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Analysis of Boundary Observability of Strongly Coupled One-dimensional Wave Equations with Mixed Boundary Conditions

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ANALYSIS OF BOUNDARY OBSERVABILITY OF STRONGLY COUPLED ONE-DIMENSIONAL WAVE EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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ANALYSIS OF BOUNDARY OBSERVABILITY OF STRONGLY COUPLED ONE-DIMENSIONAL WAVE EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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I would like to dedicate this thesis to all my close friends and family who have supported me throughout my career.

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In control theory, the time it takes to receive a signal after it is sent is referred to as the observation time. For certain types of materials, the observation time to receive a wave signal differs depending on a variety of factors, such as material density, flexibility, speed of the wave propagation, etc. Suppose we have a strongly coupled system of two wave equations describing the longitudinal vibrations on a piezoelectric beam of length L. These two wave equations have non-identical wave propagation speeds c_1 and c_2. First, we prove the exact observability inequality with the optimal observation time satisfying $T > T_1 = \frac{2L}{\min(c_1,c_2)}$ by adopting two different techniques: the multipliers method (non-spectral) and non-harmonic Fourier series (spectral). Next, we discretize the spacial variable for the system via central Finite Differences. We find that for this particular discretization, the minimal observability time approaches infinity as the discretization parameter h goes to zero, and therefore, the discretized equations lack uniform observability unlike the original equations. This is simply due to the blind use of Finite Differences which generates spurious high-frequency vibrational modes. To resolve this issue, a filtering technique, known as the direct Fourier filtering, is adopted, and an observability inequality is proved with a (sub-optimal) observation time $T > T_1 > T_2$ as the discretization parameter tends to zero. These results show that filtered finite differences can be safely applied to the system of piezoelectric beam equations in designing stabilizing controllers.

1 Introduction

Piezoelectric materials are multi-functional smart materials (most notably Lead Zirconate Titanate) used to develop electric displacement that is directly proportional to an applied mechanical stress [4, 25], see Fig. 1. This allows these materials to be used as sensors and actuators. Due to their small size and high power density, they have become more and more promising in industrial applications such as implantable biomedical devices [3, 4, 24], wearable human-machine interface for PVDF sensors [8], nano-positioners and micro-sensing [6, 7, 10], ultrasound imagers, and cleaners [26] due to the excellent advantages of the fast response time, large mechanical force, and extremely fine resolution [10]. Controlling unwanted vibrations on the host structures (or harvesting energy from ambient vibrations) via piezoelectric layers have been the major focus in cutting-edge engineering applications such as ultrasonic welders [26], micro-sensors [6, 7, 10], inchworm robots, and wearable human-machine interfaces such as PVDF sensors adhered onto the surface of skin or cardiac pacemakers under the skin of the chest [8].

These industrial applications for piezoelectric materials can make use of sensors to observe wave profiles corresponding to vibrations on a host structure, and thereby allow piezoelectric materials to actively control the vibration profile of the host structure. Since physical sensors can only take finite-dimensional measurements of displacement, velocity, or acceleration of the traveling waves at a point, it is required to work with a finite dimensional discretization of the partial differential equation system. The main challenge is not only to have the discretized model converge to the original problem for accurately representing the physical nature of the dynamics but also to mimic the control-theoretic properties, such as observability or stabilization, of the original problem.

For many applications of piezoelectricity, electrostatic (or quasi-static) approxi-

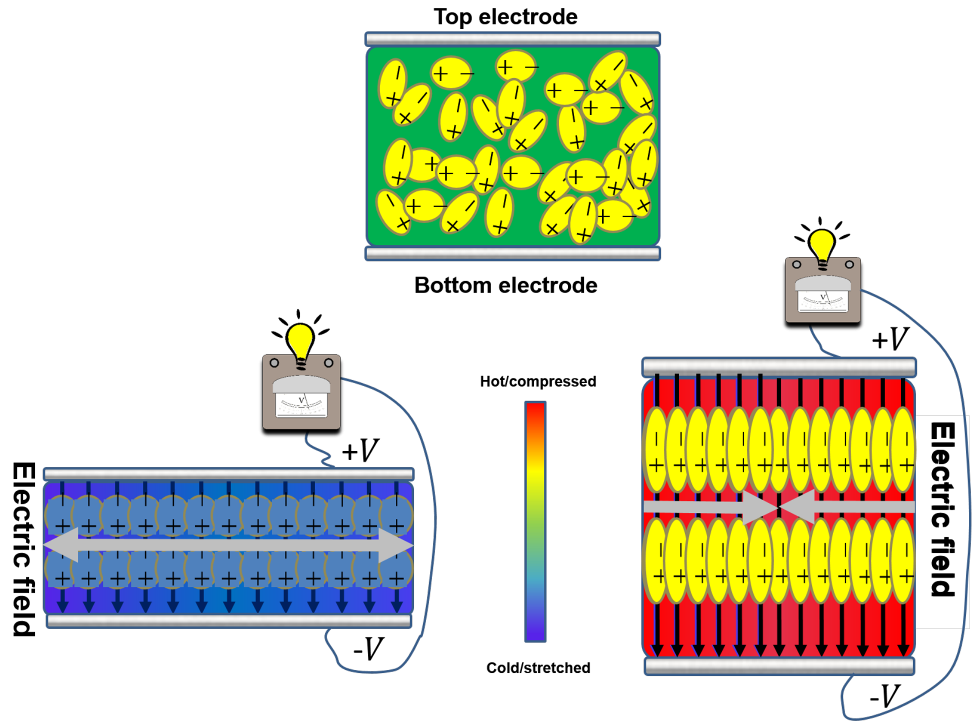

Figure 1: (a) A piezoelectric beam is an elastic beam with electrodes at their top and bottom surfaces, and connected to an external electric circuit. As voltage is applied to its electrodes, it actively (b) stretches or (c) shrinks in the longitudinal directions, therefore, causing charges to separate and line up in the vertical direction. Magnetic (and thermal) effects (stored/produced) have direct contribution to the electric field across the electrodes and longitudinal vibrations [9].

mations due to Maxwell's equations are sufficient to describe low-frequency vibrations [3, 4, 25]. Magnetic effects associated with a piezoelectric beam are traditionally omitted due to the fact that they have a minimal impact on the overall dynamics of the system, yet maximal impact in the observability/controllability of the system [16]. However, for certain piezoelectric devices, these effects can also be major. For example, for piezoelectric acoustic wave devices, there are situations in which full electromagnetic coupling needs to be considered [5, 28, 30]. As a side note, as electromagnetic waves are involved, the complete set of Maxwell equations needs to be used, coupled to the mechanical equations of motion [31].

Denoting v and p by longitudinal displacement of the centerline of the beam and total electrical displacements, a one-dimensional, strongly coupled partial differential equation model describing the longitudinal vibration profile of a piezoelectric beam of length L with the addition of magnetic effects is given in [16] as the following

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, & \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ v(0,t) = p(0,t) = 0, & \\ \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \\ p_x(L,t) - \gamma v_x(L,t) = 0, & t \in \mathbb{R}^+, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in (0,L), \\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x), & x \in (0,L), \end{cases} \tag{1.1}$$

where $\rho, \alpha, \gamma, \mu, \beta$ denote the mass density per unit volume, elastic stiffness, piezo-electric coefficient, magnetic permeability, impermittivity coefficient of the beam, respectively. Note that the wave speeds $\sqrt{\frac{\alpha}{\rho}}$ and $\sqrt{\frac{\beta}{\mu}}$ are non-identical due to the physics.

The electrostatic/quasi-static piezoelectric beam model obtained by taking the magnetic permeability constant zero, i.e. $\mu \equiv 0$ in (1.1), is a single wave equation as the following

$$\begin{cases} \rho v_{tt} - \alpha_1 v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ v(0,t) = \alpha_1 v_x(L,t) = 0, & t \in \mathbb{R}^+, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in (0,L), \end{cases} \tag{1.2}$$

where $\alpha_1 = \alpha - \gamma^2\beta$. It is known that electromagnetic effects for most piezoelectric beams are minor in comparison to the mechanical effects [25]. However, they have a dramatic effect on the observability of these materials [16] especially if there is only an electrical sensor at the boundary measuring $p_t(L,t)$ in (1.1), which is the total current accumulated at the electrodes of the beam. Then, the system is not uniformly observable and not exponentially stabilizable for high-frequency vibrational solutions [16]-[20]. Therefore, unlike the quasi-static or electrostatic models, where only one observer $v_t(L,t)$ in (1.2) is enough, two observers, i.e. $p_t(L,t)$ and $v_t(L,t)$ in (1.1), is a must for exact observability and exponential stabilizability, which are dual concepts of each other in control theory.

It is essential to note that partial differential equations are infinite dimensional systems because they have infinitely many eigenvalues, so exact observability of vibrations is an infinite-dimensional problem. For example, letting $\mathbf{z} = [v, v_t]^{\mathrm{T}}$ the system (1.2) is written in the first order form $\mathbf{z}_t = A\mathbf{z}$ where

$$A = \begin{bmatrix} 0 & I \\ \frac{\alpha_1}{\rho} D_x^2 & 0 \end{bmatrix}.$$

The eigenvalues of A can be easily computed and they are all on the imaginary axis:

$$\tilde{\lambda}_k = \pm i \sqrt{\frac{\alpha_1}{\rho}} \frac{(2k-1)\pi}{2L}, \quad k = 1, 2, \ldots, \infty. \tag{1.3}$$

For these eigenvalues, the gap between two consecutive eigenvalues, $|\tilde{\lambda}_{k+1} - \tilde{\lambda}_k|$ is uniform and is simply $\frac{\pi}{L}$. This is a desired property to prove uniform observability of (1.2). In particular, for $T > T_{min}$ one would like to prove the so-called observability inequality as the following:

$$\int_0^T |v_t(L,t)|^2 \geq C(T) \left[\int_0^L \left(|(v_0)_x|^2 + |v_1|^2 \right) dx \right] \tag{1.4}$$

where T_{min} is called the minimal observation time needed to recover initial conditions $(v_0(x), v_1(x))$ with a single measurement $v_t(L,t)$, tip velocity.

In practice, sensors (or observers) work through algorithms on the chip, and therefore they are doing calculations in finite dimensions (i.e. the computer world). Also, sensors only observe a finite number of vibrational modes. For that reason, sensor design for the observed quantity $v_t(L,t)$ in (1.4) has to be done for a numerical approximation of the system. Numerical approximation methods discretize either one or both variables of the wave equation (time and space). While discretizations have been done for both variables, or just the temporal variable over a string of length L, this project will be focusing solely on the semi-discretization of the space variable. However, the blindly approximated models do not hold for the well-known numerical approximations of the partial differential equation system, such as Finite Difference Method, Finite Element Method, or Finite Volumes Method [11, 22, 29]. The major issue in showing the observability of the discretized system is losing the uniform gap between two consecutive eigenvalues that exists for the infinite dimensional system, see Fig. 2. The existence of the uniform gap allows us to rely on vibrational observations measured by the sensors. When this gap is not present due to numerical approximation, the sensor cannot distinguish one vibrational mode from another. When this happens, the system is not observable.

To the best of our knowledge, this discrepancy was first shown in [1] for a boundary-controlled wave equation, and this was later diagnosed first for a wave equation with Dirichlet boundary conditions [11].

For example, as Finite Difference space-discretized approximations for (1.2) is utilized for the discertization parameter $h = \frac{1}{N+1} > 0$, one can show that the eigenvalues of the finite-dimensional model is precisely given as the following [27]

$$\tilde{\lambda}_k = \pm i \frac{2}{h} \sqrt{\frac{\alpha_1}{\rho}} \sin\left(\frac{(2k-1)\pi h}{2(2-h)} \right), \quad k = 1, 2, \dots, N. \tag{1.5}$$

The comparison between (1.3) and (1.5) shows that not only the discrete eigenvalues diverge away from the continuous counterparts for high-frequency eigenvalues, but also the gap between two consecutive eigenvalues $|\tilde{\lambda}_{k+1} - \tilde{\lambda}_k|$ approaches zero as h approaches zero which is not the case for the continuous eigenvalues. This is particularly not desired in control theory, since eigenvalues get extremely close to each other, and the observer may not be able distinguish one vibrational mode from another one.

