such that* $x \oplus -x = 0$

5) $c_1(c_2x) = (c_1c_2)x$ *for all* $x \in X$ *and all* $c_1, c_2 \in F$*.*

6) $(c_1 + c_2)x = c_1x \oplus c_2x$ *for all* $x \in X$ *and all* $c_1, c_2 \in F$*.*

7) $c(x \oplus y) = cx \oplus cy$ *for all* $x, y \in X$ *and all* $c \in F$*.*

$1x = x$ *for all* $x \in X$*, where* 1 *is the unit element of the scalar field* F*.*

Example 7. The spaces:

1) R^n and C^n;
2) $P =$ the space of all polynomials (real or complex valued);
3) $M_{m\times n} :=$ the space of all (real or complex valued) $m \times n$ matrices;
4) $C[a, b]$: $=$ the space of all continuous (real) function defined on the closed interval $[a, b]$.

are all vector spaces.

Definition III.28. (Linear Subspace)

Suppose X is a vector space,then a subset S of X is a linear subspace if

$$x, y \in S \Rightarrow c_1 x \oplus c_2 y \in S \;\; \textit{for all} \;\; c_1, c_2 \in F.$$

In the other words, a subspace S is closed under addition and scalar multiplication.

Definition III.29. (Linear Dependence and Independence)

Suppose $x_1, x_2, \ldots, x_n, \ldots.$ are elements a linear vector space X. If there exist scalars $c_1, c_2, \ldots, c_n, \ldots$, not all of which are zeros, such that

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = 0,$$

then $x_1, x_2, \ldots, x_n, \ldots$ are called a linearly dependent set. If no such set of scalars exist then we say that $x_1, x_2, \ldots, x_n, \ldots$ are linearly independent set.

Definition III.30. (Span of Vectors)

If $x_1, x_2, \cdots, x_n$ are elements of a linear vector space X, then

$$Span\{x_1, x_2, \ldots, x_n\} = \{x \in X : x = \Sigma_{i=1}^{n} c_i x_i;\; c_i \in F\}$$

So, $Span\{x_1, x_2, \ldots, x_n\}$ is a subspace of X and we say that it is spanned by $\{x_1, x_2, \ldots, x_n\}$.

Definition III.31. Dimension of linear vector spaces

If the linear vector space is spanned by a finite set of linearly independent vectors $\{x_1, \ldots, x_n\}$ so that $X = Span\{x_1, \ldots, x_2\}$ then X has dimension n . If there exists an infinite set of linearly independent vectors, then X is said to be infinite dimensional.

Example 8. The spaces R^n and C^n, $M_{m \times n}$ are finite dimensional, while the spaces P and $C[a, b]$ are infinite dimensional. Next, we define the norm in a linear vector space.

Definition III.32. (Norm of a Complex Vector Space)

A non-negative set function on a linear vector space X is called a norm and denoted by $\|\cdot\|$ where $\|\cdot\| : X \to R^+$ with

1) $\|x\| = 0$ if and only if $x = 0$.

2) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ which is called triangle inequality.

3) $\|cx\| = |c|\|x\|$ for all $x \in X$ and $c \in C$.

Example 9. Let $X = R^n$. For each $1 \leq p \leq \infty$ we define the functional $\|\cdot\|_p : X \to R^+$ by

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}.$$

In particular

1) $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$ (the sum norm);

2) $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ (the Euclidian norm);

3) $\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$ (the maximum norm).

Example 10. Let $X = C[a, b]$ be the space of all bounded continuous function on $[a, b]$. Define the function norm on $C[a, b]$:

$$\|f\|_{\sup} = \sup_{a \leq t \leq b} |f(t)|.$$

Then $\|\cdot\|_{\sup}$ is a norm. This is called the uniform or sup norm.

Definition III.33. (Normed Space)

Let X be a linear vector space with a norm $\|\cdot\|$ on it, then X is called a norm linear space and is denoted by $(X, \|\cdot\|)$ or sometimes we write just X .

Example 11. The spaces R^n, C^n and $C[a, b]$ with the norms defined above are the normed spaces.

Example 12. let $X = BC(-\infty, \infty)$ the space of bounded continuous functions on R with the norm

$$\|f\| = \sup_{t \in (-\infty, \infty)} |f(t)|$$

is also a normed space.

Example 13. Let $L_p(a,b)$ be the space of all p-integrable functions (i.e. $\int_a^b |f(t)|^p dt$ exists) with

$$\|f\|_p = (\int_a^b |f(t)|^p dt)^{1/p}$$

is a normed space. (That norm is called the p-norm).

Definition III.34. (Convergence of Sequences on Normed Space)

We say that $\{x_n\}$ converges to x_0 in a normed linear space $(X, \|\cdot\|)$ if

$$\lim_{n\to\infty} \|x_n - x_0\| = 0$$

With the convergence concept, we now can introduce the continuity of an operator, a map between two normed spaces.

Definition III.35. (Continuity of an Operator)

An operator F between two normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x_0) > 0$ such that

$$\|F(x) - F(x_0)\|_y < \epsilon \textit{ whenever } \|x - x_0\| < \delta.$$

An operator is said to be continuous, if it is continuous at every point in its domain.

Also, the map $F : X \longrightarrow Y$ is continuous at x_0 if and only if

$$\lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n)$$

for every sequence $\{x_n\}$ in X convergent to x_0. So, continuity and convergence are closely related.

In particular, when $X = R$, then the operators are called functions with values in Y. A function $f : R \mapsto Y$ is continuous on the domain I if

$$\lim_{t_n \to t} f(t_n) = f(t)$$

for each $t_n, t \in I$.

