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An Alternate Solution to the Two-Dimensional Ising Model

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AN ALTERNATE SOLUTION
TO THE
TWO-DIMENSIONAL ISING MODEL

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

AN ALTERNATE SOLUTION TO THE TWO-DIMENSIONAL ISING MODEL

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An alternate solution for the two-dimensional Ising model in a zero external magnetic field is presented. Following Kaufman, the partition function is written as the trace of a certain matrix product. The trace can be evaluated by computing the determinant of a related matrix. The determinant is evaluated by finding the product of its eigenvalues; in the thermodynamic limit the result is equivalent to that found by Onsager.

I. INTRODUCTION

A fundamental question in the axiomatic foundation of statistical mechanics is: Will statistical mechanics predict a phase transition for a physically reasonable situation? Mathematically, a phase transition is said to exist if either the logarithm of the partition function or one of its derivatives becomes discontinuous. Therefore, only an analytical expression for the partition function allows one to determine if a phase transition exists in any given situation.

Under the present formulation of statistical mechanics the so-called Ising model has received the most attention as a physically reasonable model which undergoes a phase transition.[1] Ernst Ising, in his dissertation of 1925, succeeded in calculating the partition function for a one-dimensional ferromagnetic system. Associated with each point on a linear chain one may assign a "spin" variable restricted to the scalar values "up" or "down," each spin interacting only with its nearest neighbor. Unfortunately, the model does not undergo a phase transition; it is now known that a phase transition will not exist for any one-dimensional situation for which the range of interaction is finite.[2] The two-dimensional model, although intended to be a simplified model of a ferromagnetic system, is perhaps more properly equivalent to a model of a binary alloy; in any case it is the two-dimensional model which exhibits a phase transition.

There are, at present, two distinct methods for calculating the partition function. Historically, an algebraic approach was the first to yield a solution for the model. In 1941 Kramers and Wannier showed that the partition function could be represented as the largest eigenvalue of a certain matrix; furthermore, by symmetry arguments, they were able to locate the transition temperature.[3] These results were not rigorous; it was Lars Onsager, who, in 1944, rigorously calculated the partition function in what is now considered one of the most important papers in theoretical physics.[4] Onsager's solution is principally based on generating an "operator element" Lie algebra which is then reduced in dimensionality by a representation in terms of two-dimensional operators from which the partition function may be calculated. Although Onsager's result is exact, it is extremely lengthy and complicated. Various means of simplifying the solution have been obtained, notably by Kaufman,[5] who also used an algebraic approach, and Thompson,[6] whose solution employs raising and lowering operators, much like those used in quantum mechanics.

A unique alternative to the algebraic approach employs combinatorial mathematics. This formulation is based on a result due to van der Waerden[7] who had shown that the problem is equivalent to counting closed graphs having a certain number of bonds. Kac and Ward,[8] in a heuristic manner, showed that this method yielded the partition function as found by Onsager. Hurst and Green,[9] using Pffafians, were then able to show the equivalence of the algebraic and the combinatorial solutions.

A large number of papers have been presented dealing with various aspects of the Ising model;[10] one may therefore be curious as to their motivation. It is to be pointed out that at the present time an exact solution does not exist for either the two-dimensional model in a magnetic field, or for any aspect of the three-dimensional model. This paper shall present yet another algebraic solution for the two-dimensional model.

II. THE SOLUTION

The two-dimensional model may be considered as being built on a rectangular lattice having m rows, numbered from one to m, with n columns, labled from one to n. A "spin" variable $\mu_k^{(j)}$, restricted to the values +1 and -1 only, is associated with the k-th point in the j-th row. Spins in adjacent rows or columns interact with energies

$$-J'\mu_k^{(j)}\mu_{k+1}^{(j)}, \quad -K'\mu_k^{(j)}\mu_k^{(j+1)}$$

respectively, J' and K' being coupling constants. For symmetry purposes, cyclic boundary conditions are imposed on the model so that rows one and m and columns one and n are "adjacent;" notationally,