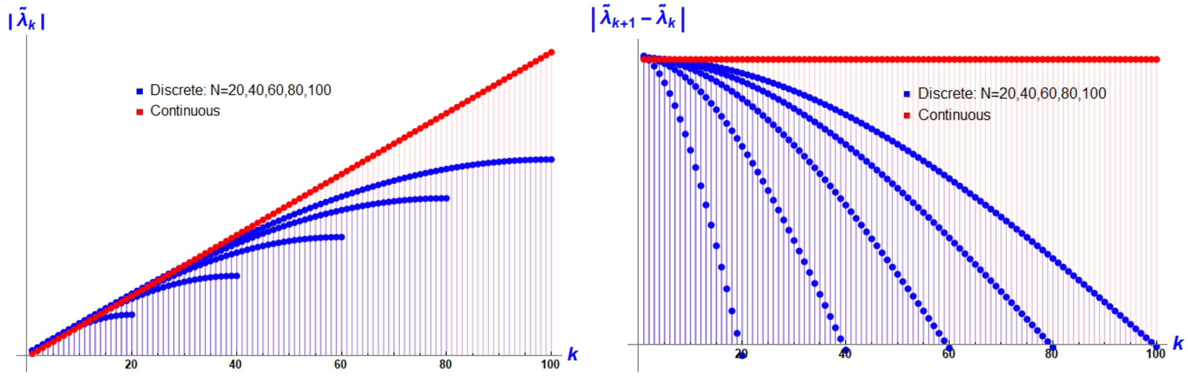

Figure 2: (a) For $\alpha_1 = \rho = 1$ in (1.2), continuous and discrete eigenvalues ($|\tilde{\lambda}_1|$,..., $|\tilde{\lambda}_k|$,...,$|\tilde{\lambda}_N|$) for $N = 20, 40, 60, 80, 100$. (b) The gap between two consecutive eigenvalues $|\tilde{\lambda}_{k+1} - \tilde{\lambda}_k| \to 0$ as $N \to \infty$ or $h \to 0$.

In fact, when describing non-observability, we look at the eigenvalues that correspond to our system, and the consequential gap between two consecutive values. For piezoelectric beam equations (1.1), there are two different branches of eigenvalues. There exists a uniform gap between any two consecutive eigenvalues for the eigenvalues on each branch, but when the spacial variable is discretized, we lose this uniform gap property as the gap becomes infinitesimally small as our discretization parameter $h \to 0$. This corresponds to two wave signals of our system that are indistinguishable from one another, which affects our ability to control the system.

In this project, we utilize the space-discretized Finite-Differences method for the piezoelectric beam equations (1.1). The first goal of this paper is to prove a lack of uniform observability for the discretized piezoelectric beam model by way of discrete multipliers. To avoid the issue demonstrated in Fig. 2, there are several methods proposed to remedy this issue, i.e. direct [2] and indirect filtering [29]. As stated before, this paper will focus on the direct Fourier filtering method to make our system exactly observable. Our second goal is to prove exact observability of the filtered solutions by directly filtering high-frequency solutions out of the system, i.e. only using a specified amount of the eigenvalues for a Fourier series, so that the eigenvalues of our discrete model closely resemble those of the continuous model. Exact observability is described here for two boundary measurements (tip beam velocity and total current accumulated at the electrodes of the beam) so that in a finite amount time, one can distinguish one solution from the other one. Proving exact observability corresponds to finding a finite value $C(T)$ for the observability inequality, as in (1.4) in the discrete setting.

To carefully demonstrate the exact observability for (1.1) and the direct Fourier filtering process for the semi-discretization of (1.1) as described above, we start with a *toy problem*; a one-dimensional wave equation with a unit wave propagation speed, i.e. $\rho = \alpha = 1$ in (1.2). In Section 2, we prove the exact observability inequality of (1.4) by the so-called multipliers method (non-spectral) and the non-harmonic Fourier series (spectral), and find the optimal observation time. All of these results are known in the literature (see [12, 13]) and we only reproduce these results for completeness. In Section 3, we semi-discretize the wave equation by Finite Differences and prove that the discretized observability inequality does not hold as the discretization parameter approaches zero. As a remedy, we apply the Fourier direct filtering technique to filter high-frequency eigenvalues. After proving several technical lemmas, we prove the observability inequality. Note that all results found in section 3 are replications of results found in [2, 11]. However, we provide all the proofs with details to prepare the reader for the rest of the thesis. In Section 4, for the novel piezoelectric beam model, we prove the exact observability inequality by the multipliers method (non-spectral) and the non-harmonic Fourier series (spectral), and find the "optimal" observation time, unlike a sub-obtimal time provided in [23]. In Section 5, we semi-discretize the piezoelectric beam equation by Finite Differences and prove that the discretized observability inequality with two observations does not hold as the discretization parameter approaches zero. We apply the Fourier direct filtering technique as in Section 3 to filter high-frequency eigenavalues, and prove the observability inequality with a sub-optimal time. The main challenge here is the non-identical wave speeds, and the non-compact coupling in (1.1).

Note that the entire novel contribution in this project lies in sections 4 and 5. The outcome is already submitted for publication [21].

2 One-dimensional Wave Equation

We will see in this section that for the single wave equation model, we generate an observability inequality with an optimal observation time, which can be proved via multipliers method [12] or by applying the so-called Ingham's Theorem [13]. These two techniques are widely used in many applications of control problems of partial differential equations. All of the results in this section have already been known in the literature (see [12, 13]), however we aim to lay the groundwork via this simplified model before the discussion of the system of coupled wave equation-model for the piezoelectric beam model in Section 4. The ideas being used in the proofs will be mimicked in Section 4 throughout several of the theorems.

2.1 Energy Solutions and Conservation of Energy

We first begin with a simple one-dimensional wave equation with clamped-free boundary conditions, modeling a string clamped on the left end, and free on the right end:

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad 0 < x < L, \quad t \geq 0 \\ u(0,t) = 0, u_x(L,t) = 0, \quad t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 < x < L \end{cases} \tag{2.1}$$

where $u(x,t)$ describes the vibration profile on the string. Define the Hilbert spaces
$\mathcal{H} = H_L^1(0,L) \times L^2(0,L)$,
$L^2(0,L) = \left\{ f(x) : \int_0^L |f(x)|^2 dx < \infty \right\}$,
$H_L^1(0,L) = \{ f(x) : f, f' \in L^2(0,L), f(0) = 0 \}$.
From [11] we know system (2.1) is well-posed in the energy space $H_L^1(0,L) \times L^2(0,L)$, i.e. for any $(u_0, u_1) \in H_L^1(0,L) \times L^2(0,L)$ there exists a unique solution
$u \in C([0,T]; H_L^1(0,L)) \cap C^1([0,T]; L^2(0,L))$. The energy of these solutions is given

by

$$E(t) = \underbrace{\frac{1}{2}\int_0^L |u_x(x,t)|^2 dx}_{\text{Potential Energy}} + \underbrace{\frac{1}{2}\int_0^L |u_t(x,t)|^2 dx}_{\text{Kinetic Energy}}, \quad t \in \mathbb{R}^+. \tag{2.2}$$

An important property of this energy is that it is conserved along the trajectories of solutions, i.e. $E(t) = E(0)$ or $\frac{d}{dt}E(t) = 0$. We pose this as in the following theorem.

Theorem 2.1 (Section 1.3, [12])**.** *For any $t \in \mathbb{R}^+$, we have that $\frac{d}{dt}E(t) = 0$.*

Proof. First, multiply (2.1) by the multiplier u_t and integrate over $[0, L]$ to get

$$\int_0^L (u_{tt} \cdot u_t - u_{xx} \cdot u_t) dt = 0. \tag{2.3}$$

Note that $u_{tt} \cdot u_t = \frac{1}{2}\frac{d}{dt}|u_t|^2$ and performing integration by parts on the second term, we get

$$\frac{1}{2}\int_0^L \frac{d}{dt}|u_t|^2 dx - u_x u_t \bigg|_0^L + \int_0^L u_x u_{xt} dx = 0. \tag{2.4}$$

Now, since $u_x u_{xt} = \frac{1}{2}\frac{d}{dt}|u_x|^2$ and $u_x u_t \bigg|_0^L = 0$ via boundary conditions,

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^L \left(|u_t|^2 + |u_x|^2\right) dx\right] = 0, \tag{2.5}$$

thus showing that energy is conserved for all $t \in \mathbb{R}^+$. □

In control theory, it is common to put PDEs in the state-space formulation, and therefore the first order form, as this allows more immediate calculations of eigenvalues and the subsequent Fourier Series. Let

$$\mathbf{z} = \begin{bmatrix} u \\ u_t \end{bmatrix}. \tag{2.6}$$

We can now re-formulate (2.1) as

$$\mathbf{z}_t = \begin{bmatrix} 0 & I \\ D_x^2 & 0 \end{bmatrix} \mathbf{z} \tag{2.7}$$

where $D_x^n = \frac{\partial^n}{\partial_x^n}$ is the second-order differential operator and $D_x^0 = I$. Now the corresponding eigenvalue problem for (2.7) is

$$\begin{bmatrix} 0 & I \\ D_x^2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{2.8}$$

(2.8) is equivalent to

$$\begin{cases} u_2 = \tilde{\lambda} u_1 \\ u_{1,xx} = \tilde{\lambda} u_2 = \tilde{\lambda}^2 u_1 \\ u_1(0) = u_{1,x}(L) = 0. \end{cases} \tag{2.9}$$

The following lemma describes the auxiliary eigenvalue problem for the operator $-D_x^2$, is necessary for finding solutions of (2.8) in the theorem after. The proofs of these results can be seen in [27].

Lemma 2.1. *The solutions to the eigenvalue problem*

$$\begin{cases} -\psi_{xx} = \lambda \psi \\ \psi(0) = \psi_x(L) = 0 \end{cases} \tag{2.10}$$

are given by

$$\omega_k = \sqrt{\lambda_k}, \quad \lambda_k = \left(\frac{(2k-1)\pi}{2L} \right)^2, \quad \psi_k = \sin\left(\omega_k x\right).$$

Since $\tilde{\lambda}_k^2 = -\lambda_k$ from (2.9) and (2.1), the following result is immediate.

Theorem 2.2 (Lemma 5, [27])**.** *The eigenvalues and eigenvectors of* (2.8) *are given* *by*

$$\tilde{\lambda}_k = \pm i\omega_k, \quad \mathbf{z}_k = \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} = \begin{bmatrix} \psi_k \\ \tilde{\lambda}_k \psi_k \end{bmatrix}.$$

We are now in position to write a Fourier series for (2.7):

$$\mathbf{z} = \begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{\infty} \left[a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}\right] \psi_k \\ \sum_{k=1}^{\infty} i\omega_k \left[a_k e^{i\omega_k t} - b_k e^{-i\omega_k t}\right] \tilde{\lambda}_k \psi_k \end{bmatrix} \tag{2.11}$$

where $a_k, b_k \in \mathbb{R}$ and can be computed explicitly in terms of the initial conditions $(u_0(x), u_1(x))$.

2.2 Proof of Uniform Observability Using Multiplier Method

The main observability inequality we seek for in this section is

$$E(0) \leq C(T) \int_0^T |u_t(L,t)|^2 \, dt \tag{2.12}$$

where $E(0)$ is the non-negative energy corresponding to the initial conditions. A physical interpretation for this inequality is that as there is no observation, i.e. $u_t(L,t) \equiv 0$, then the observer does not observe anything, meaning there is no vibration on the string, i.e. $E(0) \equiv 0$. For the exactly observable case, we will see that when $T > 2L$, the total energy, and therefore the initial state of the string, can be estimated by the boundary observation at one end. For the case of (2.1), this boundary observation is the tip velocity, $u_t(L,t)$. Here, $T = 2L$ is optimal [11].

