Spring 2018

Controllability and Observability of the Discrete Fractional Linear State-Space Model

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CONTROLLABILITY AND OBSERVABILITY OF THE DISCRETE FRACTIONAL LINEAR STATE-SPACE MODEL

A Thesis
Presented to
The Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
Of the Requirements for the Degree
Master of Science

By
Duc Nguyen

May 2018

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This thesis is dedicated to my Mom, whose unwavering spiritual support has kept me going throughout this journey.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my thesis advisor, Dr. Ferhan Atici, for her support, enthusiasm, dedication, and especially her patience. This was a challenging task, and I am grateful for her constant presence and guidance through it all. I am grateful to be her student. I would also like to thank Dr. Nezam Iraniparast and Dr. Mark Robinson, my committee members. Their insight and feedback have been invaluable as I progressed through my project.

Contents

CONTROLLABILITY AND OBSERVABILITY OF THE DISCRETE
FRACTIONAL LINEAR STATE-SPACE MODEL

Duc Nguyen | May 2018 | 38 Pages

Directed by: Dr. Ferhan Atici, Dr. Nezam Iraniparast, and Dr. Mark Robinson

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This thesis aims to investigate the *controllability* and *observability* of the discrete fractional linear time-invariant state-space model. First, we will establish key concepts and properties which are the tools necessary for our task. In the third chapter, we will discuss the discrete state-space model and set up the criteria for these two properties. Then, in the fourth chapter, we will attempt to apply these criteria to the discrete fractional model. The general flow of our objectives is as follows: we start with the first-order linear difference equation, move on to the discrete system, then the fractional difference equation, and finally the discrete fractional system. Throughout this process, we will develop the solutions to the (fractional) difference equations, which are the basis of our criteria.

Chapter 1

INTRODUCTION

A *state-space model* is a set of first-order differential or difference equations which uses *state variables* to describe a dynamical system. A state variable is, in turn, a variable that describes the "state" of that system. For example, in a mechanical system, the position of a mechanical part, as well as its velocity and direction of movement, can be state variables.

In continuous time, a time-invariant linear state-space model can be given in the form

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \\ z(t) = Cy(t) + Du(t). \end{cases}$$

The discrete time-invariant model can be given by

$$\begin{cases} y(t+1) = Ay(t) + Bu(t), \\ z(t) = Cy(t) + Du(t), \end{cases}$$

where t is a non-negative integer.

In these representations, the first equation is the state equation which describes the rate of the change of the state of the system in terms of the current state $y(t)$ and the input $u(t)$. The second equation is the output equation which describes the output in terms of the current state $y(t)$ and the input $u(t)$. A, B, C, and D are coefficient matrices, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $D \in \mathbb{R}^{r\times m}$. Also, the term $u(t)$ is the control term, or simply control, which represents our "manipulation" of the system. Without this term, the system is an *uncontrolled* system.

As technology advances, more and more tasks become automated and are handled by machines. This leads to the area of control theory gaining more attention from engineers and mathematicians alike. In control theory, two important properties of state-space models are *controllability* and *observability*. These concepts were introduced by Kalman in 1959 [5] and have played a major role in modern control theory. Essentially, controllability refers to the ability to drive the system from an initial state to some arbitrary final state, while observability refers to the ability to measure or determine the state of the system based on its outputs.

Compared to the discrete model, the discrete fractional model may offer some advantages in certain applications, such as fractional conservation of mass [10] and PID controller design, where using fractional orders can increase the controller's degree of freedom [7].

The main purpose of this thesis is to discuss the controllability and observability of a discrete fractional time-invariant linear state-space model, which is represented by

$$\begin{cases} \Delta^{\nu} y(t) = Ay(t+\nu-1) + Bu(t+\nu-1), \\ z(t) = Cy(t), \end{cases}$$

with initial condition $y(\nu - 1) = y_0$, $0 < \nu < 1$, and $t = 0, 1, 2, \ldots$. In order to achieve this, first we will establish the criteria for controllability and observability of the discrete system. The criteria, as well as their proof, are adapted from [4]. We will work with the state equation of the form $\Delta y(t) = Ay(t) + Bu(t)$, rather than $y(t + 1) = Ay(t) + Bu(t)$, in order to form a more parallel connection between the continuous equation $y'(t) = Ay(t)+Bu(t)$ and its discrete counterpart. After that, we will discuss the discrete fractional scalar equation $\Delta^{\nu} y(t) = ay(t+\nu-1)+f(t+\nu-1)$ and its solution. Once we have the solution of the scalar equation, we will proceed to establish the solution of the matrix equation $\Delta^{\nu} y(t) = Ay(t+\nu-1)+f(t+\nu-1)$ (here

f is a vector). This solution plays a central role in our investigation of controllability and observability of the system.

Chapter 2

PRELIMINARIES

In this chapter we establish a number of key concepts and theorems of discrete fractional calculus. Most of them are discussed in details in [6].

2.1 The Difference Operator

Definition 2.1.1. *Let $y(t)$ be a real-valued function. The difference operator Δ is defined by*

$$\Delta y(t) = y(t+1) - y(t). \tag{2.1}$$

This is the discrete counterpart of the derivative in continuous calculus. It is also called the *forward difference* operator, to distinguish it from the *backward difference* operator (or *nabla*) ∇, which is defined by $\nabla y(t) = y(t) - y(t-1)$.

We will make use of the following property called the product rule for the difference operator

$$\Delta(f(t)g(t)) = (\Delta f(t))g(t) + f(t+1)\Delta g(t) = (\Delta g(t))f(t) + g(t+1)\Delta f(t).$$

Also, generally, for positive integer order of Δ that is greater than 1, we have $\Delta^2 y(t) = \Delta(\Delta y(t))$, $\Delta^3 y(t) = \Delta(\Delta y^2(t))$, and so on.

2.2 The Gamma Function

Definition 2.2.1. *The gamma function of a positive real number, or a complex number with a positive real part, can be defined by the definite integral*

$$\Gamma(t) = \int_0^\infty e^{-r} r^{t-1} dr. \tag{2.2}$$

The gamma function is a generalization of the factorial function. If n is a positive integer, then

$$\Gamma(n) = (n-1)!. \tag{2.3}$$

2.3 The Discrete Exponential Function

In continuous calculus, we know that the general solution of the differential equation $y'(t) = ay(t)$ is $y(t) = Ce^{at}$, where a is any real number and C is an arbitrary constant.

In discrete calculus, the equivalent difference equation is $\Delta y(t) = ay(t)$, and the general solution of this equation is given by $y(t) = C(1+a)^t$. For $C = 1$, the function $y(t) = (1+a)^t$ is considered the exponential function of discrete calculus.

We can easily verify this result by applying the definition of the Δ-operator to $y(t) = C(1+a)^t$:

$$\begin{aligned}
\Delta y(t) &= y(t+1) - y(t) \\
&= C(1+a)^{t+1} - C((1+a)^t) \\
&= C(1+a)^t(1+a-1) \\
&= aC(1+a)^t = ay(t).
\end{aligned}$$

2.4 The Falling Factorial Power

Definition 2.4.1. *The falling factorial is defined by*

$$t^{(r)} = t^{\underline{r}} = \frac{\Gamma(t+1)}{\Gamma(t+1-r)}, \tag{2.4}$$

where $t \in \mathbb{R}$ *and* $r \in \mathbb{R}$.

