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An Analysis of a New Partizan Game Using Combinatorial Techniques

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AN ANALYSIS OF A NEW PARTIZAN GAME USING COMBINATORIAL TECHNIQUES

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AN ANALYSIS OF A NEW PARTIZAN GAME USING COMBINATORIAL TECHNIQUES

James T. Snodgrass III July 1979 40 pages

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John Horton Conway's combinatorial game theory was applied to a new partizan game with a complete analysis as the result. Mathematical values were assigned to the countably infinite number of positions in the game. Direct computation of the first eight values and extension via the Principle of Mathematical Induction made the assignments possible.

Examination of these values (which repeat with period 2) shows that the game, played on a strip of squares, can be won by the first player if the strip is of odd length and can be won by the second player if the strip is of even length. Further examination of the values leads to a completely general symmetry strategy for the first and second player wins in the appropriate cases.

INTRODUCTION

Professor John Horton Conway of Cambridge University has developed an elegant theory for the analysis of a certain class of combinatorial games. The intent of this paper is to apply that theory to a new combinatorial game and use the resulting analysis to find a winning strategy for the game. A brief introduction to Conway's theory is in order.

Games which lend themselves to this type of analysis are the two-player variety in which the players have complete information about the game. Familiar examples are chess, checkers, and tic-tac-toe. A game is completely determined by the options that the two players, henceforth called Left and Right, have in that particular game [2, pg. 71]. There is no element of probability in the games we consider.

There are two broad categories of combinatorial games, the impartial games and the partizan games. Impartial games are those in which both players have exactly the same options. The game of Nim is a familiar example. Partizan games are games in which the two players have different options. Chess, for example, is a partizan game. The theory for impartial games is older and is described by Conway [2, Chap. 11]. The theory for partizan games has been developed largely by Conway. It is of most interest to us at present.

Since the Left and Right options fix the fate of a particular game G, we adopt the notation: $G = \{G^L | G^R\}$, where G^L is the set of options available to Left and G^R has the corresponding definition for Right.

The first logical question one asks about any game is How is the game played? We adopt the normal play convention which prescribes that Left and Right move alternately until one of the players is unable to make a legal move on his turn. That player is then the loser [2, pg. 71]. There is a more subtle playing convention, the misere play, whereby the first player unable to make a legal move on his turn is the winner.

The crux of the theory for partizan games lies in assigning values to all possible positions that might be encountered in playing a game. It is well to remember at this point that for $G = \{G^L | G^R\}$ each of the attainable positions (any of the Left or Right options) is a game in itself. Now, these values we wish to assign to games should reflect the advantage that Left has over Right in that game or vice versa.

It is clear that given these values for each of the G^L and G^R, we will know the value of $G = \{G^L | G^R\}$. To find the value of a given G^L or G^R is now the problem. This is really no problem since knowing the value of some G^L or G^R requires only that we know the values of its Left and Right options. The problem, clearly, is that we need some starting point. The obvious starting point would be a position in which neither Left nor Right has any options. This is the game $\{ \ | \ \}$ which, since it provides neither Left nor Right with any advantage, is given the value 0 and called the Endgame [2, pg. 72].

Now, we use $0 = \{ \ | \ \}$ to inductively define three new games. This is done by using 0 as Left and Right options. The three new games are $\{0| \ \}$, $\{ \ |0\}$, and $\{0|0\}$. The convention adopted by Conway is that games which give Left an advantage will carry positive values, whereas games in which Right has the advantage will carry negative values. In fact, we have the more general definition:

Definition:

1. G is positive if there is a winning strategy for Left.
2. G is negative if there is a winning strategy for Right.
3. G is zero if there is a winning strategy for the second player.
4. G is fuzzy if there is a winning strategy for the first player.

We write: $G > 0$, $G < 0$, $G = 0$, and $G \, || \, 0$, respectively [2, pg. 73].

Since $\{0|\ \}$ gives Left an advantage of one move over Right, we define $\{0|\ \} = 1$. Since $\{\ |0\}$ gives Right the same advantage over Left and since in $\{\ |0\}$ Right has the same options that Left had in $\{0|\ \}$ and vice versa, we define $\{\ |0\} = -1$. In fact, if $G = \{G^L|G^R\}$, we define $-G = \{-G^R|-G^L\}$ [2, pg. 73]. Finally, in $\{0|0\}$ it is clear that the advantage goes to the first player. Therefore, $\{0|0\}$ is a fuzzy game which is given the special name $\{0|0\} = *$. Notice that 1 and -1 are numbers and * is not. A game will be a number only if no G^L is greater than or equal to any G^R.

Now, we could define new games by using 1, -1, and * as Left and Right options. Clearly the process can be continued ad infinitum. We return to the question of fixing a value for $G = \{G^L|G^R\}$. When analyzing a specific game, we will have begun with the 0 position and inductively "climbed the ladder" so that for G, we know values for all the G^L and G^R.

In $G = \{G^L|G^R\}$, consider the Left options, G^L. If $G^{L_1} \leq G^{L_2}$, Left will obviously prefer the move to G^{L_2} since he desires games with large values. We say that G^{L_1} is dominated by G^{L_2}. Similarly, considering the Right options, G^R, if $G^{R_1} \geq G^{R_2}$, then Right will prefer the move to G^{R_2} since he likes games with the smallest possible values. Again, we say G^{R_1} is dominated by G^{R_2} [2, pg. 110].

Once more, we focus our attention on the set, G^L, of Left options. Suppose some G^{L_1} has a right option, $G^{L_1R_1}$, for which the inequality, $G^{L_1R_1} \leq G$, holds. Since a move to $G^{L_1R_1}$ would benefit Right, Left might as well move directly to some $G^{L_1R_1L}$ as to move to G^{L_1}. We say that the move from G to G^{L_1} is reversible through $G^{L_1R_1}$ [2, pg. 110]. Likewise, suppose that G has some Right option G^{R_1} such that the inequality, $G^{R_1L_1} \geq G$, holds for some Left option, $G^{R_1L_1}$, of G^{R_1}. Then we say the move from G to G^{R_1} is reversible through $G^{R_1L_1}$ since Right might as well move from G to some $G^{R_1L_1R}$ as to move from G to G^{R_1} [2, pg. 110].

This discussion of dominated and reversible options brings us to:

<u>Theorem</u>:

The following changes do not affect the value of G:

1. Deleting any dominated option.
2. If G^{L_0} is reversible through $G^{L_0R_0}$, replacing G^{L_0} as a Left option of G by all the Left options $G^{L_0R_0L}$ of $G^{L_0R_0}$.
3. If G^{R_1} is reversible through $G^{R_1L_1}$, replacing G^{R_1} by all the $G^{R_1L_1R}$. [2, pg. 110-111, Th. 68]

Therefore, the first step in assigning a value to $G = \{G^L|G^R\}$ is to remove all dominated options and replace all reversible options as prescribed in the theorem above. Now, if this has been done and all the G^L and G^R are options such that every G^L is strictly less than every G^R, then G is a number. In fact, G is the "simplest" number that is greater than every G^L and less than every G^R, where "simplest" means created first with respect to the inductive process described above [2, pg. 23, pp. 81-82]. If G is not a number, then we merely apply the above theorem and write G in the resulting form.

The last notion to be discussed is that of the sum of games. Left and Right play in the (disjunctive) sum of two games G and H (write G + H) by choosing one of the component games and making a legal move in that component. If Left wishes to move in G + H, he may make a move to $G^L + H$ for some Left option G^L of G, or he may move to $G + H^L$ for some Left option H^L of H. Likewise, Right's moves in the sum are to $G^R + H$ or to $G + H^R$ where G^R and H^R are Right options in G and H, respectively. Hence, $G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$. The value assigned to a disjunctive sum of games is the sum of the values of the component games [1, pg. 420].