Theorem 2.3 (Section 3.4, [12])**.** *For any* $T > 2L$ *and* $u_0, u_1 \in \mathcal{H}$*, there exists a constant* $C(T) > 0$ *such that* (2.12) *holds for every solution of* (2.1) *uniformly.*

Proof. First we multiply (2.1) by xu_x and integrate over $[0,T] \times [0,L]$:

$$\int_0^T \int_0^L (u_{tt} - u_{xx}) x u_x \, dx dt = 0. \tag{2.13}$$

Let $X_1 = \int_0^T \int_0^L u_{tt} x u_x \, dxdt$. Now integrate X_1 by parts with respect to t to obtain

$$X_1 = \int_0^L x u_x u_t \Big|_0^T dx - \int_0^T \int_0^L x u_{xt} u_t \, dxdt. \tag{2.14}$$

Note that $x u_{xt} u_t = \frac{x}{2} \frac{d}{dx} |u_t|^2$, and applying integration by parts again with respect to x we get

$$\int_0^L \int_0^T \frac{x}{2} \frac{d}{dx} |u_t|^2 dxdt = \int_0^T \frac{x}{2} |u_t|^2 \Big|_0^L dt - \int_0^T \int_0^L \frac{1}{2} |u_t|^2 \, dxdt, \tag{2.15}$$

and therefore,

$$X_1 = \int_0^L x u_x u_t \Big|_0^T dx + \int_0^T \int_0^L \frac{1}{2} |u_t|^2 dxdt - \int_0^T \frac{L}{2} |u_t(L,t)|^2 \, dt. \tag{2.16}$$

Now let $X_2 = -\int_0^T \int_0^L x u_x u_{xx}$ and note that $x u_x u_{xx} = \frac{x}{2} \frac{d}{dx} |u_x|^2$. Integrate by parts with respect to x to obtain

$$X_2 = -\int_0^T \frac{x}{2} |u_x|^2 \Big|_0^L dt + \int_0^T \int_0^L \frac{1}{2} |u_x|^2 \, dxdt, \tag{2.17}$$

and by the boundary conditions $\frac{x}{2} |u_x|^2 \Big|_0^L = 0$. We can see that $X_1 + X_2 = 0$, and keeping in mind conservation of energy, $\frac{dE}{dt} = 0$, combine (2.16) and (2.17) to get

$$\begin{aligned} X_1 + X_2 \;\; &= \\ &\int_0^L x u_x u_t \Big|_0^T dx + \int_0^T \int_0^L \tfrac{1}{2} |u_t|^2 \, dxdt - \int_0^T \tfrac{L}{2} |u_t(L,t)|^2 dt \\ &\quad + \int_0^T \int_0^L \tfrac{1}{2} |u_x|^2 \, dxdt \\ &= 0. \end{aligned}$$

Combining the energy terms,

$$\int_0^L x u_x u_t \Big|_0^T \, dx + \int_0^T E(0) dt = \frac{L}{2} \int_0^T |u_t(L,t)|^2 \, dt. \tag{2.18}$$

Since energy is constant,

$$\int_0^L x u_x u_t \Big|_0^T \, dx + TE(0) = \frac{L}{2} \int_0^T |u_t(L,t)|^2 \, dt. \tag{2.19}$$

By Young's inequality, we can see that

$$\begin{aligned} \left| \int_0^L x u_x u_t \Big|_0^T \, dx \right| &\leq \int_0^L |x u_x u_t| \Big|_0^T \, dx \\ &\leq \int_0^L x |u_x u_t| \Big|_0^T \, dx \\ &\leq \frac{L}{2} \int_0^L (|u_x|^2 + |u_t|^2) \Big|_0^T \, dx \\ &= |LE(T) - LE(0)| \\ &\leq |LE(T)| + |LE(0)| \\ &= 2LE(0). \end{aligned} \tag{2.20}$$

Applying (2.20) to (2.18) we get

$$TE(0) - 2LE(0) \leq \frac{L}{2} \int_0^T |u_t(L,t)|^2 \, dt,$$

and therefore

$$E(0) \leq \frac{L}{2(T-2L)} \int_0^T |u_t(L,t)|^2 \, dt. \tag{2.21}$$

Note that $T > 2L$ to make the observability constant $\frac{L}{2(T-2L)}$ positive. □

2.3 Proof of Uniform Observability Using Ingham's Theorem

In this section, we will now focus on proving Theorem 8 via Ingham's Theorem. The motivation is to find an optimal observation time without the use of multipliers. Note that when applying the multiplier method to the continuous single wave equation, we already generated an optimal control time, but we show that for the coupled wave equation model, the multiplier method results in a sub-optimal observation time, and so Ingham's Theorem will be more beneficial.

We first begin this section with a fact from functional analysis that in the $L^2(0, L)$ space, both $\{\sin(\omega_k x)\}_{k\geq 1}$ and $\{\cos(\omega_k x)\}_{k\geq 1}$ form an orthogonal basis for some sequence ω_k ([14], pg. 154). In our case, we have that $\omega_k = \sqrt{\lambda_k} = \frac{2}{h}\sin\left(\frac{(2k+1)\pi h}{2(2L-h)}\right)$. The following lemma shows how the energy of solutions can be written in terms of our Fourier coefficients in (2.11).

Lemma 2.2 (Section 3.4, [13]). *For a_k and b_k, the Fourier coefficients in* (2.11), *$E(t)$ can be written as*

$$E(t) = \frac{1}{2}\sum_{k=1}^{\infty}\omega_k^2(|a_k|^2 + |b_k|^2).$$

Proof. By (2.11),

$$u_x(x,t) = \sum_{k=1}^{\infty}(a_k e^{i\omega_k t} + b_k e^{-i\omega_k t})\psi_{k,x},$$
$$u_t(x,t) = \sum_{k=1}^{\infty} i\omega_k(a_k e^{i\omega_k t} - b_k e^{-i\omega_k t})\psi_k,$$

where $\psi_{k,x} = \omega_k\cos(\omega_k x)$. It is known that $\{\sin(\omega_k x)\}_{k\geq 1}$ is an orthogonal basis in $L^2(0, L)$ [14]. Since $\{\sin(\omega_k x)\}_{k\geq 1}$ satisfies the left-end boundary condition as well, $\mathbf{z_k} = \begin{bmatrix}\sin(\omega_k x)\\ \tilde{\lambda}\sin(\omega_k x)\end{bmatrix}$ forms a basis in $H_L^1(0, L)\times L^2(0, L)$. We use this property to find the following equations where terms with $\int_0^L \sin(\omega_m x)\sin(\omega_n x)dx$ and

$\int_0^L \cos(\omega_m x)\cos(\omega_n x)dx$ are eliminated for $m \neq n$ since

$$\begin{aligned}
\int_0^L |u_x(x,0)|^2 dx &= \int_0^L \left| \sum_{k=1}^{\infty} \omega_k (a_k + b_k)\cos(\omega_k x) \right|^2 dx \\
&= \sum_{k=1}^{\infty} \omega_k^2 |a_k + b_k|^2 \underbrace{\int_0^L \cos^2(\omega_k x)\, dx}_{=\frac{1}{2}} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \omega_k^2 |a_k + b_k|^2, \qquad (2.22)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^L |u_t(x,0)|^2\, dx &= \int_0^L \left| \sum_{k=1}^{\infty} i\omega_k (a_k - b_k)\sin(\omega_k x) \right|^2 dx \\
&= \sum_{k=1}^{\infty} \omega_k^2 |a_k - b_k|^2 \underbrace{\int_0^L \sin^2(\omega_k x)\, dx}_{=\frac{1}{2}} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \omega_k^2 |a_k - b_k|^2. \qquad (2.23)
\end{aligned}$$

Now we use (2.22) and (2.23) to find $E(0)$:

$$\begin{aligned}
E(0) &= \frac{1}{2}\int_0^L |u_x(x,0)|^2\, dx + \frac{1}{2}\int_0^L |u_t(x,0)|^2\, dx \\
&= \frac{1}{4}\sum_{k=1}^{\infty} \omega_k^2 |a_k + b_k|^2 + \frac{1}{4}\sum_{k=1}^{\infty} \omega_k^2 |a_k - b_k|^2 \\
&= \frac{1}{4}\sum_{k=1}^{\infty} w_k^2 (|a_k + b_k|^2 + |a_k - b_k|^2) \\
&= \frac{1}{2}\sum_{k=1}^{\infty} w_k^2 (|a_k|^2 + |b_k|^2). \qquad \square
\end{aligned}$$

Since we know that $\sum_{k=1}^{\infty} \omega_k^2(|a_k|^2 + |b_k|^2) = 2E(0)$, and energy is finite, then $\sum_{k=1}^{\infty} \omega_k^2(|a_k|^2 + |b_k|^2) < \infty$.

Now we give a formal statement of Ingham's Theorem, as in [12].

Theorem 2.4 (Ingham's Theorem, Section 4.3, [13]). *Let* $\{\omega_k\}_{k\in K}$ *be a family of real numbers, satisfying the uniform gap condition*

$$\tau := \inf_{k\neq n}|\omega_k - \omega_n| > 0. \tag{2.24}$$

If I *is a bounded interval of length* $|I| > 2\pi/\tau$*, then there exists constants* $c_1, c_2 > 0$ *such that*

$$c_1 \sum_{k\in K} |x_k|^2 \leq \int_I |x(t)|^2 dt \leq c_2 \sum_{k\in K} |x_k|^2 \tag{2.25}$$

for all functions given by the sum

$$x(t) = \sum_{k\in K} x_k e^{i\omega_k t} \tag{2.26}$$

with square-summable complex coefficients x_k*, i.e.* $\sum\limits_{k\in K} |x_k|^2 < \infty$.

Now we provide a newer version of Theorem 2.3 using Ingham's Theorem, where we look at the boundary observation and how we can bound it below by the energy.

Theorem 2.5 (Section 4.5, [13]). *For* $T > 2L$ *there exists positive constants* $c_1, c_2 > 0$ *such that for all initial conditions* $(u^0, u^1) \in \mathcal{H}$ *we have*

$$c_1 E(0) \leq \int_0^T |u_t(L,t)|^2 dt \leq c_2 E(0).$$

Proof. First, we show the uniform gap condition, as in Ingham's Theorem, is satisfied.

$$\tau = \inf_{k\neq n} |\omega_k - \omega_n| = \inf_{k\neq n} \left|\frac{(2k-1)\pi}{2L} - \frac{(2n-1)\pi}{2L}\right| = \inf_{k\neq n}\left|\frac{(k-n)\pi}{L}\right| = \frac{\pi}{L}.$$

Let $\omega_{-k} = -\omega_k$ and $a_{-k} = -b_k$ for $k = 1, 2, ..., \mathbb{Z} \setminus \{0\} = \mathbb{Z}^*$. Hence we have

$$u_t(L,t) = \sum_{k=1}^{\infty} i\omega_k(a_k e^{i\omega_k t} - b_k e^{-i\omega_k}) \sin(\omega_k L)$$
$$= \sum_{k\in\mathbb{Z}^*} i\omega_k a_k e^{i\omega_k t} \sin(\omega_k L),$$

and keeping in mind that $\omega_k = \frac{(2k-1)\pi}{2L}$, we can rewrite $\sin(\omega_k L)$:

$$u_t(L,t) = \sum_{k\in\mathbb{Z}^*} i\omega_k a_k e^{i\omega_k t}(-1)^{2k+1}.$$

By our reference again to having finite energy in Theorem 2.1, we know coefficients are square summable as

$$\sum_{k\in\mathbb{Z}^*} |i\omega_k a_k(-1)^{2k+1}|^2 = \sum_{k=1}^{\infty} \omega_k^2(|a_k|^2 - |b_k|^2) \leq \underbrace{\sum_{k=1}^{\infty} w_k^2(|a_k|^2 + |b_k|^2)}_{E(t)} < \infty.$$

Hence, by Ingham's inequality, for $T > \frac{2\pi}{\tau} = 2L$, there exists real constants $c_1, c_2 > 0$ such that

$$c_1 \sum_{k=1}^{\infty} w_k^2(|a_k|^2 + |b_k|^2) \leq \int_0^T |u_t(L,t)|^2 dt \leq c_2 \sum_{k=1}^{\infty} w_k^2(|a_k|^2 + |b_k|^2),$$

and so

$$C_1 E(0) \leq \int_0^T |u_t(L,t)|^2 dt \leq C_2 E(0), \tag{2.27}$$

where $C_1 = 2c_1$, and $C_2 = 2c_2$. □

By bounding the observation from below only in (2.27), and taking $C(T) = \frac{1}{C_1}$ in (2.12), we generate an equal inequality which proves Theorem 2.3, with notably optimal time $T = 2L$.