$$\mu_k^{(m+1)} = \mu_k^{(1)}, \quad \mu_{n+1}^{(j)} = \mu_1^{(j)}.$$

The total interaction energy E is:

$$E = -\sum_{j=1}^{m}\sum_{k=1}^{n}\left[J'\mu_k^{(j)}\mu_{k+1}^{(j)} + K'\mu_k^{(j)}\mu_k^{(j+1)}\right]. \tag{1}$$

and the partition function Z for the model is:

$$Z = \sum \exp(-\beta E) \tag{2}$$

where β is the reciprocal of the Boltzmann constant times the temperature and the summation is taken over all $\mu_k^{(j)} = \pm 1$.

It will be convenient to write the partition function in terms of certain 2^n-dimensional matrices. Since a spin value may be either +1 or -1, it is possible to label components of vectors and matrices in 2^n dimensions with an ordered n-tuple of +1's and -1's. Thus, for example, with n = 2, the components $v(\mu)$ of a vector v are:

$$v(+1,+1) \qquad v(+1,-1) \qquad v(-1,+1) \qquad v(-1,-1)$$

It is to be understood that a μ without a subscript refers to a set of spins:

$$\mu = \{ \mu_1, \mu_2, \mu_3, \ldots, \mu_n \} .$$

Let $\delta(0) = 1$, $\delta(2) = \delta(-2) = 0$, and for $1 \leq k \leq n$, define the matrices S_k and C_k by

$$S_k(\mu, \mu') = \mu_k \prod_\alpha \delta(\mu_\alpha - \mu'_\alpha)$$
$$C_k(\mu, \mu') = \delta(\mu_k + \mu'_k) \prod_{\alpha \neq k} \delta(\mu_\alpha - \mu'_\alpha)$$

with the product over α extending from one to n. The algebra of these matrices is:

$$S_j^2 = C_j^2 = I \qquad\qquad C_j C_k = C_k C_j$$
$$S_j C_j = - C_j S_j \qquad\qquad S_j C_k = - C_k S_j \quad (j \neq k)$$
$$S_j S_k = S_k S_j$$

The partition function may be written in terms of these matrices. Noting that the product of two spins can assume the values +1 and -1 only one has

$$\begin{aligned}\left[\exp\{J S_k S_{k+1}\}\right](\mu, \mu') &= \left[I \cosh J + S_k S_{k+1} \sinh J\right](\mu, \mu') \\ &= \left[\cosh J + \mu_k \mu_{k+1} \sinh J\right] \prod_\alpha \delta(\mu_\alpha - \mu'_\alpha) \\ &= \exp\{J \mu_k \mu_{k+1}\} \prod_\alpha \delta(\mu_\alpha - \mu'_\alpha)\end{aligned}$$

obtained by expanding the exponential and using the commutation properties of the S's. Thus, one may define the matrix M_1,

$$M_1 = \exp\left\{ J \sum_{k=1}^{n} S_k S_{k+1} \right\} \tag{3}$$

having elements,

$$M_1(\mu, \mu') = \exp\left\{ J \sum_{k=1}^{n} S_k S_{k+1} \right\} \prod_{\alpha} \delta(\mu_\alpha - \mu'_\alpha) .$$

Notice that all elements of this matrix are non-negative.

Similarly, upon making the definition

$$\tanh K^* = \exp(-2K)$$

or equivalently,

$$(2 \sinh 2K)^{1/2} \sinh K^* = e^{-K}$$

$$(2 \sinh 2K)^{1/2} \cosh K^* = e^{K}$$

one finds

$$(2 \sinh 2K)^{1/2} \left[\exp\{K^* C_k\}\right](\mu, \mu') = \left[I e^{K} - C_k e^{-K} \right](\mu, \mu')$$

$$= \left[e^{K} \delta(\mu_k - \mu'_k) + e^{-K} \delta(\mu_k + \mu'_k) \right] \prod_{\alpha \neq k} \delta(\mu_\alpha - \mu'_\alpha)$$

$$= \exp\left\{ K \mu_k \mu_{k+1} \right\} \prod_{\alpha \neq k} \delta(\mu_\alpha - \mu'_\alpha)$$

