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Discrete Fractional Hermite-Hadamard Inequality

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DISCRETE FRACTIONAL HERMITE-HADAMARD INEQUALITY

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This thesis is dedicated to my Mom and Dad, I can't describe how much I feel grateful to have you. Thank you for always believing in me. I love you all so much.

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CONTENTS

DISCRETE FRACTIONAL HERMITE-HADAMARD INEQUALITY

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This thesis is comprised of three main parts: The Hermite-Hadamard inequality on discrete time scales, the fractional Hermite-Hadamard inequality, and Karush-Kuhn-Tucker conditions on higher dimensional discrete domains. In the first part of the thesis, Chapters 2 & 3, we define a convex function on a special time scale $\mathbb{T}$ where all the time points are not uniformly distributed on a time line. With the use of the substitution rules of integration we prove the Hermite-Hadamard inequality for convex functions defined on $\mathbb{T}$. In the fourth chapter, we introduce fractional order Hermite-Hadamard inequality and characterize convexity in terms of this inequality. In the fifth chapter, we discuss convexity on n–dimensional discrete time scales $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ where $\mathbb{T}_i \subset \mathbb{R}$, $i = 1, 2, ..., n$ are discrete time scales which are not necessarily periodic. We introduce the discrete analogues of the fundamental concepts of real convex optimization such as convexity of a function, subgradients, and the Karush–Kuhn–Tucker conditions.

We close this thesis by two remarks for the future direction of the research in this area.

Chapter 1

INTRODUCTION

The Hermite-Hadamard inequality [**15, 16**] states that if $f : I \to \mathbb{R}$ is a convex function, then the following inequality is satisfied:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left(\int_a^b f(t)dt\right) \le \frac{f(a)+f(b)}{2},$$

where $a, b \in I$ and I is an interval in $\mathbb{R}$.

In the theory of convex functions, the Hermite-Hadamard inequality plays an important role. It has been used as a tool to obtain many results in integral inequalities, approximation theory, optimization theory and numerical analysis. It has been developed for different classes of convexity, such as quasi-convex functions, log-convex, r-convex functions, p-functions [**13**], and recently for discrete functions [**4**]. For the history of its developments in many directions, we refer the reader to a paper by Mitrinović and Lacković [**25**]. For the generalizations and applications in probability, we refer the reader to a paper by Merkle [**22**].

In this thesis, we introduce convexity by means of a midpoint condition of a function defined on a time scale which has all the points as isolated. We state and prove the Hermite-Hadamard inequality for such a class of functions. We call this new inequality the discrete Hermite-Hadamard inequality.

Fractional calculus on time scales is an ongoing research topic where mathematicians are trying to unify time scale calculus and fractional calculus. As an application of this elegant theory we construct a fractional type Hermite-Hadamard inequality and show that it completely characterizes the convexity of functions defined on isolated time scales.

Convex optimization, a branch of mathematical optimization theory, has been developed in two directions: the real convex optimization and the discrete (or combinatorial) convex optimization. Recent developments such as interior point methods, semidefinite programming and robust optimization in convex optimization theory have stimulated new interest by mathematicians and other scientists. It is applied in areas such as automatic control systems, mathematical economics, electronic circuit design, medical imaging, etc. [**9**], [**29**], [**5**]. Other applications can be found in combinatorial optimization and global optimization where it has been used to find bounds on the optimal value or to find approximate solutions [**11**].

On the other hand, the discrete convex optimization combines ideas from real convex optimization and combinatorial optimization to provide optimization techniques for discrete functions with the convexity property. It was first developed for integer valued functions defined on integer lattice points. In [**27**] and [**23**] the discrete convexity concepts are introduced for real-valued functions defined on $\mathbb{Z}^n$. More recently, Mozyrska and Torres introduced the convexity of a function defined on a time scale (a nonempty closed subset of R) in their paper [**26**]. This short paper can be considered as the establishment of the foundation of convex functions on time scales. Their definition is as follows.

Definition 1.1. [**26**] *Let I be an interval in $\mathbb{R}$ such that the set $I_{\mathbb{T}} := I \cap \mathbb{T}$ is a nonempty subset of $\mathbb{T}$. A function f defined and continuous on $I_{\mathbb{T}}$ is called convex on $I_{\mathbb{T}}$ if for any $t_1, t_2 \in I_{\mathbb{T}}$*

$$(t_2 - t)f(t_1) + (t_1 - t_2)f(t) + (t - t_1)f(t_2) \geq 0, \; t \in I_{\mathbb{T}}.$$

More recently, Adivar and Fang defined convexity on the product of time scales [**1, 2**]. Motivated by these pioneers' work, we give a different definition of discrete convex functions on domains which are in the product form $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, where $\mathbb{T}_i \subset \mathbb{R}$, $i = 1, 2, ..., n$ are discrete time scales which are not necessarily periodic (i.e. the jump operator may not be constant).

There are some advantages of using discrete convexity. One of the advantage occurs when the objective function and constraint functions are discrete convex but not real convex. In this case one cannot apply real convex optimization methods, however it is possible to apply discrete convex optimization methods.

Chapter 2

PRELIMINARIES

Let $\mathbb{T} = \{0 = t_0, t_1, t_2, t_3, ...\}$ be a set of nonnegative real numbers such that $t_i < t_j$ for $i < j$. We assume that $|\mathbb{T}| = \aleph_0$, where $\aleph_0$ denotes the cardinality of natural numbers. We then define the operators $\sigma(t_i) = t_{i+1}$, $\mu(t_i) = \sigma(t_i) - t_i$, $\rho(t_i) = t_{i-1}$, and $\nu(t_i) = t_i - \rho(t_i)$ for $t_i \in \mathbb{T}$, which are known as the forward jump, the forward graininess, the backward jump, and the backward graininess operators respectively. The time scale $\mathbb{T}$ can be considered as a discrete time scale. If $\mathbb{T}$ is the set of integers (i.e. $\mathbb{T} = \mathbb{Z}$), then $\sigma(t) = t + 1$, $\mu(t) = 1$, $\rho(t) = t - 1$ and $\nu(t) = 1$, for all $t \in \mathbb{T}$.

Definition 2.1. *Let f be a real-valued function defined on $\mathbb{T}$. Then the Δ-derivative and the ∇-derivative of f are defined, respectively, as*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \qquad f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)},$$

where $t \in \mathbb{T}$. We define a second order derivative as $f^{\Delta^2} := f^{\Delta\Delta} = (f^{\Delta})^{\Delta}$. The Δ-integral and the ∇-integral of f are defined as

$$\int_a^b f(\tau)\Delta\tau = \sum_{s\in[a,b)\cap\mathbb{T}} f(s)\mu(s), \qquad \int_a^b f(\tau)\nabla\tau = \sum_{s\in(a,b]\cap\mathbb{T}} f(s)\nu(s),$$

respectively, where $a, b \in \mathbb{T}$.

We also use the notation $\mathbb{T}^{\kappa}$ which is defined as

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \sup\mathbb{T} < \infty \\ \mathbb{T}, & \sup\mathbb{T} = \infty. \end{cases}$$

For further reading on time scales, we refer the reader to an excellent book on the analysis of time scales [**7**].

Throughout this study, we focus on the discrete time scales. Let $\mathbb{T}$ be any discrete time scale and $a, b \in \mathbb{T}$ with $a < b$. $[a,b]_{\mathbb{T}}$ means $[a,b] \cap \mathbb{T}$. We define

$$\mathbb{T}_{[a,b]} = \left\{ t \mid t = \frac{b-u}{b-a} \text{ for } u \in \ [a,b]_{\mathbb{T}} \right\}.$$

Note that $\mathbb{T}_{[a,b]} \subset [0,1]$. We also want to point out that there is a bijective (one-to-one and onto) map between $[a,b]_{\mathbb{T}}$ and $\mathbb{T}_{[a,b]}$.

Definition 2.2. *$f : \mathbb{T} \to \mathbb{R}$ is called convex on $\mathbb{T}$ if for every $x, y \in \mathbb{T}$ with $x < y$, the following inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

is satisfied for all $\lambda \in \mathbb{T}_{[x,y]}$.

Definition 2.3. *We define the midpoint of a and b as the n^{th} element in a finite time scale $[a,b]_{\mathbb{T}}$ with cardinality $2n-1$.*

For the notation, we denote the midpoint of a and b by $m_{[a,b]}$ in $[a,b]_{\mathbb{T}}$, and the corresponding midpoint of 0 and 1 by $m_{[0,1]}$ in $\mathbb{T}_{[a,b]}$. If $\mathbb{T} = \mathbb{Z}$, then Definition 2.3 reduces to the standard definition of midpoint on $\mathbb{Z}$, namely $m_{[a,b]} = \frac{a+b}{2}$ and $m_{[0,1]} = \frac{1}{2}$.