Definition III.36. **(Derivatives)** *Suppose X is a normed space. A vector-valued function $f(t) : R \to X$ is differentiable at t_0 if*

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \textit{exists in} \quad X.$$

The function f is called differentiable in a domain I if f is differentiable at all points in I and

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

is called the derivative of f at t_0. The vector valued function F is called an antiderivative or (indefinite) integral of f if $F'(t) = f(t)$. The concept of definite integral of a vector-valued function can also be defined the same way as for a real-valued function.

Definition III.37. Open and Closed Sets

Let A be a set in a normed space. A is closed if all convergent sequences in A have their limit points in A.

A set A is open if for any point $x \in A$, there is a $\epsilon > 0$ such that the set $\{y : \|x - y\| < \epsilon\}$ is wholly contained in S. Also, A is open if its complement is closed.

Definition III.38. (Bounded Sets)

Let A be a set in a normed linear space $(X, \|\cdot\|)$. A is bounded if

$$\sup_{x \in A} \|x\| < \infty$$

Definition III.39. (Cauchy Sequence)

We say that a sequence $\{x_n\}$ of elements in a normed linear space $(X, \|\cdot\|)$ is a Cauchy sequence if

$$\|x_n - x_m\| \to 0 \quad as \quad m, n \to \infty.$$

It is clear, that each convergent sequence in a normed linear space is a Cauchy sequence, but the converse statement is not always true. If the converse statement holds, we call that the normed space is complete.

Definition III.40. (Banach Space)

A Banach space is a complete normed linear space. That means, if each Cauchy sequence converges.

Linear Operators in Banach Spaces

In this chapter we will define linear operators on Banach spaces, which are the generalization of matrices on R^n. Operators are the mappings between two Banach spaces. Linear operators play an important role, since differential operators are linear. First, we give some definitions.

Definition IV.41. *Let X and Y be two Banach spaces. The operator $A : X \longrightarrow Y$ is called linear if*

$$\begin{aligned} A(x+y) &= A(x) + A(y) \text{ and} \\ A(\lambda x) &= \lambda(Ax) \end{aligned}$$

Example 14.

1) Let A be an $m \times n$ matrix, then the operator $F : R^n \mapsto R^m$ defined by

$$Fx = A \cdot x$$

is a linear operator.

2) Suppose $E = C[a,b]$ and define a) $A : C[a,b] \mapsto C[a,b]$ by

$$A = \int_a^t f(s)ds$$

and b) $B : C^1[a,b] \mapsto C[a,b]$ by

$$Bf = f'.$$

Then both A and B are linear operators.

Next we introduce the concept of boundedness of a linear operator.

Definition IV.42. *A linear operator A is bounded if there is a number $c > 0$ such* *that*

$$\|Ax\| \leq c\|x\|$$

Example 15.

1) Suppose A is an $n \times n$ matrix and R^n is equipped with Euclidean norm. Then the operator $F : R^n \mapsto R^n$ defined by

$$Fx = A \cdot x$$

is a bounded operator. We have

$$\begin{aligned} \|Fx\|^2 &= \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j|^2 \\ &\leq \sum_{i=1}^{n}(\sum_{j=1}^{n} |a_{ij}|^2)(\sum_{j=1}^{n} |x_j|^2) \\ &= c^2\|x\|^2 \text{ where } c^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2 \end{aligned}$$

2) Refer to Example (14) we can see that $Af = \int_a^t f(s)ds$ is a bounded linear operator. Indeed,

$$\begin{aligned} \|Af\| &= \max_{a \leq t \leq b} |\int_a^t f(s)ds| \\ &\leq \max_{a \leq t \leq b} \int_a^t |f(s)|ds \\ &\leq \int_a^t \max_{a \leq s \leq b} |f(s)|ds \\ &= \int_a^t \|f(s)\|ds \\ &= \|f\|(t-a) \\ &\leq (b-a)\|f\|. \end{aligned}$$

Example 16. Consider the linear operator $Bf = f'$ on $C^1[a, b]$ is the space of continuous differentiable function. We show it is not bounded. Indeed, let $f(t) = \sin(nt)$ then $\|f\| = 1$, but $f'(t) = \cos(nt)n$ and hence, $\|Bf\| = n\|f\|$. Therefore, for each integer n we can find f such that $\|Bf\| = n\|f\|$, which makes B unbounded.

Definition IV.43. (Norm of an Operator) *The norm of bounded operator is defined by*

$$\begin{aligned} \|A\| &= \inf\{c : \|Ax\| \leq c\|x\|\} \\ &= \sup_{x \in E} \frac{\|Ax\|}{\|x\|} \\ &= \sup_{\|x\|=1} \|Ax\| \end{aligned}$$

Clearly, a number c is the norm of a bounded operator $A : X \mapsto Y$ if and only if

1) $\|Ax\| \leq c\|x\|$ for each $x \in X$ and

2) There exists an element $x \in X$ such that $\|Ax\| = c\|x\|$.

Example 17. Consider the operator $A : C[a, b] \longrightarrow C[a, b]$ with $Af(t) = \int_a^t f(s)$ where $a \leq t \leq b$. We want to find the norm of A. By Example (15) we know that

$$\|Af(t)\| \leq (b - a)\|f\|.$$

On the other hand, take $f(t) \equiv 1$, then $(Af)(t) = t - a$ and hence, $\|Af\| = (b - a) = (b - a)\|f\|$. Thus, $\|A\| = (b - a)$.

The next theorem will collect some basic properties of the norm of an operator.