If $r \in \mathbb{N}$, we also write

$$t^{(r)} = t^{\underline{r}} = t(t-1)(t-2)\ldots(t-r+1).$$

Throughout the thesis, we assume that $t^{(r)} = 0$ if $t+1-r$ is a non-positive integer.

The following are some useful properties of this factorial function.

Theorem 2.4.2. *Assuming the following factorial functions are well-defined,*

(i) $\Delta t^{(\nu)} = \nu t^{(\nu-1)}$,

(ii) $(t-\nu)t^{(\nu)} = t^{(\nu+1)}$,

(iii) $\nu^{(\nu)} = \Gamma(\nu+1)$.

These results follow immediately after applying basic properties of the difference operator and the gamma function.

2.5 The Summation Operator

Definition 2.5.1. *Let $y(t)$ be a function such that $\Delta y(t) = Y(t)$. An indefinite sum (or antidifference) of $y(t)$ is defined by*

$$\sum Y(t) = y(t) + C, \tag{2.5}$$

where C is an arbitrary constant.

The antidifference of the discrete exponential function $y(t) = (1+a)^t$, whre $a \neq 0$, is

$$\sum (1+a)^t = \frac{(1+a)^t}{a} + C.$$

We note that this result is similar to $\int e^{at} dt = \dfrac{e^{at}}{a} + C$ in continuous calculus.

2.6 Fundamental Theorem of Discrete Calculus-I

Theorem 2.6.1. *Suppose $[a, b]$ is a discrete interval with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, and $F : [a, b] \to \mathbb{R}$ is a function such that $\Delta F(t) = f(t)$ on [a, b-1]. Then*

$$\sum_{t=a}^{b-1} f(t) = F(t)\Big|_{t=a}^{b} = F(b) - F(a). \tag{2.6}$$

Proof. By using the definition of the Δ-operator, we have

$$\begin{aligned}
\sum_{t=a}^{b-1} f(t) &= \sum_{t=a}^{b-1} \Delta F(t) \\
&= \sum_{t=a}^{b-1} (F(t+1) - F(t)) \\
&= F(a+1) - F(a) + F(a+2) - F(a+1) + \cdots + F(b) - F(b-1)
\end{aligned}$$

$$= F(b) - F(a).$$

□

2.7 The First-Order Linear Difference Equation

We will establish the solution of the first order linear difference equation given by

$$\Delta y(t) = ay(t) + bu(t),$$

where a and b are constants, and $y(0) = y_0$. The formula of the solution is called the variation of constants (or variation of parameters) formula. This result helps us more easily acquire the solution of the system represented by the matrix form of this equation.

Theorem 2.7.1 (Variation of Constants)**.** *The solution of the first order linear difference equation*

$$\Delta y(t) = ay(t) + bu(t)$$

where a and b are constants, and $y(0) = y_0$, is

$$y(t) = (1+a)^t y_0 + \sum_{s=0}^{t-1} (1+a)^{t-s-1} bu(s).$$

Proof. We start by multiplying both sides by $(1+a)^{-(t+1)}$

$$(1+a)^{-(t+1)} \Delta y(t) - a(1+a)^{-(t+1)} y(t) = b(1+a)^{-(t+1)} u(t),$$

then we apply the product rule for the difference operator from Section 2.1

$$\begin{aligned}
\Delta[(1+a)^{-t}y(t)] &= bu(t)(1+a)^{-(t+1)} \\
\sum_{s=0}^{t-1}\Delta[(1+a)^{-s}y(s)] &= \sum_{s=0}^{t-1} bu(s)(1+a)^{-(s+1)} \\
(1+a)^{-s}y(s)\Big|_{s=0}^{t} &= \sum_{s=0}^{t-1} bu(s)(1+a)^{-(s+1)} \\
(1+a)^{-t}y(t) - y_0 &= \sum_{s=0}^{t-1} bu(s)(1+a)^{-(s+1)} \\
y(t) = (1+a)^t y_0 &+ \sum_{s=0}^{t-1}(1+a)^{t-s-1}bu(s).
\end{aligned}$$

□

2.8 Converting a Higher-Order Difference Equation to a First-Order System

We start with the second-order difference equation of the form

$$a_2\Delta^2 y(t) + a_1\Delta y(t) + a_0 y(t) = bu(t).$$

Let $y_1 = y$, $y_2 = \Delta y = \Delta y_1$. We have

$\Delta y_1 = \Delta y = y_2$,

$\Delta y_2 = \Delta^2 y = -\dfrac{a_0}{a_2}y - \dfrac{a_1}{a_2}\Delta y + \dfrac{b}{a_2}u(t) = -\dfrac{a_0}{a_2}y_1 - \dfrac{a_1}{a_2}y_2 + \dfrac{b}{a_2}u(t).$

The original second-order equation is then equivalent to the following two equations, written in matrix form

$$\begin{bmatrix}\Delta y_1 \\ \Delta y_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{a_0}{a_2} & -\dfrac{a_1}{a_2}\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{b}{a_2}u(t)\end{bmatrix},$$

or simply

$$\Delta y(t) = Ay(t) + f(t),$$

where $y(t) \in \mathbb{R}^2$, $f(t) \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2x2}$. This is the first-order discrete linear system, part of the discrete linear state-space model (in Δ form).

Using this technique, we can convert an n^{th}-order difference equation to a system of first-order difference equations (or a first-order discrete system).

2.9 The Fractional Difference Equation

In general, a linear discrete fractional equation can be written in the form $\Delta^r u(t) = f(t)$, where $r \in \mathbb{R}$ and $t \in \mathbb{N}_a = \{a, a+1, a+2, \dots\}$. Depending on the value or r, we have several cases as discussed below.

If n is a positive integer, then the solution of the initial value problem given by

$$\Delta^n u(t) = f(t), \quad t = a, a+1, a+2, , \dots,$$
$$u(a+j-1) = 0, \quad j = 1, 2, \dots, n,$$

is the function

$$u(t) = \Delta^{-n} f(t) = \sum_{s=a}^{t-1} \frac{(t-\sigma(s))^{(n-1)}}{(n-1)!} f(s),$$

where $\sigma(s) = s + 1$.

More generally, for a negative fractional order, we make use of the following forward fractional sum defined by Miller and Ross [8]

$$\Delta_a^{-\nu} f(t) = \sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)} f(s), \tag{2.7}$$

where $\nu > 0$ and $a \in \mathbb{R}$. We also note that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$.

For a positive fractional order, we use the following definition called the Riemann-Liouville fractional difference

$$\Delta_a^{\mu} f(t) = \Delta_a^{m-\nu} f(t) = \Delta^m(\Delta_a^{-\nu} f(t)), \tag{2.8}$$

where $0 \leq m-1 < \mu \leq m$ and $m \in \mathbb{N}$.

For the purpose of this thesis, we will discuss fractional equations of the form $\Delta^{\nu} y(t) = ay(t+\nu-1) + f(t+\nu-1)$ and its matrix counterpart, where $0 < \nu < 1$ and the coefficients are constants.