Definition 2.4. *$f : \mathbb{T} \to \mathbb{R}$ satisfies the midpoint condition if*

$$f(m_{[a,b]}) \leq m_{[0,1]} f(a) + (1 - m_{[0,1]}) f(b), \tag{2.1}$$

for every $a, b \in \mathbb{T}$ with the cardinality of $[a,b]_{\mathbb{T}}$ an odd number.

Remark 2.5. *If we want to be more precise for the number $m_{[0,1]}$ in (2.1), then we can write it as $m_{[0,1]} = \frac{b - m_{[a,b]}}{b-a}$. Note that if $\mathbb{T} = \mathbb{Z}$, then the inequality (2.1) becomes $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$, as it is stated in the paper [**4**].*

The following two theorems are crucial in the proof of the next theorem about some equivalent criteria for convexity of real-valued functions defined on $\mathbb{T}$.

THEOREM 2.6. *(Taylor's Theorem [**7**]) Let $n \in \mathbb{N}$. Suppose f is n-times differentiable on $\mathbb{T}^{\kappa^{n-1}}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and define the functions h_k by*

$$h_0(r,s) \equiv 1 \text{ and } h_{k+1}(r,s) = \int_s^r h_k(\tau,s)\Delta\tau \text{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

THEOREM 2.7. *(Mean Value Theorem [**8**]) Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is differentiable on $[a,b)_{\mathbb{T}}$. Then there exists $\xi, \tau \in [a,b)_{\mathbb{T}}$ such that*

$$f^{\Delta}(\tau) \le \frac{f(b)-f(a)}{b-a} \le f^{\Delta}(\xi).$$

THEOREM 2.8. *Let $f : \mathbb{T} \to \mathbb{R}$ be given. The following are equivalent:*

(i) f is convex on $\mathbb{T}$.

(ii) f satisfies the midpoint condition (2.1).

(iii) $f^{\Delta^2}(t) \ge 0$ for all $t \in \mathbb{T}$.

(iv) $f(x) \ge f(y) + (x-y)f^{\Delta}(y)$ for all $x, y \in \mathbb{T}$ with $x > y$,

(or $f(x) \ge f(y) + (x-y)f^{\nabla}(y)$ for all $x < y$).

PROOF. We prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (iv) \Rightarrow (iii)$.

$(i) \Rightarrow (ii)$: Let $a, b \in \mathbb{T}$ with $a < b$ and $[a,b]_{\mathbb{T}}$ have odd number of time points. This implies that $[a,b]_{\mathbb{T}}$ has a midpoint $m_{[a,b]}$. Since f is convex, by choosing $\lambda = \frac{b-m_{[a,b]}}{b-a}$, we obtain

$$f(m_{[a,b]}) \leq \left(\frac{b - m_{[a,b]}}{b-a}\right) f(a) + \left(\frac{m_{[a,b]} - a}{b-a}\right) f(b).$$

Then the midpoint condition (2.1) follows.

Next, we prove that (ii) implies (iii). Let $t \in \mathbb{T}$. Then $\sigma(t) \in \mathbb{T}$. Applying the midpoint condition at $\sigma(t)$, we have

$$f(\sigma(t)) \leq \frac{\sigma^2(t) - \sigma(t)}{\sigma^2(t) - t} f(t) + \frac{\sigma(t) - t}{\sigma^2(t) - t} f(\sigma^2(t)).$$

Simple algebra implies that $f^{\Delta^2}(t) \geq 0$.

Next, we prove that $(iii) \Rightarrow (i)$. Let $x, y \in \mathbb{T}$ with $x < y$. Fix $\lambda \in \mathbb{T}_{[x,y]}$.

Define $x_0 = \lambda x + (1-\lambda)y$. Using Taylor's theorem (Theorem 2.6) at x_0 we have

$$f(y) = \sum_{i=0}^{1} h_i(y, x_0) f^{\Delta^i}(x_0) + \sum_{\tau = x_0}^{\rho^2(y)} h_1(y, \sigma(\tau)) f^{\Delta^2}(\tau)(\sigma(\tau) - \tau).$$

Since $f^{\Delta^2}(\tau) \geq 0$ on $\mathbb{T}$ and $h_1(y, \sigma(\tau)) = y - \sigma(\tau) \geq 0$ on $\mathbb{T}$, we have

$$f(y) \geq f(x_0) + (y - x_0) f^{\Delta}(x_0). \tag{2.2}$$

Using the Mean Value Theorem (Theorem 2.7) for f on $[x, x_0]_{\mathbb{T}}$, there exists $\tau \in [x, x_0)_{\mathbb{T}}$ such that

$$\frac{f(x_0) - f(x)}{x_0 - x} \leq f^{\Delta}(\tau).$$

Since $f^{\Delta^2}(t) \geq 0$ on $\mathbb{T}$, we have $f^{\Delta}(\tau) \leq f^{\Delta}(x_0)$. Therefore we obtain

$$f(x) \geq f(x_0) + (x - x_0) f^{\Delta}(x_0). \tag{2.3}$$

If we multiply the inequality (2.2) by $1-\lambda$ and the inequality (2.3) by λ and add them side by side, we obtain

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)$$

for all $\lambda \in \mathbb{T}_{[x,y]}$.

Now we prove $(i) \Rightarrow (iv)$. Given any $x, y \in \mathbb{T}$ such that $y<x$, by convexity of f on $[y,x]$, we have

$$f(\sigma(y)) \le \frac{\sigma(y)-y}{x-y}f(x)+\frac{x-\sigma(y)}{x-y}f(y).$$

After rearranging the terms we have

$$f^{\Delta}(y) \le \frac{f(x)-f(y)}{x-y}.$$

This simplifies to $f(x) \ge f(y)+(x-y)f^{\Delta}(y)$ for all $x>y$. The same argument works to show that

$$f(x) \ge f(y)+(x-y)f^{\nabla}(y) \text{ for all } x<y.$$

Finally we prove $(iv) \Rightarrow (iii)$. Given $f(x) \ge f(y)+(x-y)f^{\Delta}(y)$ for all $x>y$. By choosing $x=\sigma^2(y)$, we obtain $f^{\Delta^2}(y) \ge 0$. Since y is arbitrary, (iii) follows.

This completes the proof. $\square$

Chapter 3

HERMITE-HADAMARD INEQUALITY ON DISCRETE TIME SCALES

In this chapter we prove the discrete Hermite-Hadamard inequality for convex functions defined on a discrete time scale $\mathbb{T}$.

Theorem 3.1. *(Substitution rule on time scales)* [**12**] *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a an rd-continuous function and ν is differentiable with rd-continuous derivative, then if $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s$$

or

$$\int_a^b f(t)\nu^{\nabla}(t)\nabla t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\nabla} s.$$

Definition 3.2. *(Dual Time Scales)*[**10**] *Given a time scale T we define the dual time scale $\overline{\mathbb{T}} = \{s \in \mathbb{R} | -s \in \mathbb{T}\}$.*

Next, we state and prove the substitution rule for a strictly decreasing function $\nu : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ can be any time scale whether isolated or not.

Theorem 3.3. *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly decreasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)(-\nu^{\Delta})(t)\,\Delta t = \int_{\nu(b)}^{\nu(a)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\nabla s}$$

and

$$\int_a^b f(t)(-\nu^{\nabla})(t)\,\nabla t = \int_{\nu(b)}^{\nu(a)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\Delta} s.$$

PROOF. We prove the second equality. In the proof we use the basics of the dual time scales introduced in [**10**](see definition 3.2) and the substitution method for a strictly increasing function (Theorem 3.1). We start on the right side of the equality:

$$\begin{aligned}\int_{\nu(b)}^{\nu(a)} \left(f\circ\nu^{-1}\right)(s)\,\widetilde{\Delta}s &= \int_{-\nu(a)}^{-\nu(b)} (f\circ\nu^{-1})^*(s)\overline{\nabla}s\\ &= \int_{-\nu(a)=(u^{-1}\circ\nu)(a)}^{-\nu(b)=(u^{-1}\circ\nu)(a)} f(u^{-1}\circ\nu)^{-1}(s)\overline{\nabla}s\\ &= \int_a^b f(t)(u^{-1}\circ\nu)^{\nabla}(t)\nabla t\\ &= \int_a^b f(t)(-\nu)^{\nabla}\nabla t,\end{aligned}$$

where $\overline{\mathbb{T}}$ represents the dual time scale, $u(s) := -s$ and

$f(\nu^{-1})^*(s) = f(\nu^{-1}(-s)) = f((\nu^{-1}\circ u)(s)) = f((u^{-1}\circ\nu)^{-1}(s)).$ □

REMARK 3.4. *We note that the statement of Theorem 2.3 (ii) in the paper* [**12**] *is not correct since* $\widetilde{\mathbb{T}}$ *was defined as* $-\nu(\mathbb{T})$ *for a strictly decreasing function* ν.