Theorem IV.44. ([1], Theorem 1.10)

1. *Suppose A and B are two linear, bounded operator from a Banach space X to a Banach space Y. Then the following statements hold:*

(i) $\|A\| \geq 0$ *and* $\|A\| = 0$ *iff* $A \equiv 0$*;*

(ii) $\|cA\| = |c|\|A\|$ *for any scalar* c*;*

(ii) $\|A+B\| \leq \|A\| + \|B\|$.

Hence, the space of all linear, bounded operators from X *to* Y *is again a normed space, and is denoted by* $L(X,Y)$*. If* $X = Y$*, then* $L(X,X)$ *is denoted by* $L(X)$*.*

2. *For each* $A :\in L(X,Y)$ *and* $B \in L(Y,Z)$ *we have*

$$\|B \cdot A\| \leq \|A\| \cdot \|B\|.$$

In particular, if $A \in L(X)$*, the we have* $\|A^n\| \leq \|A\|^n$ *for all integers* $n \geq 2$*.*

The following theorem is the "product rule" for the derivative of functions with values in normed spaces.

Theorem IV.45. *Suppose* $f(t) : I \mapsto X$ *and* $A(t) : I \mapsto L(X)$ *are continuously differentiable on a domain* I*. Then the vector-valued function* $g(t) := A(t)f(t)$ *is continuously differentiable in* I *and*

$$g'(t) = A'(t)f(t) + A(t)f'(t).$$

Proof. For each $t \in I$, consider the difference

$$\begin{aligned} & g'(t) - [A'(t)f(t) + A(t)f'(t)] \\ = & \lim_{h\to 0} \frac{A(t+h)f(t+h) - A(t)f(t)}{h} - [A'(t)f(t) + A(t)f'(t)] \\ = & \lim_{h\to 0}[\frac{A(t+h) - A(t)}{h} - A'(t)]f(t+h) + \lim_{h\to 0} A(t)[\frac{f(t+h) - f(t)}{h} - f'(t)] \\ & + A'(t)\lim_{h\to 0}[f(t+h) - f(t)] \\ = & 0, \end{aligned}$$

as each limit is zero. □

Also, Lemma I.15 holds for a function with values in a Banach space, and we state here one more time.

Lemma IV.46. *(i) Assume the vector-valued function* $f(t,s)$ *and the partial* *derivative $f_t(t,s)$ are continuous on an interval $I \times I$ and $a \in I$. Then*

$$\frac{d}{dt}\int_a^t f(t,s)ds = \int_a^t f_t(t,s)ds + f(t,t).$$

(ii) For each integrable vector-valued function f, we have

$$\| \int_a^b f(t)dt \| \le \int_a^b \|f(t)\| dt, \quad \textit{where} \quad a \le b$$

Next we want to state the contraction mapping theorem on a Banach space. An mapping (or an operator) from a Banach space X to itself is said to be a contraction mapping, if there exists a constant $\alpha : 0 \le \alpha < 1$ such that

$$\|Ay - Ax\| \le \alpha \|x - y\|$$

for all x and y in X.

Theorem IV.47. (Contraction Mapping Theorem, [4], Theorem 7.5)

If A is a contraction mapping on a Banach space then A has a unique fixed point y in X, i.e. a point y such that $Ay = y$.

We can generalize the contraction mapping theorem and will use it as the main tool to show the existence and uniqueness of a linear nonautonomous Cauchy problem on a Banach space (Theorem V.50).

Theorem IV.48. (Generalization of Contraction Mapping Theorem) *Let n be any positive integer. If A^n is a contraction mapping on a Banach space, then A has a unique fixed point.*

Proof. By the Fixed Point Theorem, A^n has a unique fixed point y with $A^n(y) = y$. We show that $Ay = y$.

From $A^n(y) = y$ it follows that $A^{n+1}y = Ay$, or equivalently, $A^n(Ay) = Ay$. That means that Ay is another fixed point of A^n. Therefore, $Ay = y$.

Next, we show that y is the only fixed point of A. In order to do it, let x be another fixed point of A, i.e. $Ax = x$, then $A^2x = Ax = x$, and $A^3x = A(A^2x) = Ax = x$, and so on... Hence, $A^nx = x$. By the uniqueness of the fixed point of A^n, we have $x = y$. □

Let us now define the spectrum and resolvent of a linear bounded operator A on Banach space, which are the generalizations of the eigenvalue set of a matrix.

Definition IV.49. (Resolvent of a Linear Operator) *If A is linear bounded operator on X then the resolvent of A, denoted by $\rho(A)$, is*

$$\rho(A) = \{\mu \in C : (\mu I - A)^{-1} \text{ exists and is bounded}\}.$$

The set $C - \rho(A)$ is called the spectrum of A which we denote by $\sigma(A)$. If $\mu \in \sigma(A)$ and $\exists x_0 \neq 0$ such that $Ax_0 = \mu x_0$, then μ is called an eigenvalue of A. The set of all eigenvalues of A is called the point spectrum of A and is denoted by $\sigma_p(A)$.

Floquet Theory on Banach Spaces

In this chapter we study the solutions of linear differential equations on a Banach space of the form

$$y'(t) = A(t)y(t),$$

where $A(t)$ is an operator-valued periodic function, i.e $A(t+\omega) = A(t)$. We study about a Nonautonomous Differential Equations and Floquet theory On Banach Spaces.

Nonautonomous Differential Equations On Banach Spaces

Let X be a Banach space. On X, we now consider the Cauchy problem or IVP

$$\begin{cases} y'(t) = A(t)y(t), \quad t \geq s, \\ y(s) = y_s \in X, \end{cases} \tag{V.12}$$

where $A(t)$ are linear, bounded operators on X. The above kind of differential equations are called evolution equations (see more details in [2]). First, we want to show the existence of the solution of the Cauchy problem (V.12).