Remembering the definition of -G, we now have a way to compare two games, G and H. If G - H = G + (-H), then we say:

1. G > H if G - H is a Left win.

2. G < H if G - H is a Right win.

3. G = H if G - H is a second player win.

4. G || H (G is fuzzy against H) if G - H is a first player win.

If G || H, then we say that G and H are not comparable [2, pp. 78-79].

Let us summarize. If L and R are two sets of games, then $G = \{L|R\}$ is a game. Furthermore, all games are constructed in this manner. If $G = \{L|R\}$, then G^L is a typical element of L and is called a Left option. Likewise, G^R is a typical element of R and is called a Right option. Left may move to any G^L and Right may move to any G^R; thus, we write $G = \{G^L|G^R\}$. We define the sum of two games, G and H, to be $G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$, and the negative of a game

$G = \{G^L | G^R\}$ is $-G = \{-G^R | -G^L\}$ [2, pg. 78].

A consideration of some impartial and partizan games will serve as an initiation to the analysis of combinatorial games. In particular, seeing an application of Conway's theory for partizan games will help us in analyzing a new partizan game.

As a matter of convention, N will be used to represent the set of natural numbers.

CHAPTER ONE

The theory for impartial games is an appropriate place to begin in looking at some actual games, since it is both older and simpler than the theory for partizan games. Certainly the most renowned impartial game is Nim, a game played with a number of heaps of beans in which a legal move is to reduce the size of any one of the heaps. Under the normal play rule, the last player to pick up a bean is the winner.

A very simple, yet elegant theory for impartial games was developed independently by R.P. Sprague and P.M. Grundy. Application of the Sprague-Grundy theory to Nim is especially enlightening since the theory elucidates the game of Nim and vice versa.

We will call the value of a Nim heap of size n, $*n$. Clearly, since Left or Right may remove from 1 to n of the beans, $*n = \{*0, *1, ---, *(n - 1) | "\}$, where the $*n$ are called Nim-numbers or impartial numbers [2, pg. 122].

The Sprague-Grundy theory asserts that every impartial game with a finite number of positions has one of the values $*0, *1, *2, ---$ (that is, every impartial game is equivalent to a Nim heap of some size); $\{*a, *b, *c, --- | "\} = *m$ where m is the least number not appearing in a, b, c, --- (m is called the mex or minimal excludent of a, b, c, ---); $*2^a + *2^b + *2^c + --- = *(2^a + 2^b + 2^c + ---)$ for a, b, c, --- distinct numbers; and $*n + *n = 0$ [1, pg. 427]. The four assertions of the Sprague-Grundy theory allow us to completely analyze the game of Nim.

Since we are dealing with impartial games, we will describe situations from Left's point of view. We lose no generality in doing so.

In a game of Nim, Left wants to move to a position which has value $*0 = 0$, since this is a second player win. Notice that a game of Nim is just the disjunctive sum of its heaps [2, pg. 122]; so to find its value, we merely sum the values of the respective heaps. The third and fourth assertions above tell us how to add those values. If, for instance, we have heaps of size n_1, n_2, ---, n_r, then their values are $*n_1$, $*n_2$, ---, $*n_r$. But, we may write each n_i as $n_i = 2^a + 2^b + 2^c + ---$. When we have expressed each n_i in this form, we have $*n_i = *(2^a + 2^b + 2^c + ---) = *2^a + *2^b + *2^c + ---$, by assertion three of the Sprague-Grundy theory. We then simply add the r numbers written in this last form except that the fourth assertion says that similar powers of 2 cancel in pairs. This entire process corresponds to writing each of the r numbers in binary notation and adding them without "carrying." The resulting number, $*m$ say, is the value of the game of Nim.

Now, for Left to make a winning move on his turn, he merely calculates the Nim-sum of the heaps and then adjusts one of the summands so that the resulting Nim-sum will be 0. Left then makes the corresponding move in the heap whose value was the number he adjusted.

Let us consider a game of Nim with heaps of sizes 3, 4, 8, and 9. In binary notation, 3, 4, 8, and 9 are 11, 100, 1000, and 1001, respectively. Nim-adding the numbers gives:

$$\begin{array}{r} 11 \\ 100 \\ 1000 \\ \underline{1001} \\ 110 \end{array}$$

Therefore, our game of Nim has the value $*6$, applying assertion three in reverse. If it is Left's turn to move, he can make the Nim-sum equal to

0 only by changing 100 to 10. This corresponds to removing 2 beans from the heap of 4 to leave 2. Careful examination will show that this analysis leads to a symmetry strategy for playing Nim. If Left found the Nim-sum of a position to be 0, then his only recourse would be to so complicate the position that Right would encounter difficulty in determining the correct move [2, pg. 126].

Let us now consider another impartial game and observe its relation to Nim as predicted by the Sprague-Grundy theory. The game of Kayles is played by two skillful bowlers with a number of rows of tenpins. The players are just accurate enough to topple any single tenpin or two adjacent ones, but cannot bowl over pins separated by any greater distance.

Clearly, any Kayles position is just the disjunctive sum of its rows. If we let K_n be the value of a row of n pins, then a legal move in that row consists of moving from K_n to $K_a + K_b$ ($a + b = n - 1$ or $n - 2$ and $a,b \geq 0$). Consider some Kayles positions:

$K_0 = \{|\} = *0 = 0$

$K_1 = \{K_0|''\} = *1 = *$

$K_2 = \{K_0,K_1|''\} = \{0,*|''\} = *2$

$K_3 = \{K_1,K_2,K_1 + K_1|''\} = \{*,*2,0|''\} = *3$

$K_4 = \{K_2,K_1 + K_1,K_3,K_1 + K_2|''\} = \{*2,0,*3|''\} = *1 = *$

where the values for K_2, K_3, and K_4 are found by applying the "mex" rule. In a similar fashion, we see that $K_5 = *4$, $K_6 = *3$, $K_7 = *2$, $K_8 = *1$, etc. In general, if the value of $K_i = *n$, we say the Grundy number of K_i is n. Therefore, the Grundy numbers of the positions K_i, $i = 1,---,8$, are 1, 2, 3, 1, 4, 3, 2, and 1, respectively [2, pg. 127]. In fact, R. K. Guy made the discovery that the values K_n repeat with period 12 after n greater than or equal to 72 [2, pg. 128].

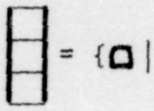
Now that we have some familiarity with Kayles, suppose Left and Right are playing the game and Left is facing the position $K_3 + K_5 + K_6 + K_8$. From our calculations above, we know that the Grundy numbers of the component rows are 3, 4, 3, and 1, respectively. Nim-adding these numbers, we see that $*3 + *4 + *3 + *1 = *5$. Clearly, Left can make a winning move by changing $*4$ to $*1$. This is readily accomplished by bowling in the row of 5 pins and toppling the pin on either end. This gives us a row with value $K_4 = *1$.

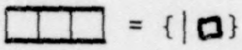
Now that we have a feel for the theory of impartial games, let us take a look at a partizan game. The game to be considered is called Domineering or Cross-Cram. Domineering is played on a checkerboard with dominoes. Left makes a legal move by placing a domino vertically to cover two adjacent squares. Right makes an analogous move in the horizontal direction. The game is over when either Left or Right cannot place a domino on the board.

We will determine how values are assigned to positions and see why some moves are better than others. Remember that the object in Domineering is to provide oneself with as many moves as possible, while at the same time depriving one's opponent of as many moves as possible.

