Counting Convex Sets on Products of Totally Ordered Sets

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COUNTING CONVEX SETS ON PRODUCTS OF TOTALLY ORDERED SETS

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I dedicate this thesis to my amazing husband, Will, for his never ending love, support and sacrifices he makes daily for me to achieve my goals and dreams. I also dedicate this work to my two incredible children who have been so patient and understanding throughout my time in graduate school. Even though they are unaware, they too have made great sacrifices.

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CONTENTS

List of Figures

List of Tables

COUNTING CONVEX SETS ON PRODUCTS OF TOTALLY ORDERED SETS

Brandy Barnette May 2015 60 Pages

Directed by: Dr. Tom Richmond, Dr. Melanie Autin, and Dr. Dominic Lanphier

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The main purpose of this thesis is to find the number of convex sets on a product of two totally ordered spaces. We will give formulas to find this number for specific cases and describe a process to obtain this number for all such spaces.

In the first chapter we briefly discuss the motivation behind the work presented in this thesis. Also, the definitions and notation used throughout the paper are introduced here.

The second chapter starts with examining the product spaces of the form $\{1, 2, \ldots, n\} \times \{1, 2\}$. That is, we begin by analyzing a two-row by n-column space for $n \in \mathbb{N}$. Three separate approaches are discussed, and verified, to find the total number of convex sets on the space. A general formula is presented to obtain this total for all n.

In the third chapter we take the same $\{1, 2, \ldots, n\} \times \{1, 2\}$ spaces from Chapter 2 and consider all the scenarios for adding a second disjoint convex set to the space. Adding a second convex set gives a collection of two mutually disjoint sets. Again, a general formula is presented to obtain this total number of such collections for all n.

The fourth chapter takes the idea from Chapter 2 and expands it to product spaces $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ consisting of more than two rows. Here the creation of convex sets having z rows from those having $z - 1$ rows is exploited to obtain a model that will give the total number of z-row convex sets on any $n \times m$ space, provided the set occupies z adjacent rows.

Finally, the fifth chapter describes all possible scenarios for convex sets to be placed in the $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ space. This chapter then explains the process needed to acquire a count of all convex sets on any such space as well. Chapter 5 ends by walking through this process with a concrete example, breaking it down into each scenario.

We conclude by briefly summarizing the results and specifying future work we would like to further investigate, in Chapter 6.

Chapter 1

INTRODUCTION

Counting the number of topologies on a finite set is an old question described in [**2**]. The answer is currently only known for sets with $x \leq 18$ elements [**6**], which speaks to the complexity of this question. By adding the stipulations that the topologies must also be on a product of two totally ordered sets and have a base of convex sets, we are able to create a more manageable problem. This also illustrates a more interesting and applicable problem, modeling computer screens and graphics used to portray a continuous image. However, they must use a finite set in order to do so. Using a known topology on the space and having a basis of convex sets is one way to obtain this portrayal of a continuous image by using only a finite-element set. Having $x = 18$ elements is equivalent to having a 6-pixel by 3-pixel computer screen, but there would be fewer convex topologies. We will begin to find all possible ways to obtain convex sets on a product of two totally ordered spaces. The goal of counting the topologies having a basis of disjoint convex sets leads to our more fundamental work here of counting the number of convex sets in a product of two totally ordered spaces. Background information on ordered spaces and counting techniques is found in [**3**] and [**5**].

Let $\underline{n} := \{1, 2, \ldots, n\}$ where n gives the number of columns in the space. Let $\underline{m} := \{1, 2, \ldots, m\}$ where m gives the number of rows in the space. The *increasing hull* of a subset, A, of a partially ordered set, X, is defined by $i(A) = \{x \in X : \exists a \in A, x \geq a\}$ [**2**]. The *decreasing hull* of a subset, A, of a partially ordered set, X, is defined by $d(A) = \{x \in X : \exists a \in A, x \leq a\}$ [**2**]. The *product order* on the Cartesian product of partially ordered sets is $(x, y) \leq (a, b)$ if and only if $x \leq a$ and $y \leq b$ [**5**].

There are many definitions and concepts for convexity. For this thesis we are using convexity in the ordered sense instead of the more widely thought of geometric sense. A set, C, is called *convex* if the intersection of the increasing hull of C and the decreasing hull of C produces exactly the set C [**5**]. An alternate form of this definition states that for any point a such that $w \leq a \leq y$ for some w and y contained in the set C, a must also be contained in the set C. Thus, in $\underline{n} \times \underline{m}$ with the product order, C is convex if $(w, x) \leq (a, b) \leq (y, z)$ and $(w, x), (y, z) \in C$ imply $(a, b) \in C$. We say a set is a *z-row set* if it occupies z adjacent rows of the m rows in a $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ space. Let $T_{cnvx}(n)_z$ be defined as the total number of possible z-row convex sets on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$, a product of ordered spaces. Let w_d denote the total number of sets having width d.

We will use the following *summation formulas*:

$$\sum_{i=1}^{n} c = cn, \text{where } c \text{ is a constant;}$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2};$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6};$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Chapter 2

TWO-ROW SPACES

2.1. Summation Method

We start by trying to find the number of convex sets on $\{1, 2, \ldots, n\} \times \{1, 2\}$ spaces. See Figure 2.1.1. This can be found by combining the total number of convex sets that take up only one row with the number of convex sets occupying two rows. Let us first look at the possible convex sets that occupy only one row.

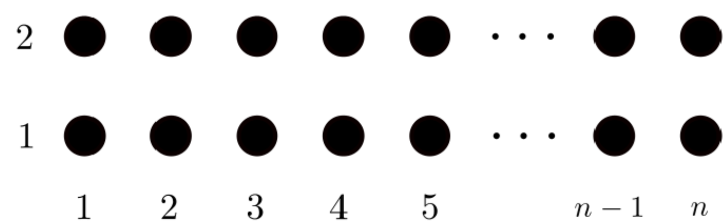

Figure 2.1.1. $\{1, 2, \ldots, n\} \times \{1, 2\}$ space.

Theorem 2.1.1. *The number of all possible convex sets on a totally ordered, one row space consisting of n points is given by*

$$T_{cnvx}(n)_1 = \frac{1}{2}n(n+1). \tag{2.1}$$

Proof. Let i denote the left endpoint and j the right endpoint of the convex set. This gives $1 \leq i \leq j \leq n$. First, we sum over all possibilities for i and then sum over all possibilities for j for each choice of i. Since $i \in \{1, \ldots, n\}$ and $j \in$

$\{i, \ldots, n\}$, we get

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n} 1 &= \sum_{i=1}^{n}(\sum_{j=1}^{n} 1 - \sum_{j=1}^{i-1} 1) \\
&= \sum_{i=1}^{n}(n - (i-1)) \\
&= \sum_{i=1}^{n} n - \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\
&= n^2 - \frac{n(n+1)}{2} + n \\
&= \frac{2n^2 - n^2 - n + 2n}{2} \\
&= \frac{n^2 + n}{2} \\
&= \frac{n(n+1)}{2}.
\end{aligned}
$$

□

Notice that $\frac{n(n+1)}{2}$ is a triangular number and that $\frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n$. The correlation between the triangular number and the number of convex set on a totally ordered, one-row space consisting of n points is easily seen. In such a space we can have a convex set, which is equivalent to an interval since we are in a one-row space, having width anywhere from 1 to n. If we consider how many convex sets we can have of each width, denoted w_i where $i \in \{1, 2, \ldots, n\}$, then the total number of convex sets in the space is found by adding the total numbers possible of each for these widths. In other words, $T_{cnvx}(n)_1$ is equal to the total number of width-1 sets plus the total number of width-2 sets, ..., plus the total number of width n sets.

Let us first consider w_1, where w_1 equals the number of width-1 sets. If our set has width 1, then it can be placed on any of the points in our set, giving n

possible distinct positions for our set to be placed. Keep in mind that we are only interested in convex sets; therefore a set of width 2 must be placed onto two adjacent points in the space. If our set has width 2, then the first point of the set can be placed on any of the first $n-1$ points of the space. Notice if the first point of our two-point set would have been placed on point n of the set, there is no point available to give a convex set that is distinct from the $n-1$ already considered. There are $n-2$ distinct positions to place the first point of a set of width 3. This pattern continues as the width of the set is increased. Thus, there are two possible places to have a convex set of width $n-1$, and only one place for a width n convex set. This gives an alternate way to prove (2.1) from Theorem 2.1.1. Now let us look at the convex sets that occupy two rows.

THEOREM 2.1.2. *The number of convex sets occupying two rows on the product of two totally ordered spaces consisting of n columns and two rows is given by*

$$T_{cnvx}(n)_2 = \frac{1}{12}n(n+1)^2(n+2). \tag{2.2}$$

PROOF. Let i denote the top left, j the top right, k the bottom left, and l the bottom right endpoints of a convex set (see Figure 2.1.2). That is, think of the convex set as intervals $[i,j]$ on the top row and $[k,l]$ on the bottom row.

In order to easily satisfy the convexity condition, we choose the endpoints in the order i, j, l, and then k. Thus, we sum across all possibilities of i, and then sum across all possibilities of j for each i. Next, we sum across all possibilities of l for each j, and finally we sum across all possibilities of k for each l. In order to

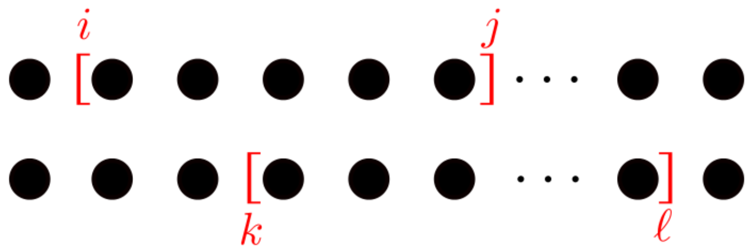

FIGURE 2.1.2. General locations of the endpoints for a two-row set.

have a convex set, the bottom right endpoint, l, cannot be any point strictly to the left of where j is located. Also, the bottom left endpoint, k, cannot be any point strictly to the left of where i is located. This is because in order for a set to be convex, if any point (a,b) satisfies $(w,x) \le (a,b) \le (y,z)$ for (w,x) and (y,z) contained in the set, (a,b) must also be contained in the set. For example, Figure 2.1.3 below portrays a two-row set where $[i,j] = [4,7]$ and $[k,l] = [2,6]$. The set contains the point $(2,1)$ and the point $(4,2)$. Since $(2,1) \le (2,2) \le (4,2)$ and $(2,2)$ is not contained in the set, the set is not convex. It is easily seen in Figure 2.1.3 that any point to the left of i that occurs directly over any point of the interval $[k,l]$ will result in a similar outcome. Now looking at the right-hand side of the set, we can see that $(6,1)$ and $(7,2)$ are both points contained in the set and $(6,1) \le (7,1) \le (7,2)$. However, because the point $(7,1)$ is not a point contained in the set, it is not convex. Once again it is easily seen in Figure 2.1.3 that any point to the right of l that is placed directly below any point of the interval $[i,j]$ will result in a set that is not convex.