Theorem V.50. (Existence and Uniqueness Theorem)

Suppose the function $t \to A(t)$ is continuous on R. Then, for every initial value $y_s \in X$ the Cauchy problem (V.12) *has a unique solution.*

Proof. We will prove the theorem by using the contraction mapping theorem. It is enough to show that, for each $T > s$, (V.12) there is a unique solution on an interval $[s, T]$.

Note that $y(t)$ is a solution of (V.12) if and only if $y(t)$ is a solution of the integral equation

$$y(t) = y_s + \int_s^t A(r)y(r)dr.$$

We now define the operator A on $C[s,T]$ by

$$(Ay)(t) = y_s + \int_s^t A(r)y(r)dr, \;\; s \le t \le T.$$

We prove that for all $t \in [s,T]$ and $n \ge 1$ we have

$$\|(A^n y - A^n x)(t)\| \le \|y - x\| M^n \frac{(t-s)^n}{n!}, \tag{V.13}$$

where $M := \max_{t\in[s,T]} \|A(t)\|$, by induction:

1) We show that it is true for $n = 1$. We have

$$\begin{aligned}
\|(Ay - Ax)(t)\| &= \|y_s + \int_s^t A(r)y(r)dr - (y_s + \int_s^t A(r)x(r)dr)\| \\
&= \|\int_s^t A(r)(y(r) - x(r))dr\| \\
&\le \int_s^t \|A(r)\|\|y(r) - x(r)\|dr \\
&\le \|y - x\| \int_s^t \|A(r)\|dr \\
&\le \|y - x\| \int_s^t \max |A(r)|dr \\
&= \|y - x\| M(t - s)
\end{aligned}$$

for all $t \in [s,T]$. So, (V.13) is true for $n = 1$.

2) Suppose that (V.13) is true for $n = k$ so that

$$\|(A^k y - A^k x)(t)\| \le M^k \|y - x\| \frac{(t-s)^k}{k!}$$

We want to show that it is true for $n = k+1$. We have:

$$\begin{aligned}
\|(A^{k+1}y - A^{k+1}x)(t)\| &= \|y_s + \int_s^t A(r)A^k y(r)dr - (y_s + \int_s^t A(r)A^k x(r)dr)\| \\
&= \| \int_s^t A(r)(A^k y(r) - A^k y(r))dr\| \\
&\leq \int_s^t \|A(r)\|\|A^k y(r) - A^k y(r)\|dr \\
&\leq \int_s^t M\|A^k y(r) - A^k y(r)\|dr. \qquad \text{(V.14)}
\end{aligned}$$

By induction assumption, $\|(A^k y - A^k x)(r)\| \leq \|y-x\|M^k\frac{(r-s)^k}{k!}$ for $r \in [s, T]$. Hence, from (V.14) we obtain

$$\begin{aligned}
\|(A^{k+1}y - A^{k+1}x)(t)\| &\leq M^{k+1}\|y-x\|\int_s^t \frac{(r-s)^k}{k!}dr \\
&= M^{k+1}\|y-x\|\frac{1}{k!}\int_0^{t-s} \tau^k d\tau \text{ (where } \tau = r-s) \\
&= M^{k+1}\|y-x\|\frac{1}{k!}[\frac{\tau^{k+1}}{k+1}]_0^{t-s} \\
&= \|y-x\|M^{k+1}\frac{1}{k!}\frac{(t-s)^{k+1}}{k+1} \\
&= \|y-x\|M^{k+1}\frac{(t-s)^{k+1}}{(k+1)!}.
\end{aligned}$$

So, (V.13) is true for $n = k+1$ and it is proved.

From (V.13) we have

$$\|A^n y - A^n x\| \leq \|y-x\|M^n\frac{(T-s)^n}{n!}$$

for all $n \geq 1$. Now, let $C_n := M^n\frac{(T-s)^n}{n!}$. Then it is easy to see that $\lim_{n\to\infty} C_n = 0$. So, there exists n_0 (big enough) such that for $C_{n_0} < 1$. For that n_0, A^{n_0} is a contraction mapping. Therefore, by the generalized contraction mapping theorem, there is a unique fixed point for A, such that $Ay = y$. Thus, the IVP (V.12) has a unique solution for every initial value $y_s \in X$. □

From the existence and uniqueness theorem, we now are able to define a family of bounded operators on X to describe the solutions of the IVP (V.12).

Definition V.51. *For each t and s in R with $t \geq s$ we define the operator $U(t,s)$ as follows: For each $y_0 \in X$, $U(t,s)y_0 := y(t)$, where $y(\cdot)$ is the unique solution of*

$$\begin{cases} y'(t) = A(t)y(t) \\ y(s) = y_0 \in X. \end{cases} \tag{V.15}$$

The family $\{U(t,s)\}_{t\geq s}$ is called an evolution family generated by family $\{A(t)\}_{t\in R}$.

We now collect the basic properties of an evolution family.

Theorem V.52. *For every t and s with $t > s$, $U(t,s)$ are linear and bounded operators on X.*

Before proving the above theorem, we state the well known Gronwall's Lemma.

Lemma V.53. (Gronwall's Lemma)

Let $I = [a,b]$ and $\beta(t)$ and $U(t)$ be real-valued continuous functions defined on I satisfying the integral inequality

$$U(t) \leq \alpha + \int_a^t \beta(r)U(r)dr$$

for all $t \in I$, then

$$U(t) \leq \alpha e^{\int_a^t \beta(r)dr}$$

for all $t \in I$.