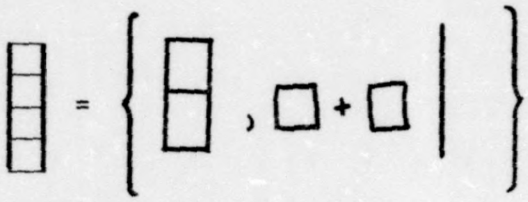
The Domineering position in which neither player has a legal move, □, has of course value 0. Suppose now that for n = 0, 1, 2, ---, Left has a move to some position of value n and Right has no moves whatever. Then, Left has an advantage of n + 1 moves and we have $\{n|\} = n + 1$ [1, pg. 420]. Similarly, if Right can move to a position of value -n for n = 0, 1, 2, ---, then $\{|-n\} = -(n + 1)$.

Consider the following positions in Domineering:

 = {0|} = 1

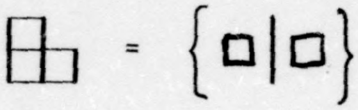
 = {| 0} = -1

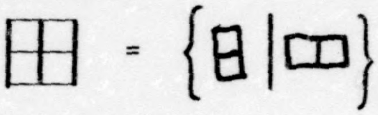
Similarly:

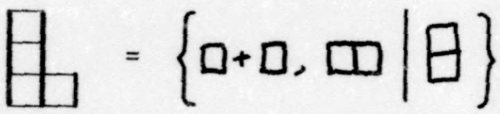
 = {1, 0 | } = {1 | } = 2

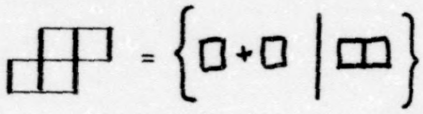
since 0 is a dominated option.

We do not have to look too far to find positions in Domineering which are not numbers. For example, consider the following:

 = {0 | 0} = *

 = {1 | -1} = ±1

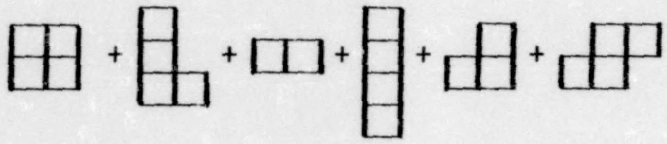
 = {0, -1 | 1} = {0 | 1} = 1/2

 = {0 | -1}

where in the second position we have replaced { 1 | -1 } with its more commonly used name. The third position derives its value of 1/2 from the simplicity theorem.

To play a game of Domineering, Left and Right always want to move in the fuzzy regions first. The difference between winning and losing also depends on choosing the correct region to move in. In regions with values {x|y} with x greater than or equal to y (i.e.-fuzzy regions), the player whose turn it is to move should move in the region with the largest value for x - y [2, pg. 424].

With that in mind, suppose Left and Right have been playing Domineering for quite a while and the checkerboard now looks like the disjunctive sum:



$= \{1|-1\} + 1/2 + (-1) + 2 + \{0|0\} + \{0|-1\}$

$= \{1|-1\} + \{0|0\} + \{0|-1\} + 3/2$

If Left starts, we have the following sequence of moves, assuming intelligent play: (L) $\{0|0\} + \{0|-1\} + 5/2$

(R) $\{0|0\} + 3/2$

(L) $3/2$

We may stop here since a positive value indicates a Left win. If Right begins, we have: (R) $\{0|0\} + \{0|-1\} + 1/2$

(L) $\{0|0\} + 1/2$

(R) $1/2$

So, even if Right starts, Left has a winning strategy. This merely says that the original sum was in Left's favor.

Now, with a bit of the theory of combinatorial games and some examples behind us, we can look at a new combinatorial game of the partizan variety and attempt an analysis.

CHAPTER TWO

We now consider a game that Left and Right play on a strip of n squares, n = 0,1,2,.... Left and Right each have a limitless supply of colored counters. Left's are either solid black or solid red and Right's are colored either red/black or black/red. A legal move in this game consists of placing one of the counters on the strip such that a single square is covered. The only restriction is that counters placed on adjacent squares must match colors on their common edge. Note that any time a counter is played in a strip, the game immediately splits into a disjunctive sum.

To analyze this game, we need to determine what type of positions may actually occur. Notice that a counter placed in a square partially determines the fate of the one or two adjacent squares. For example, if Left plays a red counter in the first square of a strip, the second square is reserved either for a red counter played by Left or a red/black counter played by Right. Likewise, if Right plays a black/red counter somewhere in the midst of the strip, the square immediately to the left is reserved for a black counter or a red/black counter, and the square immediately to the right is reserved either for a red counter or a red/black counter.

This realization suggests a very useful notational device. We will use a natural number to represent the number of squares in a strip together with a letter, R or B, to denote any "tinting." That is, R4B would indicate a strip of length four where the leftmost square is "tinted" red (reserved for a red or red/black counter) and the rightmost square is tinted black (reserved for a black or red/black counter).

Now it is possible to enumerate what positions are possible for a strip of length n. We may have: n, Rn, Bn, RnB, BnR, RnR, BnB, nR, or nB. This gives us nine positions to analyze for a strip of any length, n. However, though not a perfect symmetry, there is some degree of symmetry in this game. For instance, a red-red/black sequence appearing in a strip is the same as a black-black/red sequence appearing in the same place. We strongly suspect that there is duplication among the nine cases listed above. We formalize this in:

Lemma 1:

For a given $n \in N$, the only distinct positions in this game are n, Rn, RnB, and RnR.

Proof:

We show: 1. Rn = Bn = nB = nR
2. RnB = BnR
3. RnR = BnB

Proof is by induction on n. If n = 1, Rn = Bn = nB = nR = * since both Left and Right have a move to zero in any of those games. Also, R1B = B1R = -1 since Left has no move in either game and Right can move to zero in both. Finally, RnR = BnB = 1 since Left can move to zero in both games and Right has no move in either. This establishes the basis for the induction.

Now, suppose 1-3 hold for all $k < m$ for some $m \in N$. We show 1-3 must hold for m.

1. Consider Rm, Bm, mB, and mR.

A typical Left option of Rm is either RaR + Rb ($a + b = m - 1$, $a,b \geq 0$)

or RaB + Bb (a + b = m - 1, a $\geq$ 1, b $\geq$ 0).

Now, RaR + Rb = BaB + Bb, bB + BaB, and bR + RaR, typical Left options of Bm, mB, and mR, respectively.

Also, RaB + Bb = BaR + Rb, bR + RaB, and bB + BaR, typical Left options of Bm, mB, and mR, respectively.

Hence, every Left option of Rm is a Left option of Bm, mB, and mR.

A typical Left option of Bm must look like BaR + Rb (a + b = m - 1, a $\geq$ 1, b $\geq$ 0) or BaB + Bb (a + b = m - 1 and a,b $\geq$ 0).

Now, BaR + Rb = RaB + Bb, bR + RaB, and bB + BaR, typical Left options of Rm, mB, and mR, respectively.

Also, BaB + Bb = RaR + Rb, bB + BaB, and bR + RaR, typical Left options of Rm, mB, and mR, respectively.

Hence, every Left option of Bm is a Left option of Rm, mB, and mR.

A typical Left option of mB looks like aR + RbB (a + b = m - 1, a $\geq$ 0, b $\geq$ 1) or aB + BbB (a + b = m - 1 and a,b $\geq$ 0).

Now, aR + RbB = RbB + Ba, BbR + Ra, and aB + BbR, typical Left options of Rm, Bm, and mR, respectively.

Also, aB + BbB = RbR + Ra, BbB + Ba, and aR + RbR, typical Left options of Rm, Bm, and mR, respectively.

Hence, every Left option of mB is a Left option of Rm, Bm, and mR.

A typical Left option of mR looks like aR + RbR (a + b = m - 1, a,b $\geq$ 0) or aB + BbR (a + b = m - 1, a $\geq$ 0, b $\geq$ 1).

Now, aR + RbR = RbR + Ra, BbB + Ba, and aB + BbB, typical Left options of Rm, Bm, and mB, respectively.