These restrictions create the following sets that each endpoint must belong in: $i \in \{1,\dots,n\}$, $j \in \{i,\dots,n\}$, $l \in \{j,\dots,n\}$, and $k \in \{i,\dots,l\}$. Now using this, we

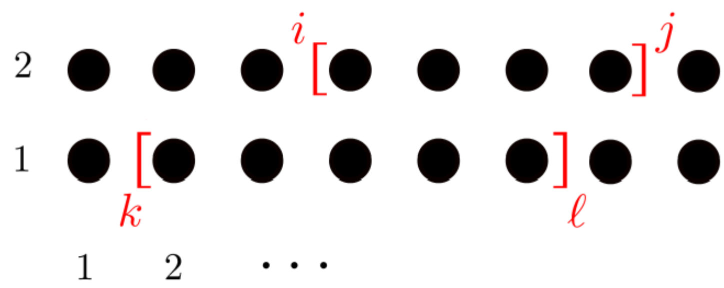

FIGURE 2.1.3. Two-row set that is not convex.

sum across all possibilities to determine the total number of two-row convex sets. This gives

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l} 1 \tag{2.3}$$

as the total number of convex sets taking up two adjacent rows. Evaluating (2.3), it is seen

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l} 1 &= \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}(\sum_{k=1}^{l} 1 - \sum_{k=1}^{i-1} 1) \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}(l-i+1) \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n}[\sum_{l=1}^{n}(l-i+1) - \sum_{l=1}^{j-1}(l-i+1)] \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n}[\frac{n(n+1)}{2} - in + n - \frac{j(j-1)}{2} + i(j-1) - (j-1)] \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n}[\frac{n^2}{2} + \frac{3n}{2} - ni - i + 1 - \frac{j^2}{2} - \frac{j}{2} + ji] \\
&= \sum_{i=1}^{n}[\sum_{j=1}^{n}(\frac{n^2}{2} + \frac{3n}{2} - ni - i + 1 - \frac{j^2}{2} - \frac{j}{2} + ji) \\
&\quad - \sum_{j=1}^{i-1}(\frac{n^2}{2} + \frac{3n}{2} - ni - i + 1 - \frac{j^2}{2} - \frac{j}{2} + ji)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\Big[\frac{n^3}{2}+\frac{3n^2}{2}-in^2-in+n-\frac{n(n+1)(2n+1)}{12}-\frac{n(n+1)}{4}+\frac{in(n+1)}{2} \\
&\qquad -(i-1)\Big(\frac{n^2}{2}+\frac{3n}{2}-ni-i+1\Big)+\frac{i(i-1)(2i-1)}{12}+\frac{i(i-1)}{4}-\frac{i^2(i-1)}{2}\Big] \\
&= \sum_{i=1}^{n}\Big[\frac{n^3}{3}+\frac{3n^2}{2}+\frac{13n}{6}-in^2-3ni+ni^2-\frac{i^3}{3}+\frac{3i^2}{2}-\frac{13i}{6}+1\Big] \\
&= \frac{n^4}{3}+\frac{3n^3}{2}+\frac{13n^2}{6}-\frac{n^3(n+1)}{2}-\frac{3n^2(n+1)}{2}+\frac{n^2(n+1)(2n+1)}{6} \\
&\qquad -\frac{n^2(n+1)^2}{12}+\frac{3n(n+1)(2n+1)}{12}-\frac{13n(n+1)}{12}+n \\
&= \frac{n^4}{12}+\frac{n^3}{3}+\frac{5n^2}{12}+\frac{n}{6} \\
&= \frac{1}{12}n(n+1)^2(n+2).
\end{aligned}
$$

□

The sequence $(T_{cnvx}(n)_2)_{n=1}^{10} = (1, 6, 20, 50, 105, 196, 336, 540, 825, 1210)$ appears as A002415 in [**6**] as 4-dimensional pyramidal numbers.

Considering the convex sets which contain points of one or both rows of a two-row n-column space, we get the following theorem.

THEOREM 2.1.3. *The total number of convex sets on the product* $\{1, 2, \ldots, n\} \times \{1, 2\}$ *of two totally ordered spaces is given by*

$$T_{cnvx}(n)_{T2} = \frac{1}{12}n(n+1)(n^2+3n+14). \tag{2.4}$$

PROOF. By assumption, we are in a two-row, n-column space. The total number of convex sets possible on this space can be found by combining the number of possible convex sets contained in only row one, only row two, and on rows one and two simultaneously. This can be found by summing equation (2.1) twice and equation (2.2) once. Thus,

$$\begin{aligned} T_{cnvx}(n)_{T2} &= 2 * \frac{n(n+1)}{2} + \frac{1}{12}n(n+1)^2(n+2) \\ &= \frac{n^4}{12} + \frac{n^3}{3} + \frac{17n^2}{12} + \frac{7n}{6} \\ &= \frac{1}{12}n(n+1)(n^2+3n+14). \end{aligned}$$

□

All possible sets are shown for $n = 1$ and $n = 2$ in Figures 2.1.4 and 2.1.5, respectively. The first ten terms of this sequence are $(T_{cnvx}(n)_{T2})_{n=1}^{10} = (3, 12, 32, 70, 135, 238, 392, 612, 915, 1320)$.

FIGURE 2.1.4. All convex sets on a $\{1\} \times \{1, 2\}$ space.

2.2. Alternative (Width) Method

An alternative way to count the total number of convex sets in a $\{1, 2, \ldots, n\} \times \{1, 2\}$ space is to look at the possible number of sets for each potential set width. Here we say a convex set has width d if the right endpoint of the bottom row, l, is located in the $d + i - 1$ position, where i is the left endpoint of the top row of the width-d convex set. It should be clear in Figure 2.2.1 below that the first set has width $d = 5$; however, notice that the second set has width $d = 5$ as well.

Similar to the discussion on the triangular numbers in the previous section, if we have a convex set occupying two rows of width 1, then we have n distinct positions where it can be placed. This gives nw_1 as the total number of possible

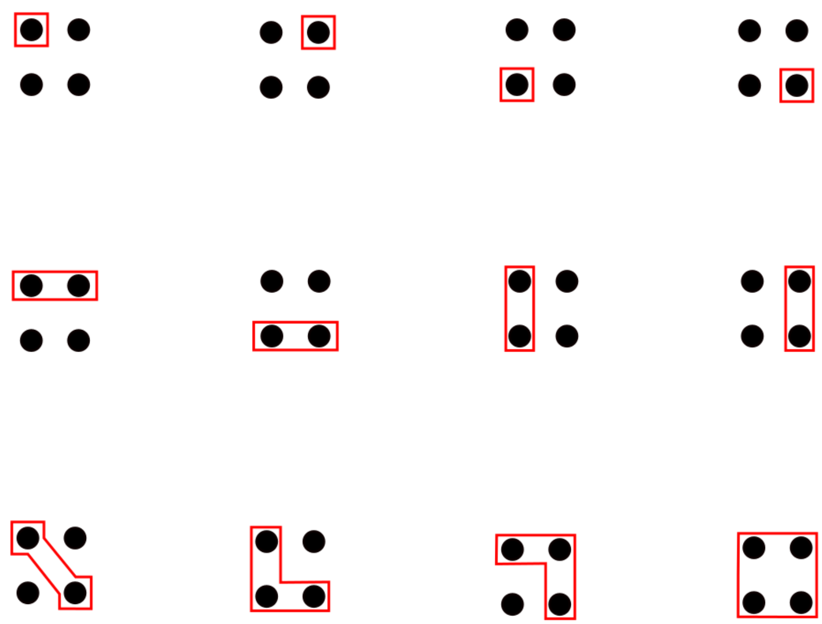

FIGURE 2.1.5. All convex sets on $\{1,2\} \times \{1,2\}$.

width-one convex sets in the space. If we have a width-two convex set, then we have $n-1$ distinct positions where it can be placed. From this we get $(n-1)w_2$ as the total number of possible width-two convex sets in the space. This process is continued for all w_d where $d \in \{1,2,\cdots,n\}$. By summing all the outcomes, we achieve the total number of convex sets in a two-row, n-column space.

THEOREM 2.2.1. *The total number of width-d sets in a two-row, n-column space is given by*

$$w_d = d^2 + 2. \tag{2.5}$$

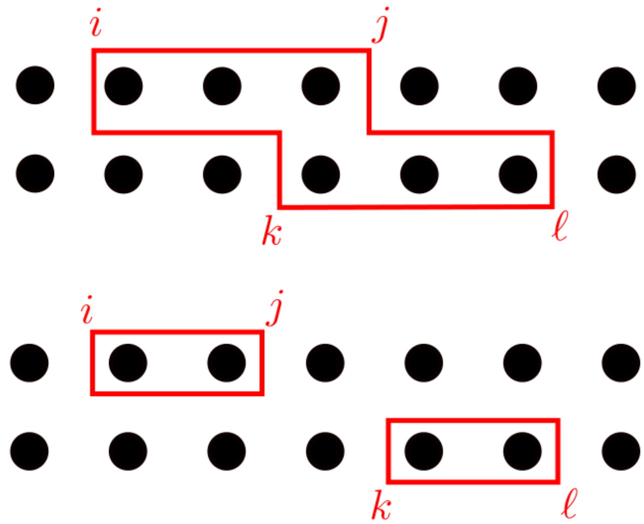

FIGURE 2.2.1. Two examples of sets with width $d = 5$.

PROOF. A convex set of width d occupying two rows gives us that our bottom right endpoint, l, is located at the $d + i - 1$ position. Since i and l are fixed, we need to determine how may possible positions there are for j and k. Recall for a set to be convex there are restrictions on l based on the location of j and on k based on the location of i. Thus, there are no restrictions on where j can be placed, giving us d distinct positions for j. Similarly, there are no restrictions on k other than that it cannot be any point strictly below where i is located. Thus, k can be placed anywhere from i to j, giving us d distinct positions for k to be placed. Therefore the total number of convex sets occupying two rows is $d * d$ or, simply d^2. Once we take into account the two one-row convex sets of width-d in our d-column space, we get $w_d = d^2 + 2$. □

This result can also be seen by geometrically extending convex sets as explained below. The total number of width-d convex sets on a two-row, n-column space can be found by extending the space to $(n+1)$ columns and finding the total number of width-$(n+1)$ sets that can be created from width-n sets. If the original width-n set is a one-row set, then it can be extended by one to create a one-row width-$(n+1)$ convex set. This operation will create two width-$(n+1)$ sets. Recall that if a two-row set is of width n in a two-row, n-column space, then i is located in position one on the top row, and l is located in position n on the bottom row. If the original width-n set, to be extended, is a two-row set, there are three operations that will result in obtaining a convex, width-$(n+1)$ set.

First, the entire $(n+1)$ column could be added in order to extend the set. If the $(n+1)$ column is added, then, for convexity, the entire top row must have been contained in the set of width n prior to the new column. This fixes the endpoints i, j and l, where $i = 1, j = n$, and $l = n$, on the original width-n convex set. This leaves only the position for k to be chosen. Since, in general, $k \in \{i, \dots, l\}$, then $k \in \{1, \dots, n\}$ giving n choices for k. Notice that the $(n+1)$ column can also be added to a one-row width-n set to create a convex set of width $(n+1)$ as long as the original one-row set to be expanded was on the top row to begin with. This operation will create $n+1$ width-$(n+1)$ sets.

Also, only the bottom point of the $(n+1)$ column could be added. There are two different ways to achieve this new set from the original width-n convex sets. First, the bottom row of the width-n set can be shifted one position to the right, which will create n^2 width-$(n+1)$ sets. This comes from the n choices for j and n choices for k of the pre-shifted set, since the width fixes i and l. Alternatively, the bottom row can be extended by one to include the bottom point of the $(n+1)$ column if the entire bottom row was contained in the original width-n set. Notice the first operation of shifting the bottom row gives sets such that it is impossible for $k = 1$. Thus, the case must be considered where the bottom row of column $n+1$ is added when i, k and l were fixed at $1, 1$ and n, respectively. This leaves only the position for j to be chosen. In general, $j \in \{i, \ldots, n\}$. Thus, there are n choices as $j \in \{1, \ldots, n\}$. This will create n width-$(n+1)$ sets. There are other routes to obtain the number of ways to extend a width-n set to width $(n+1)$, but this was found to be the best in order to not have any identical sets created using different operations that would then need to be removed from the count.

Taking the total number of width-$(n+1)$ sets created by each operation, it is found that

$$\begin{aligned} w_{n+1} &= 2 + (n+1) + n^2 + n \\ &= n^2 + 2n + 3 \\ &= (n+1)^2 + 2. \end{aligned}$$

Now, letting $n+1 = d$, the above can be written as $w_d = d^2 + 2$, which coincides with what was found in (2.5) .

Now let us consider the total number of distinct positions in which a convex set of width d can be placed. Note that i, the top left endpoint of the width-d

convex set, can be placed anywhere from the 1^{st} position to the $n-(d-1)$ position. This is because the remaining $d-1$ spaces are needed for the total width, d, of the set to be placed entirely inside the $\{1,2,\dots,n\}\times\{1,2\}$ space. Thus, there are $n-d+1$ distinct positions in which to place a convex set of width d in the space.