This proof ultimately ends the discussion for the continuous case of the wave equation as we have an optimal, uniform observation time. However, we desire a more physically applicable space-discretized model that has the same uniform observability property after applying a filtering technique, as already seen in [2], as the continuous wave equation.

3 Finite Difference Space-Discretized Wave Equation

There are several numerical methods that serve to discretize the spacial component of our wave equation such as the finite element method, Galerkin's method, finite difference method, etc. for both Dirichlet and mixed boundary conditions (see [2], [11], and the references therein). We aim to look specifically at discretizing the spacial variable of the one-dimensional wave eqution by central Finite Differences. The advantage to this method in particular is that it is not very computationally expensive, and results for observability are fairly immediate, allowing us to draw meaningful insights about the system.

In this section, the goal is to show that the discretized wave equation loses the positive exact observability result with a boundary observation. Similar to section 2, all of the results in this section have already been proved by others (see [2] for clamped-free boundary conditions, [11] for fully clamped boundary conditions). However, the precise details in [2] are laid out for each proof so that we can mimic the ideas for the coupled wave equations for the clamped-free piezoelectric beam model in Section 5. Below, we introduce the semi-discretized version of (2.1) where we discretize only the spacial variable using the aforementioned central Finite Difference Method.

3.1 Discrete Spectral Analysis and Development of Solutions

Let $u(x_j, t) \approx u_j(t)$, where $u(x_j, t)$ is the approximation of $u(x, t)$ at $x = x_j$, so given $N \in \mathbb{N}$, we set $h = \frac{L}{N+1}$ to discretize the interval $[0, L]$ as follows:

$$x_0 = 0 < x_1 = h < ... < x_N = Nh < x_{N+1} = L, \tag{3.1}$$

where $x_j = jh, j = 0, ..., N+1$. We then use the central difference formula $u_{xx}(x_j, t) \approx \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2}$, and the backwards difference formula $u_x(x_j, t) \approx \frac{u_{j+1}-u_j}{h}$ to produce the following finite-difference semi-discretization of (2.1):

$$\begin{cases} u_j'' = \frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2}, \quad 0 < t < T, j = 1, \dots, N \\ u_0 = 0, u_{N+1} = u_N, \quad 0 < t < T \\ u_j(0) = u_j^0, u_j'(0) = u_j^1, \quad j = 1, \dots, N, \end{cases} \tag{3.2}$$

where prime notation denotes derivation with respect to time t. We can see that system (3.2) is a system of N linear differential equations with N unknowns, namely $u_1, \dots, u_N$, since by virtue of our boundary conditions,$u_0 \equiv 0$ and $u_{N+1} \equiv u_N$, i.e. $u_x(L, t) \approx \frac{u_{N+1}-u_N}{h} = 0$. The discrete energy of the solutions of (3.2) is given by

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[|u_j'|^2 + \left| \frac{u_{j+1}-u_j}{h} \right|^2 \right]. \tag{3.3}$$

We see next that the discrete energy, $E_h(t)$, is also conserved along time, analogous to the conserved energy $E(t)$ in Theorem 2.1.

Lemma 3.1 (Lemma 3.1, [2])**.** *For any $h > 0$ and the solution $\mathbf{u}(t) = [u_1, \dots, u_N]^T$ of* (3.1)*, we have*

$$E_h(t) = E_h(0), \quad \forall 0 < t \le T. \tag{3.4}$$

Proof. Take (3.3) and shift index from $j = 0$ to $j = 1$, then differentiate with respect to t to produce

$$\begin{aligned} E_h'(t) &= h \sum_{j=1}^{N} \left[u_j' u_j'' + \left(\frac{u_{j+1}-u_j}{h} \right) \left(\frac{u_{j+1}'-u_j'}{h} \right) \right] \\ &+ h \left[u_0' u_0'' + \left(\frac{u_1-u_0}{h} \right) \left(\frac{u_1'-u_0'}{h} \right) \right]. \end{aligned}$$

From (3.2) we know $u_j'' = \frac{u_{j+1}-2u_j+u_{j-1}}{h^2}$, and keeping in mind the boundary conditions,

$$E_h'(t) = \frac{1}{h}\sum_{j=1}^{N}[u_j'(u_{j+1} - 2u_j + u_{j-1}) + (u_{j+1} - u_j)(u_{j+1}' - u_j')] + \frac{u_1 u_1'}{h}.$$

By rearranging the terms, we obtain

$$\begin{aligned} E_h'(t) &= \frac{1}{h}\sum_{j=1}^{N}[u_j' u_{j+1} - 2u_j' u_j + u_j' u_{j-1} + u_{j+1}u_{j+1}' - u_{j+1}u_j' - u_j u_{j+1}' + u_j u_j'] + \frac{u_1 u_1'}{h} \\ &= \frac{1}{h}\sum_{j=1}^{N}[u_j'(u_{j-1} - u_j) + u_{j+1}'(u_{j+1} - u_j)] + \frac{u_1 u_1'}{h} \\ &= \frac{1}{h}[u_0 u_1' - u_N u_{N+1}' - u_1 u_1' + u_{N+1}u_{N+1}'] + \frac{u_1 u_1'}{h} = 0. \end{aligned}$$

□

As was done for the continuous case, we put the discrete model in its state-space formulation. Choose $\mathbf{z}(t) = [u_1, u_2, ..., u_N, u_1', u_2', \ldots, u_N']^T$ so that the discretized model can be re-written in the first order form as

$$\mathbf{z}' = \begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix} \mathbf{z},$$

where A_h is given by

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix}_{N\times N.} \tag{3.5}$$

In order to analyze the issue of non-observability of the discrete model, it is necessary to look at the eigenvalue problem corresponding to (3.2), and subsequent eigenvalues and eigenvectors of that system. Consider the eigenvalue problem

$$\begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}. \tag{3.6}$$

Similar to the continuous system, the eigenvalues and eigenvectors of (3.6) are related to the eigenvalues and eigenvectors of the auxiliary problem where we only look at the discretized second-derivative operator. This new problem is given by

$$\begin{cases} A_h \boldsymbol{\varphi} = \lambda \boldsymbol{\varphi}, \\ \varphi_0 = 0, \varphi_{N+1} = \varphi_N. \end{cases} \tag{3.7}$$

Theorem 3.6 ([2, 27])**.** *The eigenvalues and eigenvectors of* (3.7) *are given by*

$$\begin{cases} \lambda_k = \frac{4}{h^2} \sin^2\left(\frac{(2k+1)\pi h}{2(2-h)}\right), & k = 1, 2, \ldots, N \\ \varphi_{k,j} = \sin\left(\frac{(2k+1)\pi j h}{2-h}\right), & j = 1, 2, \ldots, N. \end{cases} \tag{3.8}$$

These eigenvalues and eigenvectors have been computed explicitly (see [27]). It is known that the relationship between the eigenvalues and eigenvectors of (3.6) and (3.7) are the same as in the continuous case. For simplicity, let $\varphi^k = [\varphi_{k,1}, \ldots, \varphi_{k,n}], k = 1, \ldots, N$.

Theorem 3.7 (Theorem 3, [27])**.** *The eigenvalues and eigenvectors of* (3.6) *are given by*

$$\tilde{\lambda}_k = \pm i\sqrt{\lambda_k}, \quad \begin{bmatrix} \mathbf{z}_{1,k} \\ \mathbf{z}_{2,k} \end{bmatrix} = \begin{bmatrix} \varphi^k \\ \tilde{\lambda}\varphi^k \end{bmatrix}. \tag{3.9}$$

Proof. (3.9) is equivalent to

$$\begin{cases} \mathbf{z}_2 = \tilde{\lambda}\mathbf{z_1} \\ -A_h \mathbf{z}_1 = \tilde{\lambda}\mathbf{z}_2 = \tilde{\lambda}^2 \mathbf{z}_1, \end{cases} \tag{3.10}$$

therefore $\tilde{\lambda}_k = \pm i\sqrt{\lambda_k}$. □

We can now write every solution $\mathbf{u}(t) = [u_1, \ldots, u_N]^T$ of (3.2) as a Fourier series, with $\omega_k = \sqrt{\lambda_k}$:

$$\mathbf{u}(t) = \sum_{k=1}^{N} \left[a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}\right] \varphi^k, \tag{3.11}$$

for a_k and $b_k \in \mathbb{R}, k = 1 \ldots N$, which can be computed via initial data u_j^0 and u_j^1.

3.2 Lack of Uniform Observability with Respect to the Discretization Parameter

The reason we had uniform observability for the continuous wave equation lies in the fact that the gap between two consecutive eigenvalues of (2.1) is $\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \frac{(2k+1)\pi}{2L} - \frac{(2k-1)\pi}{2L} = \frac{\pi}{L}$, which is independent of k, implying uniform observability by Ingham's Theorem. However, when we look at the gap between two consecutive eigenvalues of (3.6) we get

$$\begin{aligned}\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} &= \tfrac{2}{h}\left[\sin\left(\tfrac{(2k+3)\pi h}{2(2L-h)}\right) - \sin\left(\tfrac{(2k+1)\pi h}{2(2L-h)}\right)\right] \\ &= \tfrac{2}{h}\left[\sin\left(\tfrac{k\pi h}{2L-h} + \tfrac{3\pi h}{2(2L-h)}\right) - \sin\left(\tfrac{k\pi h}{2L-h} + \tfrac{\pi h}{2(2L-h)}\right)\right].\end{aligned}$$

Now, using the sum to product trig identity

$$\begin{aligned}&= \tfrac{2}{h}\left[\sin\left(\tfrac{k\pi h}{2L-h}\right)\cos\left(\tfrac{3\pi h}{2(2L-h)}\right) + \cos\left(\tfrac{k\pi h}{2L-h}\right)\sin\left(\tfrac{3\pi h}{2(2L-h)}\right)\right. \\ &\quad \left. - \left(\sin\left(\tfrac{k\pi h}{2L-h}\right)\cos\left(\tfrac{\pi h}{2(2L-h)}\right) + \cos\left(\tfrac{k\pi h}{2L-h}\right)\sin\left(\tfrac{\pi h}{2(2L-h)}\right)\right)\right] \\ &= \tfrac{2}{h}\left[\sin\left(\tfrac{k\pi h}{2L-h}\right)\left(\cos\left(\tfrac{3\pi h}{2(2L-h)}\right) - \cos\left(\tfrac{\pi h}{2(2L-h)}\right)\right)\right. \\ &\quad \left. + \cos\left(\tfrac{k\pi h}{2L-h}\right)\left(\sin\left(\tfrac{3\pi h}{2(2L-h)}\right) - \sin\left(\tfrac{\pi h}{2(2L-h)}\right)\right)\right].\end{aligned}$$

We note that since $L = (N+1)h$, then $2L-h > 0$ since $2(N+1) > 1$. It is known that $-1 \leq \sin(x) \leq 1\ \forall x \in \mathbb{R}$, and by a quick application of the Mean Value Theorem, we see also that $\sin(x) \leq x$ for every $x \geq 0$. Using this, we see

$$\begin{aligned}\sin\left(\frac{3\pi h}{2(2L-h)}\right)-\sin\left(\frac{\pi h}{2(2L-h)}\right) &\leq \left|\sin\left(\frac{3\pi h}{2(2L-h)}\right)-\sin\left(\frac{\pi h}{2(2L-h)}\right)\right| \\ &\leq \left|\sin\left(\frac{3\pi h}{2(2L-h)}\right)\right|+\left|\sin\left(\frac{\pi h}{2(2L-h)}\right)\right| \\ &\leq \frac{2\pi h}{2L-h}.\end{aligned}$$

This now gives

$$\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k}\leq \frac{2}{h}\left(\cos\left(\frac{3\pi h}{2(2L-h)}\right)-\cos\left(\frac{\pi h}{2(2L-h)}\right)\right)+\frac{4\pi}{2L-h}\cos\left(\frac{k\pi h}{2L-h}\right).$$