We use the following notation in the proof of the next theorem: Let $a, b \in \mathbb{T}, a < b$, where the cardinality of $[a,b]_{\mathbb{T}}$ is an odd number, say $k+1$. Let $t \in \mathbb{T}_{[a,b]}$. Then there exists an $n \in \mathbb{N}\cup\{0\}$ such that $t = \sigma^n(0)$. We denote $\hat{t}$ by $\sigma^{k-n}(0)$. Similarly, let $u \in [a,b]_{\mathbb{T}}$. Then there exists a $l \in \mathbb{N}\cup\{0\}$ such that $u = \sigma^l(a)$. We denote $\hat{u}$ by $\sigma^{k-l}(a)$. We also note that $\hat{\hat{u}} = u$ and $\hat{\hat{t}} = t$.

Next we illustrate this new notation with an example.

EXAMPLE 3.5. *Let* $[a,b]_{\mathbb{T}} = \{x_0 = a, x_1, x_2, x_3, x_4, x_5, x_6 = b\}$, *where* $x_i < x_{i+1}$ *for* $0 \le i \le 5$. *Then we have* $\hat{x}_i = \sigma^{6-i}(a) = x_{6-i}$ *for* $0 \le i \le 6$. *It follows that*

$$\mathbb{T}_{[a,b]} = \{t_0 = 0, t_1 = \frac{b-x_5}{b-a}, t_2 = \frac{b-x_4}{b-a}, t_3 = \frac{b-x_3}{b-a}, t_4 = \frac{b-x_2}{b-a}, t_5 = \frac{b-x_1}{b-a}, t_6 = 1\}.$$

Hence we have $\hat{t}_i = \sigma^{6-i}(0) = t_{6-i}$ *for* $0 \le i \le 6$.

This implies that $\hat{t}_i = \dfrac{b - x_i}{b - a}$. *One simple algebra step implies that*

$$x_i = a\hat{t}_i + (1 - \hat{t}_i)b.$$

On the other hand, $\hat{x}_i = x_{6-i} = at_i + (1 - t_i)b$.

Now, we prove the main theorem of this chapter. We construct the Hermite Hadamard inequality on $\mathbb{T}$, where time points are not uniformly distributed.

THEOREM 3.6. *Suppose* $f : \mathbb{T} \to \mathbb{R}$ *is a convex function on* $[a,b]_{\mathbb{T}}$. *Then*

$$f(m_{[a,b]}) \le \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla(t) - \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t \le m_{[0,1]}f(a) + (1 - m_{[0,1]})f(b),$$

where $g : [a,b]_{\mathbb{T}} \to [a,b]_{\mathbb{T}}$ *is defined by* $g(u) = \hat{u}$ *and* $k : [a,b]_{\mathbb{T}} \to \mathbb{R}^+$ *is defined by*

$$k(x) := \begin{cases} \dfrac{g(x) - m_{[a,b]}}{g(x) - x}, & x \ne m_{[a,b]} \\ 1/2, & x = m_{[a,b]}. \end{cases} \tag{3.1}$$

PROOF. Fix $t \in \mathbb{T}_{[a,b]}$. Then there exists $x \in [a,b]_{\mathbb{T}}$ such that $x = ta + (1-t)b$. As we pointed out in Example 3.5, $\hat{x} = \hat{t}a + (1 - \hat{t})b$. Denote this $\hat{x}$ by y, i.e. $y = \hat{t}a + (1 - \hat{t})b$. Note that $m_{[a,b]} = m_{[x,y]}$ using the definition of the hat operator given in page 11.

Let $\xi : [a,b]_{\mathbb{T}} \to \mathbb{T}_{[a,b]}$ be an affine map defined as $\xi(u) = \dfrac{b-u}{b-a}$. Hence we have $\xi(x) = t$ and $\xi(y) = \hat{t}$. If $x \ne m_{[a,b]}$, then by convexity of f we have

$$f(m_{[a,b]}) \le \frac{y - m_{[a,b]}}{y - x} f(x) + \frac{m_{[a,b]} - x}{y - x} f(y). \tag{3.2}$$

If $x = m_{[a,b]}$, then it reduces to $x = y = m_{[a,b]}$. Clearly we have

$$f(m_{[a,b]}) = \frac{1}{2}f(x) + \frac{1}{2}f(y). \tag{3.3}$$

We combine (4.3) and (4.4) using the function k

$$f(m_{[a,b]}) \leq k(x)f(x) + k(y)f(y).$$

Next, we integrate each side of the above inequality from 0 to 1 on $\mathbb{T}_{[a,b]}$ and we obtain

$$\begin{aligned}
f(m_{[a,b]}) \int_{\mathbb{T}_{[a,b]}} \tilde{\Delta} t &\leq \int_{\mathbb{T}_{[a,b]}} k(x)f(x)\tilde{\Delta} t + \int_{\mathbb{T}_{[a,b]}} k(y)f(y)\tilde{\Delta} t \\
&= \int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta} t + \int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(\hat{t}))f(\xi^{-1}(\hat{t}))\tilde{\Delta} t.
\end{aligned}$$

Here we first claim that

$$\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta} t = \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla t. \tag{3.4}$$

Let us define $F := k \cdot f$. Then we have $F(\xi^{-1}(t)) = k(\xi^{-1}(t))f(\xi^{-1}(t))$.

Next, we apply the substitution rule (Theorem 3.3) to the integral on the left side of the equality in (3.4).

$$\begin{aligned}
\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta} t &= \int_{\mathbb{T}_{[a,b]}} F(\xi^{-1}(t))\tilde{\Delta} t \\
&= \int_a^b F(t)\frac{1}{b-a}\nabla t.
\end{aligned}$$

This finishes the proof of our first claim.

Next, we claim that

$$\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(\hat{t}))f(\xi^{-1}(\hat{t}))\tilde{\Delta}t = -\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t. \tag{3.5}$$

Before we start to prove the equality (3.5), we want to point out that the function g is a bijection and $g \equiv g^{-1}$ since g^2 is an identity function. As a result of this, we have $g(\hat{t}a + (1-\hat{t})b) = ta + (1-t)b$. This observation will help us to complete the proof of the claim.

By applying the substitution $w(u) = \widehat{\xi(u)}$ to the integral on the left side of the equality (3.5), we have

$$\begin{aligned}
\int_{\mathbb{T}_{[a,b]}} k(\xi^{-1}(\hat{t}))f(\xi^{-1}(\hat{t}))\tilde{\Delta}t &= \int_{\mathbb{T}_{[a,b]}} F(\xi^{-1}(\hat{t}))\tilde{\Delta}t \\
&= \int_0^1 (F \circ w^{-1})(t)\tilde{\Delta}t \\
&= -\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)F(t)\Delta t,
\end{aligned}$$

where

$$\begin{aligned}
w^{\Delta}(u) &= \frac{w(\sigma(u)) - w(u)}{\sigma(u) - u} \\
&= \frac{\xi\widehat{(\sigma(u))} - \widehat{\xi(u)}}{\sigma(u) - u} \\
&= \frac{\frac{b-\widehat{\sigma(u)}}{b-a} - \frac{b-\hat{u}}{b-a}}{\sigma(u) - u} \\
&= \frac{1}{(a-b)}\frac{(\widehat{\sigma(u)} - \hat{u})}{\sigma(u) - u} \\
&= \frac{g^{\Delta}(u)}{a-b} \geq 0,
\end{aligned}$$

since $\widehat{\sigma(u)} < \hat{u}$.

This completes the proof of our second claim.

To prove the right side of the inequality, we construct the following inequalities using the convexity of f.