THEOREM 2.2.2. *The total number of convex sets on the product of two totally ordered spaces consisting of n columns and two rows is given by*

$$T_{cnvx}(n)_{T2} = \sum_{d=1}^{n}(n-d+1)w_d. \tag{2.6}$$

PROOF.

$$\begin{aligned}
\sum_{d=1}^{n}(n-d+1)w_d &= \sum_{d=1}^{n}(n+1-d)*(d^2+2)\\
&= \sum_{d=1}^{n}[(n+1)d^2+2(n+1)-d^3-2d]\\
&= (n+1)\frac{n(n+1)(2n+1)}{6}+2n(n+1)-\frac{n^2(n+1)^2}{4}-2\frac{n(n+1)}{2}\\
&= \frac{n^4}{12}+\frac{n^3}{3}+\frac{17n^2}{12}+\frac{7n}{6}\\
&= \frac{1}{12}n(n+1)(n^2+3n+14)\\
&= T_{cnvx}(n)_{T2}.
\end{aligned}$$

Thus, $T_{cnvx}(n)_{T2}$ equals (2.4) from Theorem 2.1.3, which was proven to be the total possible number of convex sets on the product of two totally ordered spaces consisting of n columns and two rows. □

2.3. Annihilator Method

2.3.1. Definitions and Notations. A *time scale* is a subset of the real line which defines periods in which a process occurs. Let the time scale, T, be defined as $n \in T = \mathbb{N}$. Let C_i for $i \in \{1, 2, \dots\}$ represent constants. The *shift operator*, E, is defined by $E^n y(t) = y(t+n)$, where $n \in \mathbb{N}$.

The linear equation of the n^{th} order is

$$p_n(t)y(t+n) + p_{n-1}(t)y(t+n-1) + \cdots + p_1(t)(t+1) + p_0(t)y(t) = r(t), \tag{2.7}$$

where $p_0(t), \dots, p_n(t)$ and $r(t)$ are assumed to be known, and $p_0(t) \neq 0$ and $p_n(t) \neq 0$ for all t. If $r(t) \neq 0$, then we say that the equation is *nonhomogeneous*. Note that (2.7) can also be written, using the shift operator, as

$$[p_n(t)E^n + p_{n-1}(t)E^{n-1} + \cdots + p_1(t)E^1 + p_0(t)E^0]y(t) = r(t),$$

where $E^0 = I$, or as

$$(E - \lambda_1)^{\alpha_1} \cdots (E - \lambda_k)^{\alpha_k} u(t) = r(t),$$

where $\alpha_1 + \cdots + \alpha_k = n$ [**4**].

The polynomial $\lambda^n + p_{n+1}\lambda^{n-1} + \cdots + p_0$ is called the *characteristic polynomial*, and the solutions, $\lambda_1, \dots, \lambda_k$, of the characteristic equation are the *characteristic roots* [**4**].

2.3.2. Theorems. The theorems below form a basis for solving n^{th}-order, linear difference equations and can be found in standard references such as [**4**].

THEOREM 2.3.1. *Assume that $p_0(t), \ldots, p_n(t)$, and $r(t)$ are defined for $t = a, a+1, \ldots,$ and $p_0(t) \neq 0$ and $p_n(t) \neq 0$ for all t. Then for any $t_0 \in \{a, a+1, \ldots\}$ and any numbers $y_0, \ldots, y_{n-1}$, there is exactly one $y(t)$ that satisfies equation (2.7) for $t = a, a+1, \ldots,$ and $y(t_0 + k) = y_k$ for $k = 0, \ldots, n-1$.*

THEOREM 2.3.2. *If $u_1(t), \ldots, u_n(t)$ are independent solutions of equation (2.7) then every solution u_t of equation (2.7) can be written in the form*

$$u(t) = C_1 u_1(t) + \cdots + C_n u_n(t)$$

for some constants $C_1, \ldots, C_n$.

THEOREM 2.3.3. *Suppose that equation (2.7) has characteristic roots $\lambda_{1,}, \ldots, \lambda_k$ with multiplicities $\alpha_1, \ldots, \alpha_k$, respectively. Then (2.7) has n independent solutions $\lambda_1^t, \cdots, t^{\alpha_1 - 1}\lambda_1^t, \lambda_2^t, \cdots, t^{\alpha_2 - 1}\lambda_2^t, \cdots, \lambda_k^t, \cdots, t^{\alpha_k - 1}\lambda_k^t$.*

The general equation with constant coefficients,

$$y(t+n) + p_{n-1} y(t+n-1) + \cdots + p_0 y(t) = r(t),$$

can be solved by the "annihilator method" if $r(n)$ is a solution of some homogeneous equation with constant coefficients.

THEOREM 2.3.4. *Suppose that $y(t)$ solves (2.7); that is*

$$(E^n + p_{n-1}E^{n-1} + \cdots + p_0) y(t) = r(t),$$

and that $r(t)$ *satisfies*

$$(E^m + q_{m-1}E^{m-1} + \cdots + q_0)r(t) = 0.$$

Then $y(t)$ *satisfies*

$$(E^m + \cdots + q_0)(E^n + \cdots + p_0)y(t) = 0.$$

2.3.3. Model. Begin by counting the number of possible convex sets on a $\{1\} \times \{1,2\}$ space. It is easily seen that $T_{cnvx}(1)$, the total number of convex sets when $n = 1$, is 3 (see Figure 2.1.4). We continue in this manner to find $T_{cnvx}(2)$, $T_{cnvx}(3)$, and $T_{cnvx}(4)$; these are found to be 12, 32, and 70, respectively. Figure 2.1.5 shows the 12 sets for $T_{cnvx}(2)$.

After finding these values, we are able to see that the total number of convex sets on $\{1,2,\ldots,n,n+1,n+2\} \times \{1,2\}$, $T_{cnvx}(n+2)$, can be modeled by

$$T_{cnvx}(n+2) = 2 * T_{cnvx}(n+1) - T_{cnvx}(n) + (n+2)^2 + 2, \tag{2.8}$$

satisfying the following initial conditions: $T_{cnvx}(1) = 3$ and $T_{cnvx}(2) = 12$.

There are two different approaches in order to obtain (2.8). This first approach, examining the equations starting when $n = 1$ to see the recursion taking place, can be seen by:

$$\begin{aligned} T_{cnvx}(1) &= 3 \\ T_{cnvx}(2) &= 12 \\ T_{cnvx}(3) &= 32 = 12 + 12 - 3 + 3^2 + 2 \end{aligned}$$

$$T_{cnvx}(4) = 70 = 32 + 32 - 12 + 4^2 + 2$$

$$\vdots$$

$$T_{cnvx}(n+2) = 2 * T_{cnvx}(n+1) - T_{cnvx}(n) + (n+2)^2 + 2.$$

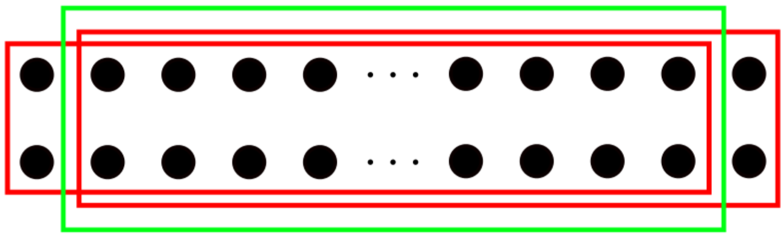

FIGURE 2.3.1. Model.

It is easier seen through a picture, shown in Figure 2.3.1, what is happening in this model. In order to obtain the total number of convex sets possible in a $\{1, 2, \ldots, n\} \times \{1, 2\}$ space, we can consider all the convex sets that are in a $\{1, 2, \ldots, n-1\} \times \{1, 2\}$ space covering the first $n - 1$ columns and covering the last $n - 1$ columns (see Figure 2.3.1). Then all the possible convex sets in the middle $n - 2$ columns that were counted in each of the $(n - 1)$-column count, need to be subtracted from the total (see Figure 2.3.1). We then must add in all possible convex sets that span the entire width of the n-column space. From (2.5), we know this to be $n^2 + 2$. Thus, the total number of possible convex set on a $\{1, 2, \ldots, n\} \times \{1, 2\}$ space is $T_{cnvx}(n) = 2 * T_{cnvx}(n-1) - T_{cnvx}(n-2) + n^2 + 2$. Since we need the total number of convex sets for $n - 1$ and $n - 2$ in order to determine $T_{cnvx}(n)$ and $n > 0$, we shift by two in order to ensure $T_{cnvx}(n)$ can be calculated for all n and give initial values for $T_{cnvx}(1)$ and $T_{cnvx}(2)$. This gives (2.8) as the total number of convex sets on a $\{1,2,\ldots,n\} \times \{1,2\}$ space for $n \geq 3$.

This gives the second-order non-homogeneous linear difference equation,

$$T_{cnvx}(n+2) - 2 * T_{cnvx}(n+1) + T_{cnvx}(n) = (n+2)^2 + 2,$$

that can be solved using the annihilator method from Theorem 2.3.3. Once solved, this will give a formula for $T_{cnvx}(n)$.

We will rewrite the equation in operator form:

$$(E^2 - 2E + 1)T_{cnvx}(n) = (n+2)^2 + 2$$

or, equivalently,

$$(E-1)^2 T_{cnvx}(n) = n^2 + 4n + 6.$$

First, we must find the solution to the homogeneous equation,

$$(E-1)^2 T_{cnvx}(n) = 0.$$

As the equation is already in shift operator form, it is easily seen that the characteristic roots are $\lambda = 1$ with multiplicity two. This gives a homogeneous solution of

$$T_{cnvx}(n)_h = C_0 1^n + nC_1 1^n$$

or simply

$$T_{cnvx}(n)_h = C_0 + nC_1.$$

Now, we can rewrite

$$(E-1)^2 T_{cnvx}(n) = n^2 + 4n + 6$$

as

$$(E-1)^3 (E-1)^2 T_{cnvx}(n) = 0$$

since $(E-1)^3$ is the "annihilator," which eliminates the nonzero function from the right-hand side of the equation.

Again, it can easily be seen that the characteristic roots are $\lambda = 1$ with multiplicity five, since our equation is in shift operator form. From Theorem 2.3.3, this gives a general solution consisting of independent solutions for $T_{cnvx}(n)$, and by Theorem 2.3.2, can be written as

$$T_{cnvx}(n)_g = C_0 + nC_1 + n^2C_2 + n^3C_3 + n^4C_4.$$

Since $C_0 + nC_1$ is the solution to the homogeneous equation,

$$T_{cnvx}(n)_p = n^2C_2 + n^3C_3 + n^4C_4$$

is a particular solution to the nonhomogeneous equation. The uniqueness of this solution is verified by Theorem 2.3.1.

Next, we will substitute $T_{cnvx}(n)_p$ into our original equation (2.8) to solve for the coefficients in the particular solution:

$$\begin{aligned} (n+2)^2C_2 &+ (n+2)^3C_3 + (n+2)^4C_4 - 2[(n+1)^2C_2 \\ &+ (n+1)^3C_3 + (n+1)^4C_4] + n^2C_2 + n^3C_3 + n^4C_4 = n^2 + 4n + 6 \end{aligned}$$

or

$$n^2(12C_4) + n(6C_3 + 24C_4) + (2C_2 + 6C_3 + 14C_4) = n^2 + 4n + 6.$$

This gives the following system of equations:

$$12C_4 = 1,$$

$$6C_3 + 24C_4 = 4,$$

$$2C_2 + 6C_3 + 14C_4 = 6.$$

Solving this system, we get

$$C_4 = \frac{1}{12}, C_3 = \frac{1}{3}, \text{ and } C_2 = \frac{17}{12},$$

giving the general solution to be,

$$T_{cnvx}(n)_g = C_0 + nC_1 + \frac{17}{12}n^2 + \frac{1}{3}n^3 + \frac{1}{12}n^4.$$

Finally, we use the initial conditions to solve for the remaining coefficients. $T_{cnvx}(1)$ = 3 and $T_{cnvx}(2) = 12$ give:

$$C_0 + C_1 = \frac{7}{6},$$

$$C_0 + 2C_1 = \frac{7}{3};$$

thus,

$$C_0 = 0 \text{ and } C_1 = \frac{7}{6}.$$

We now have a formula for the total number of convex sets on the product $\{1, 2, \ldots, n\} \times \{1, 2\}$ of totally ordered spaces, for all $n \in \mathbb{N}$:

$$T_{cnvx}(n)_g = \frac{7}{6}n + \frac{17}{12}n^2 + \frac{1}{3}n^3 + \frac{1}{12}n^4.$$

This agrees with the result found by other means in (2.4) and (2.6).