2.10 The R-Transform

One of the most important tools used in this thesis is a method of transformation called R-transform (or the so-called "discrete transform") defined in [1] by

$$R_{t_0}((f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t), \tag{2.9}$$

where f is defined on $\mathbb{N}_{t_0}$. If $t_0 = 0$, $R_0((f(t))(s)$ is the Laplace transform on the time scale of integers [3].

We will make use of the following lemmas, the first two proved in [1]:

Lemma 2.10.1. *For any* $\nu \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$,

$$R_{\nu-1}(t^{(\nu-1)})(s) = \frac{\Gamma(\nu)}{s^{\nu}}.$$

Lemma 2.10.2. *For $0 < \nu < 1$ and f defined on $\nu - 1, \nu, \nu + 1, \ldots,$*

$$R_0(\Delta^\nu f(t))(s) = s^\nu R_{\nu-1} f(t) - f(\nu - 1).$$

Lemma 2.10.3. *For $0 < \nu < 1$ and f defined on $\nu - 1, \nu, \nu + 1, \ldots,$*

$$R_0(f(t + \nu - 1)) = \frac{1}{(s+1)^{1-\nu}} R_{\nu-1} f(t).$$

Proof. Using the definition of R-transform given by (2.9) with $t_0 = 0$, we have

$$R_0(f(t + \nu - 1)) = \sum_{t=0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t + \nu - 1).$$

Making the change of variable $t + \nu - 1 = u$, we have

$$\begin{aligned} R_0(f(t + \nu - 1)) &= \sum_{u=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{u-\nu+2} f(u) \\ &= \frac{1}{(s+1)^{1-\nu}} \sum_{u=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{u+1} f(u) \\ &= \frac{1}{(s+1)^{1-\nu}} R_{\nu-1} f(u) \\ &= \frac{1}{(s+1)^{1-\nu}} R_{\nu-1} f(t). \end{aligned}$$

□

2.11 A Convolution Product on $\mathbb{N}_{\nu-1}$

We define a convolution product of two functions defined on $\mathbb{N}_{\nu-1}$ as follows

$$(f *_{\nu-1} g)(t) = \sum_{s=\nu-1}^{t} f(t - s + \nu - 1) g(s). \tag{2.10}$$

The following lemma and its proof are adapted from [2], in which a similar convolution product, $(f *_{\nu-2} g)(t)$, was introduced.

Lemma 2.11.1.

$$R_{\nu-1}((f *_{\nu-1} g)(t)) = \frac{1}{(s+1)^{-\nu}} R_{\nu-1}((f(t)) R_{\nu-1}((g(t)). \tag{2.11}$$

Proof. Using the definition of R-transform

$$\begin{aligned} R_{\nu-1}((f *_{\nu-1} g)(t)) &= \sum_{t=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} \sum_{\tau=\nu-1}^{t} f(t-\tau+\nu-1) g(\tau) \\ &= \sum_{\tau=\nu-1}^{\infty} \sum_{t=\tau}^{\infty} \left[\left(\frac{1}{s+1}\right)^{t+1} f(t-\tau+\nu-1) g(\tau) \right]. \end{aligned}$$

To perform this change of summation limits, we first identify the region bounded between the lines $\tau = t$ and $\tau = \nu - 1$ (limits of the second summation) as t goes from $\nu - 1$ to ∞ (limits of the first summation) in the τ-t plane. This region can also be described as bounded between the lines $t = \tau$ and $\tau = \nu - 1$ as t and τ go to ∞ in the same plane.

Let $t - \tau + \nu - 1 = u$, then:

$$\begin{aligned} R_{\nu-1}((f *_{\nu-1} g)(t)) &= \sum_{\tau=\nu-1}^{\infty} \sum_{u=\nu-1}^{\infty} \left[\left(\frac{1}{s+1}\right)^{u+1+\tau+1-\nu} f(u) g(\tau) \right] \\ &= \frac{1}{(s+1)^{-\nu}} \sum_{u=\nu-1}^{\infty} \left[\left(\frac{1}{s+1}\right)^{u+1} f(u) \right] \sum_{\tau=\nu-1}^{\infty} \left[\left(\frac{1}{s+1}\right)^{\tau+1} g(\tau) \right] \\ &= \frac{1}{(s+1)^{-\nu}} R_{\nu-1}((f(t)) R_{\nu-1}((g(t)). \end{aligned}$$

□

2.12 Rank of a Matrix

Definition 2.12.1. *The rank of a matrix is the maximum number of linearly independent rows (or columns) in the matrix.*

We can find the rank of a matrix by finding its row echelon form. The maximum number of rows that are not of all zeros is the rank of the matrix.

For an $n \times n$ square matrix A, $rankA = n \Leftrightarrow det(A) \neq 0$ (or A is non-singular).

For an $m \times n$ matrix, its rank cannot exceed its smaller dimension.

Chapter 3

THE DISCRETE LINEAR STATE-SPACE MODEL

3.1 Variation of Constants Formula

We first verify the solution of the system represented

$$\Delta y(t) = Ay(t) + f(t), \tag{3.1}$$

where $y(t)$ is an $n \times 1$ matrix, $y(0) = y_0$ (initial state), A is an $n \times n$ matrix, and $f(t)$ can also be written as $Bu(t)$ where B is an $n \times m$ matrix and $u(t)$ (control term) is an $m \times 1$ matrix. These dimensions and notation meanings will be the same from now on, unless stated otherwise. This solution will play a key role in establishing our criteria for both controllability and observability.

Theorem 3.1.1 (Variation of Constants)**.** *The solution of the discrete linear state-space model given by*

$$\Delta y(t) = Ay(t) + f(t),$$

where $t \in \mathbb{N}_0$, A is an $n \times n$ matrix, $y(t)$ and $f(t)$ are $n \times 1$ matrices, and $y(0) = y_0$, is

$$y(t) = (I + A)^t y_0 + \sum_{s=0}^{t-1} (I + A)^{t-s-1} f(s), \tag{3.2}$$

where I is the identity matrix I_n.

Proof. Using the definition of the difference operator with the solution, we have

$$\Delta y(t) = y(t+1) - y(t)$$

$$
\begin{aligned}
&= (I+A)^{t+1}y_0 + \sum_{s=0}^{t}(I+A)^{t-s}f(s) - (I+A)^t y_0 - \sum_{s=0}^{t-1}(I+A)^{t-s-1}f(s) \\
&= A(I+A)^t y_0 + \Big(\sum_{s=0}^{t-1}(I+A)^{t-s}f(s) + f(t)\Big) - \sum_{s=0}^{t-1}(I+A)^{t-s-1}f(s) \\
&= A(I+A)^t y_0 + \sum_{s=0}^{t-1} A(I+A)^{t-s-1}f(s) + f(t) \\
&= A\Big[(I+A)^t y_0 + \sum_{s=0}^{t-1}(I+A)^{t-s-1}f(s)\Big] + f(t) \\
&= Ay(t) + f(t).
\end{aligned}
$$

Plugging $t = 0$ into (3.2), we get $y(0) = y_0$. The proof is complete. □

3.2 Controllability

We consider the system mentioned in the previous section, rewriting $f(t)$ as $Bu(t)$

$$\Delta y(t) = Ay(t) + Bu(t). \tag{3.3}$$

Now that we have the formula of the solution of this system, let us look into the controllability of it. We recall that controllability refers to the ability to take the system from some initial state y_0 to some final state y_f.