Recalling that $x-y\leq|y-x|$ for every $x,y\in\mathbb{R}$ we get

$$\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k}\leq \frac{2}{h}\left|\cos\left(\frac{\pi h}{2(2L-h)}\right)-\cos\left(\frac{3\pi h}{2(2L-h)}\right)\right|+\frac{4\pi}{2L-h}\cos\left(\frac{k\pi h}{2L-h}\right).$$

Using the triple angle identity $\cos 3x=4\cos^3 x-3\cos x$, we see that $\cos x-\cos 3x=4\cos x\sin^2 x$. Here, $x=\frac{\pi h}{2(2L-h)}\geq 0$. For all positive real x values, we generate an upper bound for $\cos x$ as x^2+1, and also recall that $\sin x\leq x$. Using this, we observe an upper bound for $4\cos x\sin^2 x$ as $4(x^2+1)x^2=4x^4+4x^2$. This gives

$$\begin{aligned}\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k} &\leq \frac{2}{h}\left|\frac{4\pi^4h^4}{16(2L-h)^4}+\frac{4\pi^2h^2}{4(2L-h)^2}\right|+\frac{4\pi}{2L-h}\cos\left(\frac{k\pi h}{2L-h}\right) \\ &= \frac{\pi^4h^3}{2(2L-h)^4}+\frac{2\pi^2 h}{4(2L-h)^2}+\frac{4\pi}{2L-h}\cos\left(\frac{k\pi h}{2L-h}\right).\end{aligned}$$

Looking at the remaining cosine term, we add and subtract $\frac{\pi}{2}$ from the inside, so

$$
\begin{aligned}
\tfrac{4\pi}{2L-h}\cos\left(\tfrac{k\pi h}{2L-h}\right) &= \tfrac{4\pi}{2L-h}\cos\left(\tfrac{\pi}{2}+\tfrac{k\pi h-L\pi+\frac{h\pi}{2}}{2L-h}\right)\\
&= \tfrac{4\pi}{2L-h}\sin\left(\tfrac{L\pi-k\pi h-\frac{h\pi}{2}}{2L-h}\right)\\
&= \tfrac{4\pi}{2L-h}\sin\left(\tfrac{(\frac{L}{h}-k)h\pi-\frac{h\pi}{2}}{2L-h}\right)\\
&= \tfrac{4\pi}{2L-h}\sin\left(\tfrac{((N+1)-k)h\pi-\frac{h\pi}{2}}{2L-h}\right).
\end{aligned}
$$

As soon as $N+1-k \leq j \iff k \geq N+1-j$ for $j \in \mathbb{N}$,

$$
\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k} \leq \tfrac{\pi^2 h}{(2L-h)^2}\left(4j-2+\tfrac{\pi^2h^2}{2(2L-h)^2}+2\right) = \tfrac{\pi^2 h}{(2L-h)^2}\left(4j+\tfrac{\pi^2h^2}{2(2L-h)^2}\right), \quad (3.12)
$$

and so we can now bound the gap between the two largest eigenvalues as follows:

$$
\sqrt{\lambda_N}-\sqrt{\lambda_{N-1}} \leq \frac{\pi^2 h}{(2L-h)^2}\left(8+\frac{\pi^2h^2}{2(2L-h)^2}\right) \to 0 \text{ as } h \to 0. \quad (3.13)
$$

The goal of this section is to analyze the discrete version of (2.12) given by

$$
E_h(0) \leq C(T,h)\int_0^T |u_N'(t)|^2 dt, \quad (3.14)
$$

and ultimately show that the constant $C(T,h)$ blows up as $h \to 0$. It has already been proven using the multiplier technique that for high frequency solutions $u_N(t)$ of (3.2) corresponding to eigenvalue λ_N, as $h \to 0, C(T,h) \to \infty$ [2]. To understand the development of the proof, we first turn to the following lemmas.

Lemma 3.2 (Lemma 2.1, [2]). *For any eigenvector $\boldsymbol{\varphi} = [\varphi_1, \ldots, \varphi_N]^T$ with eigenvalue λ of system* (3.7) *the following identities hold:*

$$
\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}-\varphi_j}{h}\right|^2 = \lambda\sum_{j=1}^{N}\varphi_j^2, \quad (3.15)
$$

$$h\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}-\varphi_j}{h}\right|^2=\frac{\lambda h^2(2-h)}{4-\lambda h^2}\left|\frac{\varphi_N}{h}\right|^2. \tag{3.16}$$

Proof. Begin by multiplying (3.7) by φ_j to get

$$-\frac{1}{h^2}\sum_{j=1}^{N}(\varphi_{j+1}-2\varphi_j+\varphi_{j-1})\varphi_j=\lambda\sum_{j=1}^{N}\varphi_j^2.$$

Distributing,

$$-\frac{1}{h^2}\sum_{j=1}^{N}(\varphi_{j+1}\varphi_j-2\varphi_j^2+\varphi_{j-1}\varphi_j)=\lambda\sum_{j=1}^{N}\varphi_j^2.$$

Keeping in mind the boundary conditions, we take into account that

$$\begin{aligned}\sum_{j=1}^{N}(\varphi_{j+1}\varphi_j+\varphi_{j-1}\varphi_j)&=(\varphi_2\varphi_1+\varphi_0\varphi_1)+(\varphi_3\varphi_2+\varphi_1\varphi_2)+\ldots+(\varphi_{N+1}\varphi_N+\varphi_{N-1}\varphi_N)\\&=2\sum_{j=1}^{N}\varphi_{j+1}\varphi_j-\frac{1}{h^2}\varphi_N^2.\end{aligned}$$

Now adding like-terms,

$$\frac{1}{h^2}\sum_{j=1}^{N}(2\varphi_j^2-2\varphi_{j+1}\varphi_j)+\frac{\varphi_N^2}{h^2}=\lambda\sum_{j=1}^{N}\varphi_j^2. \tag{3.17}$$

Rewriting the $2\varphi_j^2$ term as $\varphi_{j+1}^2+\varphi_j^2$, we absorb the boundary term and get

$$\frac{1}{h^2}\sum_{j=0}^{N}(\varphi_{j+1}^2-2\varphi_{j+1}\varphi_j+\varphi_j^2)=\lambda\sum_{j=1}^{N}\varphi_j^2,$$

which is exactly (3.15). □

Proof. To prove (3.16) multiply (3.1) by $-j(\varphi_{j+1}-\varphi_{j-1})$, which is the discrete version of $x\varphi_x$, to obtain

$$\frac{1}{h^2}\sum_{j=1}^{N} j(\varphi_{j+1}-2\varphi_j+\varphi_{j-1})(\varphi_{j+1}-\varphi_{j-1}) = -\lambda\sum_{j=1}^{N} j\varphi_j(\varphi_{j+1}-\varphi_{j-1}).$$

Rearranging the terms,

$$\begin{aligned}\frac{1}{h^2}\sum_{j=1}^{N}[(j-1)\varphi_j^2 - 2j\varphi_{j+1}\varphi_j + 2(j+1)\varphi_{j+1}\varphi_j - (j+1)\varphi_j^2] - \frac{\varphi_N^2}{h^2}\\ = -\lambda\sum_{j=1}^{N}[j\varphi_{j+1}\varphi_j - (j+1)\varphi_{j+1}\varphi_j] - \lambda(N+1)\varphi_N^2.\end{aligned}$$

Keeping in mind that $(N+1)=\frac{L}{h}$, we get

$$\frac{1}{h^2}\sum_{j=1}^{N}[-2\varphi_j^2+2\varphi_{j+1}\varphi_j] - \frac{\varphi_N^2}{h^2} = \lambda\sum_{j=1}^{N}\varphi_{j+1}\varphi_j - \frac{\lambda L}{h}\varphi_N^2.$$

Now, combining like-terms,

$$\left(\frac{\lambda L}{h}-\frac{1}{h^2}\right)\varphi_N^2 = \frac{2}{h^2}\sum_{j=1}^{N}\varphi_j^2 + \left(\lambda-\frac{2}{h^2}\right)\sum_{j=1}^{N}\varphi_{j+1}\varphi_j, \tag{3.18}$$

and using (3.17), we can deduce that

$$\frac{2}{h^2}\sum_{j=1}^{N}\varphi_{j+1}\varphi_j = \left(\frac{2}{h^2}-\lambda\right)\sum_{j=1}^{N}\varphi_j^2 + \frac{\varphi_N^2}{h^2}. \tag{3.19}$$

Next, we normalize the eigenvector $\boldsymbol{\varphi}$ by the relation $h\sum_{j=1}^{N}\varphi_j^2 = 1$, and substitute into (3.18) and (3.19) to obtain

$$\sum_{j=1}^{N}\varphi_{j+1}\varphi_j = \frac{h}{2}\left(\frac{2}{h^2}-\lambda\right) + \frac{\varphi_N^2}{2}. \tag{3.19*}$$

$$\left(\lambda - \frac{2}{h^2}\right) \sum \varphi_{j+1}\varphi_j = -\frac{2}{h^3} + \left(\frac{\lambda}{h} - \frac{1}{h^2}\right) \varphi_N^2. \tag{3.18*}$$

Combining (3.15), (3.18*), and (3.19*) we get that

$$\frac{\lambda h^2(2-h)}{4-\lambda h^2} \left|\frac{\varphi_N}{h}\right|^2 = \lambda = h \sum_{j=0}^{N} \left|\frac{\varphi_{j+1} - \varphi_j}{h}\right|^2,$$

which is precisely (3.16) □

We are now ready to state the main theorem in this section.

Theorem 3.8 (Theorem 2.2, [2])**.** *For any $T > 0$, we have*

$$\underset{u \ sol \ of \ (3.2)}{sup} \left[\frac{E_h(0)}{\int_0^T |u_N'(t)|^2 dt}\right] \to \infty \ as \ h \to 0. \tag{3.20}$$

Proof. Consider the particular solution of (3.1) corresponding to the Nth eigenvector:

$$\boldsymbol{u}(t) = e^{i\omega_N t}\varphi^N,$$

so

$$u'(t) = i\omega_N e^{i\omega_N t}\varphi^N,$$

and

$$u_N'(t) = i\omega_N e^{i\omega_N t}\varphi^{N,N}.$$

Keeping in mind $|e^{ix}| = |i| = 1 \ \forall x \in \mathbb{R}$, and φ^k independent of t, we get that

$$\int_0^T \left|u_N'(t)\right|^2 dt = T\omega_N^2|\varphi_{N,N}|^2 = T\lambda_N|\varphi_{N,N}|^2. \tag{3.21}$$

Since energy is conserved, we have $E_h(t) = E_h(0)$, so we write the energy in terms of our solution:

$$\begin{aligned} E_h(0) &= \frac{h}{2}\sum_{j=0}^{N}\left[\left|i\sqrt{\lambda_N}e^{i\sqrt{\lambda_N}(0)}\varphi^N\right|^2 + \left|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\right|^2\right] \\ &= \frac{h}{2}\sum_{j=0}^{N}\left[\lambda_N|\varphi_{N,j}|^2 + \left|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\right|^2\right], \end{aligned}$$

but using (3.15) we get that

$$E_h(0) = h\sum_{j=0}^{N}\left[\left|\frac{\varphi_{N,j}-\varphi_{N,j+1}}{h}\right|^2\right] = \frac{\lambda_N h^2(2-h)}{4-\lambda_N h^2}\left|\frac{\varphi_{N,N}}{h}\right|^2. \tag{3.22}$$

Taking now the ratio between energy and boundary observation we get

$$\frac{E_h(0)}{\int_0^T |u_N'|^2 dt} = \frac{2-h}{T(4-\lambda_N h^2)}, \tag{3.23}$$

but looking at $\lambda_N h^2$,

$$\begin{aligned} \lambda_N h^2 &= 4\sin^2\left(\tfrac{(2N+1)\pi h}{2(2L-h)}\right) \\ &= 4\sin^2\left(\tfrac{N\pi h}{2L-h} + \tfrac{\pi h}{2(2L-h)}\right) \\ &= 4\sin^2\left(\tfrac{(\frac{L}{h}-1)\pi h}{2L-h} + \tfrac{\pi h}{2(2L-h)}\right) \\ &= 4\sin^2\left(\tfrac{L\pi-\pi h}{2L-h} + \tfrac{\pi h}{2(2L-h)}\right) = 4\sin^2(\tfrac{\pi}{2}) = 4 \text{ as } h \to 0, \end{aligned}$$

so we get that

$$\lambda_N h^2 \to 4 \quad \text{as} \quad h \to 0 \tag{3.24}$$

and thus $\frac{E_h(0)}{\int_0^T |u_N'|^2 dt} \to \infty$ as $h \to 0$. □

3.3 Uniform Observability of Filtered Solutions with Respect to the Discretization Parameter

By considering a particular solution of (3.2), we showed that for small h, $\int_0^T |u'_N(t)|^2 \; dt \to 0$ which makes our $C(T)$ blow up, which was exactly done in [2]. To remedy this problem, one technique has been proposed [11], called direct Fourier filtering, whereby we eliminate the spurious eigen-solutions of (3.2). This is done by introducing suitable classes of solutions $C_h(\gamma, T)$ generated by eigenvectors of (3.2) associated with eigenvalues such that $\lambda h^2 < \gamma$, which corresponds to an appropriate truncation of the Fourier series of (3.2) by filtering the spurious high-frequency eigenvalues due to a blind use of Finite Differences, see figure 2 and [2] for more details. Here, γ can also be referred to as the filtering parameter. Define the class of initial data of (3.2) generated by eigenvectors of (3.6) as:

$$C_h(\gamma) := \left\{ \sum_{\omega_k^2 h^2 < \gamma} a_k \varphi_k \right\} \tag{3.25}$$

where $0 < \gamma < 4$.