Chapter 3

TWO DISJOINT CONVEX SETS

Now let us look at how to find the total number of possible collections of two disjoint convex sets on $\{1, 2, \ldots, n\} \times \{1, 2\}$. In order to begin, we will use the summation method discussed in Section 2.1. It is important to note that if two given disjoint sets are the same as another collection of two disjoint sets, other than the name or label of the two, then this is considered as only one collection of two disjoint convex sets on the space and is only accounted for once. In other words, given two disjoint convex sets, A and B, on the space and two more disjoint sets, A' and B', on the space such that $A = B'$ and $B = A'$, then we consider this to be only one distinct collection of disjoint convex sets. Figure 3.0.1 illustrates this.

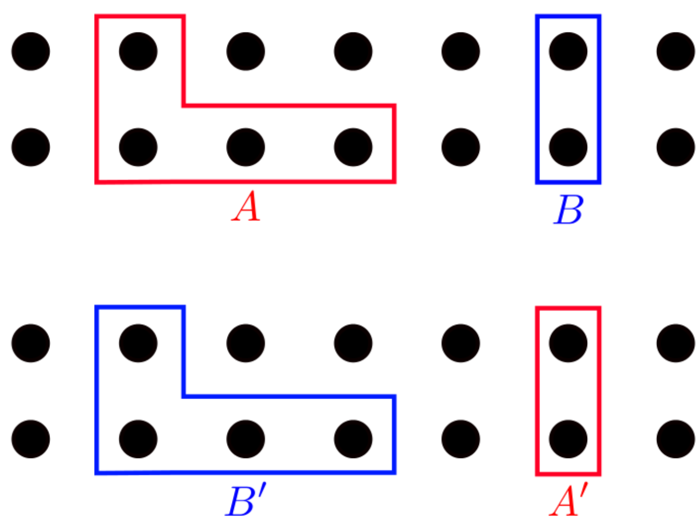

Figure 3.0.1. Two collections of disjoint sets that differ only by name.

Now, in order to look at the total number of collections of two disjoint convex sets on the space, we must look at each case that is possible separately.

3.1. Two One-Row Sets

If each of the two convex sets only occupy a single row in a two-row space, then either both sets are on the same row (Figure 3.1.1), or there is one set per row (Figure 3.1.2). Let us first consider the case of both sets being on a single row together.

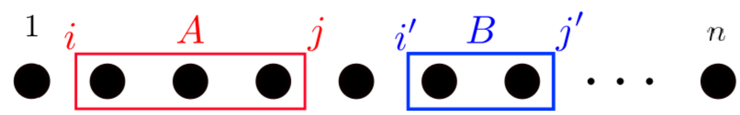

FIGURE 3.1.1. Two one-row sets occupying a single row.

THEOREM 3.1.1. *The total number of collections of two one-row disjoint convex sets together on a single row of length n is*

$$\frac{1}{24}(n-1)n(n+1)(n+2). \tag{3.1}$$

PROOF. Let i denote the left endpoint and j the right endpoint of the first convex set. Also, let i' denote the left endpoint and j' the right endpoint of the second convex set; this gives $1 \le i \le j < i' \le j' \le n$. See Figure 3.1.1 to visualize set placement. We first sum over all possibilities for i and then over all possibilities for j for each i. We continue this summation process for i' and j' over all possibilities of j and i', respectively. Since $i \in \{1, \dots, n\}$, $j \in \{i, \dots, n\}$, $i' \in \{j+1, \dots, n\}$, and $j' \in \{i', \dots, n\}$ we get

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{i'=j+1}^{n} \sum_{j'=i'}^{n} 1.$$

Using the summation formulas and Mathematica to verify, it is found that

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{i'=j+1}^{n}\sum_{j'=i'}^{n} 1 = \frac{1}{24}(n-1)n(n+1)(n+2).$$

□

Equation (3.1) generates the sequence $0, 1, 5, 15, 35, 70, 126, 210, 330, 495$ for $n = 1, 2, \ldots, 10$, which is A000332 in [**6**]. Notice that (3.1) is equivalent to $\binom{n+2}{4}$. This is because finding the total possible number of two one-row disjoint sets occupying a single row is equivalent to finding the total number of two disjoint intervals in the space. The first interval is of the form $[i, j + \epsilon)$ and the second of the form $[i', j' + \epsilon')$, where ϵ and ϵ' are positive and infinitesimally small. We choose the four endpoints from $n + 2$, where the two left endpoints, i and i', are in the original n points and the two right endpoints, $j + \epsilon$ and $j' + \epsilon'$, are in the "+2" points and are discarded to create our intervals of interest. The two additional points are added to the set of values the four endpoints are chosen from in order to ensure disjoint intervals in choosing four distinct numbers, yet still being able to obtain a single element interval. Once the four points are chosen from least to greatest, the numbers are assigned to i, j, i' and j', respectively.

Equation (3.1) is derived from the formula for the number $C(n)$ of collections of disjoint convex sets in a totally ordered set as given in [**1**]:

$$C(n) = 1 + \sum_{p=1}^{n}\sum_{j=1}^{p}\left[\binom{n-p+j}{j}\binom{p-1}{j-1}\right].$$

There the 1 counted the empty collection and j was the number of disjoint sets, so (3.1) is obtained by fixing $j = 2$ in the double sum to get

$$\sum_{p=1}^{n} \binom{n-p+2}{2}\binom{p-1}{1} = \frac{1}{24}(n-1)n(n+1)(n+2).$$

Now let us consider when one of the one-row sets is on the first row of the space, and the other one-row set is on the second row of the space.

THEOREM 3.1.2. *The total possible number of collections of two one-row disjoint convex sets, where the two sets are split between two rows of length n is*

$$\frac{1}{4}n^2(n+1)^2. \tag{3.2}$$

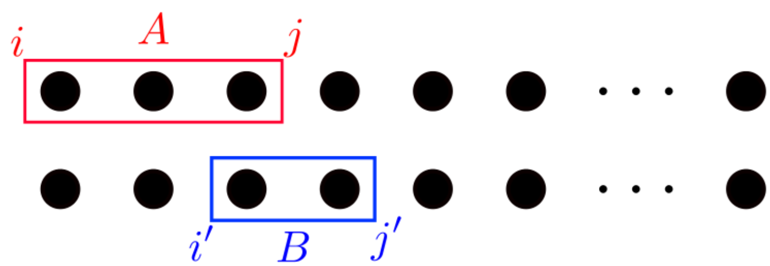

FIGURE 3.1.2. Two one-row sets occupying two seperate rows.

PROOF. Without loss of generality, we can assume that the first one-row convex set is on the first row and the second one-row convex set is on the second row. See Figure 3.1.2 to visualize set placement. Notice that because each set is isolated to a row separate from the other, the location of the second set with respect to the location of the first set will not affect the convexity of the either set. Thus, the total possible number of ways to have two one-row disjoint convex sets on separate rows in the space is the number of possible one-row convex sets on the first row times the numbers of possible convex sets on the second row. It was found in

(2.1) that the total number of one-row convex sets on $\{1,2,\ldots,n\}\times\{1,2\}$ spaces is $\frac{n(n+1)}{2}$. This gives the total possible number of collections of two one-row disjoint convex sets on separate rows to be

$$\begin{aligned}\frac{n(n+1)}{2} * \frac{n(n+1)}{2} &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= \frac{1}{4}n^2(n+1)^2.\end{aligned}$$

□

3.2. Two Sets: 1 One-Row and 1 Two-Row Set

Another way to have a collection of two disjoint convex sets on $\{1,2,\ldots,n\}\times\{1,2\}$ spaces is if one set is a two-row set and the second is a one-row set.

THEOREM 3.2.1. *The total number of collections of two disjoint convex sets where one set occupies two rows in the space and the other occupies one row is*

$$\frac{1}{360}(n-1)n(n+1)^2(n+2)(11n+18). \tag{3.3}$$

PROOF. Without loss of generality, we are able to first place the two-row set in the space and then the one-row set. To determine the total possible number of collections of sets arranged this way in the space, we must sum across all possibilities for the one-row set for each possible two-row set. All possible two-row sets can be found using the sum given by (2.3) in the proof of Theorem 2.1.2. Once the two-row convex set is placed in the space, it creates four different areas that a separate one-row convex set can be placed. These areas are shown as A, B, C and D in Figure 3.2.1. Notice that when we consider the total number of ways

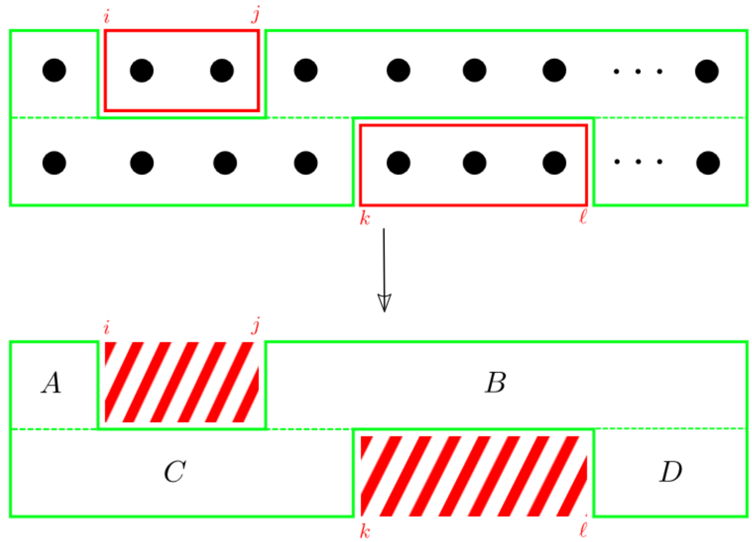

FIGURE 3.2.1. Areas created to place a one-row set after a two-row set is placed in the space.

to place a one-row set in any of these areas, where i' denotes the left endpoint and j' the right endpoint of the set, it is equivalent to determining the number of ways to place an interval of the form $[i', j')$, where $1 \le i' \le j' \le n+1$. The lower bound of i' and upper bound of j' are further restricted based on which of the four areas the one-row set is placed in. If the one-row set is placed in the area to the left of the two-row set on the top row (i.e., A in Figure 3.2.1), then we must determine the number of possible intervals to place in the space. Because the one-row set must be disjoint from the two-row set, we obtain the following bounds: $1 \le i' \le j' \le i$. Thus, the total number of possible intervals in this area can be found by all possible ways of choosing the two endpoints of $[i', j')$, where $i', j' \in \{1, 2, \cdots, i\}$. Mathematically this is $\binom{i}{2}$. Similarly, if the one-row convex set is placed in area C to the left of the two-row set on the bottom row, then we obtain the bounds $1 \le i' \le j' \le k$. Here we will choose our two endpoints of $[i', j')$, where $i', j' \in \{1, 2, \cdots, k\}$, which can be done in $\binom{k}{2}$ ways. If the one-row set is placed in areas B or D to the right of the two-row set, then we must choose our two endpoints, i', j' where $i', j' \in \{j+1, \cdots, n+1\}$ for the top row and $i', j' \in \{l+1, \cdots, n+1\}$ for the bottom row. This gives the total number of possible one-row sets in each of these areas to be $\binom{n+1-j}{2}$ and $\binom{n+1-l}{2}$, respectively. Recall that we must sum across all possibilities for the one-row set for each possible two-row set, giving us the following:

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\left[\binom{i}{2}+\binom{k}{2}+\binom{n+1-j}{2}+\binom{n+1-l}{2}\right] \tag{3.4}$$

Using the summation formulas and Mathematica to verify, it is found that

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\left[\frac{i(i-1)}{2}+\frac{k(k-1)}{2}+\frac{(n+1-j)(n-j)}{2}+\frac{(n+1-l)(n-l)}{2}\right]$$
$$=\frac{1}{360}(n-1)n(n+1)^2(n+2)(11n+18).$$

□

For $n = 1, \ldots, 10$, (3.4) generates the sequence $0, 8, 68, 310, 1022, 2744, 6384, 13356, 25740, 46464$, respectively.