Proof. of Theorem V.52:

We first show that $U(t,s)$ is linear, i.e.:

$$U(t,s)(ax+by) = aU(t,s)x + bU(t,s)y$$

for all x and y in X. To do that, let $z(t) = U(t,s)(ax+by) - aU(t,s)x - bU(t,s)y$, then

$$\begin{aligned} z'(t) &= A(t)U(t,s)(ax+by) - aA(t)U(t,s)x - bA(t)U(t,s)y \\ &= A(t)(U(t,s)(ax+by) - aU(t,s)x - bU(t,s)y) \\ &= A(t)z(t). \end{aligned}$$

Moreover, by definition of $U(t,s)$ we have $z(s) = (ax+by) - (ax+by) = 0$ (because $U(s,s) = I$, see theorem (V.54)). So, $z(t)$ is the unique solution of

$$\begin{cases} y'(t) = A(t)y(t) \\ y(s) = 0. \end{cases}$$

Since $y(t) \equiv 0$ is a solution, we have $z(t) \equiv 0$.

To show that $U(t,s)$ is a bounded operator, we use the integral form of the solution:

$$y(t) = y(s) + \int_s^t A(r)y(r)dr$$

or

$$U(t,s)y = y + \int_s^t A(r)U(r,s)ydr,$$

for every vector $y \in X$. Hence,

$$\|U(t,s)y\| \leq \|y\| + \int_s^t \|A(r)\|\|U(r,s)y\|dr.$$

We now use Gronwall's Lemma and obtain

$$\|U(t,s)y\| \leq \|y\| \cdot e^{\int_s^t \|A(r)\|dr}$$

for every $y \in X$, which shows that $U(t,s)$ is bounded. $\square$

Theorem V.54. *Let $\{U(t,s)\}_{t\geq s}$ be an evolution family generated by $\{A(t)\}_{t\in R}$. Then the following statements hold:*

1) *$U(t,t) = Id.$ for all $t \in R$;*
2) *$U(t,r)U(r,s) = U(t,s)$ for $s \leq r \leq t$;*
3) *For each $s \in R$ and $y \in X$ the function $t \mapsto U(t,s)y$ is continuously differentiable and*

$$\frac{d}{dt}U(t,s)y = A(t)U(t,s)y.$$

4) *For each $t \in R$ and $y \in X$ the function $s \mapsto U(t,s)y$ is continuously differentiable and*

$$\frac{d}{ds}U(t,s)y = -U(t,s)A(s)y.$$

5) *The solution of non-homogenous problem*

$$\begin{cases} y'(t) = A(t)y(t) + f(t) \\ y(s) = y_s \end{cases} \tag{V.16}$$

is

$$y(t) = U(t,s)y_s + \int_s^t U(t,r)f(r)dr.$$

Proof.

1) We know that $y(t) = U(t,s)y$ is the solution of (V.15). Hence, let $t = s$, then we have that $U(s,s)y = y(s) = y$, which means $U(s,s)$ is the identity operator.
2) We want to show that

$$U(t,r)U(r,s)y = U(t,s)y$$

for all $y \in X$. Clearly, for $t \geq r$, $y_1(t) = U(t,s)y$ is a solution of

$$\begin{cases} y'(t) = A(t)y(t) \\ y(r) = U(r,s)y. \end{cases} \tag{V.17}$$

Let $y_2(t) = U(t,r)U(r,s)y$ for $t \geq r$, then

$$y_2'(t) = A(t)U(t,r)U(r,s)y = A(t)y_2(t)$$

and $y_2(r) = U(r,r)U(r,s)y = U(r,s)y$, too. Thus, by the uniqueness of the solution of (V.17), we have

$$y_1(t) = y_2(t).$$

Therefore,

$$U(t,r)U(r,s) = U(t,s).$$

3) It is obvious from the definition of $U(t,s)$.

4) We want to prove that

$$\frac{d}{ds}U(t,s)y = -U(t,s)A(s)y$$

for all $y \in X$. To do it, we fix t and r and let $r \leq s \leq t$, so we have

$$\frac{d}{ds}U(t,r)y = \frac{d}{ds}U(t,s)U(s,r)y.$$

Since $U(t,r)y$ is constant with respect to s, we get

$$\begin{aligned} 0 &= \frac{d}{ds}U(t,s)U(s,r)y \\ &= [\frac{d}{ds}U(t,s)]U(s,r)y + U(t,s)\frac{d}{ds}U(s,r)y. \end{aligned}$$

From part 3) we have $\frac{d}{ds}U(s,r)y = A(s)U(s,r)y$. Hence we obtain

$$0 = (\frac{d}{ds}U(t,s))U(s,r)y + U(t,s)A(s)U(s,r)y.$$

Now, let $r = s$, we have

$$0 = \frac{d}{ds}U(t,s)y + U(t,s)A(s)y,$$

or equivalently,

$$\frac{d}{ds}U(t,s)y = -U(t,s)A(s)y.$$

5) First, let $t = s$ we have

$$\begin{aligned} y(s) &= U(s,s)y_s + \int_s^s U(t,s)f(r)d(r) \\ &= y_s. \end{aligned}$$

Next, let's take a derivative of

$$y(t) = U(t,s)y_s + \int_s^t U(t,s)f(r)dr,$$

then we have

$$\begin{aligned} y'(t) &= (\frac{dy}{dt}U(t,s))y_s + \frac{d}{dt}\int_s^t U(t,r)f(r)dr \\ &= A(t)U(t,s)y_s + \frac{d}{dt}\int_s^t U(t,r)f(r)dr \\ &= A(t)U(t,s)y_s + \int_s^t A(t)U(t,r)f(r)dr + f(t) \\ &= A(t)[U(t,s)y_s + \int_s^t U(t,r)f(r)dr] + f(t) \\ &= A(t)y(t) + f(t). \end{aligned}$$

Here we used Lemma IV.46. Hence, $y(t)$ is a solution of (V.16).