To show the main result in this section, we require the following three lemmas.

Lemma 3.3 (Lemma 3.2, [2]). *For any solution u of* (2.1)*, we have*

$$\frac{h}{2} \sum_{j=0}^{N} \int_0^L \left[u'_j u'_{j+1} \left| \frac{u_j - u_{j+1}}{h} \right|^2 \right] \; dt + X_h(t) \Big|_0^T = \frac{L}{2} \int_0^T |u'_N|^2 \; dt, \tag{3.26}$$

where $X_h(t) = h \sum_{j=1}^{N} j u'_j \left(\frac{u_{j+1} - u_{j-1}}{2} \right)$.

Proof. Multiplying (3.2) by $j \frac{u_{j+1} - u_j}{2}$ and integrating over $[0, T]$, we obtain

$$\begin{aligned} &\sum_{j=1}^{N} \int_0^T j u''_j \left(\tfrac{u_{j+1} - u_{j-1}}{2} \right) \; dt \\ &= \tfrac{1}{h^2} \sum_{j=1}^{N} \int_0^T j \left(\tfrac{u_{j+1} - u_{j-1}}{2} \right) (u_{j+1} - 2u_j + u_{j-1}) \; dt. \end{aligned} \tag{3.27}$$

Keeping in mind $X_h(t)$, we integrate by parts to obtain

$$\begin{aligned}\sum_{j=1}^{N}\int_0^T ju_j''\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\,dt &= \frac{1}{h}X_h(t)\Big|_0^T - \frac{1}{2}\sum_{j=1}^{N}\int_0^T ju_j'(u_{j+1}'-u_{j-1}')\,dt \\ &= \frac{1}{h}X_h(t)\Big|_0^T - \frac{1}{2}\sum_{j=1}^{N}\int_0^T (ju_j'u_{j+1}'-(j+1)u_j'u_{j+1}')\,dt \\ &\quad -\frac{N+1}{2}\int_0^T u_N'u_{N+1}'\,dt \\ &= \frac{1}{h}X_h(t)\Big|_0^T + \frac{1}{2}\sum_{j=1}^{N}\int_0^T u_j'u_{j+1}'\,dt - \frac{N+1}{2}\int_0^T |u_N'|^2\,dt.\end{aligned} \tag{3.28}$$

We also see

$$\begin{aligned}&\frac{1}{h^2}\sum_{j=1}^{N}\int_0^T j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)(u_{j+1}-2u_j+u_{j-1})\,dt \\ &= \frac{1}{2h^2}\sum_{j=1}^{N}\int_0^T (ju_{j+1}^2-ju_{j-1}^2-2ju_{j+1}u_j+2u_{j-1}u_j)\,dt \\ &= \frac{1}{2h^2}\sum_{j=1}^{N}\int_0^T (-2u_j^2+2u_ju_{j+1})\,dt - \frac{1}{2h^2}\int_0^T |u_N|^2\,dt \\ &= -\frac{1}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{u_{j+1}-u_{j-1}}{h}\right|^2\,dt.\end{aligned} \tag{3.29}$$

Combining (3.27), (3.28), and (3.29), we get the reult. □

Lemma 3.4 (Lemma 3.3, [2])**.** *For any u solution of* (3.2)*, and $h > 0$,*

$$-h\sum_{j=0}^{N}\int_0^T \left|\frac{u_{j+1}-u_j}{h}\right|^2\,dt + Y_h(t)\Big|_0^T = 0,$$

where $Y_h(t) = h\sum_{j=0}^{N} u_j'u_j$.

Proof. Multiplying (3.2) by u_j, we obtain

$$\sum_{j=1}^{N}\int_0^T u_j''u_j\,dt = \frac{1}{h^2}\sum_{j=1}^{N}\int_0^T u_j(u_{j+1}-2u_j+u_{j-1})\,dt. \tag{3.30}$$

Keeping in mind $Y_h(t)$,

$$\sum_{j=1}^{N}\int_0^T u_j''u_j\,dt = \frac{1}{h}Y_h(t)\Big|_0^T - \sum_{j=1}^{N}\int_0^T |u_j'|^2\,dt. \tag{3.31}$$

Looking now at the right hand side,

$$\begin{aligned}\tfrac{1}{h^2}\sum_{j=1}^{N}\int_0^T u_j(u_{j+1}-2u_j+u_{j-1})\ dt &= \tfrac{1}{h^2}\sum_{j=1}^{N}\int_0^T (u_{j+1}u_j-2u_j^2+u_{j-1}u_j)\ dt\\ &= -\tfrac{1}{h^2}\sum_{j=0}^{N}\int_0^T (u_{j+1}^2-2u_{j+1}u_j+u_j^2)\ dt \qquad (3.32)\\ &= -\sum_{j=0}^{N}\int_0^T \left|\tfrac{u_{j+1}-u_j}{h}\right|^2\ dt.\end{aligned}$$

Combining (3.30), (3.31), and (3.32), the result is established. □

Lemma 3.5 (Lemma 3.4, [2])**.** *We have that the following inequality holds*

$$\left|X_h(t)-\frac{\gamma}{8}Y_h(t)\right| \le \sqrt{L^2+\frac{3\gamma}{16\lambda_0}E_h(u,0)}.$$

Proof. Referring back to our terms $X_h(t)$ and $Y_h(t)$, we have

$$X_h(t)-\frac{\gamma}{8}Y_h(t) = h\sum_{j=1}^{N}u_j'\left[j\frac{u_{j+1}-u_{j-1}}{2}-\frac{\gamma}{8}u_j\right].$$

Then by Hölder's Inequality,

$$|X_h(t)-\frac{\gamma}{8}Y_h(t)| \le \left[h\sum_{j=1}^{N}|u_j'|^2\right]^{\frac{1}{2}}\left[h\sum_{j=1}^{N}\left|j\frac{u_{j+1}-u_{j-1}}{2}-\frac{\gamma}{8}u_j\right|^2\right]^{\frac{1}{2}} \qquad (3.33)$$

On the other hand,

$$\begin{aligned}&h\sum_{j=1}^{N}\left|j\tfrac{u_{j+1}-u_{j-1}}{2}-\tfrac{\gamma}{8}\right|^2\\ &= h\sum_{j=1}^{N}\left[\tfrac{j^2}{4}|u_{j+1}-u_{j-1}|^2+\tfrac{\gamma^2}{64}u_j^2-\tfrac{\gamma j}{8}(u_{j+1}-u_{j-1})u_j\right].\end{aligned} \qquad (3.34)$$

By the Triangle Inequality, we get

$$\begin{aligned}&h\sum_{j=1}^{N}\left|j\tfrac{u_{j+1}-u_{j-1}}{2}-\tfrac{\gamma}{8}\right|^2\\ &\le h\sum_{j=1}^{N}\left[\tfrac{j^2}{2}|u_{j+1}-u_j|^2+\tfrac{j^2}{2}|u_j-u_{j-1}|^2+\tfrac{\gamma^2}{64}u_j^2-\tfrac{\gamma j}{8}u_{j+1}u_j+\tfrac{\gamma j}{8}u_ju_{j-1}\right].\end{aligned} \qquad (3.35)$$

Now, substituting $\frac{L}{h}$ for j, since $j \leq N$, and rearranging,

$$
\begin{aligned}
&h \sum_{j=1}^{N} \left| j \tfrac{u_{j+1}-u_{j-1}}{2} - \tfrac{\gamma}{8} \right|^2 \\
&\leq h \sum_{j=0}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2 + \tfrac{\gamma^2 h}{64} h \sum_{j=0}^{N} u_j^2 + \tfrac{\gamma h}{8} h \sum_{j=0}^{N} u_{j+1} u_j - \tfrac{\gamma}{8} u_N^2 \\
&\leq h \sum_{j=0}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2 + \left(\tfrac{\gamma^2}{64} + \tfrac{\gamma}{8} \right) h \sum_{j=0}^{N} u_j^2 - \tfrac{\gamma h}{16} \sum_{j=0}^{N} (2u_j^2 - 2u_{j+1}u_j) - \tfrac{\gamma}{8} u_N^2.
\end{aligned}
\tag{3.36}
$$

Keeping in mind that $\gamma < 4$, we see

$$
\begin{aligned}
&h \sum_{j=1}^{N} \left| j \tfrac{u_{j+1}-u_{j-1}}{2} - \tfrac{\gamma}{8} \right|^2 \\
&\leq h \sum_{j=0}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2 + \left(\tfrac{3\gamma}{16} \right) h \sum_{j=0}^{N} u_j^2 - \tfrac{\gamma h}{16} \sum_{j=0}^{N} |u_{j+1} - u_j|^2 + \tfrac{\gamma h}{16} u_{N+1}^2 - \tfrac{\gamma}{8} u_N^2.
\end{aligned}
\tag{3.37}
$$

Taking $\lambda = \lambda_1$, from (3.15) we can see

$$
\begin{aligned}
&h \sum_{j=1}^{N} \left| j \tfrac{u_{j+1}-u_{j-1}}{2} - \tfrac{\gamma}{8} \right|^2 \\
&\leq \left(L^2 - \tfrac{\gamma h^2}{16} \right) h \sum_{j=0}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2 + \tfrac{3\gamma}{16\lambda_1} h \sum_{j=0}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2 + \left(\tfrac{\gamma h}{16} - \tfrac{\gamma}{8} \right) u_N^2 \\
&\leq \left(L^2 - \tfrac{\gamma h^2}{16} + \tfrac{3\gamma}{16\lambda_1} \right) h \sum_{j=0}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2.
\end{aligned}
\tag{3.38}
$$

Combining (3.33) and (3.3), we deduce by Hölder's Inequality that

$$
\begin{aligned}
\left| X_h(t) - \tfrac{\gamma}{8} Y_h(t) \right| &\leq \sqrt{L^2 - \tfrac{\gamma h^2}{16} + \tfrac{2\gamma}{16\lambda_1}} \left[h \sum_{j=1}^{N} |u_j'|^2 \right]^{\frac{1}{2}} \left[h \sum_{j=1}^{N} \left| \tfrac{u_{j+1}-u_j}{h} \right|^2 \right]^{\frac{1}{2}} \\
&\leq \sqrt{L^2 + \tfrac{3\gamma}{16\lambda_1}} E_h(0).
\end{aligned}
$$

□

Theorem 3.9 (Theorem 3.5, [2]). *For $0 < \gamma < 4$, there exists $T_1(\gamma) > 2L$ such that for all $T > T_1(\gamma)$, there exists $C_1(T, \gamma)$ such that*

$$
E_h(u, 0) \leq C_1(T, \gamma) \int_0^T |u_N'(t)|^2 \, dt, \tag{3.39}
$$

for every solution, with $(u_0, u_1) \in C_h(\gamma)$ *and all* h.