3.3. Two Two-Row Sets

The final way to have a collection of two disjoint convex sets on $\{1, 2, \ldots, n\} \times \{1, 2\}$ spaces is if both sets are two-row sets.

Theorem 3.3.1. *The total number of collections of two disjoint convex sets where both sets occupy two rows in the space is*

$$\frac{1}{20160}(n-1)n(n+1)^2(n+2)(n+3)(3n-2)(3n+8). \tag{3.5}$$

Proof. Without loss of generality, we are able to first place a two-row set in the space and then the top row of the second two-row set to the right of the top row of the first set. To determine the total possible number of collections of sets arranged this way in the space, we must sum across all possibilities for the second two-row set for each possible first two-row set. All possible ways to place the first two-row set can be found using (2.3). Once the first two-row convex set is placed in the space, it creates two different areas that the second two-row convex set can be placed, illustrated in Figure 3.3.1.

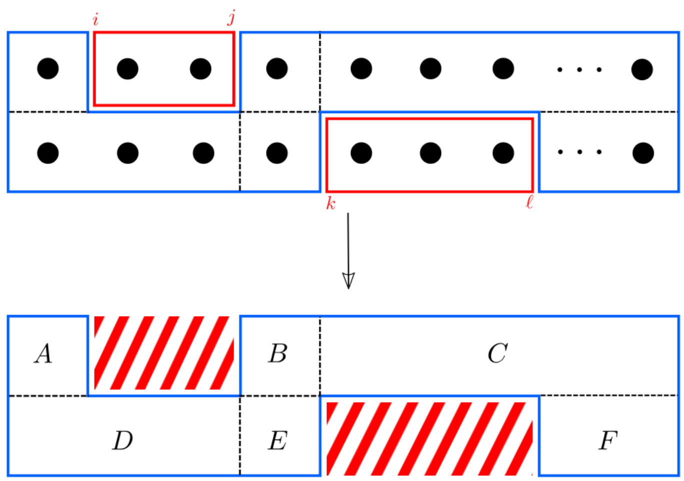

Figure 3.3.1. Areas created to place a second two-row set after the first two-row set is placed in the space.

Notice that placing a two-row set in any area combinations other than the ones listed here will guarantee a set that is *not* convex: $A \cup D, A \cup E, A \cup F, A \cup D \cup E, B \cup E$, and $B \cup C \cup F$. We will consider our first two-row set to be the one such that the top row is to the left of the top row of the second two-row set. Thus, the only combinations of areas where a second two-row convex set can be placed where the top row of the second occurs to the right of the top row of the first are $B \cup E$ and $B \cup C \cup F$.

Let i', j', k', and l' denote the top left, top right, bottom left, and bottom right endpoints, respectively, of the second set. If the second set is placed in the area created by $B \cup E$, then the following restrictions are placed in order to keep the set convex: $i' \in \{j+1, \ldots, k-1\}$, $j' \in \{i', \ldots, k-1\}$, $l' \in \{j', \ldots, k-1\}$, and $k' \in \{i', \ldots, l'\}$. Using this, we sum across all possibilities to determine the total number of two-row convex sets that can be placed in this area:

$$\sum_{i'=j+1}^{k-1} \sum_{j'=i'}^{k-1} \sum_{l'=j'}^{k-1} \sum_{k'=i'}^{l'} 1. \tag{3.6}$$

Now let us consider all possible ways to place the second set in the area created by $B \cup C \cup F$. Up until this point we have been summing across all endpoints of a given set in a particular order to ensure convexity. If we sum in that manner for this area, then the interval that l' is contained in will differ depending on whether j' falls before or after l of the first convex set in the space. To avoid having multiple sums that lead to double counts or having an index defined as a maximum, we will assign the bottom row and then the top row of our second convex set whenever it falls in this area. Thus, we choose our endpoints in the following order to ensure convexity while simplifying our sum: k', l', i' and j'. The following restrictions are placed in order to keep the set convex: $k' \in \{l+1, \ldots, n\}$, $l' \in \{k', \ldots, n\}$, $i' \in \{j+1, \ldots, k'\}$, and $j' \in \{i', \ldots, l'\}$. Using this, we sum across all possibilities to determine the total number of two-row convex sets that can be placed in the area created by $B \cup C \cup F$. This gives

$$\sum_{k'=l+1}^{n} \sum_{l'=k'}^{n} \sum_{i'=j+1}^{k'} \sum_{j'=i'}^{l'} 1. \tag{3.7}$$

Recall that we must sum across all possibilities for the second two-row set for each possible first two-row set to determine the total possible numbers of collections of two disjoint, two-row convex sets. Doing this gives

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\left[\left(\sum_{i'=j+1}^{k-1} \sum_{j'=i'}^{k-1} \sum_{l'=j'}^{k-1} \sum_{k'=i'}^{l'} 1\right) + \left(\sum_{k'=l+1}^{n} \sum_{l'=k'}^{n} \sum_{i'=j+1}^{k'} \sum_{j'=i'}^{l'} 1\right)\right]. \tag{3.8}$$

Using the summation formulas and Mathematica to verify, it is found that (3.8) is

$$\frac{1}{20160}(n-1)n(n+1)^2(n+2)(n+3)(3n-2)(3n+8).$$

□

The first several terms of the sequence generated by (3.5) are $0, 1, 16, 115, 544, 1974, 5952$. Taking all of these cases together will give the following theorem. In order to find the formula in the following theorem we must add together (3.1), (3.2), (3.3), and (3.5). However, notice that since the space has 2 rows, (3.1) needs to be added twice as this gives the total possible number of collections of two one-row disjoint convex sets together on a single row.

THEOREM 3.3.2. *The total possible number of collections of two disjoint convex sets on* $\{1, 2, \ldots, n\} \times \{1, 2\}$ *is given by*

$$\frac{1}{6720}n(n+1)(3n^6 + 21n^5 + 245n^4 + 735n^3 + 2632n^2 + 1484n - 1760). \tag{3.9}$$

The first ten terms of the sequence generated by the above equation were found to be $1, 20, 131, 565, 1915, 5509, 14032, 32496, 69675$, and 140140.

Chapter 4

CONVEX SETS ON M ADJACENT ROWS IN N COLUMNS

4.1. Sets on 3 Adjacent Rows

We are able to expand on the idea from Chapter 2, where we used (2.3) to sum across all possibilities and find the total number of convex sets taking up two adjacent rows. We now add a one-row set below the two-row convex set in such a way that the resulting three-row set satisfies the convexity requirement. Figure 4.1.1 below illustrates the resulting three-row set. As before, let i denote the top left, j the top right, k the bottom left, and l the bottom right endpoints of the original two-row convex set. Let p denote the left endpoint and q denote the right endpoint of the one-row set being used to extend our two-row set to a three-row set.

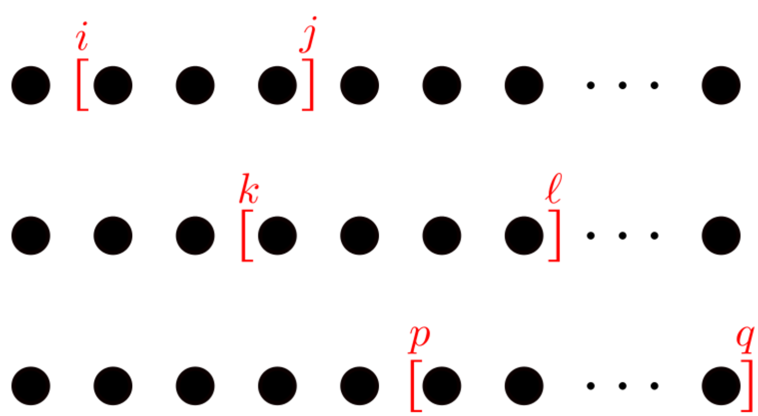

Figure 4.1.1. General locations of endpoints for a three-row set.

In order to easily satisfy the convexity condition, we choose the endpoints in this order: i, j, l, k, q and then p. Thus, we sum across all possibilities of i, and then sum across all possibilities of j for each i. Continue this process similarly as we did for the two-row set, except adding in all possibilities of our two new endpoints, p and q as well. Recall that the convexity stipulation placed restrictions on some of the endpoints, which results in creating the following sets that each endpoint must belong in: $i \in \{1, \dots, n\}$, $j \in \{i, \dots, n\}$, $l \in \{j, \dots, n\}$, $k \in \{i, \dots, l\}$, $q \in \{l, \dots, n\}$, and $p \in \{k, \dots, q\}$. Now using this we sum across all possibilities to determine the total number of three-row convex sets. This gives

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\sum_{q=l}^{n}\sum_{p=k}^{q} 1 \tag{4.1}$$

as the total number of convex sets taking up three adjacent rows. Using the summation formulas and Mathematica to verify, it is found that

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\sum_{q=l}^{n}\sum_{p=k}^{q} 1 = \frac{1}{144}n(n+1)^2(n+2)^2(n+3).$$

The above argument proves the following theorem.

THEOREM 4.1.1. *The number of all possible convex sets occupying three adjacent rows on the product of two totally ordered spaces consisting of n columns and 3 rows is given by*

$$T_{cnvx}(n)_3 = \frac{1}{144}n(n+1)^2(n+2)^2(n+3). \tag{4.2}$$

The sequence $(T_{cnvx}(n)_3)_{n=1}^{10} = (1, 10, 50, 175, 490, 1176, 2520, 4950, 9075, 15730)$ appears as A006542 in [**6**].

4.2. Sets on 4 and 5 Adjacent Rows

We are able to continue this process of adding a one-row set to an already convex set such that the resulting set still satisfies the convexity requirement. We will add a one-row convex set to the three-row set discussed in the previous section to obtain a formula for four-row convex sets. Then we add a one-row convex set to this four-row set to find a formula for five-row convex sets.