It is now possible to define the matrix

$$M_2 = \exp\left\{ K^* \sum_{k=1}^{n} C_k \right\} \tag{4}$$

with (non-negative) elements

$$M_2(\mu, \mu') = (2 \sinh 2K)^{-n/2} \exp\left\{ K \sum_{k=1}^{n} \mu_k \mu'_k \right\}$$

Also,

$$[M_1M_2](\mu,\mu') = (2\sinh 2K)^{-n/2}\exp\left\{\sum_{k=1}^{n}(J\mu_k\mu_{k+1}+K\mu_k\mu'_k)\right\}.$$

Finally, replacing J and K by $\beta J'$ and $\beta K'$, respectively, one may calculate:

$$[M_1M_2]^m(\mu^{(1)},\mu^{(1)}) = \sum_{\mu^{(2)}}\cdots\sum_{\mu^{(m)}}\prod_{k=1}^{m}[M_1M_2](\mu^{(\alpha)},\mu^{(\alpha+1)})$$

$$=(2\sinh 2K)^{-mn/2}\sum_{\mu^{(2)}}\cdots\sum_{\mu^{(m)}}\exp(-\beta E),$$

where $\sum_{\mu}$ indicates the sum over the 2^n n-tuples of +1's and -1's. The partition function is then

$$Z = (2\sinh 2K)^{mn/2}\,\mathrm{Tr}[M_1M_2]^m \qquad (5)$$

The rather mixed commutation properties of the S's and C's will prove inconvenient. It will therefore be advantageous to introduce the Jordan-Wigner transformation, which, for $1\le k\le 2n$ is:

$$\Gamma_1 = S_1 \qquad \Gamma_2 = -i\,C_1S_1$$

$$\Gamma_3 = C_1S_2 \qquad \Gamma_4 = -i\,C_1C_2S_2$$

$$\Gamma_5 = C_1C_2S_3 \qquad \Gamma_6 = -i\,C_1C_2C_3S_3$$

$$\cdots \qquad \cdots$$

$$\Gamma_{2k-1} = C_1C_2\cdots C_{k-1}S_k \qquad \Gamma_{2k} = -i\,C_1C_2\cdots C_kS_k$$

The factor i has been included to maintain hermiticity. The algebra of these matrices is:

$$\Gamma_j^2 = I$$

$$\Gamma_j\Gamma_k = -\Gamma_k\Gamma_j \qquad (j\ne k)$$

$$\Gamma_j\Gamma_k+\Gamma_k\Gamma_j = 2\delta_{jk}$$

It will prove useful to have a knowledge of the trace, not only for an arbitrary Γ, but also for a product of the Γ's. In the former, by using the relations above and noting that the trace of a matrix product is unchanged by a cyclic permutation of its factors, one has:

$$Tr(\Gamma_j) = 0$$

In a similar fashion, by using the anti-commutativity relation, for $j \neq k$, one has:

$$Tr(\Gamma_j \Gamma_k) = 0$$

It is also possible to express the C's and S's in terms of the Γ-matrices. For example, by forming the product $\Gamma_2 \Gamma_3$ one finds that $S_1 S_2 = -i\Gamma_2\Gamma_3$. In general,

$$S_k S_{k+1} = -i \; \Gamma_{2k} \Gamma_{2k+1}$$

and in a like fashion,

$$C_k = -i \, \Gamma_{2k-1} \Gamma_{2k}$$

The above results enable one to write the matrices M_1 and M_2 in terms of the Γ-matrices. First, by (4) it is immediate that

$$M_2 = (2 \sinh 2K)^{-n/2} \exp\left\{-iK \sum_{k=1}^{n} \Gamma_{2k-1} \Gamma_{2k}\right\} .$$