Proof. Let u be a solution of (3.2) where $u^0, u^1 \in C_h(\gamma)$. Keeping in mind conservation of energy, we can write the equality (3.3) as

$$TE_h(u,0) + \frac{h}{2}\sum_{j=0}^{N}\int_0^T \left[u_j' u_{j+1}' - |u_j'|^2\right]\,dt + X_h(t)\Big|_0^T = \frac{L}{2}\int_0^T |u_N'|^2\,dt. \tag{3.40}$$

For the second term in (3.40), we have

$$\begin{aligned}\sum_{j=0}^{N}\left[u_j' u_{j+1}' - |u_j'|^2\right] &= -\tfrac{1}{2}\sum_{j=0}^{N}|u_{j+1}' - u_j'|^2 + \tfrac{L}{2}|u_N'|^2\\ &= -\tfrac{1}{2}\sum_{\mu_k^2 h^2 \le \gamma}|a_k|^2\mu_k^4 h^2\sum_{j=1}^{N}|\varphi_{k,j}|^2 + \tfrac{L}{2}|u_N'|^2\\ &\ge -\tfrac{1}{2}\gamma\sum_{\mu_k^2 h^2 \le \gamma}|a_k|^2\mu_k^2\sum_{j=1}^{N}|\varphi_{j,k}|^2 + \tfrac{L}{2}|u_N'|^2.\end{aligned} \tag{3.41}$$

Hence,

$$\sum_{j=0}^{N}\left[u_j' u_{j+1}' - |u_j'|^2\right] \ge -\frac{1}{2}\gamma\sum_{j=0}^{N}|u_j'|^2 + \frac{L}{2}|u_N'|^2.$$

From (3.40) and the last estimate, we deduce that

$$TE_h(u,0) - \frac{\gamma}{4}h\sum_{j=0}^{N}\int_0^T |u_j'|^2\,dt + \frac{h}{4}\int_0^T |u_N'|^2 + X_h(t)\Big|_0^T \le \frac{L}{2}\int_0^T |u_N'|^2\,dt. \tag{3.42}$$

By adding and subtracting $h\sum\limits_{j=0}^{N}|u_j'|^2$, Lemma 3.4 implies that

$$h\sum_{j=1}^{N}\int_0^T |u_j'|^2\,dt = TE_h(u,0) + \frac{1}{2}Y_h(t)\Big|_0^T. \tag{3.43}$$

Combining (3.42) and (3.43), we get

$$T\left(1-\frac{\gamma}{4}\right)E_h(u,0) - \frac{\gamma}{8}Y_h(t)\Big|_0^T + X_h(t)\Big|_0^T \le \frac{2L-h}{4}\int_0^T |u_N'|^2\,dt. \tag{3.44}$$

Now, adding (3.44) and Lemma 3.5 we obtain

$$\left[T\left(1-\frac{\gamma}{4}\right)-2\sqrt{L^2+\frac{3\gamma}{16\lambda_1}}\right]E_h(u,0)\leq\frac{L}{2}\int_0^T|u'_N|^2\;dt, \tag{3.45}$$

which implies that

$$E_h(u,0)\leq\frac{L}{2\left(T\left(1-\frac{\gamma}{4}\right)-2\sqrt{L^2+\frac{3\gamma}{16\lambda_1}}\right)}\int_0^T|u'_N|^2\;dt,$$

for

$$T>\frac{2\sqrt{L^2+\frac{3\gamma}{16\lambda_1}}}{1-\frac{\gamma}{4}}.$$

Thus, Theorem 3.9 holds with

$$T_1(\gamma)=\frac{2\sqrt{L^2+\frac{3\gamma}{16\lambda_1}}}{1-\frac{\gamma}{4}},$$

and

$$C_1(T,\gamma)=\frac{L}{2\left(T\left(1-\frac{\gamma}{4}\right)-2\sqrt{L^2+\frac{3\gamma}{16\lambda_1}}\right)}.$$

□

4 One-dimensional Model for Piezoelectric Beam Equations

In this section, we introduce the type of model where the main interest lies, namely the strongly coupled system of partial differential equations representing longitudinal vibrations of the centerline of a piezoelectric beam retaining magnetic effects. The analysis here will mimic the previous two sections, however there are significantly more nuanced proofs due to the strong coupling and the non-identical wave speeds. We first prove the exact observability inequality of the model by the the multipliers method [12] providing a sub-optimal observation time. This later is improved to obtain the optimal observation time by applying the so-called Ingham's Theorem [13].

4.1 Spectral Analysis and Development of Solutions

Recall the coupled partial differential equation-model (1.1) for the piezoelectric beam:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, & \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ v(0,t) = p(0,t) = 0, & \\ \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & \\ p_x(L,t) - \gamma v_x(L,t) = 0, & t \in \mathbb{R}^+, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in (0,L), \\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x), & x \in (0,L). \end{cases} \tag{4.1}$$

Define $\mathbb{X} = (\mathbb{L}^2(0,L))^2$ where $\mathbb{L}^2(0,L) = \{v : \int_0^L |v|^2 dx < \infty\}$, and

$$H^1_L(0,L) = \{v \in H^1(0,L) : v(0) = 0\},$$

and the complex linear space $\mathrm{H} = \left(H^1_L(0,L)\right)^2 \times \mathbb{X}$. The boundary conditions associated with this model are referred to as the clamped-free natural boundary conditions for a beam, coming out of a thorough variational approach [16], and $\alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0$ and $p_x(L,t) - \gamma v_x(L,t) = 0$ are referred to as the strain and voltage boundary conditions. The energy associated with (4.1) is

$$\begin{aligned}
\mathrm{E}(t) &= \frac{1}{2}\int_0^L \left\{\rho|v_t|^2 + \mu|p_t|^2 + \alpha|v_x|^2 - \gamma^2\beta|v_x|^2 + \gamma^2\beta|v_x|^2 - 2\beta\gamma v_x p_x + \beta|p_x|^2\right\} dx \\
&= \frac{1}{2}\int_0^L \left\{ \underbrace{\rho|v_t|^2 + \mu|p_t|^2}_{\text{Kinetic Energy}} + \underbrace{(\alpha v_x - \gamma\beta p_x)v_x}_{\text{Potential Energy}} + \underbrace{\beta(-\gamma v_x + p_x)p_x}_{\text{Electrical Energy}} \right\} dx.
\end{aligned}$$

However, we can use the fact that $\alpha_1 = \alpha - \gamma^2\beta$ to write the energy in a more compact form such that it is immediately recognized that energy is non negative:

$$\mathrm{E}(t) = \frac{1}{2}\int_0^L \left\{\rho|v_t|^2 + \mu|p_t|^2 + \alpha_1|v_x|^2 + \beta\left|\gamma v_x - p_x\right|^2\right\}\ dx. \tag{4.2}$$

A more in depth look at the derivation of (4.1) is given in [16] and [18], and the references therein.

To reformulate (4.1), let $\boldsymbol{\varphi} = (v, p, v_t, p_t)^{\mathrm{T}}$ with $\boldsymbol{\varphi}^0 = (v_0, p_0, v_1, p_1)^{\mathrm{T}}$. Now, the system (4.1) can be put into the following state-space formulation:

$$\varphi_t = \mathcal{A}\varphi = \begin{pmatrix} 0 & I_{2\times 2} \\ -A & 0 \end{pmatrix} \varphi, \quad \boldsymbol{\varphi}(0) = \boldsymbol{\varphi}^0 \tag{4.3}$$

where A is an unbounded matrix operator defined by

$$A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X} \quad A = \begin{pmatrix} -\frac{\alpha}{\rho}D_x^2 & \frac{\gamma\beta}{\rho}D_x^2 \\ \frac{\gamma\beta}{\mu}D_x^2 & -\frac{\beta}{\mu}D_x^2 \end{pmatrix} \tag{4.4}$$

with $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathrm{H} \to \mathrm{H}$, and

$$\mathrm{Dom}(\mathcal{A}) = \{\psi \in \mathrm{H} \cap ((H^2(0,L))^2 \times (H^1_L(0,L))^2); \psi_{1x}(L) = \psi_{2x}(L) = 0\} \tag{4.5}$$

is densely defined in H. Now define the constants

$$\begin{aligned}
\zeta_1 &= \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{\alpha\mu}{\alpha_1\beta} + \tfrac{\rho}{\alpha_1} + \sqrt{\left(\tfrac{\alpha\mu}{\alpha_1\beta} + \tfrac{\rho}{\alpha_1}\right)^2 - \tfrac{4\rho\mu}{\beta\alpha_1}}}\\
\zeta_2 &= \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{\alpha\mu}{\alpha_1\beta} + \tfrac{\rho}{\alpha_1} - \sqrt{\left(\tfrac{\alpha\mu}{\alpha_1\beta} + \tfrac{\rho}{\alpha_1}\right)^2 - \tfrac{4\rho\mu}{\beta\alpha_1}}}\\
b_1 &= \tfrac{1}{\gamma\mu}(\alpha_1\zeta_1^2 - \rho) = \tfrac{1}{2}\left(\gamma + \tfrac{\alpha_1}{\gamma\beta} - \tfrac{\rho}{\gamma\mu} + \sqrt{\left(\gamma + \tfrac{\alpha_1}{\gamma\beta} - \tfrac{\rho}{\gamma\mu}\right)^2 + \tfrac{4\rho}{\mu}}\right)\\
b_2 &= \tfrac{1}{\gamma\mu}(\alpha_1\zeta_2^2 - \rho) = \tfrac{1}{2}\left(\gamma + \tfrac{\alpha_1}{\gamma\beta} - \tfrac{\rho}{\gamma\mu} - \sqrt{\left(\gamma + \tfrac{\alpha_1}{\gamma\beta} - \tfrac{\rho}{\gamma\mu}\right)^2 + \tfrac{4\rho}{\mu}}\right).
\end{aligned} \tag{4.6}$$

Obviously, $\zeta_1, \zeta_2 \in \mathbb{R}^+$ since by $\alpha = \alpha_1 + \gamma^2\beta$ we have that $\sqrt{\left(\frac{\alpha\mu}{\alpha_1\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1}} = \sqrt{\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1}}$, and $\left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} + \frac{\rho}{\alpha_1}\right)^2 - \frac{4\rho\mu}{\beta\alpha_1} = \left(\frac{\gamma^2\mu}{\alpha_1} + \frac{\mu}{\beta} - \frac{\rho}{\alpha_1}\right)^2 + \frac{4\rho\mu\gamma^2}{\alpha_1^2} > 0$. Additionally, $b_1, b_2 \neq 0, \quad b_1 \neq b_2, \quad b_1 b_2 = -\frac{\rho}{\mu}$. Moreover, b_1 and b_2 solve the quadratic equation

$$b^2 - \left(\frac{\alpha}{\gamma\beta} - \frac{\rho}{\gamma\mu}\right) b - \frac{\rho}{\mu} = 0. \tag{4.7}$$

Theorem 4.10. *(Theorem 3, [18]). Let $\sigma_j = \frac{(2j-1)\pi}{2L}, j \in \mathbb{N}$. The eigenvalue problem $\mathcal{A}\mathbf{Y} = \lambda\mathbf{Y}$ has distinct eigenvalues*

$$\tilde{\lambda}^{\mp}_{1j} = \mp i\sqrt{\lambda_{1j}} = \tfrac{\mp i\sigma_j}{\zeta_1}, \quad \tilde{\lambda}^{\mp}_{2j} = \mp i\sqrt{\lambda_{2j}} = \tfrac{\mp i\sigma_j}{\zeta_2}. \tag{4.8}$$