THEOREM 4.2.1. *The number of all possible convex sets occupying four adjacent rows on the product of two totally ordered spaces consisting of 4 rows and n columns is given by*

$$T_{cnvx}(n)_4 = \frac{1}{2880}n(n+1)^2(n+2)^2(n+3)^2(n+4). \tag{4.3}$$

PROOF. Let i, j, k, l, p and q be the endpoints of the three-row convex set defined in the previous section, and let r and s be the left and right endpoints, respectively, of the one-row convex set to be added below the three-row set to create a four-row set convex occupying four adjacent rows. The restrictions on i, j, k, l, p and q are the same as before. For the convexity stipulation the bottom right endpoint, s, cannot be any point strictly below where q is located, and r cannot be any point strictly below where i is located. These restrictions create the following sets that each endpoint must belong in: $i \in \{1, \ldots, n\}$, $j \in \{i, \ldots, n\}$, $l \in \{j, \ldots, n\}$, $k \in \{i, \ldots, l\}$, $q \in \{l, \ldots, n\}$, $p \in \{k, \ldots, q\}$, $s \in \{q, \ldots, n\}$, and $r \in \{p, \ldots, s\}$. Now using this we sum across all possibilities to determine the total number of four-row convex sets. This gives

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\sum_{q=l}^{n}\sum_{p=k}^{q}\sum_{s=q}^{n}\sum_{r=p}^{s}1 \tag{4.4}$$

as the total number of convex sets taking up four adjacent rows. Using the summation formulas and Mathematica to verify, it is found that (4.4) is

$$\frac{1}{2880}n(n+1)^2(n+2)^2(n+3)^2(n+4).$$

□

THEOREM 4.2.2. *The number of all possible convex sets occupying five adjacent rows on the product of two totally ordered spaces consisting of 5 rows and n columns is given by*

$$T_{cnvx}(n)_5 = \frac{1}{86400}n(n+1)^2(n+2)^2(n+3)^2(n+4)^2(n+5). \tag{4.5}$$

PROOF. Using the sum defined in (4.4) for the four-row convex sets occupying four adjacent rows, we add a one-row convex set below in order to create a five-row convex set from the four-row set. Let t be the left endpoint and u the right endpoint of the one-row set to be added. For the newly created five-row set to be convex, the following restrictions must be placed: $u \in \{s, \dots, n\}$ and $t \in \{r, \dots, u\}$. Once again we use the restrictions on these ten endpoints to sum across all possibilities to determine the number of five-row convex sets. This gives

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{l=j}^{n}\sum_{k=i}^{l}\sum_{q=l}^{n}\sum_{p=k}^{q}\sum_{s=q}^{n}\sum_{r=p}^{s}\sum_{u=s}^{n}\sum_{t=r}^{u}1 \tag{4.6}$$

as the total number of convex sets occupying five adjacent rows. Using the summation formulas and Mathematica to verify, it is found that (4.6) is

$$\frac{1}{86400}n(n+1)^2(n+2)^2(n+3)^2(n+4)^2(n+5).$$

□

The sequences $(T_{cnvx}(n)_4)_{n=1}^{10} = (1, 15, 105, 490, 1764, 5292, 13860, 32670, 70785, 143143)$ and $(T_{cnvx}(n)_5)_{n=1}^{10} = (1, 21, 196, 1176, 5292, 19404, 60984, 169884, 429429, 1002001)$ appear as A006857 and A108679 in [**6**], respectively.

4.3. Sets on z Adjacent Rows

Examining the number of possible one-row sets given by $T_{cnvx}(n)_1$, two-row sets given by $T_{cnvx}(n)_2$, three-row sets given by $T_{cnvx}(n)_3$, four-row sets given by $T_{cnvx}(n)_4$, and five-row sets given by $T_{cnvx}(n)_5$, it is seen that the number of possible z-row sets occupying z adjacent rows on a $m \times n$ space can be modeled as shown in the following Theorem.

THEOREM 4.3.1. *The number of possible convex sets occupying z adjacent rows on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, z\}$ can be found by*

$$T_{cnvx}(n)_z = \prod_{m=1}^{z} \frac{(n+m-1)(n+m)}{m(m+1)} \tag{4.7}$$

for $z = 1, 2, 3, 4,$ *and* 5.

We conjecture that this formula holds for any positive integer z.

CHAPTER 5

GENERAL CASE

5.1. Ways to Split Sets Within the Space

In order to find the total number of possible convex sets on $\{1,2,\ldots,n\} \times \{1,2,\ldots,m\}$, we must look at all of the ways to have a convex set placed in the space. First, we are able to have convex sets consisting of z adjacent rows as defined in Chapter 4. Note that $z \in \{1,2,\ldots,m\}$. The number of possible z-adjacent-row convex sets is determined by the number of rows in the space, m. For example, there are m possible ways to place a one-row convex set in a $\{1,2,\ldots,n\} \times \{1,2,\ldots,m\}$ space and $\frac{1}{2}n(n+1)$ possible one-row convex sets. Thus, the total number of possible one-row convex sets in an $\underline{n} \times \underline{m}$ space is given by

$$m * \frac{1}{2}n(n+1). \tag{5.1}$$

Table 5.1.1 gives the total number of one-row convex sets determined by (5.1) for all combinations of $m \in \{1,2,3,4,5\}$ and $n \in \{1,2,3,4,5\}$.

$m \backslash n$	1	2	3	4	5
1	1	3	6	10	15
2	2	6	12	20	30
3	3	9	18	30	45
4	4	12	24	40	60
5	5	15	30	50	75

TABLE 5.1.1. The total number of one-row convex sets in an n x m space for m and n between 1 and 5.

Looking at all of the ways to place a convex set within two adjacent rows in the space, it is seen that there are $m - 1$ possible ways to pick the two adjacent rows. This can be seen by considering where the top row of the two-row set is placed. It may be placed on any row in the set except for the bottom row, as there is at least one row needed below where it is placed to accommodate the second row of the two-adjacent-row set. Thus, the total number of possible two-row convex sets in an $\underline{n} \times \underline{m}$ space is given by

$$(m-1) * \frac{1}{12} n(n+1)^2(n+2). \tag{5.2}$$

This is the total number of ways to place a two-row set times the total possible number of two-row sets found by (2.2). Table 5.1.2 gives the total number of one-row convex sets determined by (5.2) for all combinations of $m \in \{1,2,3,4,5\}$ and $n \in \{1,2,3,4,5\}$. This idea continues for all z-row convex sets in a space to give the following theorem.

$m \backslash n$	1	2	3	4	5
1	0	0	0	0	0
2	1	6	20	50	105
3	2	12	40	100	210
4	3	18	60	150	315
5	4	24	80	200	420

TABLE 5.1.2. The total number of two-adjacent-row convex sets in an n x m space for m and n between 1 and 5.

Theorem 5.1.1. *The total number of possible z-row convex sets that occupy z adjacent rows in an $\underline{n} \times \underline{m}$ space can be found by*

$$(m - z + 1) * T_{cnvx}(n)_z. \tag{5.3}$$

Proof. The total number of ways to place a z-row convex set that occupies z adjacent rows in an m-row space can be determined by looking at the total number of ways to place the first row of the set in the space. Since our set is on z adjacent rows, we must have $z - 1$ available rows in the space after our first row of the z-row set is placed in order to accommodate the entire z-row set in the space. This tells us that we can place this first row anywhere from the first row of the space until the $m - (z - 1)^{th}$ row of the space. This gives us $m - z + 1$ possible positions to place a z-row convex set occupying z adjacent rows. Once we consider the number of possible convex sets occupying z adjacent rows given by (4.7) times all possible ways to place this z-row set, we get

$$(m - z + 1) * T_{cnvx}(n)_z$$

as the total number of possible z-row convex sets that occupy z adjacent rows in a $\underline{n} \times \underline{m}$ space. □

Let us look at an example where m is known to better understand this idea. Consider $m = 4$; this gives a $\{1, 2, \ldots, n\} \times \{1, 2, 3, 4\}$ space. In this space, we are able to place one-, two-, three-, and four-adjacent-row sets. It is obvious that since we are in a four-row space, that a one-row set can be placed on any of the four rows giving $4 * T_{cnvx}(n)_1$ as the total possible number of one-row convex sets in a four-row, n-column space. Similarly, it can be seen that when a two-row set occupying two adjacent rows is placed in a four-row space, there are three ways that it can be placed. It could be placed on rows 1 and 2, rows 2 and 3, or rows 3 and 4, giving $3 * T_{cnvx}(n)_2$ as the total possible number of two-row convex sets occupying two adjacent rows in a four-row, n-column space. Also, $2 * T_{cnvx}(n)_3$ and $1 * T_{cnvx}(n)_4$ will give the total possible number of three- and four-adjacent row sets, respectively, in a $\{1, 2, \ldots, n\} \times \{1, 2, 3, 4\}$ space. Combining all of these possibilities, it is seen that the total number of possible convex sets on $\{1, 2, \ldots, n\} \times \{1, 2, 3, 4\}$ space, such that all z-row convex sets occupy z-adjacent rows, is

$$4 * T_{cnvx}(n)_1 + 3 * T_{cnvx}(n)_2 + 2 * T_{cnvx}(n)_3 + 1 * T_{cnvx}(n)_4.$$

Generalizing this idea to an m-row space leads to the following theorem.

THEOREM 5.1.2. *The total number of possible convex sets on* $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ *spaces, such that no convex set has an empty row contained inside the set, is given by*

$$\sum_{z=1}^{m} (m - z + 1) * T_{cnvx}(n)_z. \tag{5.4}$$

The other way to place a z-row set on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ spaces, would be to place the z-row set on z nonadjacent rows. For example, looking at our previous scenario where $m = 4$, a two-row convex set could be split between rows 1 and 3, 1 and 4, or 2 and 4. Similarly a three-row set could be split and placed among rows 1, 2 and 4 or rows 1, 3 and 4. Since we are in a four-row space, it is not possible to place a four-row convex set that occupies four nonadjacent rows.

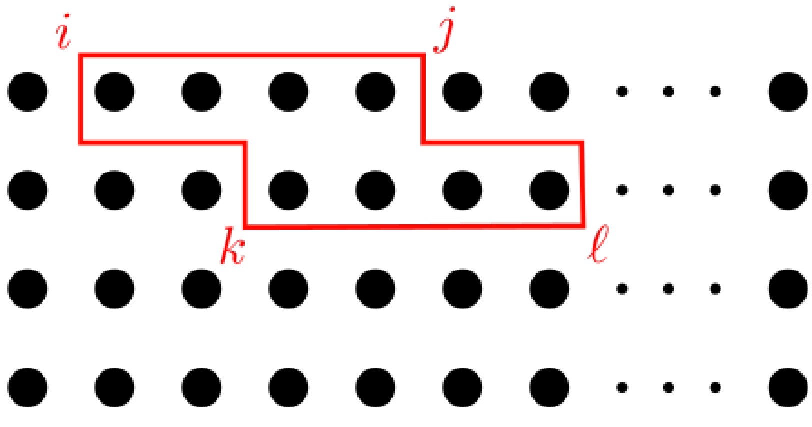

FIGURE 5.1.1. A two-row set that is convex when placed on two adjacent rows.

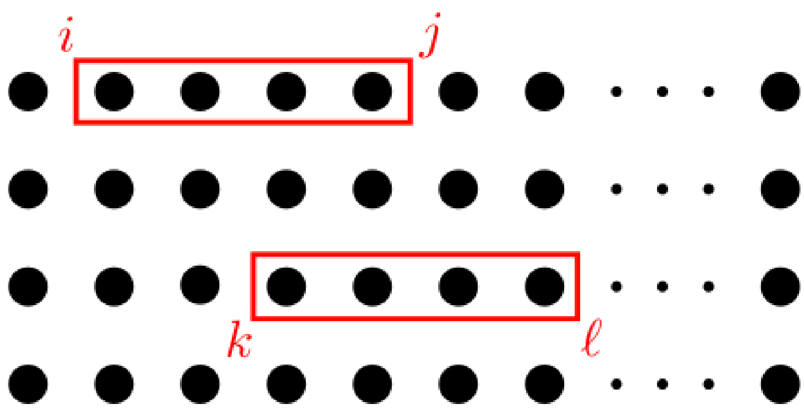

FIGURE 5.1.2. Same two-row set (from Figure 5.1.1) that is not convex when placed on two nonadjacent rows.

Once we start placing z-row sets on z nonadjacent rows, the way we count will change. The restrictions placed on the indices for the endpoints, stipulated previously to ensure convexity, will no longer guarantee convexity once one or more empty rows are placed within the interior of the z-row set. Recall from Section

2.1 that in order to have a two-row convex set, the bottom right endpoint, l, cannot be any point strictly to the left of where j is located. Also, the bottom left endpoint k cannot be any point strictly to the left of where i is located. In that section the number of rows in the space was specified as two; thus it was not possible to have a two-row convex set occupying two-nonadjacent rows. If these same stipulations are used once the set is split between two nonadjacent rows, then sets that are no longer convex are obtained. For example, Figure 5.1.1 depicts a two-row convex set on two adjacent rows. If we take that same two-row set and move the second row of the set down in the space such that at least one empty row appears between the two rows of the set, the set in Figure 5.1.2 is obtained. It is easily seen from Figure 5.1.2 that this is no longer convex when split on two nonadjacent rows.