There is, however, a difficulty in transforming the matrix M_1 due to the boundary conditions, namely, the term $S_n S_1$. From the definition of Γ_{2j} one has

$$S_n S_1 = i \, U \, \Gamma_{2j} \Gamma_1$$

where the operator U has been defined by:

$$U = \prod_{k=1}^{n} C_k \tag{6}$$

Thus, M_1 may be written

$$M_1 = \exp\left\{-iJ\sum_{j=1}^{n-1} \Gamma_{2j}\Gamma_{2j-1} + i\,UJ\,\Gamma_{2j}\Gamma_1\right\}.$$

Upon introducing projection operators Λ_+ and Λ_- defined by:

$$\Lambda_+ = \tfrac{1}{2}(I + U), \qquad \Lambda_- = \tfrac{1}{2}(I - U) \tag{7}$$

and noting that since $U^2 = I$, $\Lambda_+^2 = \Lambda_+$, $\Lambda_-^2 = \Lambda_-$ while $\Lambda_+ + \Lambda_- = I$,

one has

$$\begin{aligned}\exp(iJU\Gamma_{2j}\Gamma_1) &= \cosh J + iUJ\Gamma_{2j}\Gamma_1 \sinh J \\ &= (\Lambda_+ + \Lambda_-)\cosh J + i(\Lambda_+ - \Lambda_-)\Gamma_{2j}\Gamma_1 \sinh J\end{aligned}$$

and so

$$\exp(iJU\Gamma_{2j}\Gamma_1) = \Lambda_+ \exp(iJ\Gamma_{2j}\Gamma_1) + \Lambda_- \exp(-iJ\Gamma_{2j}\Gamma_1).$$

Using this result in M_1 one arrives at the conclusion

$$M_1 = \Lambda_+ M_1^+ + \Lambda_- M_1^-$$

with the matrices M_1^+ and M_1^- being defined as:

$$M_1^+ = \exp\left\{-iJ\sum_{j=1}^{n-1}\Gamma_{2j}\Gamma_{2j+1} + iJ\Gamma_{2j}\Gamma_1\right\},$$

$$M_1^- = \exp\left\{-iJ\sum_{j=1}\Gamma_{2j}\Gamma_{2j+1}\right\}.$$

The partition function (5) may now be written:

$$Z = (2\sinh 2K)^{-mn/2}\, \mathrm{Tr}\left[\Lambda_+ M_1^+ M_2 + \Lambda_- M_1^- M_2\right]^m$$

or, since $\Lambda_+^m = \Lambda_+$, and $\Lambda_-^m = \Lambda_-$, while $\Lambda_+\Lambda_- = 0$,

$$Z = (2\sinh 2K)^{-mn/2}\,\mathrm{Tr}\left[\Lambda_+(M_1^+M_2)^m + \Lambda_-(M_1^-M_2)^m\right] \quad (8)$$

Thus, in place of the term $\mathrm{Tr}(M_1M_2)^m$, one must now evaluate two separate traces in order to compute the partition function.

To this point, all of the calculations are exact. However, the traces involved are very difficult to evaluate. One can construct a lengthy argument, whose details contribute little to the present discussion, which shows that in the thermodynamic limit one obtains the correct result by evaluating $\mathrm{Tr}(M_1^+M_2)^m$ only. The argument is based on known properties of matrices with non-negative elements. Therefore, in place of considering the partition function (8), one now considers:

$$Z = (2\sinh 2K)^{-mn/2}\,\mathrm{Tr}(M_1^+M_2)^m$$

and it will be this expression which will be discussed in the remainder of this paper.