Definition 3.2.1. *The system (3.3) is* ***completely controllable*** *if any desired final state can be achieved from any given initial state in finite time.*

In other words, given any initial state $y_0 = y(0)$ and any final state y_f, there exists a non-negative integer $T < \infty$, and a control function $u(t)$, $0 \leq t \leq T$, such that $y(T) = y_f$.

In order to establish our theorem for the controllability of this system, we will define the *controllability matrix* $\mathbf{W}$ as follows:

$$\mathbf{W} = [B \quad (I+A)B \quad (I+A)^2B \quad \dots \quad (I+A)^{n-1}B]. \tag{3.4}$$

Before discussing the controllability theorem, let us take a look at the following lemma.

Lemma 3.2.2. *Given* $\mathbf{W} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$*, for any* $N \geq n$*, we have* $rank\mathbf{W}(N) = rank\mathbf{W}$*, where* $\mathbf{W}(N) = [B \quad AB \quad A^2B \quad \dots \quad A^{N-1}B]$.

Proof. This proof is adapted from [4]. Let $P(\lambda) = \lambda^n + P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots + P_n$ be the characteristic polynomial of matrix A. By applying Cayley-Hamilton Theorem, which states that every square matrix satisfies its own characteristic equation, we have that $P(A) = 0$, or

$$A^n + P_1A^{n-1} + P_2A^{n-2} + \cdots + P_nI = 0$$

$$A^n = -\sum_{i=1}^{n} P_iA^{n-i}$$

$$A^nB = -\sum_{i=1}^{n} P_iA^{n-i}B. \qquad (1)$$

This means that the columns of A^nB are linearly dependent on the columns of $\mathbf{W}$. Thus $rank\mathbf{W}(n+1) = rank\mathbf{W}$.

Now if we multiply (1) by A, we get

$$A^{n+1}B = -(P_1A^nB + P_2A^{n-1}B + \cdots + P_nAB).$$

This means $rank\mathbf{W}(n+2) = rank\mathbf{W}(n+1) = rank\mathbf{W}$. Repeating this process iteratively, we conclude that $rank\mathbf{W}(N) = rank\mathbf{W}$ for any $N \geq n$. □

Now let us prove the controllability theorem.

Theorem 3.2.3. *The system* $\Delta y(t) = Ay(t) + Bu(t)$ *is completely controllable* $\Leftrightarrow$ $rank\mathbf{W} = n$

Again, n is the dimension of the square matrix A.

Proof. First, suppose $rank\mathbf{W} = n$, $t_0 = 0$. Plugging n into the solution of the system, given by the variation of constants formula (3.2), we have

$$y(n) = (I + A)^n y_0 + \sum_{s=0}^{n-1} (I + A)^{n-s-1} Bu(s)$$

$$y(n) - (I + A)^n y_0 = \mathbf{W}u(n)$$

where $u(n) = \begin{bmatrix} u(n-1) \\ u(n-2) \\ \dots \\ u(0) \end{bmatrix}$.

We have $rank\mathbf{W} = n$, so $range\mathbf{W} = \mathbb{R}^n$, and $y(n) - (I + A)^n y_0 \in range\mathbf{W} = \mathbb{R}^n$.

Thus for any given initial state y_0 and final state y_f, if we let $y_f = f(n)$, then there exists a control $u(n)$ such that

$$y_f = y_n = (I + A)^n y_0 + \mathbf{W}u(n)$$

and the system is completely controllable by definition.

Now suppose the system is completely controllable, and $rank\mathbf{W} < n$ (as $\mathbf{W} \in \mathbb{R}^{n \times nm}$, its rank cannot exceed n).

For any $N > n$, by Lemma 3.2.2 we have that $rank\mathbf{W}(N) = rank\mathbf{W} < n$. This means $range\mathbf{W}(N) \subset \mathbb{R}^n$, and thus there exists some final state $y_f \in \mathbb{R}^n$ that cannot be reached regardless of initial state y_0 and control $u(n)$.

This contradicts the initial assumption that the system is completely controllable, therefore $rank\mathbf{W}$ must be n. $\square$

Example 3.2.4. *A linear system is represented by*

$$\Delta y(t) = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} y(t) + \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} u(t).$$

We want to know whether this system is completely controllable or not. The controllablity matrix of this system, by definition, is

$$\mathbf{W} = [B \quad (I+A)B \quad (I+A)^2B] = \begin{bmatrix} 2 & 1 & 4 & 4 & 10 & -1 \\ 0 & -1 & -2 & -3 & -8 & -10 \\ 1 & 2 & 4 & -1 & 2 & -1 \end{bmatrix}.$$

It is easy to check that the first three columns are linearly independent, so the row rank of $\mathbf{W}$ *is 3, which means rank*$\mathbf{W} = 3$*, and the system is completely controllable.*

3.3 Observability

Definition 3.3.1. *The system*

$$\begin{cases} \Delta y(t) = Ay(t) + Bu(t), \\ z(t) = Cy(t), \end{cases} \tag{3.5}$$

where $z(t) = Cy(t)$ *is the output measurement,* $z(t) \in \mathbb{R}^r$*, and* $C \in \mathbb{R}^{r\times n}$ *is a constant matrix, is* ***completely observable*** *if any initial state* y_0 *can be uniquely determined from the output* $z(t)$ *and input u(t).*

Here we omit the term $Du(t)$ as it has no effect on the validity of the criterion or how it is proved.

We define the *observability matrix* of the system as

$$\mathbf{O} = \begin{bmatrix} C \\ C(I+A) \\ C(I+A)^2 \\ \cdots \\ C(I+A)^{n-1} \end{bmatrix}.$$

Theorem 3.3.2. *The system*

$$\begin{cases} \Delta y(t) = Ay(t) + Bu(t), \\ z(t) = Cy(t), \end{cases}$$

is completely observable $\Leftrightarrow$ $rank\mathbf{O} = n$

Proof. First, using the variation of constants formula (3.2), we have

$$z(t) = Cy(t) = C\Big[(I+A)^t y_0 + \sum_{s=0}^{t-1}(I+A)^{t-s-1}Bu(s)\Big]$$
$$C(I+A)^t y_0 = z(t) - C\sum_{s=0}^{t-1}(I+A)^{t-s-1}Bu(s).$$

Letting $z(t) - C\sum_{s=0}^{t-1}(I+A)^{t-s-1}Bu(s) = z^*(t)$ and $t = 0, 1, 2, \ldots, n-1$, we have

$$\begin{bmatrix} C \\ C(I+A) \\ C(I+A)^2 \\ \cdots \\ C(I+A)^{n-1} \end{bmatrix} y_0 = \begin{bmatrix} z^*(0) \\ z^*(1) \\ z^*(2) \\ \cdots \\ z^*(n-1) \end{bmatrix}.$$

Now suppose $rank\mathbf{O} = n$. Then, with known control $u(t)$ and output $z(t)$,

y_0 can be uniquely determined, and $y_0 \in \mathbb{R}^n$ since $range\mathbf{O} = \mathbb{R}^n$. This means the system is completely observable.