In order to ensure that a two-row set that occupies nonadjacent rows is convex, the following restrictions must be placed on the four endpoints: $i \in \{1, \ldots, n\}$, $j \in \{i, \ldots, n\}, k \in \{j+1, \ldots, n\}$, and $j \in \{k, \ldots, n\}$. Now summing across all possibilities for each endpoint, we get

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=j+1}^{n} \sum_{j=k}^{n} 1$$

as the total of two-row convex sets split between nonadjacent rows. Using summation formulas and Mathematica to verify the above sum, proves the following theorem.

THEOREM 5.1.3. *The total possible number of two-row convex sets occupying two nonadjacent rows in* $\{1, \ldots, n\} \times \{1, \ldots, m\}$ *is given by*

$$T_{cnvx}(n)_{2:1,1} = \frac{1}{24}(n-1)n(n+1)(n+2). \tag{5.5}$$

Notice that this is the same as (3.1), which gives the total possible number of collections of two one-row disjoint sets together on a single row. When a two-row set is placed on two nonadjacent rows, consider the restrictions placed on endpoints i, j, k and l to ensure convexity: $1 \leq i \leq j < k \leq l \leq n$. These restrictions mimic the restrictions placed on the two one-row sets to ensure disjoint sets on a single row ($1 \leq i \leq j < i' \leq j' \leq n$), differing only by the label placed on each endpoint. Thus, it is apparent that determining the total number of two-row convex sets occupying two nonadjacent rows is equivalent to determining the total number of two disjoint one-row sets in a single row. This is because in order for the two-row set placed on two nonadjacent rows to be convex, the projection of the set onto the top row of the space must produce two disjoint convex sets. In general, in order for a z-row set placed on z nonadjacent rows to be convex, the projection of the set onto the top row of the space must produce a disjoint set for every piece of the set that was separated within the z-row set by an empty row.

Recalling the example we have been working with where the number of rows in the space is four, there is one more type of set that can be placed in the $\underline{n} \times \underline{m}$ space that has not yet been addressed. This set is a three-row set that is split among three nonadjacent rows. Two approaches can be taken in order to find the formula that will give the total number possible. One way is to look at the summation method frequently used throughout this work. Once again, the indices need to be altered to ensure convexity since the break is being added. The new restrictions placed on the endpoints previously defined in Section 4.1 for three-row sets create the following sets that each endpoint must be contained in: $i \in \{1, \ldots, n\}, j \in \{i, \ldots, n\}, k \in \{j+1, \ldots, n\}, l \in \{k, \ldots, n\}, q \in \{l, \ldots, n\}$ and $p \in \{k, \ldots, q\}$. Summing across all possibilities, it is found that

$$T_{cnvx}(n)_{3:1,2} = \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j+1}^{n}\sum_{l=k}^{n}\sum_{q=l}^{n}\sum_{p=k}^{q} 1 = \frac{1}{360}(n-1)n(n+1)^2(n+2)(n+3) \qquad (5.6)$$

gives the total possible number of three-row convex sets occupying three nonadjacent rows in the four-row space.

Notice that the above restrictions on the endpoints give a total based on the one-row piece of the three-row set occupying a row before the break and the two-row piece of the set occupying the rows after the break. It is easily verified that by altering the indices so that the two-row piece happens before the break, the outcome of $\frac{1}{360}(n-1)n(n+1)^2(n+2)(n+3)$ is still obtained. The reasoning to why which piece is placed before the break is insignificant is easier seen through the alternate way to find this formula. Also notice (5.6) and these new restrictions specifically give the number of three-row convex sets split such that one of the three rows is separated out, but the remaining two rows still occupy adjacent rows. This is because in the example we are specifically looking at ways to place a three-row set in a four-row space; thus there are not enough rows to split a three-row set such that each row of the set is on a nonadjacent row from every other row in the set. Therefore in a five- or more row space, the total possible number of three-row sets split this way would also have to be determined. This could be done by altering the indices across the sums to account for the additional break or by looking at the number of ways to have three one-row disjoint convex sets on a single row, as this would be the projection of the set onto the top row of the space.

The other way to determine the total number of possible three-row sets occupying three nonadjacent rows is to look at the projection of the set onto the top row, similar to the way it was done with the two-row sets previously. Note that this is still specifically for the case in the four-row space; thus two of the three rows of the set are still on adjacent rows with only one row of the set being separated. Consider the projection of such a set onto the top row. Then in order for the set to be convex, the projection of the one-row piece of the set must be disjoint from the projection of the two-row piece of the three-row set. This is equivalent to looking at the total number of ways to have two disjoint sets in a two-row space, where one set is a two-row set and one set is a one-row set, such that the one-row set is completely to the left of the top row of the two-row set, or it is completely to the right of the bottom row of the two-row set. In this scenario it is easy to see it will not matter if the one-row set is placed to the left or right of the two-row set; the count across all possibilities of endpoints will yield the same result. Thus, as far as a total number of three-row disjoint sets split into two sets is concerned, the placement of the one break doubles the outcome of the total.

Going back to the example, we can now calculate the total possible numbers of convex sets that can be placed in a $\{1,2,\dots,n\} \times \{1,2,3,4\}$ space. To do this we will combine the total number of z-row convex set for all $z \in \{1,2,3,4\}$. We must consider all z-row sets on z adjacent rows and non-adjacent rows, and we must multiply each of these by the total number of ways to place them in the space. Thus, (5.4) with $m = 4$ plus three times (5.5) plus two times (5.6) will give us the following total number of possible convex sets on a four-row, n-column space:

$$\begin{aligned}
& 4 * \frac{1}{2}n(n+1) + 3 * \frac{1}{12}n(n+1)^2(n+2) + 2 * \frac{1}{144}n(n+1)^2(n+2)^2(n+3) \\
+ \quad & 1 * \frac{1}{2880}n(n+1)^2(n+2)^2(n+3)^2(n+4) + 3 * \frac{1}{24}(n-1)n(n+1)(n+2) \\
+ \quad & 2 * \frac{1}{360}(n-1)n(n+1)^2(n+2)(n+3).
\end{aligned}$$

For example, in order to find the total number of possible convex sets in a four-row, five-column space, evaluate the above equation when n=5 to get a total of 3,448 sets.

5.2. The General Process

When considering a $\{1,2,\dots,n\} \times \{1,2,\dots.m\}$ space, one process to determine the total number of possible convex sets that can be placed in the space is to section the space into *blocks* and run through all possible scenarios of the placement, size, and number of the blocks on a space. More specifically, the blocks are created by placing a z-row convex set in the space. For example, if a six-row set is placed on nonadjacent rows in a $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ space such that four adjacent rows of the set are separated from the remaining two (which are on adjacent rows), this creates two blocks of the set in the space based on the projections of the set onto the top row and left column (see Figure 5.2.1). If a six-row set is placed in the space such that it is split into three two-adjacent-row pieces on non-adjacent rows, then this creates three blocks in the space based on the projections of the set onto the top row and left column (see Figure 5.2.2).

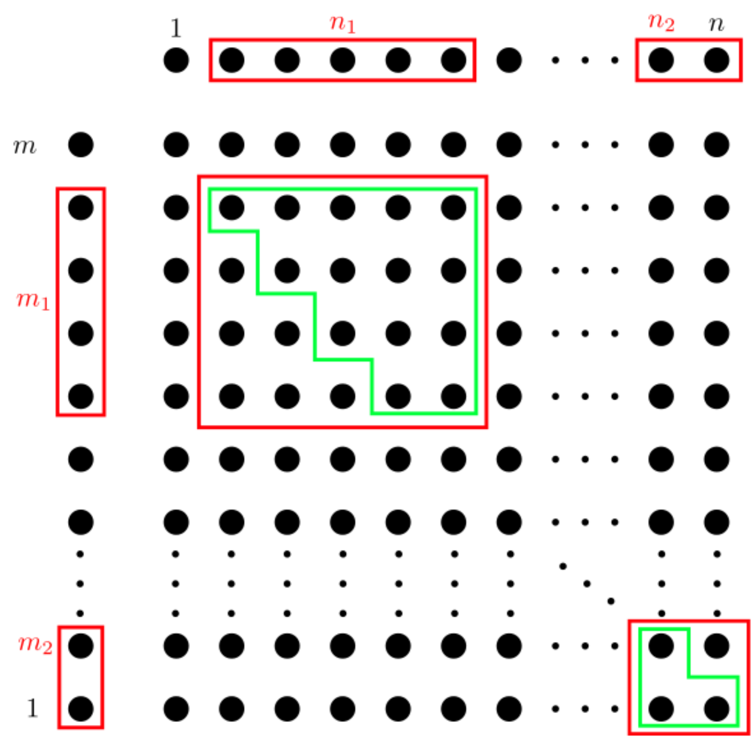

FIGURE 5.2.1. A single set on nonadjacent rows that creates 2 blocks of space.

This idea comes from using the projection of the width of a set onto the top row of a space and its height projected onto the left column in the space. The number of blocks on the space tells how many pieces a single z-row set has been

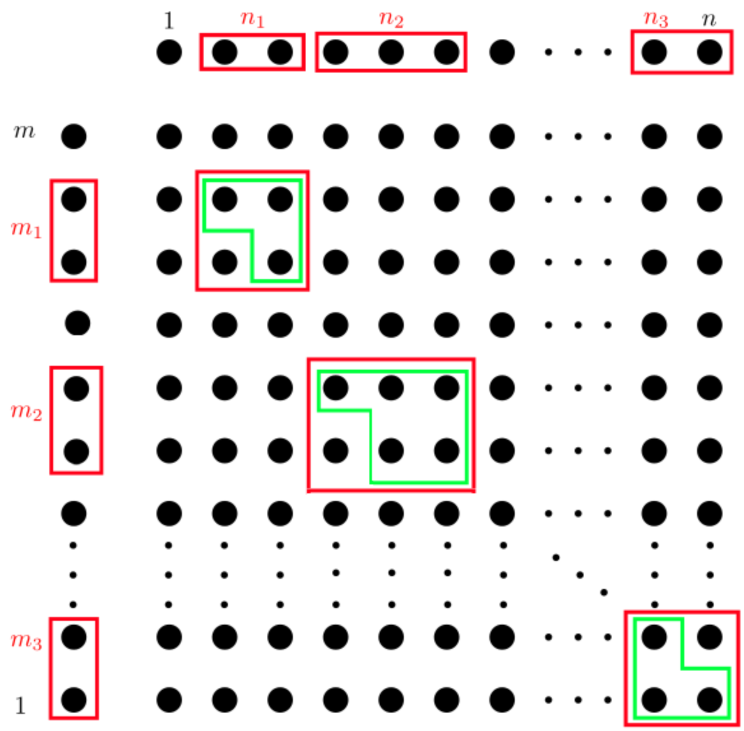

FIGURE 5.2.2. A single set on nonadjacent rows that creates 3 blocks of space.

split into along z nonadjacent rows. For example, a z-row convex set on z adjacent rows will create one interval projection for width along the top and one interval projection for height along the left column. Notice that a block of a set that is projected onto the top interval can create more than one disjoint interval if only those actual points are projected, but here the projection of interest is the one of the total width per block and, therefore, will only produce one interval per block. The projections give us the width and height values of a subspace of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ of which a piece of a z-row set spans the entire width and height of the subspace. Thus, in order to determine the total number of convex sets on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ spaces, calculating the total possible number of convex sets within each size block times the total number of ways to position each size block in the space will give us the total of interest. That is, the total number of ways to place each possible size of a single block in the space times the total number of possible convex sets within each of these blocks will account for all of the z-row convex sets that are on z adjacent rows. The total number of ways to place two blocks in the space times the total possible convex sets contained within each, such that the projection of the widths and lengths produce disjoint intervals along the projection axes, will account for all the z-row convex sets that are split into two separate pieces. As the number of blocks placed in the space increases, it will account for all possible ways to split a z-row convex set on m adjacent rows.

The total number of possible convex sets in a given block is found by calculating the total number of possible convex sets that have width of n_j for all $j \in \{1, 2, \ldots, n\}$ and height m_i for all $i \in \{1, 2, \ldots, m\}$. This is done by summing across all options of endpoints as done previously. However, since the sets span the entire width of the subspace being calculated over, the top left-most endpoint will be fixed at the first point of the subspace and the bottom right-most endpoint will be fixed at the last point of the subspace. Figures 5.2.1 and 5.2.2 show that a skipped row creates an additional box; however skipping a column does not create a new box. Figure 5.2.3 illustrates this. Figure 5.2.3 also portrays two different sets that can be placed in the same width block, and that in order to have a set the width of the block, the two endpoints discussed above must remain fixed.