Also, aB + BbR = RbB + Ba, BbR + Ra, and aR + RbB, typical Left options of Rm, Bm, and mB, respectively.

Hence, every Left option of mR is a Left option of Rm, Bm, and mB.

Hence, we have shown that Rm, Bm, mB, and mR all have the same set of Left options.

A typical Right option of Rm looks like RaR + Bb ($a + b = m - 1$, $a,b \geq 0$) or RaB + Rb ($a + b = m - 1$, $a \geq 1$, $b \geq 0$).

Now, RaR + Bb = BaB + Rb, bR + BaB, and bB + RaR, typical Right options of Bm, mB, and mR, respectively.

Also, RaB + Rb = BaR + Bb, bB + RaB, and bR + BaR, typical Right options of Bm, mB, and mR, respectively.

Hence, every Right option of Rm is a Right option of Bm, mB, and mR.

A typical Right option of Bm looks like BaR + Bb ($a + b = m - 1$, $a \geq 1$, $b \geq 0$) or BaB + Rb ($a + b = m - 1$, $a,b \geq 0$).

Now, BaR + Bb = RaB + Rb, bB + RaB, and bR + BaR, typical Right options of Rm, mB, and mR, respectively.

Also, BaB + Rb = RaR + Bb, bR + BaB, and bB + RaR, typical Right options of Rm, mB, and mR, respectively.

A typical Right option of mB looks like aR + BbB ($a + b = m - 1$, $a,b \geq 0$) or aB + RbB ($a + b = m - 1$, $a \geq 0$, $b \geq 1$).

Now, aR + BbB = RbR + Ba, BbB + Ra, and aB + RbR, typical Right options of Rm, Bm, and mR, respectively.

Also, aB + RbB = RbB + Ra, BbR + Ba, and aR + BbR, typical Right options of Rm, Bm, and mR, respectively.

Hence, every Right option of mB is a Right option of Rm, Bm, and mR.

A typical Right option of mR looks like aR + BbR ($a + b = m - 1$, $a \geq 0$, $b \geq 1$) or aB + RbR ($a + b = m - 1$, $a,b \geq 0$).

Now, aR + BbR = RbB + Ra, BbR + Ba, and aB + RbB, typical Right options of Rm, Bm, and mB, respectively.

Also, aB + RbR = RbR + Ba, BbB + Ra, and aR + BbB, typical Right options of Rm, Bm, and mB, respectively.

Hence, every Right option of mR is a Right option of Rm, Bm, and mB.

Hence, Rm, Bm, mB, and mR all have the same set of Right options.

Hence, Rm = Bm = mB = mR and 1 is established.

2. Consider RmB and BmR.

A typical Left option of RmB looks like RaR + RbB (a + b = m - 1 and a $\geq$ 0, b $\geq$ 1).

But, RaR + RbB = BaB + BbR, a typical Left option of BmR.

A typical Left option of BmR looks like BaR + RbR (a + b = m - 1, a $\geq$ 1, b $\geq$ 0).

But, BaR + RbR = RbR + RaB, a typical Left option of RmB.

Hence, RmB and BmR have the same set of Left options.

A typical Right option of RmB looks like RaR + BbB (a + b = m - 1, a,b $\geq$ 0) or RaB + RbB (a + b = m - 1 and a,b $\geq$ 1).

Now, RaR + BbB = BbB + RaR and RaB + RbB = BaR + BbR, typical Right options of BmR.

A typical Right option of BmR looks like BaR + BbR (a + b = m - 1, a,b $\geq$ 1) or BaB + RbR (a + b = m - 1 and a,b $\geq$ 0).

Now, BaR + BbR = RaB + RbB and BaB + RbR = RbR + BaB, typical Right options of RmB.

Hence, RmB and BmR have the same set of Right options.

Hence, RmB = BmR and 2 is established.

3. Consider RmR and BmB

A typical Left option of RmR looks like RaR + RbR (a + b = m - 1, a,b $\geq$ 0) or RaB + BbR (a + b = m - 1 and a,b $\geq$ 1).

Now, RaR + RbR = BaB + BbB and RaB + BbR = BaR + RbB, typical Left options of BmB.

A typical Left option of BmB looks like BaR + RbB (a + b = m - 1, a,b $\geq$ 1) or BaB + BbB (a + b = m - 1 and a,b $\geq$ 0).

Now, BaR + RbB = RaB + BbR and BaB + BbB = RaR + RbR, typical Left options of RmR.

Hence, RmR and BmB have the same set of Left options.

A typical Right option of RmR looks like RaR + BbR (a + b = m - 1 and a $\geq$ 0, b $\geq$ 1).

Now, RaR + BbR = BaB + RbB, a typical Right option of BmB.

A typical Right option of BmB looks like BaR + BbB (a + b = m - 1 and a $\geq$ 1, b $\geq$ 0).

Now, BaR + BbB = RaB + RbR, a typical Right option of RmR.

Hence, RmR and BmB have the same set of Right options.

Hence, RmR = BmB and 3 is established.

Hence, by induction, we have 1-3 true for all n ϵ N.

Q.E.D.

So, the task of analyzing this game becomes a more reasonable proposition. For any natural number, n, we need only consider strips which look like n, Rn, RnB, and RnR.

We begin the analysis by directly computing the values of the four distinct positions of length n, n = 1, 2, 3, and 4. Of course, the values we are most interested in are the values for the "untinted" strips of length n. However, to compute these values we must know the values for the positions Rn, RnB, and RnR. This will become apparent as the analysis proceeds.

In the analysis of a specific position, we will list Left's options in the following manner: The options obtained by playing a red counter in all legal squares moving left to right across the strip will be listed first. The options obtained by playing a black counter in all legal squares moving left to right across the strip will follow. The same convention will be used in listing Right's options with the red/black counter being considered first.

We begin the analysis with:

n = 1: 1 = {0 | 0} = *

R1 = {0 | 0} = *

R1B = { | 0} = -1

R1R = {0 | } = 1

n = 2: 2 = {R1 | R1}

= {* | *} = 0

R2 = {R1, R1R, R1B | B1, R1R, R1B}

= {R1, R1R, R1B | R1, R1R, R1B}

= {*, 1, -1 | *, 1, -1}

= {1 | -1} = ±1 since * and -1 are dominated options for Left and * and 1 are dominated options for Right.

R2B = {R1B | B1B}

= {R1B | R1R}

= {-1 | 1} = 0

R2R = {R1R | B1R}

= {R1R | R1B}

= {1 | -1} = ±1

n = 3 3 = {R2, 1R + R1 | B2, 1R + B1}

= {R2, R1 + R1 | R2, R1 + R1}

= {±1, * + * | ±1, * + *}

= {±1, 0 | ±1, 0}

= {0 | 0} = * since ±1 is a reversible option in either case.

$R3 = \{R2, R1R + R1, R2R, R1B + B1, R2B \mid B2, R1R + B1, R2R, R1B + R1, R2B\}$

$= \{R2, R1R + R1, R2R, R1B + R1, R2B \mid R2, R1R + R1, R2R, R1B + R1, R2B\}$

$= \{\pm 1, 1 + *, \pm 1, -1 + *, 0 \mid \pm 1, 1 + *, \pm 1, -1 + *, 0\}$

$= \{1 + * \mid -1 + *\} = * + \pm 1$, since $-1 + *$ and 0 are dominated Left options; $1 + *$ and 0 are dominated Right options; and ± 1 is reversible in either case. The last equality follows by definition of $* + \pm 1$ and is easily verified.

$R3B = \{R2B, R1R + R1B \mid B2B, R1R + B1B, R1B + R1B\}$

$= \{R2B, R1R + R1B \mid R2R, R1R + R1R, R1B + R1B\}$

$= \{0, 0 \mid \pm 1, 2, -2\}$

$= \{0 \mid -2\}$ since 2 and ± 1 are dominated Right options.