On the other hand, suppose the system is completely observable. This means we can uniquely determine y_0 from the system of equation

$$\begin{bmatrix} C \\ C(I+A) \\ C(I+A)^2 \\ \cdots \\ C(I+A)^{n-1} \end{bmatrix} y_0 = \begin{bmatrix} z^*(0) \\ z^*(1) \\ z^*(2) \\ \cdots \\ z^*(n-1) \end{bmatrix}.$$

This system represents rn equations of n unknowns $y_{01}, y_{02}, \ldots, y_{0n}$. Since the solution set is unique, there exist at least n rows in $\mathbf{O}$ that are linearly independent, and it follows that $rank\mathbf{O} = n$. $\square$

Example 3.3.3. *A linear system is represented by*

$$\Delta y(t) = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 4 \\ -2 & 1 & 1 \end{bmatrix} y(t) + Bu(t),$$

$$z(t) = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} y(t),$$

and we want to know whether this system is completely observable or not. The observability matrix of this system, by definition, is

$$\mathbf{O} = \begin{bmatrix} C \\ C(I+A) \\ C(I+A)^2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -5 & 0 & 6 \\ -22 & -4 & 17 \end{bmatrix}$$

and

$$z(t) = Cy(t) = C(I+A)^t y_0 + C\sum_{s=0}^{t-1}(I+A)^{t-s-1}Bu(s)$$

$$z(t) - C\sum_{s=0}^{t-1}(I+A)^{t-s-1}Bu(s) = C(I+A)^t y_0.$$

Let $z(t) - C\sum_{s=0}^{t-1}(I+A)^{t-s-1}Bu(s) = z^*(t)$*, which can be calculated at every point* t *if* B*,* $u(t)$*, and* $z(t)$ *are known. Let* $t = 0, 1, 2$*, then we have a first-order system of linear equations*

$$\begin{bmatrix} z^*(0) \\ z^*(1) \\ z^*(2) \end{bmatrix} = \begin{bmatrix} C \\ C(I+A) \\ C(I+A)^2 \end{bmatrix} y_0 = \begin{bmatrix} -2 & 1 & 0 \\ -5 & 0 & 6 \\ -22 & -4 & 17 \end{bmatrix} y_0 = \mathbf{O}y_0.$$

This system gives a unique solution for y_0 *since* $\det(\mathbf{O}) \neq 0$ *(which is equivalent to* $\text{rank}\mathbf{O} = 3$*), and the original system is completely observable.*

Chapter 4

THE DISCRETE FRACTIONAL MODEL

4.1 Variation of Constants Formula

4.1.1 The Scalar Equation

Before looking into the matrix equation, we first consider the scalar version of the fractional difference equation of order ν

$$\Delta^{\nu} y(t) = ay(t+\nu-1) + f(t+\nu-1), \tag{4.1}$$

where $a \in \mathbb{R}$, $\nu \in \mathbb{R}$, $0 < \nu < 1$, and $t \in \mathbb{N}_0$. We recall that $\Delta^{\nu} y(t) = \Delta(\Delta^{-(1-\nu)} y(t))$ by the Riemann-Liouville formula from Section 2.9. The solution to this equation helps us establish a similar formula for the matrix equation, on which the criteria for controllability and observability are based.

Theorem 4.1.1 (Variation of Constants)**.** *The solution of the equation*

$$\Delta^{\nu} y(t) = ay(t+\nu-1) + f(t+\nu-1),$$

where $t \in \mathbb{N}_0$, $a \in \mathbb{R}$, $\nu \in \mathbb{R}$, $0 < \nu < 1$, *and* $y(\nu - 1) = y_0$ *is*

$$y(t) = \hat{y}(t) y_0 + (\hat{y} *_{\nu-1} f)(t-1), \tag{4.2}$$

where

$$\hat{y}(t) = \sum_{i=0}^{\infty} \frac{a^i}{\Gamma((i+1)\nu)} (t + i(\nu - 1))^{((i+1)\nu - 1)}. \tag{4.3}$$

Proof. To verify this is indeed the solution of (4.1), we first apply R_0-transform to both sides of (4.1).

$$R_0(\Delta^\nu y(t)) = aR_0(y(t + \nu - 1)) + R_0(f(t + \nu - 1)).$$

Applying Lemmas 2.10.2 and 2.10.3 to the right-hand side and left-hand side respectively, we get

$$\begin{aligned}
s^\nu R_{\nu-1}(y(t)) - y(\nu - 1) &= \frac{a}{(s+1)^{1-\nu}} R_{\nu-1}(y(t)) + \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}(f(t)) \\
\left(s^\nu - \frac{a}{(s+1)^{1-\nu}}\right) R_{\nu-1}(y(t)) &= y_0 + \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}(f(t)) \\
R_{\nu-1}(y(t)) &= \frac{y_0}{s^\nu - \frac{a}{(s+1)^{1-\nu}}} + \frac{R_{\nu-1}(f(t))}{(s+1)^{1-\nu}\left(s^\nu - \frac{a}{(s+1)^{1-\nu}}\right)}.
\end{aligned}$$

Now if we apply $R_{\nu-1}$ to both sides of the variation of constants formula (4.2), we have

$$R_{\nu-1}(y(t)) = R_{\nu-1}(\hat{y}(t))y_0 + R_{\nu-1}((\hat{y} *_{\nu-1} f)(t - 1)).$$

So to complete the verification, we need to show that

$$R_{\nu-1}(\hat{y}(t)) = \frac{1}{s^\nu - \frac{a}{(s+1)^{1-\nu}}}$$

and

$$\begin{aligned}
R_{\nu-1}((\hat{y} *_{\nu-1} f)(t - 1)) &= \frac{R_{\nu-1}(f(t))}{(s+1)^{1-\nu}\left(s^\nu - \frac{a}{(s+1)^{1-\nu}}\right)} \\
&= \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}(\hat{y}(t)) R_{\nu-1}(f(t)).
\end{aligned}$$

Applying the definitions of R_{t_0} and $\hat{y}(t)$ given by (2.9) and (4.3), respectively, with $t_0 = \nu - 1$, we have

$$R_{\nu-1}(\hat{y}(t)) = \sum_{i=0}^{\infty}\left[\sum_{t=\nu-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\frac{a^i}{\Gamma((i+1)\nu)}(t+i(\nu-1))^{((i+1)\nu-1)}\right].$$

Here with the change of variable $t + i(\nu-1) = u$, or $t = u - i\nu + i$, we get

$$\begin{aligned} R_{\nu-1}(\hat{y}(t)) &= \sum_{i=0}^{\infty}\left[\sum_{u=(i+1)\nu-1-i}^{\infty}\left(\frac{1}{s+1}\right)^{u+1-i\nu+i}\frac{a^i}{\Gamma((i+1)\nu)}u^{((i+1)\nu-1)}\right] \\ &= \sum_{i=0}^{\infty}\left[\sum_{u=(i+1)\nu-1}^{\infty}\left(\frac{1}{s+1}\right)^{u+1}\frac{u^{((i+1)\nu-1)}}{\Gamma((i+1)\nu)}\frac{a^i}{(s+1)^{i(1-\nu)}}\right]. \end{aligned}$$

The assumption that $t^{(r)} = 0$ if $t + 1 - r \in \{0, -1, -2, \ldots\}$ from Section 2.4 means $u^{((i+1)\nu-1)} = 0$ if $u = ((i+1)\nu - 1 - i$ and $i \geq 1$. This allows us to change the lower limit of the inner summation to $u = (i+1)\nu - 1$.