Now, we want to show that that solution is the only solution of (V.16). Assume that there are two solutions $y_1(t), y_2(t)$, we will show that $y_1(t) = y_2(t)$.
Let

$$y(t) = y_1(t) - y_2(t).$$

Then

$$\begin{aligned} y'(t) &= A(t)y_1(t) + f(t) - A(t)y_2(t) - f(t) \\ &= A(t)(y_1(t) - y_2(t)) \\ &= A(t)y(t) \end{aligned}$$

and $y(s) = 0$. So, $y(t)$ is the solution of

$$\begin{cases} y'(t) = A(t)y(t) \\ y(s) = 0 \end{cases}$$

So, $y(t) \equiv 0$ by uniqueness. Therefore, $y_1(t) = y_2(t)$.

$\square$

Floquet Theory

We now study the IVP (V.12), in which $A(t)$ is periodic with the period ω, i.e. $A(t+\omega) = A(t)$. That system is called a Floquet system on Banach spaces. We have the following observation on a Floquet system.

Theorem V.55. *Suppose $\{U(t,s)\}_{t\geq s}$ is an evolution family generated by $\{A(t)\}_{t\in R}$ in a Floquet system. Then we have*

$$U(t+\omega, s+\omega) = U(t,s).$$

for all t and s with $t \geq s$.

Proof. We show that

$$U(t+\omega, s+\omega)y_0 = U(t,s)y_0$$

for all $y_0 \in X$. Note that $U(t,s)y_0$ is the solution of

$$(IVP)\begin{cases} y'(t) = A(t)y(t) \\ y(s) = y_0, \end{cases} \tag{V.18}$$

for $t \geq s$. Let $z(t) := U(t+\omega, s+\omega)y_0$. Then

$$\begin{aligned} z'(t) &= A(t+\omega)U(t+\omega, s+\omega)y_0 \\ &= A(t)U(t+\omega, s+\omega)y_0 \\ &= A(t)z(t). \end{aligned}$$

Moreover, $z(s) = U(s+\omega, s+\omega)y_0 = y_0$. So, $z(t)$ is also the solution of (V.18). By the uniqueness of the solution, we have $z(t) = U(t,s)y_0$, which implies that

$$U(t+\omega, s+\omega) = U(t,s).$$

□

Next we define the following operator valued function:

$$P(t) := U(t+\omega, t)$$

and the operator

$$V := P(0) = U(\omega, 0).$$

The operator V is called the **monodromy** of a Floquet system.

Theorem V.56. *We have*

1) *The function $P(t)$ is ω-periodic i.e. $P(t+\omega) = P(t)$;*

2) *The point spectrum set of $P(t)$ is independent of t. In other words,*

$$\sigma_p(P(t)) = \sigma_p(V)$$

for all $t \in R$.

Proof. We first prove

1) We have

$$\begin{aligned} P(t+\omega) &= U(t+\omega+\omega, t+\omega) \\ &= U(t+\omega, t) \ \ (\text{since } \ U(t,s) \ \ \text{is } \omega-\text{periodic}) \\ &= P(t). \end{aligned}$$

Thus, $P(t)$ is ω-periodic.

2) First we show that for $s < t$ we have

$$\sigma_p(P(s)) \subseteq \sigma_p(P(t)).$$

To do that, suppose $s < t$ and let $\mu \in \sigma_p(P(s))$. We will show that

$\mu \in \sigma_p(P(t))$. This means that $P(s)x_0 = \mu x_0$ for some $x_0 \neq 0$. We show that $P(t)y_0 = \mu y_0$ for some $y_0 \neq 0$. Take $y_0 := U(t,s)x_0$

$$\begin{aligned} P(t)y_0 &= P(t)U(t,s)x_0 \\ &= U(t+\omega,t)U(t,s)x_0 \\ &= U(t+\omega,s)x_0 \\ &= U(t+\omega,s+\omega)U(s+\omega,s)x_0 \\ &= U(t,s)P(s)x_0 \\ &= U(t,s)\mu x_0 \\ &= \mu U(t,s)x_0 \\ &= \mu y_0. \end{aligned}$$

Thus, $P(t)y_0 = \mu y_0$ and μ is eigenvalue of $P(t)$. Therefore,
$\sigma_p(P(s)) \subseteq \sigma_p(P(t))$.
On the other hand, take an integer n_0 big enough, such that $s + n_0\omega > t$. By the same argument, we have

$$\sigma_p(P(t)) \subseteq \sigma_p(P(s+n_0\omega)).$$

But, $P(s+n_0\omega) = P(s)$, since $P(t)$ is ω-periodic. Hence, we have

$$\sigma_p(P(s)) \subseteq \sigma_p(P(t)) \subseteq \sigma_p(P(s+n_0\omega)) = \sigma_p(P(s)),$$

or

$$\sigma_p(P(s)) = \sigma_p(P(t)).$$

□

Theorem V.57. *The number μ is an eigenvalue of $V = U(\omega,0)$ if and only if the*

Floquet system

$$y'(t) = A(t)y(t) \tag{V.19}$$

has a nontrivial solution $y(t)$ *with*

$$y(t+\omega) = \mu y(t)$$

for all $t \in R$.