$= -1 + \pm 1$

$R3R = \{R2R, R1R + R1R, R1B + B1R \mid B2R, R1R + B1R\}$

$= \{R2R, R1R + R1R, R1B + R1B \mid R2B, R1R + R1B\}$

$= \{\pm 1, 2, -2 \mid 0, 0\}$

$= \{2 \mid 0\}$ since ± 1 and -2 are dominated Left options.

$= 1 + \pm 1$

$n = 4$ $\quad 4 = \{R3, 1R + R2 \mid B3, 1R + B2\}$

$= \{R3, R1 + R2 \mid R3, R1 + R2\}$

$= \{* + \pm 1, * + \pm 1 \mid * + \pm 1, * + \pm 1\}$

$= 0$

$R4 = \{R3, R1R + R2, R2R + R1, R3R, R1B + B2, R2B + B1, R3B \mid B3, R1R + B2, R2R + B1, R3R, R1B + R2, R2B + R1, R3B\}$

$= \{R3, R1R + R2, R2R + R1, R3R, R1B + R2, R2B + R1, R3B \mid$

R3, R1R + R2, R2R + R1, R3R, R1B + R2, R2B + R1, R3B}

= {* + ±1, 1 + ±1, * + ±1, 1 + ±1, -1 + ±1, *, -1 + ±1 |
* + ±1, 1 + ±1, * + ±1, 1 + ±1, -1 + ±1, *, -1 + ±1}

= {* + ±1, 1 + ±1, *, -1 + ±1 | * + ±1, 1 + ±1, *, -1 + ±1}

= {1 + ±1 | -1 + ±1} = 0, since * + ±1, *, and -1 + ±1 are dominated Left options while * + ±1, 1 + ±1, and * are dominated Right options.

R4B = {R3B, R1R + R2B, R2R + R1B | B3B, R1R + B2B, R1B + R2B}

= {R3B, R1R + R2B, R2R + R1B | R3R, R1R + R2R, R1B + R2B}

= {-1 + ±1, 1, -1 + ±1 | 1 + ±1, 1 + ±1, -1}

= {1 | -1} = ±1 since -1 + ±1 is a dominated Left option and 1 + ±1 is a dominated Right option.

R4R = {R3R, R1R + R2R, R1B + B2R | B3R, R1R + B2R, R2R + B1R}

= {R3R, R1R + R2R, R1B + R2B | R3B, R1R + R2B, R2R + R1B}

= {1 + ±1, 1 + ±1, -1 | -1 + ±1, 1, -1 + ±1}

= {1 + ±1 | -1 + ±1} = 0 since -1 is a dominated Left option and 1 is a dominated Right option.

In summary of the analysis to this point, we have shown by direct computation that:

	n	Rn	RnB	RnR
1	*	*	-1	1
2	0	±1	0	±1
3	*	* + ±1	-1 + ±1	1 + ±1
4	0	0	±1	0

At this juncture, there appears to be no particular pattern to the values that various positions take on, with one exception. The value of the position we are really interested in, a strip of length n, seems to be oscillating between * and 0. We continue the analysis by direct computation to see if our suspicions are borne out.

Before proceeding, take notice of some interesting relations in the preliminary analysis. For a fixed n, the Left options in n and Rn were essentially the same as the Right options in the same games. Also, for fixed n, the Left options in RnR were just the Right options in RnB, and the Right options in RnR were just the Left options in RnB. These results come as a consequence to Lemma 1. We have:

Corollary 1:

In n and Rn, Right has the same options as Left. The Left options in RnR are the Right options in RnB and vice versa.

Proof:

A typical Left option of n is aR + Rb ($a + b = n - 1$, $a,b \geq 0$) and a typical Right option of n is aR + Bb ($a + b = n - 1$, $a,b \geq 0$). By Lemma 1, aR + Rb = aR + Bb. Hence, the Left and Right options for n are the same.

A typical Left option of Rn is RaR + Rb ($a + b = n - 1$, $a,b \geq 0$) or RaB + Bb ($a + b = n - 1$, $a \geq 1$, $b \geq 0$). A typical Right option of Rn is RaR + Bb ($a + b = n - 1$, $a,b \geq 0$) or RaB + Rb ($a + b = n - 1$, $a \geq 1$, $b \geq 0$). By Lemma 1, RaR + Rb = RaR + Bb and RaB + Bb = RaB + Rb. Hence, the Left and Right options in Rn are the same.

A typical Left option of RnB is RaR + RbB ($a + b = n - 1$, $a \geq 0$, $b \geq 1$). A typical Right option of RnR is RaR + BbR ($a + b = n - 1$, $a \geq 0$, $b \geq 1$). By Lemma 1, RaR + RbB = RaR + BbR. Hence, the Left options in RnB and the Right options in RnR are the same.

A typical Right option in RnB is RaR + BbB (a + b = n - 1, a,b $\geq$ 0) or RaB + RbB (a + b = n - 1, a $\geq$ 1, b $\geq$ 1). A typical Left option in RnR is RaR + RbR (a + b = n - 1, a,b $\geq$ 0) or RaB + BbR (a + b = n - 1, a $\geq$ 1, b $\geq$ 1). By Lemma 1, RaR + BbB = RaR + RbR and RaB + RbB = RaB + BbR. Hence, the Right options in RnB and the Left options in RnR are the same. Q.E.D.

Obviously, the application of this corollary will greatly reduce the time and pain involved in the direct computation of these positions. Calculation of the values for n = 1, 2, 3, and 4 was carried out in detail for the purpose of illustration, but we become a little more expedient in these subsequent calculations:

n = 5: 5 = {R4, 1R + R3, 2R + R2 | "}

= {R4, R1 + R3, R2 + R2 | "}

= {0, * + * + ±1, ±1 + ±1 | "}

= {0, ±1 | "}

= {0 | 0} = * since ±1 is a reversible option in either case.

R5 = {R4, R1R + R3, R2R + R2, R3R + R1, R4R, R1B + B3, R2B + B2, R3B + B1, R4B | "}

= {R4, R1R + R3, R2R + R2, R3R + R1, R4R, R1B + R3, R2B + R2, R3B + R1, R4B | "}

= {0, 1 + * + ±1, ±1 + ±1, * + 1 + ±1, 0, -1 + * + ±1, ±1, * + -1 + ±1, ±1 | "}

= {0, * + 1 + ±1, * + -1 + ±1, ±1 | "}

= {* + 1 + ±1 | * + -1 + ±1} = * since ±1 is a reversible option and the others are dominated in either case.

R5B = {R4B, R1R + R3B, R2R + R2B, R3R + R1B | B4B, R1R + B3B,

R2R + B2B, R1B + R3B, R2B + R2B}

= {R4B, R1R + R3B, R2R + R2B, R3R + R1B | R4R, R1R + R3R, R2R + R2R, R1B + R3B, R2B + R2B}

= {±1, 1 + -1 + ±1, ±1, -1 + 1 + ±1 | 0, 1 + 1 + ±1, ±1 + ±1, -1 + -1 + ±1, 0}

= {±1 | 0, 2 + ±1, -2 + ±1}

= {±1 | -2 + ±1} = -1

R5R = {0, 2 + ±1, -2 + ±1 | ±1}

= {2 + ±1 | ±1} = 1

n = 6: 6 = {R5, 1R + R4, 2R + R3 | "}

= {R5, R1 + R4, R2 + R3 | "}

= {*, *, ±1 + * + ±1 | "}

= {* | *} = 0

R6 = {R5, R1R + R4, R2R + R3, R3R + R2, R4R + R1, R5R, R1B + B4, R2B + B3, R3B + B2, R4B + B1, R5B | "}

= {R5, R1R + R4, R2R + R3, R3R + R2, R4R + R1, R5R, R1B + R4, R2B + R3, R3B + R2, R4B + R1, R5B | "}

= {*, 1, ±1 + * + ±1, 1 + ±1 + ±1, *, 1, -1, * + ±1, -1 + ±1 + ±1, ±1 + *, -1 | "}

= {*, 1, -1, * + ±1 | "}

= {1 | -1} = ±1, removing dominated options.