Since $\exp(i\theta\Gamma_j\Gamma_k) = \Gamma_j(\Gamma_j\cosh\theta + i\Gamma_k\sinh\theta)$, matrices M_1^+ and M_2 may be written in the form:

$$M_1^+ = (\cosh J)^n \prod_{j=1}^{n} \Gamma_{2j}(\Gamma_{2j} + y\epsilon_j\Gamma_{2j+1}).$$

$$M_2 = (\cosh K^*)^n \prod_{j=1}^{n} \Gamma_{2j-1}(\Gamma_{2j-1} + x\Gamma_{2j}).$$

where $x = -i\tanh K^*$, $y = -i\tanh J$, $\epsilon_n = -1$, and $\epsilon_j = 1$ otherwise. Therefore,

$$Z = \left[\cosh J\cosh K^*(2\sinh 2K)^{-1/2}\right]^{mn}\,\mathrm{Tr}\left\{\prod_{j=1}^{4mn} A_j\right\}$$

where, for $1 \leq r \leq n$ and $0 \leq s \leq m-1$,

$$A_{2r-1+4ns} = \Gamma_{2r}$$

$$A_{2r+4ns} = \Gamma_{2r} + y\epsilon_r \Gamma_{2r+1}$$

$$A_{2r-1+2n+4ns} = \Gamma_{2r-1}$$

$$A_{2r+2n+4ns} = \Gamma_{2r-1} + x\Gamma_{2r} .$$

It is known that the trace of a product of factors linear in the Γ's can be expressed in terms of the determinant of an anti-symmetric matrix.[11,12] In the present case, one has

$$\mathrm{Tr}\left\{\prod_{j=1}^{4mn} A_j\right\} = 2^{2n}(\det D)^{1/2}$$

Here D, which is 4mn by 4mn, can be partitioned into 4n by 4n blocks. For $1 \leq r \leq m$ and $r < s$, the blocks are

$$D_{rr} = E, \qquad D_{rs} = -D_{sr} = F$$

while for $1 \leq j \leq 4n$ and $j < k$,

$$E_{jj} = 0 \qquad F_{jj} = 2^{-2n}\mathrm{Tr}(A_j)^2$$

$$E_{jk} = -E_{kj} = 2^{-2n}\,\mathrm{Tr}(A_j A_k),$$

$$F_{jk} = F_{kj} = E_{jk}$$

The matrices E and F are explicitly calculated for the case n = 5 and are listed as Figures 1 and 2, respectively.

Matrix D is readily transformed to block-diagonal form. With D written in block form and T the unitary matrix

$$T_{jk} = m^{-1/2} \exp\left[i\pi k\left(\frac{2j-1}{m}\right)\right], \qquad 1 \leq j,k \leq m$$

$$
\begin{matrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -y & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -y & -x & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y & -x & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -y & -x & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & -x & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{matrix}
$$

Figure 1. Matrix E for $n = 5$.

$$
\begin{array}{cccccccccccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & -y & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & 1 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & y & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v
\end{array}
$$

Figure 2. Matrix F for $n = 5$.

Here, $u = 1 + y^2$ and $v = 1 + x^2$.

one computes

$$D'_{rs} \equiv (T^{-1} D T)_{rs} = \delta_{rs}(E + a_r F),$$

where δ_{rs} is the Kronecker delta and $a_r = -i \cot \frac{2r-1}{m}\pi$.

At this point, the 4mn-dimensional problem has been decomposed into m, 4n-dimensional problems; that is

$$\det D = \prod_{r=1}^{m} \det D'_{rr}$$

Furthermore, each of these 4n-dimensional problems can be decomposed into n, 4-dimensional problems. Let V be the 4n by 4n permutation matrix which arranges the components of a vector in the order

1,2,2n+1,2n+2,3,4,2n+3, ..., 2n-1,2n,4n-1,4n.