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b),$$

$$f(y) \leq \frac{b-y}{b-a} f(a) + \frac{y-a}{b-a} f(b).$$

Next, we multiply both inequality by $k(x)$ and $k(y)$ respectively. We obtain

$$k(x)f(x) \leq \frac{b-x}{b-a} k(x)f(a) + \frac{x-a}{b-a} k(x)f(b),$$

$$k(y)f(y) \leq \frac{b-y}{b-a} k(y)f(a) + \frac{y-a}{b-a} k(y)f(b).$$

Simple algebra implies the following identities.

$$\frac{b-x}{b-a} k(x) + \frac{b-y}{b-a} k(y) = \frac{b - m_{[a,b]}}{b-a},$$

$$\frac{x-a}{b-a} k(x) + \frac{y-a}{b-a} k(y) = \frac{m_{[a,b]} - a}{b-a}.$$

Recall that x and y both depend on t. We let t vary over $\mathbb{T}_{[a,b]}$ and integrate each side of the last two inequalities on $\mathbb{T}_{[a,b]}$ and we add them side by side, we obtain

$$\int_{\mathbb{T}_{[a,b]}} k(\nu^{-1}(t)) f(\nu^{-1}(t)) \tilde{\Delta} t + \int_{\mathbb{T}_{[a,b]}} k(\nu^{-1}(\hat{t})) f(\nu^{-1}(\hat{t})) \tilde{\Delta} t$$

$$\leq \frac{b - m_{[a,b]}}{b-a} f(a) + \frac{m_{[a,b]} - a}{b-a} f(b) = m_{[0,1]} f(a) + (1 - m_{[0,1]}) f(b)$$

.

where the last equality is being achieved by means of Remark 2.5. $\square$

COROLLARY 3.7. *Suppose $f : h\mathbb{Z} \to \mathbb{R}$ is a convex function with $h > 0$, $a, b \in h\mathbb{Z}, a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left[\int_{[a,b]_{h\mathbb{Z}}} f(t)\Delta t + \int_{[a,b]_{h\mathbb{Z}}} f(t)\nabla t\right] \le \frac{f(a)+f(b)}{2}. \tag{3.6}$$

PROOF. Here g and k simplify into $g(x) = a + b - x$ and $k(x) = 1/2$ and $g^{\Delta}(x) = -1$. Hence we have the desired inequality. $\square$

When $h = 1$, we obtain the Hermite-Hadamard inequality on $\mathbb{Z}$.

COROLLARY 3.8. [**4**] *Suppose $f : \mathbb{Z} \to \mathbb{R}$ is a convex function on $[a, b]_{\mathbb{Z}}$ with $a, b \in \mathbb{Z}$, $a < b$, and $a + b$ an even number. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left[\int_a^b f(t)\Delta t + \int_a^b f(t)\nabla t\right] \le \frac{f(a)+f(b)}{2}. \tag{3.7}$$

Next, we give an alternate proof of the Hermite-Hadamard inequality on $\mathbb{R}$ (called as continuous Hermite-Hadamard inequality) by using the main result of this chapter. For this purpose, we first state the following lemma without giving its proof.

LEMMA 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function on $\mathbb{R}$. Then for any $h > 0$, its restriction to $h\mathbb{Z}$ is also a convex function.*

COROLLARY 3.10. *Let f be a real convex function on the finite interval $[a, b] \subset \mathbb{R}$. Then f satisfies the continuous Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)}\left(\int_a^b f(t)dt\right) \le \frac{f(a)+f(b)}{2}.$$

PROOF. By restricting f to $h\mathbb{Z}$ we obtain the inequality (3.6). Since f is convex on $[a,b]$, it is continuous on (a,b), hence integrable on $[a,b]$. When h tends to zero, the $\Delta-$integral and $\nabla-$integral converge to the Riemann integral of f on $[a,b]$. In other words,

$$\lim_{h\to 0}\sum_{t\in[a,b)_{h\mathbb{Z}}} f(t)h = \int_a^b f(t)dt \text{ and } \lim_{h\to 0}\sum_{t\in(a,b]_{h\mathbb{Z}}} f(t)h = \int_a^b f(t)dt.$$

Hence the result follows. □

3.1. Applications

(i). Let $f(t) = (1+h)^{\frac{t}{h}}$ be a function on $h\mathbb{Z}$ for some positive real number h. Since $f^{\Delta^2}(t) = f(t) \geq 0$, f satisfies the Hermite-Hadamard inequality on the interval $[a,b]_{h\mathbb{Z}}$, where $a, b \in h\mathbb{Z}$.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_{[a,b]_{h\mathbb{Z}}} f(t)\Delta t + \int_{[a,b]_{h\mathbb{Z}}} f(t)\nabla t\right] \leq \frac{f(a)+f(b)}{2}, \tag{3.8}$$

where

$$\int_{[a,b]_{h\mathbb{Z}}} f(t)\Delta t = \sum_{t\in[a,b)_{h\mathbb{Z}}} (1+h)^{\frac{t}{h}} h = (1+h)^{\frac{b}{h}} - (1+h)^{\frac{a}{h}},$$

$$\int_{[a,b]_{h\mathbb{Z}}} f(t)\nabla t = \sum_{t\in(a,b]_{h\mathbb{Z}}} (1+h)^{\frac{t}{h}} h = (1+h)^{1+\frac{b}{h}} - (1+h)^{1+\frac{a}{h}}.$$

Hence it follows that

$$\begin{aligned}(1+h)^{\frac{a+b}{2h}} &\leq \tfrac{1}{2(b-a)}\left(\left[(1+h)^{\frac{b}{h}} - (1+h)^{\frac{a}{h}}\right] + \left[(1+h)^{1+\frac{b}{h}} - (a+h)^{1+\frac{a}{h}}\right]\right)\\ &\leq \frac{(1+h)^{\frac{a}{h}} + (1+h)^{\frac{b}{h}}}{2}.\end{aligned}$$

Now, if $x = f(a)$ and $y = f(b)$, then the above inequality (3.8) simplifies into

$$\sqrt{xy} \leq \frac{2+h}{h}\Big[\frac{y-x}{f^{-1}(y)-f^{-1}(x)}\Big] \leq \frac{x+y}{2}, \tag{3.9}$$

for all $x, y \in f(h\mathbb{Z})$.

Note that inequality (3.9) holds for all $x, y \in f(h\mathbb{Z})$ for a given $h > 0$. Now we let h vary and take the limit as h goes to zero. Then $f(t) = (1+h)^{\frac{t}{h}}$ converges to e^t and the inequality turns into the well-known geometric-logarithmic-arithmetic mean inequality:

$$\sqrt{xy} \leq \frac{y-x}{\ln y - \ln x} \leq \frac{x+y}{2},$$

for all x, y positive real numbers.

(ii). Let $f : h\mathbb{N} \to \mathbb{R}$ be defined as $f(t) = \frac{1}{t}$ for some positive real number h. Since $f^{\Delta^2}(t) = \frac{2}{t(t+h)(t+2h)} \geq 0$, f is convex on $h\mathbb{N}$. Hence the Hermite-Hadamard inequality holds and we obtain

$$\frac{2}{a+b} \leq \frac{H(b,a)}{b-a} - \frac{h}{2ab} \leq \frac{\frac{1}{a}+\frac{1}{b}}{2},$$

where $a, b \in h\mathbb{N}$ and $H(b,a) = \int_{[a,b]_{h\mathbb{Z}}} \frac{1}{t}\Delta t$.

We take the limit as h goes to 0. Then we have

$$\frac{2}{a+b} \leq \frac{\ln(b)-\ln(a)}{b-a} \leq \frac{\frac{1}{a}+\frac{1}{b}}{2},$$

for all a, b positive real numbers.

Theorem 3.11. *Let $\mathbb{T}$ be a discrete time scale and f be function on $\mathbb{T}$, not necessarily convex, satisfying $\alpha \leq f^{\Delta^2}(t) \leq \beta$. Then we get*

$$\alpha U \le \Big(\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla t - \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t\Big) - f(m_{[a,b]}) \le \beta U$$

and

$$\alpha V \le m_{[0,1]}f(a)+(1-m_{[0,1]})f(b)-\Big(\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} k(t)f(t)\nabla t-\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} g^{\Delta}(t)k(t)f(t)\Delta t\Big) \le \beta V,$$

where $U = \frac{1}{(b-a)}\Big[\int_{[a,b]_{\mathbb{T}}} k(t)h_2(t)\nabla t - \int_{[a,b]_{\mathbb{T}}} k(t)g^{\Delta}(t)h_2(t)\Delta t\Big] - h_2(m_{[a,b]})$ *and*

$$V = m_{[0,1]}h_2(a)+(1-m_{[0,1]})h_2(b)-\frac{1}{(b-a)}\Big[\int_{[a,b]_{\mathbb{T}}} k(t)h_2(t)\nabla t-\int_{[a,b]_{\mathbb{T}}} k(t)g^{\Delta}(t)h_2(t)\Delta t\Big].$$

PROOF. Let $h_2(t)$ be the Taylor monomial with $s = 0$. In other words it is a function on $\mathbb{T}$ whose second $\Delta-$derivative is 1. We define Taylor monomials in Definition 4.1. Let $F(t) := f(t) - \alpha h_2(t)$ and $G(t) := \beta h_2(t) - f(t)$. Since $\alpha \le f^{\Delta^2}(t) \le \beta$ we have $F^{\Delta^2}(t) \ge 0$ and $G^{\Delta^2}(t) \ge 0$. Therefore F and G are convex. If we apply the Hermite-Hadamard inequality for both F and G, then we derive the desired inequalities. □

COROLLARY 3.12. *If* $\mathbb{T} = \mathbb{Z}$, *then* $U = \frac{(b-a)^2+2}{24}$ *and* $V = \frac{(b-a)^2-1}{12}$.