R6B = {R5B, R1R + R4B, R2R + R3B, R3R + R2B, R4R + R1B | B5B, R1R + B4B, R2R + B3B, R1B + R4B, R2B + R3B}

= {R5B, R1R + R4B, R2R + R3B, R3R + R2B, R4R + R1B | R5R, R1R + R4R, R2R + R3R, R1B + R4B, R2B + R3B}

= {-1, 1 + ±1, ±1 + -1 + ±1, 1 + ±1, -1 | 1, 1, ±1 + 1 + ±1, -1 + ±1, -1 + ±1}

$= \{-1, 1 + \pm 1 \mid 1, -1 + \pm 1\}$

$= \{1 + \pm 1 \mid -1 + \pm 1\} = 0$

$R6R = \{1, -1 + \pm 1 \mid -1, 1 + \pm 1\} = \{1 \mid -1\} = \pm 1$

n = 7: $7 = \{R6, 1R + R5, 2R + R4, 3R + R3 \mid "\}$

$= \{R6, R1 + R5, R2 + R4, R3 + R3 \mid "\}$

$= \{\pm 1, * + *, \pm 1, (* + \pm 1) + (* + \pm 1) \mid "\}$

$= \{\pm 1, 0 \mid \pm 1, 0\} = *$ since ± 1 is reversible

$R7 = \{R6, R1R + R5, R2R + R4, R3R + R3, R4R + R2, R5R + R1,$
$R6R, R1B + B5, R2B + B4, R3B + B3, R4B + B2, R5B + B1,$
$R6B \mid "\}$

$= \{R6, R1R + R5, R2R + R4, R3R + R3, R4R + R2, R5R + R1,$
$R6R, R1B + R5, R2B + R4, R3B + R3, R4B + R2, R5B + R1,$
$R6B \mid "\}$

$= \{\pm 1, 1 + *, \pm 1, 1 + \pm 1 + * + \pm 1, \pm 1, * + 1, \pm 1, -1 +$
$*, 0, -1 + \pm 1 + * + \pm 1, \pm 1 + \pm 1, -1 + *, 0 \mid "\}$

$= \{\pm 1, * + 1, * - 1, 0 \mid "\}$

$= \{* + 1 \mid * - 1\} = * + \pm 1$ removing dominated options
and the reversible option, ± 1.

$R7B = \{R6B, R1R + R5B, R2R + R4B, R3R + R3B, R4R + R2B,$
$R5R + R1B \mid B6B, R1R + B5B, R2R + B4B, R3R + B3B,$
$R1B + R5B, R2B + R4B, R3B + R3B\}$

$= \{R6B, R1R + R5B, R2R + R4B, R3R + R3B, R4R + R2B,$
$R5R + R1B \mid R6R, R1R + R5R, R2R + R4R, R3R + R3R, R1B$
$+ R5B, R2B + R4B, R3B + R3B\}$

$= \{0, 0, \pm 1 + \pm 1, 1 + \pm 1 + -1 + \pm 1, 0, 1 + -1 \mid \pm 1,$
$2, \pm 1, 1 + \pm 1 + 1 + \pm 1, -2, \pm 1, -2\}$

$= \{0 \mid \pm 1, 2, -2\} = -1 + \pm 1$ since ± 1 and 2 are dominated

Right options.

R7R = {±1, 2, -2 | 0} = 1 + ±1 since ±1 and -2 are dominated

Left options.

n = 8: 8 = {R7, 1R + R6, 2R + R5, 3R + R4 | "}

= {R7, R1 + R6, R2 + R5, R3 + R4 | "}

= {* + ±1, * + ±1, * + ±1, * + ±1 | "}

= {* + ±1 | * + ±1} = 0

R8 = {R7, R1R + R6, R2R + R5, R3R + R4, R4R + R3, R5R + R2, R6R + R1, R7R, R1B + B6, R2B + B5, R3B + B4, R4B + B3, R5B + B2, R6B + B1, R7B | "}

= {R7, R1R + R6, R2R + R5, R3R + R4, R4R + R3, R5R + R2, R6R + R1, R7R, R1B + R6, R2B + R5, R3B + R4, R4B + R3, R5B + R2, R6B + R1, R7B | "}

= {* + ±1, 1 + ±1, * + ±1, 1 + ±1, * + ±1, 1 + ±1, * + ±1, 1 + ±1, -1 + ±1, *, -1 + ±1, ±1 + * + ±1, -1 + ±1, *, -1 + ±1 | "}

= {* + ±1, 1 + ±1, -1 + ±1, * | "}

= {1 + ±1 | -1 + ±1} = 0

R8B = {R7B, R1R + R6B, R2R + R5B, R3R + R4B, R4R + R3B, R5R + R2B, R6R + R1B | B7B, R1R + B6B, R2R + B5B, R3R + B4B, R1B + R6B, R2B + R5B, R3B + R4B}

= {R7B, R1R + R6B, R2R + R5B, R3R + R4B, R4R + R3B, R5R + R2B, R6R + R1B | R7R, R1R + R6R, R2R + R5R, R3R + R4R, R1B + R6B, R2B + R5B, R3B + R4B}

= {-1 + ±1, 1, -1 + ±1, 1 + ±1 + ±1, -1 + ±1, 1, -1 + ±1 | 1 + ±1, 1 + ±1, 1 + ±1, 1 + ±1, -1, -1, ±1 + -1 + ±1}

$= \{-1 + \pm 1, 1 \mid 1 + \pm 1, -1\}$

$= \{1 \mid -1\} = \pm 1$

$R8R = \{-1, 1 + \pm 1 \mid -1 + \pm 1, 1\}$

$= \{1 + \pm 1 \mid -1 + \pm 1\} = 0$

Summarizing:

	n	Rn	RnB	RnR
5	*	*	-1	1
6	0	±1	0	±1
7	*	* + ±1	-1 + ±1	1 + ±1
8	0	0	±1	0

and we see that this table is identical to the one for n = 1, 2, 3, and 4. Not only are the values for a strip of length n repeating as suspected, but values for the other positions have also made another cycle. We intend to prove that this pattern must hold in general.

CHAPTER THREE

The claim at the close of the preceding chapter is that the cycle established through n = 4 and repeated once through n = 8 is indeed general. We offer:

Theorem 1:

For this game, we have the following values:

1. n = * for n = 4k + 1, k = 0, 1, 2, ...

   Rn = *

   RnB = -1

   RnR = 1

2. n = 0 for n = 4k + 2, k = 0, 1, 2, ...

   Rn = ±1

   RnB = 0

   RnR = ±1

3. n = * for n = 4k + 3, k = 0, 1, 2, ...

   Rn = * + ±1

   RnB = -1 + ±1

   RnR = 1 + ±1

4. n = 0 for n = 4k + 4, k = 0, 1, 2, ...

   Rn = 0

   RnB = ±1

   RnR = 0

Proof is accomplished by applying the strong principle of induction.

The result has been shown for $k = 0$ and $k = 1$ (corresponding to $n = 1, ---, 8$) by direct computation. Thus, the basis is established.

We now assume the result holds for all $k < K$, and show that it must hold for K.

First we examine the options, in general, for n, Rn, RnB, and RnR. We have:

$n = \{Ra + Rb,\ 0 \leq a,b \leq n-1,\ a+b = n-1 \mid "\}$ where Right's options are the same as Left's by Corollary 1.