Proof. $\Rightarrow$: Assume that μ is an eigenvalue of V, that is, there is a nonzero vector $y_0 \in X$ such that

$$\begin{aligned} \mu y_0 &= V y_0 \\ &= U(\omega, 0) y_0. \end{aligned}$$

Take $y(t) = U(t,0)y_0$ to be a nontrivial solution of the Floquet system (V.19). We have

$$\begin{aligned} y(t+\omega) &= U(t+\omega, 0)y_0 \\ &= U(t+\omega, \omega)U(\omega, 0)y_0 \\ &= U(t,0)Vy_0 \\ &= U(t,0)\mu y_0 \\ &= \mu U(t,0)y_0 \\ &= \mu y(t). \end{aligned}$$

Thus,

$$y(t+\omega) = \mu y(t).$$

$\Leftarrow$: Assume that $y(t)$ is a nontrivial solution of the Floquet system with $y(t+\omega) = \mu y(t)$. Let $y_0 \in X$ such that $y(t) = U(t,0)y_0$. We have

$$\begin{aligned} y(t+\omega) &= U(t+\omega,\omega)U(\omega,0)y_0 \\ y(t+\omega) &= U(t,0)U(\omega,0)y_0 \\ y(t+\omega) &= U(t,0)Vy_0 \\ \mu y(t) &= U(t,0)Vy_0 \\ \mu U(t,0)y_0 &= U(t,0)Vy_0 \\ U(t,0)\mu y_0 &= U(t,0)Vy_0. \end{aligned}$$

Take $t = 0$, we have $U(0,0) = Id$ and then $\mu y_0 = Vy_0$.

Hence, μ is eigenvalue of V. □

We now show that the spectrum of the monodromy V will determine the stability of the Floquet system

Theorem V.58. *The Floquet system is asymptotically stable, i.e.* $\lim_{t\to\infty} y(t) = 0$ *for all solutions* y*, if* $\lim_{m\to\infty} V^m = 0$*.*

Proof. Suppose $\lim_{m\to\infty} V^m = 0$. That means $\lim_{m\to\infty} V^m y_0 = 0$ for each $y_o \in E$. We show that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} U(t,0)y_0 = 0$$

for all $y_0 \in E$. Let $t = m\omega + t_0$ for some integer m and $0 \le t_0 \le \omega$.

Then

$$\begin{aligned} y(t) = U(t,0)y_0 &= U(t_0 + m\omega, 0)y_0 \\ &= U(t_0 + m\omega, \omega)U(\omega, 0)y_0 \\ &= U(t_0 + m\omega, \omega)Vy_0 \\ &= U(t_0 + m\omega, 2\omega)U(2\omega, \omega)Vy_0 \\ &= U(t_0 + m\omega, 2\omega)V^2 y_0 \text{ (since } U(2\omega,\omega) = U(\omega, 0) = V) \\ &= ... \\ &= U(t_0 + m\omega, m\omega)V^m y_0 \\ &= U(t_0, 0)V^m y_0. \end{aligned}$$

Hence,

$$\begin{aligned} \|y(t)\| &= \|U(t_0,0)V^m y_0\| \\ &\leq \|U(t_0,0)\| \cdot \|V^m y_0\| \\ \|y(m\omega + t_0)\| &\leq M \cdot \|V^m y_0\| \\ &\longrightarrow 0 \ \text{ as } m \to \infty \ \textit{but } m \to \infty \textit{ when } t \to \infty, \end{aligned}$$

where $M = \max_{0 \leq t_0 \leq \omega}\{\|U(t_0, 0)\|\}$, and the theorem is complete. □

We know that for the spectrum radius R of V, where $R := \sup\{|\lambda| : \quad \lambda \in \sigma(V)\}$ we have

$$R = \lim_{m\to\infty} \|V^m\|^{1/m}.$$

See [1] for more details. So, $R < 1$ if and only if $\lim_{m\to\infty} V^m = 0$. Hence, from the above Theorem, we have:

Corollary V.59. *If the spectrum radius R of V is less than 1, then the Floquet* *system is asymptotically stable.*

The next lemma is concerning the periodic solution of an inhomogeneous initial value problem with the ω-periodic inhomogeneity.

Lemma V.60. *Consider the inhomogeneous initial value problem*

$$\begin{cases} y'(t) = A(t)y(t) + f(t) \\ y(s) = y_s, \end{cases} \tag{V.20}$$

where $f(t)$ is an ω-periodic function, with corresponding solution

$$y(t) = U(t,s)y_s + \int_s^t U(t,r)f(r)dr.$$

Then $y(t)$ is ω-periodic if and only if $y(s+\omega) = y_s$.

Proof. Clearly, if $y(t)$ is ω-periodic, then $y(s+\omega) = y_s$. Now, assume that $y(s+\omega) = y_s$ and we want to show that $y(t) = y(t+\omega)$ for all $t \geq s$. We have

$$\begin{aligned}
y(t+\omega) &= U(t+\omega,s)y_s + \int_s^{t+\omega} U(t+\omega,r)f(r)dr \\
&= U(t+\omega,s)y_s + \int_s^{s+\omega} U(t+\omega,r)f(r)dr + \int_{s+\omega}^{t+\omega} U(t+\omega,r)f(r)dr \\
&= U(t+\omega,s+\omega)U(s+\omega,s)y_s + \int_s^{s+\omega} U(t+\omega,s+\omega)U(s+\omega,s)f(r)dr + \\
&\qquad + \int_{s+\omega}^{t+\omega} U(t+\omega,r)f(r)dr \\
&= U(t,s)U(s+\omega,s)y_s + \int_s^{s+\omega} U(s+\omega,r)f(r)dr + \int_{s+\omega}^{t+\omega} U(t+\omega,r)f(r)dr \\
&= U(t,s)y(s+\omega) + \int_{s+\omega}^{t+\omega} U(t+\omega,r)f(r)dr \\
&= U(t,s)y_s + \int_{s+\omega}^{t+\omega} U(t+\omega,r)f(r)dr \text{ (since } y(s+\omega) = y_s) \\
&= U(t,s)y_s + \int_s^t U(t+\omega,r'+\omega)f(r'+\omega)dr',
\end{aligned}$$

where $r' = r - \omega$. Since $f(t)$ is a ω-periodic($f(s'+\omega) = f(s')$) and

$U(t+\omega, r'+\omega) = U(t, r')$, so we have that

$$\begin{aligned} y(t+\omega) &= U(t,s)y_s + \int_s^t U(t,r')f(r')dr' \\ &= y(t), \end{aligned}$$

for all $t \geq s$. Thus, $y(t)$ is an ω-periodic solution.