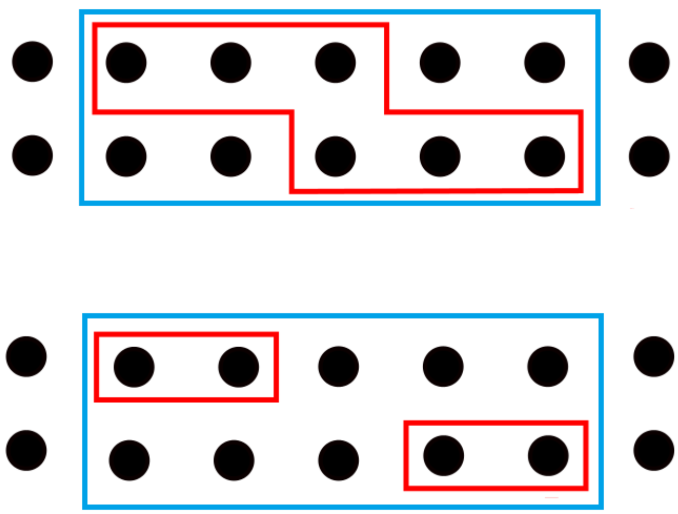

FIGURE 5.2.3. Two possible sets placed in a block of width 5 and height 2.

Thus, instead of summing across $2z$ endpoints for a z-row set, the sum will only be across $2z - 2$ endpoints, throwing out the first and last summation from the original $2z$-sum and changing the indices of the new first and last summations. This total will be denoted $S(m_i, n_i)$ and must be calculated separately once m_i and n_i are chosen. As the number of blocks being placed on a $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ space simultaneously is increased, m_i and n_i will be further restricted. Recall that two blocks in the space have to be placed in such a way to count the total number of ways to split any z-row convex set into two nonadjacent pieces. This gives that once m_1 is chosen, when choosing m_2, i must be in the set $\{1, 2, \ldots, m - (m_1 + 1)\}$. Note that any choice for m_1 that creates an empty set for m_2 to be chosen from is invalid as this creates a single-block scenario that would have already been accounted for previous to accounting for all two-block scenarios. Thus, once it is increased to looking at g blocks being placed in the space, then $m_1 + m_2 + \cdots + m_g \leq m - (g - 1)$. Also, recall that for z-row convex sets to be split into two nonadjacent pieces, their projection intervals of the total length must be disjoint from one another. Thus, for g blocks being placed in the space, then $n_1 + n_2 + \cdots + n_g \leq n$.

Now we need to find the total number of ways that g blocks can be placed in the space. This is equivalent to finding the total number of ways $m_1, m_2, \ldots, m_g$ and $n_1, n_2, \ldots, n_g$ can be placed in the space for all i and j. This can be determined by looking at the number of ways to partition the projection of $m_1, m_2, \ldots .m_g$, the height intervals, along the first column of the space. For g blocks, $m_1, m_2, \ldots .m_{g-1}$ require at least one row be skipped after each. Therefore, g blocks account for $(m_1 + 1) + (m_2 + 1) + (m_3 + 1) + \cdots + (m_{g-1} + 1) + (m_g)$ points, giving $m - (m_1 + m_2 + m_3 + \cdots + m_g + g - 1)$ leftover points. Thus, we are choosing g blocks from g plus the leftover points. The total number of ways to position $m_1, m_2, \ldots, m_g$ is

$$P(m_1, m_2, \ldots, m_g) = \binom{m - (m_1 + m_2 + m_3 + \cdots + m_g) + 1}{g}. \tag{5.7}$$

Similarly, looking at the number of ways to partition the projection of $n_1, n_2, \ldots, n_g$, the width interval, along the top row of the space will determine the total possible number of ways to place the width of the blocks in the space. For g blocks, $n_1 + n_2 + \cdots + n_g$ points are accounted for, giving $n - (n_1 + n_2 + \cdots + n_g)$ leftover points. Thus, we are choosing g from g plus the leftover points. The total number of ways to position $n_1, n_2, \ldots, n_g$ is

$$Q(n_1, n_2, \ldots, n_g) = \binom{g + n - (n_1 + n_2 + \cdots + n_g)}{g}. \tag{5.8}$$

This describes the proof for the following theorem.

THEOREM 5.2.1. *The total possible number of convex sets on* $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ *spaces can be found by*

$$\sum_{m_i, n_j, g} P(m_1, m_2, \ldots, m_g) * Q(n_1, n_2, \ldots, n_g) * S(m_1, n_1) * S(m_2, n_2) * \cdots * S(m_g, n_g), \tag{5.9}$$

where the sum is across all allowable g *and all allowable* m_i *and* n_j *for each allowable* g.

5.3. Example

Recall the example from Section 5.1 of a four-row, five-column space that was found to have 3,448 possible convex sets. Below is this same example using the process described in Section 5.2. Notice that $g \in \{1, 2\}$ in this example, since if we tried to place 3 blocks such that the z-row convex set was split into 3 nonadjacent sections, then at minimum we need three rows (one per block) and two rows for the breaks. This gives that we need a minimum of five rows in a four-row space; thus $g \le 2$. For $g = 1$, $m_1 \in \{1, \ldots, m\}$ and $n_1 \in \{1, \ldots, n\}$. For $g = 2$, $m_1 \in \{1, 2\}, m_2 \in \{1, \ldots m - (m_1 + 1)\}, n_1 \in \{1, 2, 3, 4\}$ and $n_2 \in \{1, \ldots n - n_1\}$, since $m_1 + m_2 + \cdots + m_i \le m$ and $n_1 + n_2 + \cdots + n_j \le n$. The totals for each possible number of widths, lengths, and convex sets per block size for a $\{1,2,3,4,5\} \times \{1,2,3,4\}$ space are listed in Table 5.3.1 and Table 5.3.2 below.

	m_1	n_1	$P(m_1)$	$Q(n_1)$	$S(m_1,n_1)$	Total
1 Block						
	1	1	4	5	1	20
	1	2	4	4	1	16
	1	3	4	3	1	12
	1	4	4	2	1	8
	1	5	4	1	1	4
	2	1	3	5	1	15
	2	2	3	4	4	48
	2	3	3	3	9	81
	2	4	3	2	16	96
	2	5	3	1	25	75
	3	1	2	5	1	10
	3	2	2	4	8	64
	3	3	2	3	31	186
	3	4	2	2	85	340
	3	5	2	1	190	380
	4	1	1	5	1	5
	4	2	1	4	13	52
	4	3	1	3	76	228
	4	4	1	2	295	590
	4	5	1	1	889	889
Total						3119

Table 5.3.1. The totals for each possible number of widths, lengths, and convex sets for 1 block in a 5-column, 4-row space.

	m_1	m_2	n_1	n_2	$P(m_1, m_2)$	$Q(n_1, n_2)$	$S(m_1, n_1)$	$S(m_2, n_2)$	Total
2 Blocks									
	1	1	1	1	3	10	1	1	30
	1	1	1	2	3	6	1	1	18
	1	1	1	3	3	3	1	1	9
	1	1	1	4	3	1	1	1	3
	1	1	2	1	3	6	1	1	18
	1	1	2	2	3	3	1	1	9
	1	1	2	3	3	1	1	1	3
	1	1	3	1	3	3	1	1	9
	1	1	3	2	3	1	1	1	3
	1	1	4	1	3	1	1	1	3
	1	2	1	1	1	10	1	1	10
	1	2	1	2	1	6	1	4	24
	1	2	1	3	1	3	1	9	27
	1	2	1	4	1	1	1	16	16
	1	2	2	1	1	6	1	1	6
	1	2	2	2	1	3	1	4	12
	1	2	2	3	1	1	1	9	9
	1	2	3	1	1	3	1	1	3
	1	2	3	2	1	1	1	4	4
	1	2	4	1	1	1	1	1	1
	2	1	1	1	1	10	1	1	10
	2	1	1	2	1	6	1	1	6
	2	1	1	3	1	3	1	1	3
	2	1	1	4	1	1	1	1	1
	2	1	2	1	1	6	4	1	24
	2	1	2	2	1	3	4	1	12
	2	1	2	3	1	1	4	1	4
	2	1	3	1	1	3	9	1	27
	2	1	3	2	1	1	9	1	9
	2	1	4	1	1	1	16	1	16
Total									329

TABLE 5.3.2. The totals for each possible number of widths, lengths, and convex sets for 2 blocks in a 5-column, 4-row space.

Now by adding all possible 1 and 2 blocks, the total number of possible convex sets on a four-row, five-column space is found to be 3,448, which matches what was found in Section 5.1. Recall that the sum $S(m_i, n_i)$ must be calculated separately after each m_i and n_j are chosen. Thus, for this example we need to sum across all possible one-, two-, three-, and four-row sets that have width n_j. Notice that $S(1, n_j) = 1$ since there is only one way to choose a set with height 1 that spans the entire n_j width. Using the summation process across all endpoints for each z-row set, the following are obtained:

$$\begin{aligned}
S(2, n_j) &= \sum_{j=1}^{n_j} \sum_{k=1}^{n_j} 1 = n_j^2; \\
S(3, n_j) &= \sum_{j=1}^{n_j} \sum_{l=j}^{n_j} \sum_{k=1}^{l} \sum_{p=k}^{n_j} 1 = \frac{1}{24}(5n_j^4 + 10n_j^3 + 7n_j^2 + 2n_j); \\
S(4, n_j) &= \sum_{j=1}^{n_j} \sum_{l=j}^{n_j} \sum_{k=1}^{l} \sum_{q=l}^{n_j} \sum_{p=k}^{q} \sum_{r=p}^{n_j} 1 = \frac{1}{360}(7n_j^6 + 42n_j^5 + 100n_j^4 + 120n_j^3 + 73n_j^2 + 18n_j).
\end{aligned}$$

These, along with the position formulas found in the previous section, are used to calculate the entries of Tables 5.3.1 and 5.3.2. For the 1-block, m_1 and n_1 are all possible heights and widths, respectively, that a single block placed in the space can obtain. For the 2-blocks, m_1 and m_2 are all the possible ways to add two numbers (heights of two blocks) to get a value less than or equal to three. The value must be less than or equal to three because having 2 blocks gives that at least one row must be skipped between the two sections of blocks; thus the projection of heights can only occupy at most three of the four rows in the space.

The values for n_1 and n_2 are all possible widths the two blocks can be from the total n width of the space. Since neither n_1 nor n_2 can equal zero, which would give a 1-block space already accounted for separately, then n_1 can be any value between 1 and 4, with n_2 being any value between 1 and $n - n_1$ for each choice of n_1.

Chapter 6

CONCLUSION

In this thesis, convex sets on products of two totally ordered spaces were considered. We analyzed the ways to obtain and count convex sets on $\{1, 2, \ldots, n\} \times \{1, 2\}$ spaces by many different approaches and methods, including summation and modeling. Additionally, we examined the ways to obtain and count a collection of two mutually disjoint sets on the same space. Next, we demonstrated how to obtain and count convex sets on adjacent rows when increasing the number of rows giving $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ spaces. Then we described the different ways a z-row convex set could be placed on an $\underline{n} \times \underline{m}$ space. Finally, we generalized to give a process that, when followed, will obtain the total number of convex sets on any $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ space. Although many formulas and results were found, there is still plenty of room for exploration.

In the future, the ways to obtain and count the number of z-row convex sets on z nonadjacent rows with a single general formula will be analyzed. How to take the process we found for obtaining the total number of convex sets on any $m \times n$ space and generalize it to a single formula will be considered. Also, the ways to obtain and count the collections of k mutually disjoint sets on $\{1, 2, \ldots, n\} \times \{1, 2\}$ spaces and then extending to obtain and count on $\underline{n} \times \underline{m}$ will be studied.

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