By Lemma 2.10.1, we have

$$\sum_{u=(i+1)\nu-1}^{\infty}\left(\frac{1}{s+1}\right)^{u+1}\frac{u^{((i+1)\nu-1)}}{\Gamma((i+1)\nu)} = \frac{R_{(i+1)\nu-1}(u^{((i+1)\nu-1)})}{\Gamma((i+1)\nu)} = \frac{1}{s^{(i+1)\nu}},$$

and thus

$$\begin{aligned} R_{\nu-1}(\hat{y}(t)) &= \sum_{i=0}^{\infty}\frac{a^i}{(s+1)^{i(1-\nu)}s^{(i+1)\nu}} \\ &= \frac{1}{s^{\nu}}\sum_{i=0}^{\infty}\left(\frac{a}{(s+1)^{1-\nu}s^{\nu}}\right)^i. \end{aligned}$$

Here we will assume that $\left|\dfrac{a}{(s+1)^{1-\nu}s^{\nu}}\right| < 1$, and by applying the property

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ for $|r| < 1$ of the geometric series, we have

$$\begin{aligned} R_{\nu-1}(\hat{y}(t)) &= \frac{1}{s^\nu} \frac{1}{1 - \frac{a}{(s+1)^{1-\nu} s^\nu}} \\ &= \frac{1}{s^\nu - \frac{a}{(s+1)^{1-\nu}}}. \end{aligned}$$

Next, applying $R_{\nu-1}$ to $(\hat{y} *_{\nu-1} f)(t-1)$, we have

$$R_{\nu-1}((\hat{y} *_{\nu-1} f)(t-1)) = \sum_{t=\nu-1}^{\infty} \left[\left(\frac{1}{s+1} \right)^{t+1} (\hat{y} *_{\nu-1} f)(t-1) \right].$$

Evaluating $(\hat{y} *_{\nu-1} f)(t-1)$ at $t = \nu - 1$ using the definition of the convolution product given by (2.10), we find that it is 0. Thus we have

$$\begin{aligned} R_{\nu-1}((\hat{y} *_{\nu-1} f)(t-1)) &= \sum_{t=\nu}^{\infty} \left[\left(\frac{1}{s+1} \right)^{t+1} (\hat{y} *_{\nu-1} f)(t-1) \right] \\ &= \sum_{t=\nu-1}^{\infty} \left[\left(\frac{1}{s+1} \right)^{t+2} (\hat{y} *_{\nu-1} f)(t) \right] \\ &= \frac{1}{s+1} \sum_{t=\nu-1}^{\infty} \left[\left(\frac{1}{s+1} \right)^{t+1} (\hat{y} *_{\nu-1} f)(t) \right] \\ &= \frac{1}{s+1} R_{\nu-1}((\hat{y} *_{\nu-1} f)(t)), \end{aligned}$$

and by Lemma 2.11.1 we have

$$R_{\nu-1}((\hat{y} *_{\nu-1} f)(t-1)) = \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}((\hat{y}(t)) R_{\nu-1}((f(t)).$$

We have verified that the $R_{\nu-1}$-transform of the variation of constants formula agrees with the $R_{\nu-1}$-transform of the original equation term-by-term, and thus completed this proof. □

Note: The function $\hat{y}(t)$ is specified by the parameter a, therefore we may use the notation $\hat{y}_a(t)$ when we want to distinguish between different $\hat{y}(t)$ functions that are specified by different parameter a's. Also, in the scope of this thesis, we assume that a and ν are chosen so that $\hat{y}(t)$ converges. The condition under which $\hat{y}(t)$ converges is discussed in [2].

4.1.2 The Discrete Fractional System

Now that we have verified the solution of the scalar discrete fractional equation (4.1), we will use a similar approach to verify the solution of the matrix equation

$$\Delta^\nu y(t) = Ay(t+\nu-1) + f(t+\nu-1), \tag{4.4}$$

where $y(\nu-1) = y_0$, A is an $n \times n$ matrix, and $y(t+\nu-1)$ and $f(t+\nu-1)$ are $n \times 1$ matrices.

Theorem 4.1.2 (Variation of Constants)**.** *The solution of the system given by*

$$\Delta^\nu y(t) = Ay(t+\nu-1) + f(t+\nu-1),$$

where $t \in \mathbb{N}_0$, $\nu \in \mathbb{R}$, $0 < \nu < 1$, A is an $n \times n$ matrix, $y(t+\nu-1)$ and $f(t+\nu-1)$ are $n \times 1$ matrices, and $y(\nu-1) = y_0$, is

$$y(t) = \hat{y}_A(t)y_0 + (\hat{y}_A *_{\nu-1} f)(t-1), \tag{4.5}$$

where

$$\hat{y}_A(t) = \sum_{i=0}^{\infty} \frac{A^i}{\Gamma((i+1)\nu)} (t + i(\nu-1))^{((i+1)\nu-1)}. \tag{4.6}$$

Proof. Similar to what we did in the scalar case, we start by applying R_0 to both sides of (4.4), then apply Lemmas 2.10.2 and 2.10.3.

$$R_0(\Delta^\nu y(t)) = AR_0(y(t+\nu-1)) + R_0(f(t+\nu-1))$$

$$s^\nu R_{\nu-1}(y(t)) - y(\nu-1) = \frac{A}{(s+1)^{1-\nu}} R_{\nu-1}(y(t)) + \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}(f(t))$$

$$\left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right) R_{\nu-1}(y(t)) = y_0 + \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}(f(t))$$

$$R_{\nu-1}(y(t)) = \left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1} y_0 + \left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1} \frac{R_{\nu-1}(f(t))}{(s+1)^{1-\nu}}.$$

Then, we want to show that

$$R_{\nu-1}(\hat{y}_A(t)) = \left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1}$$

and

$$\begin{aligned} R_{\nu-1}((\hat{y}_A *_{\nu-1} f)(t-1)) &= \left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1} \frac{R_{\nu-1}(f(t))}{(s+1)^{1-\nu}} \\ &= \frac{1}{(s+1)^{1-\nu}} R_{\nu-1}(\hat{y}_A(t)) R_{\nu-1}(f(t)). \end{aligned}$$

These results can be acquired by going through essentially the same steps as in the scalar case, and the verification is complete. □

Note: As we work through $R_{\nu-1}(\hat{y}_A(t))$, we will run into $\sum_{i=0}^{\infty}\left(\frac{A}{(s+1)^{1-\nu}s^\nu}\right)^i$, which is a *geometric series of a matrix*. It can be proved that the series $\sum_{i=0}^{\infty} A^i$ of a matrix A converges if and only if the *spectral radius* of A is less than 1, through working with the *Jordan canonical form* of A [9], and it converges to $(I-A)^{-1}$. This is the same for $\hat{y}_A(t)$. In the scope of this thesis, we proceed under the assumption that $\hat{y}_A(t)$ converges.

In order to investigate the controllability and observability of the matrix system, we want to express $\hat{y}_A(t)$ in a different form than an infinite sum. First, we notice that $\frac{1}{(s+1)^{1-\nu}}\left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1}$ is in fact $R_0(\hat{y}_A(t))$ by Lemma 2.10.3. By taking the inverse R_0-transform of this expression, we can express $\hat{y}_A(t)$ in terms of several scalar functions $\hat{y}_a(t)$, where a is a number. To demonstrate this process, let us work through the following example.