$Rn = \{RaR + Rb,\ 0 \leq a,b \leq n-1,\ a+b = n-1;\ RaB + Rb,\ 1 \leq a \leq n-1,$
$0 \leq b \leq n-2,\ a+b = n-1 \mid "\}$

$RnB = \{RaR + RbB,\ 0 \leq a \leq n-2,\ 1 \leq b \leq n-1,\ a+b = n-1 \mid RaR + RbR,$
$0 \leq a,b \leq n-1,\ a+b = n-1;\ RaB + RbB,\ 1 \leq a,b \leq n-2,\ a+b = n-1\}$

$RnR = \{$RnB's Right options $\mid$ RnB's Left options$\}$ by Corollary 1.

Note that there are duplicates among the options for particular positions. For instance, in the Right option set for RnB, R1R + R(n-2)R is exactly the same as R(n-2)R + R1R. However, the option lists for each position are exhaustive, and this is all that is necessary for our purposes.

Now, consider $n = 4K + 1$. If we look at the options in the four basic positions, we see that the indices a and b must sum to $n - 1 = 4K$.

Therefore, $a + b \equiv 0 \bmod 4$. This can happen in four ways:

1. $a \equiv 0 \bmod 4$ and $b \equiv 0 \bmod 4$
2. $a \equiv 1 \bmod 4$ and $b \equiv 3 \bmod 4$
3. $a \equiv 2 \bmod 4$ and $b \equiv 2 \bmod 4$
4. $a \equiv 3 \bmod 4$ and $b \equiv 1 \bmod 4$

For all positions of length $n = 4k + 1$, $k > 1$, each of these four instances will occur. For $k = 0$ and $k = 1$, we already verified the assertions of the theorem.

So, for n: Ra + Rb = 0, ±1, 0, or ±1 (corresponding respectively to 1-4)

Therefore, n = {0, ±1 | 0, ±1} = {0 | 0} = *

for Rn: RaR + Rb = 0, 1 + * + ±1, 0, 1 + * + ±1

RaB + Rb = ±1, -1 + * + ±1, ±1, -1 + * + ±1

Therefore, Rn = {0, 1 + * + ±1, -1 + * + ±1, ±1|"}

= {1 + * + ±1 | -1 + * + ±1} = *

for RnB: RaR + RbB = ±1, ±1, ±1, ±1

RaR + RbR = 0, 2 + ±1, 0, 2 + ±1

RaB + RbB = 0, -2 + ±1, 0, -2 + ±1

Therefore, RnB = {±1 | 0, 2 + ±1, -2 + ±1}

= {±1 | -2 + ±1} = -1

Also, RnR = {0, 2 + ±1, -2 + ±1 | ±1}

= {2 + ±1 | ±1} = 1

Suppose now that n = 4K + 2. Then a + b = n - 1 implies a + b ≡ 1 mod 4.

Now, this can happen in four ways:

1. a ≡ 0 mod 4 and b ≡ 1 mod 4
2. a ≡ 1 mod 4 and b ≡ 0 mod 4
3. a ≡ 2 mod 4 and b ≡ 3 mod 4
4. a ≡ 3 mod 4 and b ≡ 2 mod 4

So, for n: Ra + Rb = *, *, *, *

Therefore, n = {* | *} = 0

for Rn: RaR + Rb = *, 1, *, 1

RaB + Rb = * + ±1, -1, * + ±1, -1

Therefore, Rn = {*, * + ±1, -1, 1 | "} = {1 | -1} = ±1

for RnB: RaR + RbB = -1, 1 + ±1, -1, 1 + ±1

RaR + RbR = 1, 1, 1, 1

$RaB + RbB = -1 + \pm 1, -1 + \pm 1, -1 + \pm 1, -1 + \pm 1$

Therefore, $RnB = \{-1, 1 + \pm 1 \mid 1, -1 + \pm 1\} = \{1 + \pm 1 \mid -1 + \pm 1\} = 0$

Also, $RnR = \{1, -1 + \pm 1 \mid -1, 1 + \pm 1\} = \{1 \mid -1\} = \pm 1$

If $n = 4K + 3$, then $a + b = n - 1$ implies that $a + b \equiv 2 \bmod 4$.

Hence, we must have one of:

1. $a \equiv 0 \bmod 4$ and $b \equiv 2 \bmod 4$
2. $a \equiv 1 \bmod 4$ and $b \equiv 1 \bmod 4$
3. $a \equiv 2 \bmod 4$ and $b \equiv 0 \bmod 4$
4. $a \equiv 3 \bmod 4$ and $b \equiv 3 \bmod 4$

So, for n: $Ra + Rb = \pm 1, 0, \pm 1, 0$

Therefore, $n = \{\pm 1, 0 \mid \pm 1, 0\} = \{0 \mid 0\} = *$

for Rn: $RaR + Rb = \pm 1, * + 1, \pm 1, * + 1$

$RaB + Rb = 0, * - 1, 0, * - 1$

Therefore, $Rn = \{\pm 1, 0, * - 1, * + 1 \mid "\}$

$= \{* + 1 \mid * - 1\} = * + \pm 1$

for RnB: $RaR + RbB = 0, 0, 0, 0$

$RaR + RbR = \pm 1, 2, \pm 1, 2$

$RaB + RbB = \pm 1, -2, \pm 1, -2$

Therefore, $RnB = \{0 \mid \pm 1, 2, -2\} = \{0 \mid -2\} = -1 + \pm 1$

Also, $RnR = \{\pm 1, 2, -2 \mid 0\} = \{2 \mid 0\} = 1 + \pm 1$

If $n = 4K + 4$, then $a + b = n - 1$ implies that $a + b \equiv 3 \bmod 4$.

Hence, we must have one of:

1. $a \equiv 0 \bmod 4$ and $b \equiv 3 \bmod 4$
2. $a \equiv 1 \bmod 4$ and $b \equiv 2 \bmod 4$
3. $a \equiv 2 \bmod 4$ and $b \equiv 1 \bmod 4$
4. $a \equiv 3 \bmod 4$ and $b \equiv 0 \bmod 4$

So, for n: $Ra + Rb = * + \pm 1, * + \pm 1, * + \pm 1, * + \pm 1$

Therefore, $n = \{* + \pm 1 \mid * + \pm 1\} = 0$

for Rn: $RaR + Rb = * + \pm 1, 1 + \pm 1, * + \pm 1, 1 + \pm 1$

$RaB + Rb = *, -1 + \pm 1, *, -1 + \pm 1$

Therefore, $Rn = \{* + \pm 1, 1 + \pm 1, *, -1 + \pm 1 \mid "\}$

$= \{1 + \pm 1 \mid -1 + \pm 1\} = 0$

for RnB: $RaR + RbB = -1 + \pm 1, 1, -1 + \pm 1, 1$

$RaR + RbR = 1 + \pm 1, 1 + \pm 1, 1 + \pm 1, 1 + \pm 1$

$RaB + RbB = -1, -1, -1, -1$

Therefore, $RnB = \{-1 + \pm 1, 1 \mid -1, 1 + \pm 1\} = \{1 \mid -1\}$

$= \pm 1$

Also, $RnR = \{-1, 1 + \pm 1 \mid -1 + \pm 1, 1\} = \{1 + \pm 1 \mid -1 + \pm 1\} = 0$

Note that we could assign actual values to the positions of length $4K + 1$ because each option of a given one of these positions had length $4k + 1$, $4k + 2$, $4k + 3$, or $4k + 4$ for $k < K$ and therefore, by the induction hypothesis, could be assigned an appropriate value.

From that point, it was necessary to consider the four cases in the order exhibited. That is, obtaining the result for $n = 4K + 4$ necessitated having the result for $n = 4K + 3$ since, for instance, one Left option of Rn ($n = 4K + 4$) is Rn' ($n' = 4K + 3$).