□

We now state the main theorem of this chapter, which finds the conditions such that for each ω-periodic function f, the Floquet system has a unique ω-periodic solution. This theorem has been proved first time in [5], 1999 with a different proof, in which the evolution semigroups were used. In this thesis, we show a simpler proof with only basic knowledge.

Theorem V.61. (Existence and Uniqueness of Periodic Solution) *Consider the Floquet system*

$$\begin{cases} y'(t) = A(t)y(t) + f(t) \\ y(s) = y_s. \end{cases} \tag{V.21}$$

Then the following statements are equivalent:

i) For every ω-periodic function $f(t)$, Equation (V.21) *has a unique ω-periodic solution $y(t)$.*

ii) For the monodromy V we have $1 \in \rho(V)$.

Proof. ii) ⇒ i) Assume that $1 \in \rho(V)$. Let $y(t)$ be a solution of (V.21), that means,

$$y(t) = U(t,s)y_s + \int_s^t U(t,r)f(r)dr$$

We need to show that for each ω-periodic $f(t)$, there is only one value y_s that makes $y(t)$ to become ω-periodic. Let $y(t)$ be an ω-periodic solution of (V.21). Then

$y(0) = y(\omega)$, or in other words,

$$\begin{aligned} y(0) &= U(\omega,0)y(0) + \int_0^\omega U(\omega,r)f(r)dr \\ &= Vy(0) + \int_0^\omega U(\omega,r)f(r)dr. \end{aligned}$$

Thus,

$$(1-V)y(0) = \int_0^\omega U(\omega,r)f(r)dr.$$

Since $1 \in \rho(V)$, we have

$$y(0) = (1-V)^{-1}\int_0^\omega U(\omega,r)f(r)dr.$$

So, there is a unique $y(0)$ that makes $y(\omega) = y(0)$. By Lemma V.60, it means there is a unique $y(0)$ that makes $y(t)$ to be ω-periodic.

i) $\Rightarrow$ ii): Assume that for each function $f(t)$ which is ω-periodic, there is a unique $y(t)$ which is an ω-periodic solution. We need to show that $1 \in \rho(V)$, or equivalently, we need to show that $(1-V)$ is both injective and surjective.

To show that $(1-V)$ is injective, assume, by the contraction, that 1 is an eigenvalue of V, that means $Vy_0 = y_0$ for the corresponding eigenvector y_0. We show that the system

$$y'(t) = A(t)y(t), \tag{V.22}$$

(i.e., we choose $f(t) = 0$, which is a periodic function), has two different ω-periodic solutions.

Consider the function

$$y(t) = U(t,0)y_0.$$

Clearly, $y(t)$ is a solution of (V.22). Moreover,

$$y(\omega) = U(\omega, 0)y_0 = Vy_0 = y_0 = y(0).$$

Hence, by Lemma V.60, $y(t)$ is a ω-periodic solution of (V.22). On the other hand, $y(t) \equiv 0$ is another ω-periodic solution of system (V.22). This is a contradiction. Hence, $(1 - V)$ is injective.

To show $(1 - V)$ is surjective, we show that for each $y_0 \in X$, there is a vector $u_0 \in X$, such that $(1 - V)u_0 = y_0$.

To do that, take a real-valued function $g(t)$ with the following properties:

1) $g(\omega) = g(0) = 0$;
2) $\int_0^\omega g(s)ds = 1$.

For example, if $\omega = \pi$, then $g(t) = \frac{1}{2}\sin t$ satisfies the above conditions. For any ω, then $g(t) = \frac{\pi}{2\omega}\sin\frac{\pi t}{\omega}$ does. Take $f(t) := g(t)U(t, 0)y_0$, then $f(t)$ is an ω-periodic function. Let $y(t)$ be the unique solution of the system (V.21) corresponding to f. Then we have

$$\begin{aligned} y(t) &= U(t,0)y(0) + \int_0^t U(t,r)f(r)dr \\ &= U(t,0)y(0) + \int_0^t U(t,r)g(r)U(r,0)y_0 dr \\ &= U(t,0)y(0) + \int_0^t g(r)U(t,0)y_0 dr \\ &= U(t,0)y(0) + [\int_0^t g(r)dr]U(t,0)y_0. \end{aligned}$$

Hence, if $t = \omega$ we have

$$\begin{aligned} y(\omega) &= U(\omega,0)y(0) + [\int_0^\omega g(r)dr]U(\omega,0)y_0 \\ y(0) &= Vy(0) + Vy_0, \end{aligned}$$

which implies

$$(1 - V)(y(0) + y_0) = y_0.$$

Hence, $(1 - V)$ is surjective, and the proof is complete. □

In conclusion, the Floquet theory is a very important tool to study the stability of solutions of linear differential equations. We can apply Floquet theory to some differential equations from physics or biology, such as the population equations and transportation equations.

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