Example 4.1.3. *Given* $A = \begin{bmatrix} 0.5 & 0.1 \\ -0.2 & 0.2 \end{bmatrix}$, *with eigenvalues* $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, *we want to find* $R_0^{-1}\left(\frac{1}{(s+1)^{1-\nu}}\left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1}\right)$.

We have:

$$s^\nu I - \frac{A}{(s+1)^{1-\nu}} = \frac{1}{(s+1)^{1-\nu}}\begin{bmatrix} s^\nu(s+1)^{1-\nu} - 0.5 & -0.1 \\ 0.2 & s^\nu(s+1)^{1-\nu} - 0.2 \end{bmatrix}.$$

As $M^{-1} = \frac{adj(M)}{|M|}$ *for a square matrix* M, *we have*

$$\left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1} =$$

$$= \frac{(s+1)^{1-\nu}}{(s^\nu(s+1)^{1-\nu} - 0.3)(s^\nu(s+1)^{1-\nu} - 0.4)}\begin{bmatrix} s^\nu(s+1)^{1-\nu} - 0.2 & 0.1 \\ -0.2 & s^\nu(s+1)^{1-\nu} - 0.5 \end{bmatrix}.$$

Therefore

$$\frac{1}{(s+1)^{1-\nu}}\left(s^\nu I - \frac{A}{(s+1)^{1-\nu}}\right)^{-1} =$$

$$= \frac{1}{(s^\nu(s+1)^{1-\nu} - 0.3)(s^\nu(s+1)^{1-\nu} - 0.4)}\begin{bmatrix} s^\nu(s+1)^{1-\nu} - 0.2 & 0.1 \\ -0.2 & s^\nu(s+1)^{1-\nu} - 0.5 \end{bmatrix}$$

$$= \frac{1}{s^{\nu}(s+1)^{1-\nu} - 0.3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} +$$

$$+ \frac{1}{(s^{\nu}(s+1)^{1-\nu} - 0.3)(s^{\nu}(s+1)^{1-\nu} - 0.4)} \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.1 \end{bmatrix}$$

$$= \frac{1}{s^{\nu}(s+1)^{1-\nu} - 0.3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} +$$

$$+10 \left(\frac{1}{s^{\nu}(s+1)^{1-\nu} - 0.4} - \frac{1}{s^{\nu}(s+1)^{1-\nu} - 0.3} \right) \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.1 \end{bmatrix}$$

Here we apply R_0^{-1} to each term and get the final result:

$$\hat{y}_A(t) = \hat{y}_{0.3}(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 10 \left(\hat{y}_{0.4}(t) - \hat{y}_{0.3}(t) \right) \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.1 \end{bmatrix}$$

$$= \hat{y}_{0.3}(t) \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} + \hat{y}_{0.4}(t) \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

Note: We notice that there is an analog between the final result and the result we acquire when applying the Putzer Algorithm to calculate the matrix exponential e^{At}.

4.2 Controllability

In this section we establish the criterion for controllability of the matrix discrete fractional equation

$$\Delta^{\nu} y(t) = Ay(t+\nu-1) + f(t+\nu-1),$$

where $y(\nu-1) = y_0$ and A is an $n \times n$ matrix. Also, $f(t+\nu-1)$ can be written as $Bu(t+\nu-1)$, where B is an $n \times m$ matrix and $u(t+\nu-1)$ is an $m \times 1$ matrix.

The concept of controllability for this system is similar to that of the discrete scalar system, which is the ability to transfer any initial state y_0 to any desired final state $y_f = y(T)$ where T is finite.

We define the *controllability matrix* of this system as:

$$\hat{\mathbf{W}} = [\hat{y}(\nu-1)B \quad \hat{y}(\nu)B \quad \hat{y}(\nu+1)B \quad \dots \quad \hat{y}(n+\nu-2)B].$$

The dimension of $\hat{\mathbf{W}}$ is $n \times nm$.

Theorem 4.2.1. *The system represented by $\Delta^\nu y(t) = Ay(t+\nu-1) + Bu(t+\nu-1)$ is* ***completely controllable*** $\Leftrightarrow$ *$rank\hat{\mathbf{W}} = n$.*

Proof. First, suppose $rank\hat{\mathbf{W}} = n$. Plugging $t = n+\nu-1$ into the variation of constants formula yields

$$y(n+\nu-1) = \hat{y}_A(n+\nu-1)y_0 + (\hat{y}_A *_{\nu-1} f)(n+\nu-2)$$

$$y(n+\nu-1) = \hat{y}_A(n+\nu-1)y_0 + \sum_{s=\nu-1}^{n+\nu-2} \hat{y}(n+\nu-2-s+\nu-1)Bu(s)$$

$$y(n+\nu-1) - \hat{y}_A(n+\nu-1)y_0 = \hat{\mathbf{W}}u(n),$$

where $u(n) = \begin{bmatrix} u(n+\nu-2) \\ u(n+\nu-3) \\ \dots \\ u(\nu-1) \end{bmatrix}$.

$rank\hat{\mathbf{W}} = n$ implies that $range\hat{\mathbf{W}} = \mathbb{R}^n$, and therefore $y(n+\nu-1) - \hat{y}_A(n+\nu-1)y_0 \in range\hat{\mathbf{W}}$. This means for any given initial state y_0 and final state y_f, if we let $y_f = y(n+\nu-1)$, then there exists $u(n)$ such that $y_f - \hat{y}_A(n+\nu-1)y_0 = \hat{\mathbf{W}}u(n)$, and the system is completely controllable.

On the other hand, suppose the system is completely controllable, and $rank\hat{\mathbf{W}} < n$. $rank\hat{\mathbf{W}} < n$ implies that $range\hat{\mathbf{W}} \subset \mathbb{R}^n$. This means there exists some $y_f \in \mathbb{R}^n$ that cannot be reached regardless of y_0 and $u(n)$. This is a contradiction to the initial assumption that the system is completely controllable. Therefore, $rank\hat{\mathbf{W}} = n$, and the proof is complete. □

Example 4.2.2. *Consider the system* $\Delta^\nu y(t) = Ay(t+\nu-1) + Bu(t+\nu-1)$ *where* $A = \begin{bmatrix} 0.5 & 0.1 \\ -0.2 & 0.2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The controllability matrix is

$$\hat{\mathbf{W}} = [\hat{y}_A(\nu-1)B \quad \hat{y}_A(\nu)B].$$

From the previous example, we know that the eigenvalues of A *are* $\lambda_1 = 0.3$ *and* $\lambda_2 = 0.4$, *and*

$$\hat{y}_A(t) = \hat{y}_{0.3}(t)\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} + \hat{y}_{0.4}(t)\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned}
\hat{\mathbf{W}} &= [\hat{y}_A(\nu-1)B \quad \hat{y}_A(\nu)B] \\
&= \left[\hat{y}_{0.3}(\nu-1)\begin{bmatrix} -3 \\ 6 \end{bmatrix} + \hat{y}_{0.4}(\nu-1)\begin{bmatrix} 4 \\ -4 \end{bmatrix} \quad \hat{y}_{0.3}(\nu)\begin{bmatrix} -3 \\ 6 \end{bmatrix} + \hat{y}_{0.4}(\nu)\begin{bmatrix} 4 \\ -4 \end{bmatrix}\right] \\
&= \begin{bmatrix} -3\hat{y}_{0.3}(\nu-1) + 4\hat{y}_{0.4}(\nu-1) & -3\hat{y}_{0.3}(\nu) + 4\hat{y}_{0.4}(\nu) \\ 6\hat{y}_{0.3}(\nu-1) - 4\hat{y}_{0.4}(\nu-1) & 6\hat{y}_{0.3}(\nu) - 4\hat{y}_{0.4}(\nu) \end{bmatrix},
\end{aligned}$$

and

$$det(\hat{\mathbf{W}}) = 12(\hat{y}_{0.4}(\nu-1)\hat{y}_{0.3}(\nu) - \hat{y}_{0.4}(\nu)\hat{y}_{0.3}(\nu-1)),$$

which is non-zero since $\hat{y}_{0.3}(\nu)$ *and* $\hat{y}_{0.4}(\nu)$ *are linearly independent. This implies that* $\hat{\mathbf{W}}$ *has rank* n*. Therefore the given system is completely controllable.*