COROLLARY 3.13. *If* $\mathbb{T} = q^{\mathbb{N}}$, *then*

$$U = \frac{1}{q(1+q)^2(b-a)}\Big[q^2\sqrt{ab}(b^2-a^2) + 2n(q^2-1)(ab)^{3/2} - 2(q^2+q)ab(b-a)\Big] \text{ and}$$
$$V = \frac{1}{q(1+q)^2(b-a)}\Big[q\sqrt{ab}(b^2-a^2) - 2n(q^2-1)(ab)^{3/2}\Big].$$

Chapter 4

FRACTIONAL HERMITE-HADAMARD INEQUALITY

In this chapter, we define nabla time scale monomials and time scale power functions. Then, we define Riemann-Liouville fractional integrals, and we state substitution rules in time scales calculus. Next, we prove the fractional Hermite-Hadamard inequality in Theorem 4.6.

Definition 4.1. *(Nabla time scale monomials) Suppose f is n-times differentiable on $\mathbb{T}^{\kappa^{n-1}}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and define the functions $\hat{h}_k$ by*

$$\hat{h}_0(t,s) \equiv 1 \text{ and } \hat{h}_{k+1}(t,s) = \int_s^t \hat{h}_k(\tau,s)\nabla\tau \text{ for } k \in \mathbb{N}_0.$$

Definition 4.2. *Let $\mathbb{T}$ be a time scale. A collection of functions $\{\hat{h}_\alpha(\cdot,\cdot) : \tilde{\mathbb{T}} \times \tilde{\mathbb{T}} \to \mathbb{R}\}$ for $-1 < \alpha < \infty$ are called time scale power functions if they satisfy*

(i) For all $\alpha > -1$, $\hat{h}_\alpha(t,s)$ is a positive ld-continuous function in both variables when $t > s$ and $\hat{h}_\alpha(t,s) \equiv 0$ whenever $t \leq s$.

(ii) Whenever $\alpha \in \mathbb{N}_0$ and $t \geq s$, $\hat{h}_\alpha(t,s)$ corresponds with the nabla time scale monomials.

(iii) For all $\alpha, \beta > -1$ one has

$$\int_s^t \hat{h}_\alpha(t,\rho(\tau))\hat{h}_\beta(\tau,s)\nabla(\tau) = \hat{h}_{\alpha+\beta+1}(t,s),$$

for $t, s \in \tilde{\mathbb{T}}$ and $s < t$.

Remark 4.3. *If we pick $\mathbb{T} = \mathbb{Z}$, then we have $\hat{h}_{\alpha-1}(t,s) = \dfrac{(t-s)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$.*

For example, if $\alpha = 3/2$ *and* $s = 0$*, then* $\hat{h}_{\alpha-1}(t,s) = \dfrac{t^{1/2}}{\sqrt{\pi}}$.

Definition 4.4. *The time scale Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$ *with lower limit* a *is defined by*

$$J_a^\alpha f(t) = \int_a^t \hat{h}_{\alpha-1}(t,\rho(s))f(s)\nabla(s), \tag{4.1}$$

where $\hat{h}_{\alpha-1}(t,s)$ *is a time scale power function.*

Remark 4.5. *In Definition 4.4 we used* $J_a^\alpha f(t)$ *notation instead of* $({}_a\nabla_t^{-\alpha} f)(t)$ *to emphasize that we are working on a time scale.*

Now, we prove the main theorem of this chapter. We construct a fractional order Hermite-Hadamard inequality on a special time scale $\mathbb{T}$, where not all of its points are uniformly distributed.

Theorem 4.6. *Suppose* $f : \mathbb{T} \to \mathbb{R}$ *is a convex function on* $[a,b]_\mathbb{T}$ *where* $a, b \in \mathbb{T}, a < b$*, and* $[a,b]_\mathbb{T}$ *has odd cardinality. Then*

$$f(m_{[a,b]}) \leq \frac{1}{\beta(b-a)} J_a^\alpha[(kf)(t) + (kf)(\hat{t})]\Big|_{t=b} \leq m_{[0,1]}f(a) + (1 - m_{[0,1]})f(b), \tag{4.2}$$

where $\beta = \dfrac{\hat{h}_\alpha(b,a)}{b-a}$*, and* α *is a positive number.*

Proof. Fix $t \in \mathbb{T}_{[a,b]}$. Then there exists $x \in [a,b]_\mathbb{T}$ such that $x = ta + (1-t)b$. As we pointed out above, we have $\hat{x} = \hat{t}a + (1-\hat{t})b$. Denote this $\hat{x}$ by y, i.e. $y = \hat{t}a + (1-\hat{t})b$. Note that $m_{[a,b]} = m_{[x,y]}$ using the definition of the hat operator.

Let $\xi : [a,b]_{\mathbb{T}} \to \mathbb{T}_{[a,b]}$ be an affine map defined as $\xi(u) = \dfrac{b-u}{b-a}$. Hence we have $\xi(x) = t$ and $\xi(\hat{x}) = \xi(y) = \hat{t}$. If $x \neq m_{[a,b]}$, then by convexity of f we have

$$f(m_{[a,b]}) \leq \frac{y - m_{[a,b]}}{y-x} f(x) + \frac{m_{[a,b]} - x}{y-x} f(y). \tag{4.3}$$

If $x = m_{[a,b]}$, then it reduces to $x = y = m_{[a,b]}$. Clearly we have

$$f(m_{[a,b]}) = \frac{1}{2} f(x) + \frac{1}{2} f(y). \tag{4.4}$$

We combine (4.3) and (4.4) using the function k

$$f(m_{[a,b]}) \leq k(x) f(x) + k(y) f(y).$$

Next we multiply both sides by $P(t) = \hat{h}_{\alpha-1}(b, \rho(\xi^{-1}(t)))$. Note that since $\xi(x) = t$ we have $\hat{h}_{\alpha-1}(b, \rho(\xi^{-1}(t))) = \hat{h}_{\alpha-1}(b, \rho(x))$. We integrate each side of the above inequality from 0 to 1 on $\mathbb{T}_{[a,b]}$ and we obtain

$$\begin{aligned}
f(m_{[a,b]}) \int_{\mathbb{T}_{[a,b]}} P(t) \tilde{\Delta} t &\leq \int_{\mathbb{T}_{[a,b]}} P(t) k(x) f(x) \tilde{\Delta} t + \int_{\mathbb{T}_{[a,b]}} P(t) k(y) f(y) \tilde{\Delta} t \\
&= \int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(b, \rho(\xi^{-1}(t))) \cdot k(\xi^{-1}(t)) \cdot f(\xi^{-1}(t)) \tilde{\Delta} t \\
&+ \int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(b, \rho(\xi^{-1}(t))) \cdot k(\xi^{-1}(\hat{t})) \cdot f(\xi^{-1}(\hat{t})) \tilde{\Delta} t.
\end{aligned}$$

Here we first claim that

$$\begin{aligned}
&\int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(b, \rho(\xi^{-1}(t))) \cdot k(\xi^{-1}(t)) \cdot f(\xi^{-1}(t)) \tilde{\Delta} t \\
&\quad = \frac{1}{b-a} \int_{[a,b]_{\mathbb{T}}} \hat{h}_{\alpha-1}(b, \rho(t)) k(t) f(t) \nabla t = \frac{1}{b-a} J_a^{\alpha} (kf)(t) \Big|_{t=b}
\end{aligned} \tag{4.5}$$

If we define $F(t)$ as $\hat{h}_{\alpha-1}(b,\rho(t))\cdot k(t)\cdot f(t)$, then we have

$$F(\xi^{-1}(t)) = \hat{h}_{\alpha-1}(b,\rho(\xi^{-1}(t)))k(\xi^{-1}(t))f(\xi^{-1}(t)).$$

Next, we apply substitution rule to the integral on the left side of the equality in (4.5).

$$\begin{aligned}\int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(\rho(\xi^{-1}(t)))k(\xi^{-1}(t))f(\xi^{-1}(t))\tilde{\Delta}t &= \int_{\mathbb{T}_{[a,b]}} F(\xi^{-1}(t))\tilde{\Delta}t \\ &= \int_a^b F(t)\frac{1}{b-a}\nabla t \\ &= \frac{1}{b-a}J_a^{\alpha}(kf)(t)\Big|_{t=b}\end{aligned}$$

This finishes the proof of our first claim.