Then, with

$$D''_{rr} \equiv V^{-1} D'_{rr} V = E'' + a_r F''$$

where $E'' = V^{-1}EV$ and $F'' = V^{-1}FV$, one finds that both E" and F" have a simple 4 by 4 block structure, namely, for $1 \leq j \leq n$

$$E''_{jj} = P \qquad E''_{j,j+1} = Q, \qquad E''_{j+1,j} = Q^T$$

$$E''_{1n} = Q^T, \qquad E''_{n1} = -Q,$$

$$F''_{jj} = R, \qquad F''_{j,j+1} = Q, \quad F''_{j+1,j} = Q^T$$

$$F''_{1n} = -Q^T, \qquad F''_{n1} = -Q$$

and all other blocks are zero. Here, explicitly,

$$P = \begin{pmatrix} 0 & 1 & 0 & x \\ -1 & 0 & 0 & x \\ 0 & 0 & 0 & 1 \\ -x & -x & -1 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 1 & 0 & x \\ 1 & 1+y^2 & 0 & x \\ 0 & 0 & 0 & 1 \\ x & x & 1 & 1+x^2 \end{pmatrix}$$

The matrices E" and F" are shown in Figures 3 and 4, respectively, for the case n = 5. Both E" and F" are brought onto block-diagonal form by the unitary transformation

$$W_{jk} = n^{-1/2} \exp\left[\pi i k \left(\frac{2j-1}{n}\right)\right], \quad 1 \le j, k \le n$$

The result is:

$$\left[W^{-1}(E'' + a_r F'')W\right]_{st} = \left[P + a_r R + (a_r+1) b_s Q + (a_r - 1) b_s^{-1} Q^T\right] \delta_{st}$$

where $b_s = \exp[i\pi \frac{2s-1}{n}]$.

The 4 by 4 determinants are readily evaluated; thus:

$$\det D = \prod_{r=1}^{n} \prod_{s=1}^{m} \left\{ (1+y^2)(1+x^2) + (a_r^2 - 1)\left[x^2 + y^2 - xy(b_s + b_s^{-1})\right]\right\}.$$

Finally, using $\prod^{n} \sin\left(\frac{2s-1}{2n}\right)\pi = 2^{-n+1}$,

$$\frac{1}{mn} \ln Z = \ln 2 + \frac{1}{2^{mn}} \sum_{r=1}^{m} \sum_{s=1}^{n} \ln\left(\cosh 2J \cosh 2K - \sinh 2J \cos\left(\frac{2s-1}{n}\pi\right) - \sinh 2K \cos\left(\frac{2r-1}{m}\pi\right)\right)$$

which in the limit $n \to \infty$ and $m \to \infty$ is equivalent to the Onsager result.

0	1	0	y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	y	0	0	x	x	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	x	0	0
-y	-y	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	x	0	0
0	0	0	0	0	1	0	y	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	y	0	0	x	x	0	0	0	0	0	0	0	0
0	-x	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	-x	0	0	-y	-y	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	y	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	y	0	0	x	x	0	0	0	0
0	0	0	0	0	-x	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	-x	0	0	-y	-y	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	y	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	y	0	0	x	x
0	0	0	0	0	0	0	0	0	-x	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	-x	0	0	-y	-y	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	y
0	0	-x	-x	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	y
0	0	0	0	0	0	0	0	0	0	0	0	0	-x	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	-x	0	0	-y	-y	-1	0

Figure 3. Matrix E reordered for n = 5.

$$
\begin{matrix}
1 & 1 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & u & 0 & x & 0 & 0 & y & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 & 0 \\
x & x & 1 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & u & 0 & x & 0 & 0 & y & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y & 0 & 0 & x & x & 1 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & 0 & x & 0 & 0 & y & y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y & 0 & 0 & x & x & 1 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & 0 & x & 0 & 0 & y & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & x & x & 1 & v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & x \\
0 & 0 & -y & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & 0 & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & x & x & 1 & v
\end{matrix}
$$

Figure 4. Matrix F reordered for $n = 5$.

Here $u = 1 + y^2$ and $v = 1 + x^2$.

FOOTNOTES

1 See for example, the review article: S. G. Brush, Rev. Mod. Phys. 39, 883 (1967).

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BIBLIOGRAPHICAL SKETCH

The writer, born December 5, 1947, was educated in the public schools of Baltimore Maryland, receiving his B. S. in Physics from Towson State College in June, 1973. Since that time he has been a graduate student at Western Kentucky University.