4.3 Observability

Now we will discuss the observability of the system represented by

$$\begin{cases} \Delta^\nu y(t) = Ay(t+\nu-1) + Bu(t+\nu-1), \\ z(t) = Cy(t), \end{cases} \tag{4.7}$$

where $z(t) = Cy(t)$ is the output measurement, $z(t)$ is an $r \times 1$ matrix, and C is an $r \times n$ matrix.

We define the *observability matrix* of this system as:

$$\hat{\mathbf{O}} = \begin{bmatrix} C\hat{y}(\nu-1) \\ C\hat{y}(\nu) \\ C\hat{y}(\nu+1) \\ \cdots \\ C\hat{y}(n+\nu-2) \end{bmatrix}.$$

Theorem 4.3.1. *The system represented by*

$$\begin{cases} \Delta^\nu y(t) = Ay(t+\nu-1) + Bu(t+\nu-1), \\ z(t) = Cy(t), \end{cases}$$

is ***completely observable*** $\Leftrightarrow rank\hat{\mathbf{O}} = n$

Proof. We start by plugging the variation of constants formula into the equation of the output:

$$z(t) = Cy(t) = C\left[\hat{y}_A(t)y_0 + \sum_{s=\nu-1}^{t-1} \hat{y}_A(t-1-s+\nu-1)Bu(s)\right]$$

$$\begin{aligned} C\hat{y}_A(t)y_0 &= z(t) - C\sum_{s=\nu-1}^{t-1} \hat{y}_A(t-1-s+\nu-1)Bu(s) \\ &= z_1(t). \end{aligned}$$

Let $t = \nu-1, \nu, \ldots, n+\nu-2$, we have:

$$\begin{bmatrix} C\hat{y}(\nu-1) \\ C\hat{y}(\nu) \\ \cdots \\ C\hat{y}(n+\nu-2) \end{bmatrix} y_0 = \begin{bmatrix} z_1(\nu-1) \\ z_1(\nu) \\ \cdots \\ z_1(n+\nu-2) \end{bmatrix}.$$

Now suppose $rank\hat{\mathbf{O}} = n$. Then given $u(t)$ and a known $z(t)$, y_0 can be determined uniquely and $y_0 \in \mathbb{R}^n$. This means the system is completely observable.

On the other hand, suppose the system is completely observable. Then y_0 can be uniquely determined from the system of rn equations from above. This system yields a unique solution for y_0, which means there exists at least n rows in $\hat{\mathbf{O}}$ that are linearly independent. This in turn means $rank\hat{\mathbf{O}} = n$, and the proof is complete. □

Chapter 5

CONCLUSION AND FUTURE WORK

Control theory has gained more and more attention as the world has become more and more automated. Given a system, we want to know how well-behaved it is and we want to have as much control over it as possible. In this regard, two important properties of a system are controllability and observability.

We first provided key concepts and properties in the subject of discrete fractional calculus. In the third chapter we established the criteria for controllability and observability of the discrete system represented by

$$\begin{cases} \Delta y(t) = Ay(t) + Bu(t), \\ z(t) = Cy(t). \end{cases}$$

In the fourth chapter, we first discussed the discrete fractional scalar equation

$$\Delta^\nu y(t) = ay(t+\nu-1) + f(t+\nu-1)$$

and formulated its solution. With this formula, as well as the criteria for the discrete system, we proceeded to establish the criteria for the discrete fractional system represented by

$$\begin{cases} \Delta^\nu y(t) = Ay(t+\nu-1) + Bu(t+\nu-1), \\ z(t) = Cy(t), \end{cases}$$

where $0 < \nu < 1$. These criteria turned out to be much similar to those in the discrete case. A notable result is that the solution for the state of the system, on which the criteria are based, become more involved in the fractional case. In the solution of the fractional system, we made use of a special function given by

$$\hat{y}_A(t) = \sum_{i=0}^{\infty} \frac{A^i}{\Gamma((i+1)\nu)}(t + i(\nu - 1))^{((i+1)\nu - 1)}.$$

This is the matrix counterpart of the so-called discrete Mittag-Leffler function [2].

Here, a promising idea is to follow up with investigation of the discrete time-variant system, which is given by

$$\begin{cases} \Delta y(t) = A(t)y(t) + B(t)u(t), \\ z(t) = C(t)y(t) + D(t)u(t), \end{cases}$$

where the coefficient matrices are time-dependent. It would be interesting to see how different the solution of the state equation becomes and what special function it relies on.

These results would then be our basis to move on to the discrete fractional time-variant system

$$\begin{cases} \Delta^{\nu} y(t) = A(t)y(t + \nu - 1) + B(t)u(t + \nu - 1), \\ z(t) = C(t)y(t) + D(t)u(t), \end{cases}$$

where t is a positive integer.

Bibliography

[1] F. M. Atici and P. W. Eloe, A transform method in discrete fractional calculus, *International Journal of Difference Equation*, **2**, 2(2007), 165-176.

[2] F. M. Atici and P. W. Eloe, Linear forward fractional difference equations, *Communications in Applied Analysis*, 19(2015), 31-42.

[3] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser Boston Inc. Boston, MA, 2001.

[4] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, NY, 2005.

[5] R. E. Kálmán, On the general theory of control system, *IRE Transactions on Automatic Control*, **4**, 3(1959), 110.

[6] W. G. Kelly and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, San Diego, CA, 2001.

[7] J. A. T. Machado, M. F. Silva, R. S. Barbosa, I. S. Jesus, C. M. Reis, M. G. Marcos, and A. F. Galhano, Some applications of fractional calculus in engineering, *Mathematical Problems in Engineering*, **2010**, 1-34.

[8] K. S. Miller and B. Ross, Fractional difference calculus, in *Univalent Functions, Fractional Calculus, and Their Applications (Kōriyama, 1988)*, Ellis Horwood Ser. Math. Appl. pages 139-152, Chichester, 1989.

[9] C. Watson, The geometric series of matrix, http://www.math.uvic.ca/ dcwatson/work/geometric.pdf, retrieved April 10, 2018.

[10] S. Wheatcraft and M. M. Meerschaert, Fractional conservation of mass, *Advances in Water Resources*, **31**, 10(2008), 1377-1381.