Now, we claim that

$$\begin{aligned}&\int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(b,\rho(\xi^{-1}(t)))\cdot k(\xi^{-1}(\hat{t}))\cdot f(\xi^{-1}(\hat{t}))\tilde{\Delta}t \\ &= \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} \hat{h}_{\alpha-1}(b,\rho(t))k(\hat{t})f(\hat{t})\nabla t \\ &= \frac{1}{b-a}J_a^{\alpha}(kf)(\hat{t})\Big|_{t=b}\end{aligned} \tag{4.6}$$

Let us define $G(t) := \hat{h}_{\alpha-1}(b,\rho(t))\cdot k(\hat{t})\cdot f(\hat{t})$. Now we will apply substitution to the integral on the left side of the equality (4.6).

$$\int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(b,\rho(\xi^{-1}(t)))\cdot k(\xi^{-1}(\hat{t}))\cdot f(\xi^{-1}(\hat{t}))\tilde{\Delta}t = \int_{\mathbb{T}_{[a,b]}} G(\xi^{-1}(t))\tilde{\Delta}t$$

$$\begin{aligned}&= \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} G(t)\nabla t = \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} \hat{h}_{\alpha-1}(b,\rho(t))k(\hat{t})f(\hat{t})\nabla t \\ &= \frac{1}{b-a}J_a^{\alpha}(kf)(\hat{t})\Big|_{t=b}.\end{aligned}$$

This completes the proof of our second claim.

If we combine these two integrals and we get the left side of the fractional Hermite-Hadamard inequality

$$f(m_{[a,b]})\beta \leq \frac{1}{(b-a)}\Big(J_a^{\alpha}(kf)(t)\Big|_{t=b} + J_a^{\alpha}(kf)(\hat{t})\Big|_{t=b}\Big). \tag{4.7}$$

To prove the right side of the inequality, we construct the following inequalities using convexity of f.

$$f(x) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b),$$

$$f(y) \leq \frac{b-y}{b-a}f(a) + \frac{y-a}{b-a}f(b).$$

Next, we multiply both inequality by $P(t)k(x)$ and $P(t)k(y)$ respectively. We obtain

$$P(t)k(x)f(x) \leq \frac{b-x}{b-a}P(t)k(x)f(a) + \frac{x-a}{b-a}P(t)k(x)f(b),$$

$$P(t)k(y)f(y) \leq \frac{b-y}{b-a}P(t)k(y)f(a) + \frac{y-a}{b-a}P(t)k(y)f(b).$$

Simple algebra implies the following identities

$$\frac{b-x}{b-a}k(x) + \frac{b-y}{b-a}k(y) = \frac{b-m_{[a,b]}}{b-a},$$

$$\frac{x-a}{b-a}k(x) + \frac{y-a}{b-a}k(y) = \frac{m_{[a,b]}-a}{b-a}.$$

Recall that x and y both depend on t. We let t vary over $\mathbb{T}_{[a,b]}$ and integrate each side of the last two inequalities on $\mathbb{T}_{[a,b]}$ and we add them side by side, we obtain

$$\frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} F(t)\nabla t - \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} G(t)\Delta t$$

$$\leq \left(\frac{b-m_{[a,b]}}{b-a}f(a) + \frac{m_{[a,b]}-a}{b-a}f(b)\right)\int_{\mathbb{T}_{[a,b]}} P(t)\tilde{\Delta}t = \left(m_{[0,1]}f(a) + (1-m_{[0,1]})f(b)\right)\beta.$$

Next, we divide both sides by β and write the integrals in fractional form and we get the desired fractional inequality.

$$f(m_{[a,b]}) \leq \frac{1}{\beta(b-a)}\left(J_a^{\alpha}(kf)(t)\Big|_{t=b} + J_a^{\alpha}(kf)(\hat{t})\Big|_{t=b}\right) \leq m_{[0,1]}f(a) + (1-m_{[0,1]})f(b).$$

Lastly, we calculate β.

$$\beta = \int_{\mathbb{T}_{[a,b]}} P(t)\tilde{\Delta}t = \int_{\mathbb{T}_{[a,b]}} \hat{h}_{\alpha-1}(b,\rho(\xi^{-1}(t)))\tilde{\Delta}t$$

$$= \frac{1}{b-a}\int_{[a,b]_{\mathbb{T}}} \hat{h}_{\alpha-1}(b,\rho(t))\nabla t = \frac{1}{b-a}\hat{h}_{\alpha}(b,a).$$

The last equality is proved in [**14**] ,Theorem 3.57. □

Corollary 4.7. [**4**] *Suppose $f:\mathbb{Z}\to\mathbb{R}$ is a convex function on $[a,b]_{\mathbb{Z}}$, where $a,b\in\mathbb{Z}, a<b,$ and $a+b$ an even number. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{2\beta(b-a)}\left[{}_{b-1}\Delta^{-\alpha}(f)(t)\Big|_{t=a-\alpha} + \nabla_{a+1}^{-\alpha}(f)(t)\Big|_{t=b}\right] \leq \frac{f(a)+f(b)}{2} \tag{4.8}$$

where $\beta = \int_{\mathbb{T}_{[a,b]}} ((b-a)t+(\alpha-1))^{\underline{\alpha-1}}\tilde{\Delta}t$ *and α is a positive real number.*

Proof. In Theorem 4.6 we proved $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$.

$(ii) \Rightarrow (i):$

Let us pick $a = \rho(m)$ and $b = \sigma(m)$. Then LFHHI (4.11) implies

$$f(m)\frac{\hat{h}_\alpha(b,a)}{b-a} \leq 2f(m)k(m)\hat{h}_{\alpha-1}(b,a)\frac{m-a}{b-a} + \hat{h}_{\alpha-1}(b,m)\frac{b-m}{b-a}[k(a)f(a)+k(b)f(b)],$$

$$f(m)[\frac{\hat{h}_\alpha(b,a)}{b-a} - \hat{h}_{\alpha-1}(b,a)\frac{m-a}{b-a}] \leq \hat{h}_{\alpha-1}(b,m)\frac{b-m}{b-a}[k(a)f(a)+k(b)f(b)].$$

Since $\hat{h}_\alpha(b,a) = \hat{h}_{\alpha-1}(b,a)(m-a) + \hat{h}_{\alpha-1}(b,m)(b-m)$ we get

$$f(m) \leq k(a)f(a) + k(b)f(b).$$

Therefore $f^\Delta(m) \geq 0$ for all m.

$(iii) \Rightarrow (i):$

Let $a = \rho(m)$ and $b = \sigma(m)$. Then RFHHI (4.12) will give us

$$\begin{aligned} 2k(m)f(m)\hat{h}_{\alpha-1}(b,a)\frac{m-a}{b-a} + \hat{h}_{\alpha-1}(b,m)\frac{b-m}{b-a}[k(a)f(a)+k(b)f(b)] \\ \leq \frac{\hat{h}_\alpha(b,a)}{(b-a)}[k(a)f(a)+k(b)f(b)]. \end{aligned} \tag{4.9}$$

This simplifies to

$$f(m)\hat{h}_{\alpha-1}(b,a)\frac{m-a}{b-a} \leq \Big(\frac{\hat{h}_\alpha(b,a)}{(b-a)} - \hat{h}_{\alpha-1}(b,m)\frac{b-m}{b-a}\Big)[k(a)f(a)+k(b)f(b)]. \tag{4.10}$$

Since $\hat{h}_\alpha(b,a) = \hat{h}_{\alpha-1}(b,a)(m-a) + \hat{h}_{\alpha-1}(b,m)(b-m)$ we get

$$f(m) \le k(a)f(a) + k(b)f(b).$$

Therefore f is convex on $\mathbb{T}$.

$\square$

4.1. Characterization of Convexity via the Fractional Hermite-Hadamard Inequality

In this chapter we define left and right fractional Hermite-Hadamard inequalities. Then we show that each of these inequalities characterizes convexity.

Definition 4.8. *We define left fractional Hermite-Hadamard inequality (* LFHHI*) and right fractional Hermite-Hadamard inequality (* RFHHI*) as the first and second inequality in (4.2).*

$$f(m_{[a,b]}) \le \frac{1}{\beta(b-a)} J_a^\alpha [(kf)(t) + (kf)(\hat{t})]\Big|_{t=b}, \tag{4.11}$$

$$\frac{1}{\beta(b-a)} J_a^\alpha [(kf)(t) + (kf)(\hat{t})]\Big|_{t=b} \le m_{[0,1]} f(a) + (1 - m_{[0,1]}) f(b), \tag{4.12}$$

where $\beta = \dfrac{\hat{h}_\alpha(b,a)}{b-a}$.

Theorem 4.9. *Let $\mathbb{T}$ be a discrete time scale and let $f : \mathbb{T} \to \mathbb{R}$ be a function on $[a,b]_\mathbb{T}$. Then the following are equivalent:*

(i) *f is a convex function.*

(ii) *The left fractional Hermite-Hadamard inequality,* LFHHI*, holds.*

(iii) *The right fractional Hermite-Hadamard inequality,* RFHHI*, holds.*

PROOF. In Theorem 4.6 we proved $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$.