Hence, we have shown that if the result holds for all $k < K$, then the result holds for K. Hence, the result holds for all $k \in N$.

Q.E.D.

Now our goal is to learn to actually play the game successfully. That is, we know from Theorem 1 that any game beginning with a strip of odd length has value * and is thus a first player win. We would like to find a winning strategy for the first player. Likewise, any game beginning with a strip of even length has value 0 and is therefore a second player win. We would also like to find a winning strategy for the second player here. That is the problem we tackle in the final chapter.

CHAPTER FOUR

We begin our search for a strategy by considering strips of even length with Left starting. According to the theorem, this should be a second player win, so we attempt to find the winning Right reply to any Left move.

As is true for many combinatorial games, the strategy for this game is based on symmetry. Right's reply to a Left move will in some sense restore symmetry to the original position.

Suppose that Left makes his first move by placing a counter of either color in either one of the end squares. Right will place his counter in the opposite end and orient it in the following manner: If the resulting strip will have length congruent to 2 mod 4, Right will color the ends differently, and if the resulting strip will have length congruent to 0 mod 4, he will color the ends the same. Since we started with a strip of even length, it is clear that after Left and Right make their first moves the resulting position will still have even length and that length will necessarily be congruent to either 0 or 2 mod 4. Right makes these responses because (from the theorem) $RnB = 0$ for $n \equiv 2 \bmod 4$ and $RnR = 0$ for $n \equiv 0 \bmod 4$.

Now suppose that Left makes his first move in a square other than one of the ends. The result is a position which looks like $Ra + Rb$, $a + b = n-1$, where n is the length of the original strip. Since n was even, a is (without loss of generality) less than b. By the theorem, Rn + Rn = 0 for all n, so Right will make his move in the component Rb in such a fashion as to cancel Ra. When he has done this, the original position will be split into three components, two of which have length a. Hence, the remaining strip must have even length. If that length is congruent to 2 mod 4, Right will place his counter so that the ends of the enclosed strip are colored differently, and if that length is congruent to 0 mod 4, Right will place his counter so that the ends have the same color. These moves lead to positions which have value 0 and are therefore winning moves.

So, regardless of Left's beginning move, Right can always make a winning response.

Now we consider what may happen when Left makes his second move. If Left moved in one of the end squares on his first move, then his second move will be made in either RaR, $a \equiv 0 \bmod 4$, or RaB, $a \equiv 2 \bmod 4$. Before we examine Left's options, note the following equalities which follow directly from Theorem 1:

1. RnR + RnR = 0, n even
2. RnB + RnB = 0, n even
3. RnR + RnB = 0, n odd

Suppose Left is moving in RaR, $a \equiv 0 \bmod 4$. His options are:

1. RcR + RdR, $d<c$, d even
2. RcR + RdR, $d<c$, d odd
3. RcB + RdB, $d<c$, d even
4. RcB + RdB, $d<c$, d odd, where $c + d = a - 1$

In each case, Right will cancel the shorter component by applying one of the equalities listed above. His responses are:

1. RdR + ReB + RdR, $e \equiv 2 \bmod 4$

2. RdB + ReR + RdR, $e \equiv 0 \mod 4$

3. RdB + ReB + RdB, $e \equiv 2 \mod 4$

4. RdR + ReR + RdB, $e \equiv 0 \mod 4$,

respectively. Note that each of these moves is a winning move since the resulting position has value 0.

Now, suppose that Left is making his second move in a position which looks like RaB, $a \equiv 2 \mod 4$. His options are:

1. RcR + RdB, $d<c$, d even

2. RcR + RdB, $d<c$, d odd

3. RcB + RdR, $d<c$, d even

4. RcB + RdR, $d<c$, d odd where $c + d = a - 1$.

Again, Right will cancel the component of length d by moving in the longer component and applying one of the equalities listed above. These moves are:

1. RdB + ReR + RdB, $e \equiv 0 \mod 4$

2. RdR + ReB + RdB, $e \equiv 2 \mod 4$

3. RdR + ReR + RdR, $e \equiv 0 \mod 4$

4. RdB + ReB + RdR, $e \equiv 2 \mod 4$

respectively. Again, each of these moves leads to a position of value 0.

Finally, we consider the cases where Left made his first move in the interior of the strip. Having discussed Right's first reply, we know that on his second move Left will be facing: Ra + Ra + RmB, $m \equiv 2 \mod 4$, or Ra + Ra + RmR, $m \equiv 0 \mod 4$. From the discussion just completed we know Right's correct reply if Left moves in the component of length m in either case. Hence, we need only consider Right's response if Left makes his second move in one of the components of length a.

If Left moves in Ra, he has the following options:

1. RcR + Rd, c + d = a-1

2. RcB + Rd, c + d = a-1

Right will respond in the other Ra component. He makes the following responses to 1.: RcR + Rd if c is even and RcB + Rd if c is odd. His responses to 2. are: RcB + Rd if c is even and RcR + Rd if c is odd. These are moves to 0 and hence, winning moves for Right.

So, given any second move for Left, Right can always return the game to a position with value 0. We are now ready to consider what positions Left has to face on his third move. Careful examination shows that any position Left may now encounter involves summands for which we have already developed strategies or else involves one of the partial sums:

1. RaR + RaR, a even

2. RaB + RaB, a even

3. RaB + RaR, a odd.

Left's options in RaR + RaR, a even, are:

1. RcR + RdR + RaR

2. RcB + RdB + RaR

Since a was even, c and d are necessarily (without loss of generality) odd and even, respectively. Hence, Right will respond by moving in RaR to:

1. RcB + RdR

2. RcR + RdB, respectively. These are moves to 0.

Left's option in RaB + RaB, a even, is: RcR + RdB + RaB, where c and d are (without loss of generality) odd and even, respectively. Hence, Right's correct reply is a move in RaB to RcB + RdB.

Left's options in RaB + RaR, a odd, are:

1. RcR + RdB + RaR

2. RcR + RdR + RaB

3. RcB + RdB + RaB.

Since a is odd and c + d = a-1, we have that c and d are either both even or both odd. Right will respond in the component of length a each time. His responses are:

1. RcR + RdB, c and d even
   RcB + RdR, c and d odd

2. RcR + RdR, c and d even
   RcB + RdB, c and d odd

3. RcB + RdB, c and d even
   RcR + RdR, c and d odd.

From this point on, any move Left can make is merely a duplicate of a move we have already considered. Hence, if Left begins a game on a strip of even length, Right can always respond to him with a winning move. Now, it is easy to see that if Right starts a game on an even strip, the situation is essentially the same as the one just discussed and Left will always be able to respond with a winning move.

Therefore, we have developed the strategy for the second player win on a strip of even length. We may now consider the strategy for the first player win on a strip of odd length.

Suppose a strip is composed of an odd number of squares, say k. Since Rn + Rn = 0 for any n, it is clear that either Left or Right can make a winning first move by placing a counter in square (k + 1)/2. The first player then becomes the second player in the game R(k - 1)/2, + R(k - 1)/2, a game for which the second player has winning strategy. Hence, in a strip of odd length, the first player can win by moving in the middle square and then by following the second player win strategy in Rn + Rn.

Three realizations about this strategy are in order. First, we had to begin at some point (strips of even length, Left starting) and develop the strategy step by step to make sure that a winning move could always be made. The remaining details then followed with relatively little effort. Secondly, past a certain point (the third move) we encountered only positions for which the strategy had already been worked out. Finally, the entire strategy is predicated on six equalities obtained from Theorem 1:

1. $Rn + Rn = 0$ for all n
2. $RnB = 0$ for $n \equiv 2 \mod 4$
3. $RnR = 0$ for $n \equiv 0 \mod 4$
4. $RnR + RnR = 0$ for n even
5. $RnB + RnB = 0$ for n even
6. $RnB + RnR = 0$ for n odd

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