$(ii) \Rightarrow (i)$:

Let us pick $a = \rho(m)$ and $b = \sigma(m)$. Then LFHHI (4.11) will give us

$$f(m)\frac{\hat{h}_\alpha(b,a)}{b-a} \leq 2f(m)k(m)g_{\alpha-1}(b,a)\frac{m-a}{b-a} + g_{\alpha-1}(b,m)\frac{b-m}{b-a}[k(a)f(a)+k(b)f(b)],$$

$$f(m)[\frac{\hat{h}_\alpha(b,a)}{b-a} - g_{\alpha-1}(b,a)\frac{m-a}{b-a}] \leq g_{\alpha-1}(b,m)\frac{b-m}{b-a}[k(a)f(a)+k(b)f(b)].$$

Since $\hat{h}_\alpha(b,a) = g_{\alpha-1}(b,a)(m-a) + g_{\alpha-1}(b,m)(b-m)$ we get

$$f(m) \leq k(a)f(a) + k(b)f(b).$$

Therefore $f^\Delta(m) \geq 0$ for all m.

$(iii) \Rightarrow (i)$:

Let $a = \rho(m)$ and $b = \sigma(m)$. Then RFHHI (4.12) will give us

$$\begin{aligned} 2k(m)f(m)g_{\alpha-1}(b,a)\frac{m-a}{b-a} + g_{\alpha-1}(b,m)\frac{b-m}{b-a}[k(a)f(a)+k(b)f(b)] \\ \leq \frac{g_\alpha(b,a)}{(b-a)}[k(a)f(a)+k(b)f(b)]. \end{aligned} \tag{4.13}$$

This simplifies to

$$f(m)g_{\alpha-1}(b,a)\frac{m-a}{b-a} \leq \Big(\frac{g_\alpha(b,a)}{(b-a)} - g_{\alpha-1}(b,m)\frac{b-m}{b-a}\Big)[k(a)f(a)+k(b)f(b)]. \tag{4.14}$$

Since $\hat{h}_\alpha(b,a) = g_{\alpha-1}(b,a)(m-a) + g_{\alpha-1}(b,m)(b-m)$ we get

$$f(m) \leq k(a)f(a) + k(b)f(b).$$

Therefore f is convex on $\mathbb{T}$. $\square$

Chapter 5

CONVEX OPTIMIZATION ON DISCRETE TIME SCALES

In this chapter, we discuss convexity on n–dimensional discrete time scales $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ where $\mathbb{T}_i \subset \mathbb{R}$, $i = 1, 2, ..., n$ are discrete time scales which are not necessarily periodic. We introduce the discrete analogues of the fundamental concepts of real convex optimization such as convexity of a function, subgradients, and the Karush–Kuhn–Tucker conditions. In the last section we illustrate our result in an example.

An optimization problem, or mathematical programming problem, is minimizing the objective function under the given constraints.

$$\text{minimize } f(x) \text{ subject to } g_i(x) \leq 0, x \in X \text{ for } i = 1, 2, ..., m. \tag{5.1}$$

Here X could be any of the following sets; $X = \mathbb{R}^n$, $X = \{x \in \mathbb{R}^n | x \geq 0\}$, $X = \mathbb{Z}^n$, or $X = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, where $\mathbb{T}_i \subset \mathbb{R}$, $i = 1, 2, ..., n$ are discrete time scales, not necessarily periodic.

Definition 5.1. *An optimization problem (5.2) is called a convex optimization problem or a convex programming problem if f and g_i are real convex functions for $i = 1, 2, ..., m$ and $X = \mathbb{R}^n$.*

$$\textit{minimize } f(x) \textit{ subject to } g_i(x) \leq 0, x \in X \textit{ for } i = 1, 2, ..., m. \tag{5.2}$$

The Lagrangian function corresponding to the objective function $f(x)$ is defined as follows.

$$L(x, u) = f(x) + u^T g(x)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))$.

Definition 5.2. *h is called a subgradient for f at $x^0 \in \mathbb{R}^n$ if it satisfies the following inequality*

$f(x) \geq f(x^0) + \langle x - x^0, h \rangle$ *for all* $x \in$ *Domain*(f).

where $\langle \cdot, \cdot \rangle$ *is the dot product.*

A function defined on $\mathbb{R}^n$ is called a real convex function if it has a subgradient at each point of its domain. We define discrete convexity in the next section using an analogue of the subgradient property of real convex functions and some techniques from time scales calculus such as partial delta and partial nabla derivatives. Partial delta and partial nabla derivatives are introduced in [**6**]. For further reading on time scales, we refer the reader to an excellent book on the analysis of time scales [**7**]. Let μ_i, and ν_i be the graininess functions on $\mathbb{T}_i$ and e_i be the i^{th} basis element of the n–dimensional Euclidean space. The partial delta and nabla derivatives are defined as follows.

$$\Delta_i f(x^0) := \frac{f(x^0 + e_i\mu_i(x^0)) - f(x^0)}{\mu_i(x^0)}, \qquad \nabla_i f(x^0) := \frac{f(x^0) - f(x^0 - e_i\nu_i(x^0))}{\nu_i(x^0)}.$$

5.1. Discrete Convex Functions

Definition 5.3. *Let* $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, *where* $\mathbb{T}_i \subset \mathbb{R}$ *is a discrete time scale. A function* $f : \mathbb{T} \to \mathbb{R}$ *is called discrete convex if given any point* $a = (a_1, a_2, ..., a_n) \in \mathbb{T}$, *we have*

$$f(x) \geq f(a) + \langle x - a, \nabla^D f(x, a) \rangle \text{ for all } x \in \mathbb{T},$$

$$\nabla^D f(x, a) := (f_{x_1}(x, a), f_{x_2}(x, a), ..., f_{x_n}(x, a)),$$

$$f_{x_i}(x, a) := \begin{cases} \Delta_i f(a), & if\ x_i \geq a_i \\ \nabla_i f(a), & if\ x_i \leq a_i. \end{cases}$$

Note that the discrete gradient vector of a function, $(\nabla^D f)(x, y)$, is a function of two vectors, x and y. The definition depends on the difference of the components of these two vectors.

Theorem 5.4. *Any finite sum of discrete convex function is also discrete convex.*

$\square$

Remark 5.5. *Note that discrete convexity is not necessarily a weaker structure than real convexity. In other words real convexity does not imply discrete convexity. For instance,* $f(x,y) = 25(2y-x)^2 + 1/4(2-x)^2$ *is real convex however one can show that it does not satisfy the discrete convexity condition.*

Remark 5.6. *On the other hand, there is a discrete convex function which is not real convex. To construct such a function we assume that the domain of the function is bounded by an interval of length* $M > 0$*. From the definition of discrete and real convexity one can obtain*

$f(x,y) = (x+y)^2 - kx^2$ *is discrete convex if and only if* $k \leq \frac{M^2+M+1}{M(M-1)}$,

$f(x,y) = (x+y)^2 - kx^2$ *is real convex if and only if* $k \leq \frac{(M+1)^2}{M^2}$.

Therefore, for values of $k \in (\frac{M^2+M+1}{M(M-1)}, \frac{(M+1)^2}{M^2})$, $f(x,y)$ *is a discrete convex function, but not real convex.*

Remark 5.7. *Adivar and Fang* [**1, 2**] *defined the discrete convex function on* $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$*, where* $\mathbb{T}_i \subset \mathbb{R}$, $i = 1, 2, ..., n$ *are time scales, as a function whose epigraph is convex. Therefore the discrete restriction* $f|\mathbb{Z}^n$ *of a convex function* f *on the real domain is convex on* $\mathbb{Z}^n$*. Conversely, every convex function on a discrete domain can be extended to a convex function on the real domain. However, the discrete convexity in the sense of this paper is not weaker than convexity on the real domain as pointed out in the two abovementioned remarks. Nonetheless, these two definitions match in* $\mathbb{T} \subset \mathbb{R}$*, a special time scale where the time points are not necessarily uniformly distributed on a time line.*

5.2. Karush–Kuhn–Tucker Conditions on Discrete Time Scales

Definition 5.8. *A discrete convex programming problem is an optimization problem with f and g_i are discrete convex functions for $i = 1, 2, ..., m$ and $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$.*

$$\text{minimize } f(x) \text{ subject to } g_i(x) \le 0, x \in \mathbb{T} \text{ for } i = 1, 2, \ldots, m. \tag{5.3}$$

The set $S = \{x \in \mathbb{T} | g_i(x) \le 0 \text{ for } i = 1, 2, \ldots, m\}$ is called the feasible set. The Lagrangian associated with this programming problem is a function $L : \mathbb{T} \times \mathbb{R}^m \to \mathbb{R}$ defined as

$$L(x, u) = f(x) + u_1 g_1(x) + \cdots + u_m g_m(x). \tag{5.4}$$

Definition 5.9. *A point $(x^0, u^0) \in \mathbb{T} \times \mathbb{R}^m$ is called a saddle point of L if*

$x^0 \ge 0, u^0 \ge 0$ and $L(x^0, u) \le L(x^0, u^0) \le L(x, u^0)$ for all $x \ge 0, u \ge 0$ and $x \in \mathbb{T}$.

Theorem 5.10. *Let (x^0, u^0) be a saddle point of the Lagrangian function L. Then x^0 is a solution to the convex programming problem and $f(x^0) = L(x^0, u^0)$.*

Proof. The condition $L(x^0, u) \le L(x^0, u^0)$ yields

$$u_1 g_1(x^0) + ... + u_m g_m(x^0) \le u_1^0 g_1(x^0) + ... + u_m^0 g_m(x^0).$$

By keeping $u_2, ..., u_m$ fixed and taking the limit $u_1 \to \infty$, we infer that $g_1(x^0) \le 0$. Similarly, one gets $g_2(x^0) \le 0, ..., g_m(x^0) \le 0$. Thus x^0 belongs to the feasible set S. From $L(x^0, 0) \le L(x^0, u^0)$ and the definition of S we infer $0 \le u_1^0 g_1(x^0) + ... + u_m^0 g_m(x^0) \le 0$, hence $u_1^0 g_1(x^0) + ... + u_m^0 g_m(x^0) = 0$ and $f(x^0) = L(x^0, u^0)$. Since $L(x^0, u^0) \le L(x, u^0)$ for all $x \ge 0$ this implies $f(x^0) \le f(x) + u_1^0 g_1(x) + ... + u_m^0 g_m(x)$. We also have $f(x) + u_1^0 g_1(x) + ... + u_m^0 g_m(x) \le f(x)$ for all $x \in S$.

If we combine the last two inequalities we get $f(x^0) \le f(x)$ for all x in the feasible set S. Therefore x^0 is a solution to the convex programming problem (5.3). $\square$

THEOREM 5.11. *Suppose* $f, g_1, \dots, g_{m-1}$, *and* g_m *are discrete convex functions on* $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$. *Then* (x^0, u^0) *is a saddle point of the Lagrangian* L *if and only if*

$x^0 \geq 0$ $\qquad$ $u^0 \geq 0$

$\Delta_{x_i} L(x^0, u^0) \geq 0$ *if* $x_i^0 = 0$ $\qquad$ $\dfrac{\partial L}{\partial u_j}(x^0, u^0) = g_j(x^0) \leq 0$ *if* $u_j^0 = 0$

$\Delta_{x_i} L(x^0, u^0) \geq 0,\ \nabla_{x_i} L(x^0, u^0) \leq 0$ *if* $x_i^0 > 0$ $\qquad$ $\dfrac{\partial L}{\partial u_j}(x^0, u^0) = g_j(x^0) = 0$ *whenever* $u_j^0 > 0$.

PROOF. If (x^0, u^0) is a saddle point of L, then clearly we have $x^0, u^0 \geq 0$.

If $x_i^0 = 0$, then $\Delta_{x_i} L(x^0, u^0) = \dfrac{L(x^0 + e_i\mu_i(x^0), u^0) - L(x^0, u^0)}{\mu_i(x^0)} \geq 0$ since (x^0, u^0) is saddle point.

If $x^0 > 0$, then $\Delta_{x_i} L(x^0, u^0) \geq 0$ and $\nabla_{x_i} L(x^0, u^0) \leq 0$ since $L(x^0, u^0) \leq L(x, u^0)$ for all x.

If $u_j^0 = 0$, then $L(x^0, u^0 + te_j) \leq L(x^0, u^0)$ for all $t \geq -u_j^0$.
Therefore, $\dfrac{\partial L}{\partial u_j}(x^0, u^0) = \lim_{x \to 0^+} \dfrac{L(x^0, u^0 + te_j) - L(x^0, u^0)}{t} \leq 0$

If $u_j^0 > 0$, then $\dfrac{\partial L}{\partial u_j}(x^0, u^0) = 0$ since (x^0, u^0) is a saddle point.

Suppose the conditions in the theorem are satisfied. Since f and g_i are discrete convex functions on $\mathbb{T}$, for a fixed u^0, $L(x, u^0)$ is a discrete convex function too. By convexity of $L(x, u^0)$ we have

$$L(x, u^0) \geq L(x^0, u^0) + \langle (x - x^0), (\nabla^D L)(x, x^0, u^0) \rangle.$$

By the conditions on x and using the definition of discrete gradient we obtain

$$\langle (x - x^0), (\nabla^D L)(x, x^0, u^0) \rangle \geq 0\ .$$

Therefore, we have $L(x, u^0) \geq L(x^0, u^0)$ for all x.

To show the other side of the inequality, we consider $L(x^0, u)$ as a linear function in $\mathbb{R}^m$ on variables $u_1, ..., u_m$. Since it is a linear function on u–coordinates, we have

$L(x^0,u) = L(x^0,u^0) + \sum_{j=1}^{m}(u_j - u_j^0)\frac{\partial L}{\partial u_j}(x^0,u^0) \le L(x^0,u^0) + \sum_{j=1}^{m} u_j \frac{\partial L}{\partial u_j}(x^0,u^0) \le L(x^0,u^0).$

Hence we have $L(x^0,u) \le L(x^0,u^0) \le L(x,u^0)$ for all $x,u \ge 0$. This concludes that (x^0,u^0) is a saddle point. $\square$

5.3. An Example

In this section we demonstrate our theory on a nonlinear programming problem.

Consider

$$z^* = \min_{x,y} f(x_1,x_2) = 6(x_1-10)^2 + 4(x_2-12.5)^2$$

$$\text{subject to } x_1^2 + (x_2-5)^2 \le 50$$

$$x_1^2 + 3x_2^2 \le 200$$

$$(x_1-6)^2 + x_2^2 \le 37$$

$$x_i \in \mathbb{Z}^{\ge 0} \text{ for } i = 1,2.$$

Since both $6(x_1-10)^2$ and $4(x_2-12.5)^2$ are discrete convex, $f(x,y)$ is discrete convex too. For this problem, the Lagrangian is

$L(x_1,x_2,u_1,u_2) = 6(x_1-10)^2 + 4(x_2-12.5)^2 + u_1(x_1^2 + (x_2-5)^2 - 50) + u_2(x_1^2 + 3x_2^2 - 200) + u_3((x_1-6)^2 + x_2^2 - 37)$.

By the Karush-Kuhn-Tucker conditions we have $u_1(x_1^2 + (x_2-5)^2 - 50) = 0$

$u_2(x_1^2 + 3x_2^2 - 200) = 0$ and $u_3((x_1-6)^2 + x_2^2 - 37) = 0$

Clearly, we have $x_1 \ge 0$ and $x_2 \ge 0$. From Theorem 5.11, we deduce $\Delta_{x_i} L \ge 0$ and $\nabla_{x_i} L \le 0$ for $i = 1,2$. If we combine all these conditions one can reach the optimal solution $(x_1^*, x_2^*) = (7,6)$. Note that here (u_1,u_2,u_3) are not necessarily unique since the Karush-Kuhn-Tucker conditions in Theorem 5.11 involves inequalities. Yet, one can choose $u_2 = 0$ and $u_1 = 2, u_3 = 14$ values to justify the above inequalities.

CHAPTER 6

CONCLUSION AND FUTURE WORK

In the theory of convex functions, the Hermite-Hadamard inequality plays an important role. It has been used as a tool to obtain many results in integral inequalities, approximation theory, optimization theory and numerical analysis [**13**], [**4**], [**25**],[**22**]. In this study, we presented fundamental definitions and formulas in discrete fractional calculus for the convenience of the reader. In the third chapter, we defined a convex real-valued function on a discrete time scale $\mathbb{T}$ where not all the time points are uniformly distributed on a time line. We stated the midpoint condition for a function defined on $\mathbb{T}$. We then proved four equivalent statements for convex functions on $\mathbb{T}$. Then with the use of the substitution rules of fractional calculus we proved the Hermite-Hadamard inequality for convex functions defined on $\mathbb{T}$. As a corollary, we gave an alternate proof to the Hermite-Hadamard inequality for functions defined on the set of real numbers $\mathbb{R}$. The last section of this chapter was devoted to some interesting examples of the Hermite-Hadamard inequality. In the fourth chapter, we carried the previous results to the fractional case. We started by giving a brief introduction to fractional order integration, and then we constructed the discrete fractional Hermite-Hadamard inequality and proved that this inequality completely characterizes discrete convexity.

The following two remarks will state some open problems for the researchers who are interested in working in this area.

REMARK 6.1. *The midpoint condition plays an important role to prove the main result of Chapter 3, Theorem 3.5. Even though, the convexity of the function on any time scale has been defined in* [**4, 26**]*, it is still an open problem to define the midpoint condition for such a function.*

Remark 6.2. *As a further research topic one can define a higher dimensional version of the fractional Hermite-Hadamard inequality on domains which are products of isolated time scales.*

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