Since $\tilde{\lambda}_{1j}^- = -\tilde{\lambda}_{1j}^+, \quad \tilde{\lambda}_{2j}^- = -\tilde{\lambda}_{2j}^+$, *the corresponding eigenfunctions are*

$$\mathbf{Y}_{1j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^+} \\ \frac{b_1}{\tilde{\lambda}_{1j}^+} \\ 1 \\ b_1 \end{pmatrix} \sin \sigma_j x, \mathbf{Y}_{-1j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^+} \\ \frac{b_1}{\tilde{\lambda}_{1j}^+} \\ -1 \\ -b_1 \end{pmatrix} \sin \sigma_j x,$$

$$\mathbf{Y}_{2j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{2j}^+} \\ \frac{b_2}{\tilde{\lambda}_{2j}^+} \\ 1 \\ b_2 \end{pmatrix} \sin \sigma_j x, \;\; \mathbf{Y}_{-2j} = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{2j}^+} \\ \frac{b_2}{\tilde{\lambda}_{2j}^+} \\ -1 \\ -b_2 \end{pmatrix} \sin \sigma_j x, \tag{4.9}$$

where ζ_1, ζ_2, b_1 *and* b_2 *are defined by (4.6). We are now able to write the Fourier series for* (4.1). *The function*

$$\varphi(x,t) = \sum_{j\in\mathbb{N}} \left[c_{1j}\mathbf{Y}_{1j} e^{\tilde{\lambda}_{1j}^+ t} + d_{1j}\mathbf{Y}_{-1j} e^{-\tilde{\lambda}_{1j}^+ t} + c_{2j}\mathbf{Y}_{2j} e^{\tilde{\lambda}_{2j}^+ t} + d_{2j}\mathbf{Y}_{-2j} e^{-\tilde{\lambda}_{2j}^+ t} \right] \tag{4.10}$$

solves (4.3) for the initial data

$$\begin{aligned} \varphi^0 &= \sum_{j\in\mathbb{N}} \left[c_{1j}\mathbf{Y}_{1j} + d_{1j}\mathbf{Y}_{-1j} + c_{2j}\mathbf{Y}_{2j} + d_{2j}\mathbf{Y}_{-2j} \right] \\ &= \sum_{j\in\mathbb{N}} \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^+}(c_{1j}+d_{1j}) + \frac{1}{\tilde{\lambda}_{2j}^+}(c_{2j}+d_{2j}) \\ \frac{b_1}{\tilde{\lambda}_{1j}^+}(c_{1j}+d_{1j}) + \frac{b_2}{\tilde{\lambda}_{2j}^+}(c_{2j}+d_{2j}) \\ (c_{1j}-d_{1j}) + (c_{2j}-d_{2j}) \\ b_1(c_{1j}-d_{1j}) + b_2(c_{2j}-d_{2j}) \end{pmatrix} \sin \sigma_j x \end{aligned}$$

where $\{c_{kj}, d_{kj}, \quad k=1,2, \quad j\in\mathbb{N}\}$ *are complex numbers such that*

$$\begin{aligned} &\|\varphi^0\|_{\mathrm{H}}^2 \asymp \sum_{j\in\mathbb{N}} \left(|c_{1j}|^2 + |d_{1j}|^2 + |c_{2j}|^2 + |d_{2j}|^2\right), \text{i.e.} \\ &\tilde{C}_1 \;\|\varphi^0\|_{\mathrm{H}}^2 \le \sum_{i=1}^{2}\sum_{j\in\mathbb{N}} \left(|c_{ij}|^2 + |d_{ij}|^2\right) \le \tilde{C}_2 \;\|\varphi^0\|_{\mathrm{H}}^2 \end{aligned} \tag{4.11}$$

with two positive constants $\tilde{C}_1, \tilde{C}_2$ which are independent of the particular choice of $\varphi^0 \in \mathrm{H}$.

4.2 Proof of Uniform Observability Using Multiplier Method

In this section, we prove the exact boundary observability for (4.1) using multipliers. However, the observability time is sub-optimal. This is simply due to the strong coupling of equations. Let's now state the main observability theorem.

Theorem 4.11. *The operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathrm{H} \to \mathrm{H}$ in (4.4) is the generator of a unitary semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. For given $\varphi_0 \in \mathrm{H}$, $\varphi \in C[\mathbb{R}, \mathrm{H}]$, and $\frac{d}{dt}E(t) = 0$. Moreover, letting $T > \frac{2L}{\sigma}$ there exists a constant $C(T)$ such that*

$$\int_0^T \left(\rho|v_t(L,t)|^2 + \mu|p_t(L,t)|^2\right)\ dt \geq C(T)\|\varphi_0\|_E^2 \tag{4.12}$$

where σ depends on the wave speed of each equation and is defined in the proof.

Here, note that the observations are physical and meaningful. In fact, $v_t(L,t)$ is the tip velocity of the beam, and $p_t(L,t)$ is the total current accumulated at the electrodes of the beam.

Proof. Consider (4.1). Multiply the first and second equations in (4.1) by the multipliers xv_x and xp_x, respectively, and integrate over $[0,L] \times [0,T]$

$$\begin{aligned} &\textstyle\int_0^T \int_0^L \left(\rho v_{tt} - \alpha_1 v_{xx} - \gamma\beta\left(\gamma v_{xx} - p_{xx}\right)\right) x v_x dx\ dt = 0 \\ &\textstyle\int_0^T \int_0^L \left(\mu p_{tt} + \beta(\gamma v_{xx} - p_{xx})\right) x p_x dx\ dt = 0 \end{aligned} \tag{4.13}$$

where we used $\alpha = \alpha_1 + \gamma^2\beta$. Adding these two equations

$$\textstyle\int_0^T \int_0^L \left(\rho v_{tt} x v_x + \mu p_{tt} x p_x - \alpha_1 x v_x v_{xx} - \beta x(\gamma v_x - p_x)\left(\gamma v_{xx} - p_{xx}\right)\right) dx\ dt = 0, \tag{4.14}$$

and integrating (4.14) by parts yields

$$\left[\int_0^L (\rho v_t x v_x + \mu p_t x p_x)\right]_0^T dx - \tfrac{L}{2}\int_0^T (\rho v_t^2(L,t) + \mu p_t^2(L,t))\ dt + TE(0) = 0. \tag{4.15}$$

By the Hölder's and Cauchy's inequalities, we deduce that

$$\left|\int_0^L (\rho v_t x v_x + \mu p_t x p_x)\, dx\right| = \left|\int_0^L (\rho v_t x v_x - \mu p_t x p_x)\, dx\right|. \tag{4.16}$$

By the triangle inequality,

$$\left|\int_0^L (\rho v_t x v_x + \mu p_t x p_x)\, dx\right| \le L\int_0^L \rho\,|v_t v_x|\, dx + L\int_0^L \mu\,|p_t p_x|\, dx.$$

By the Cauchy Inequality,

$$\begin{aligned}
&\left|\int_0^L (\rho v_t x v_x + \mu p_t x p_x)\, dx\right| \\
&\le L\left(\int_0^L \rho|v_t|^2 dx\right)^{1/2}\left(\frac{\rho}{\alpha_1}\int_0^L \alpha_1|v_x|^2 dx\right)^{1/2} + \left(\int_0^L \mu|p_t|^2 dx\right)^{1/2}\left(\mu\int_0^L |p_x|^2 dx\right)^{1/2} \\
&= L\left(\int_0^L \rho|v_t|^2 dx\right)^{1/2}\left(\frac{\rho}{\alpha_1}\int_0^L \alpha_1|v_x|^2 dx\right)^{1/2} \\
&\quad + \left(\int_0^L \mu|p_t|^2 dx\right)^{1/2}\left(\mu\int_0^L |p_x - \gamma v_x + \gamma v_x|^2 dx\right)^{1/2}.
\end{aligned}$$

Adding and subtracting γv_x in $\mu\int_0^L |p_x|^2$ we obtain

$$\begin{aligned}
&\left|\int_0^L (\rho v_t x v_x + \mu p_t x p_x)\, dx\right| \\
&\le L\left(\int_0^L \rho|v_t|^2 dx\right)^{1/2}\left(\frac{\rho}{\alpha_1}\int_0^L \alpha_1|v_x|^2 dx\right)^{1/2} \\
&\quad + \left(\int_0^L \mu|p_t|^2 dx\right)^{1/2}\left(\left(\int_0^L \mu|\gamma v_x - p_x|^2 dx\right)^{1/2} + \left(\mu\int_0^L |\gamma v_x|^2 dx\right)^{1/2}\right).
\end{aligned}$$

Bringing the constants out,

$$\begin{aligned}&\left|\int_0^L (\rho v_t x v_x + \mu p_t x p_x)\, dx\right| \\ &\leq \sqrt{\frac{\rho}{\alpha_1}}\frac{L}{2}\int_0^L \rho|v_t|^2 dx + \left(\sqrt{\frac{\rho}{\alpha_1}} + \sqrt{\frac{\mu\gamma^2}{\alpha_1}}\right)\frac{L}{2}\int_0^L \alpha_1|v_x|^2 dx \\ &\quad + \left(\sqrt{\frac{\mu}{\beta}} + \sqrt{\frac{\mu\gamma^2}{\alpha_1}}\right)\frac{L}{2}\int_0^L \mu|p_t|^2 dx + \sqrt{\frac{\mu}{\beta}}\frac{L}{2}\int_0^L \beta|\gamma v_x - p_x|^2 dx \\ &\leq L\max\left(\sqrt{\frac{\rho}{\alpha_1}} + \sqrt{\frac{\mu\gamma^2}{\alpha_1}}, \sqrt{\frac{\mu}{\beta}} + \sqrt{\frac{\mu\gamma^2}{\alpha_1}}\right)E(0),\end{aligned}$$

and therefore,

$$(T - \tfrac{2L}{\sigma})E(0) \leq \tfrac{L}{2}\textstyle\int_0^T \left(\rho|v_t(L,t)|^2 + \mu|p_t(L,t)|^2\right)\ dt \tag{4.17}$$

where $\sigma = \min\left(\frac{1}{\sqrt{\frac{\rho}{\alpha_1}}+\sqrt{\frac{\mu\gamma^2}{\alpha_1}}}, \frac{1}{\sqrt{\frac{\mu}{\beta}}+\sqrt{\frac{\mu\gamma^2}{\alpha_1}}}\right)$. The proof concludes by choosing $C(T) = \frac{2(T-\frac{2L}{\sigma})}{L}$.

Note that the proof of Theorem 4.11 is based on the use of multipliers [12] so the observation time may be suboptimal yet it is still an improvement of the one in [23].

Next, we apply the non-harmonic Fourier series, and in particular the Ingham's theorem [13], to obtain the optimal observation time. The existence of the uniform gap across each branch of the eigenvalues of the operator $\mathcal{A}$ in (4.3) is utilized.

4.3 Proof of Uniform Observability Using Ingham's Theorem

Similar to before, in this section we apply Ingham's inequality for (4.1) to get an optimal observation time T in terms of the wave speeds.

Let $s_{1j} = \frac{\sigma_j}{\zeta_1} = \frac{(2j-1)\pi}{2L\zeta_1}$ and $s_{2j} = \frac{\sigma_j}{\zeta_2} = \frac{(2j-1)\pi}{2L\zeta_2}$ for $j \in \mathbb{N}$. The set of eigenvalues (5.11) can be rewritten as

$$\tilde{\lambda}_{kj}^{\mp} = \mp i s_{kj}, \quad k = 1, 2, \quad j \in \mathbb{N}. \tag{4.18}$$

Since $\mathcal{A}^* = -\mathcal{A}$, the function $\varphi = e^{\mathcal{A}^* t}\varphi^0$, given explicitly by (4.10), and

$$\varphi(x,t) = \sum_{k=1}^{2} \sum_{j\in\mathbb{N}} \left[c_{kj} Y_{kj} e^{i s_{kj} t} + d_{kj} Y_{-kj} e^{-i s_{kj} t} \right]. \tag{4.19}$$

Using this solution, we look at the observation and note that $\sin^2 \sigma_j x = (-1)^{2j+1}$

$$\begin{aligned} &\int_0^T \left(\rho |v_t(L,t)|^2 + \mu |p_t(L,t)|^2\right)\, dt \\ &= \int_0^T \rho \left| \sum_{j\in\mathbb{N}} (-1)^{2j+1} \left(c_{1j} e^{i s_{1j} t} - d_{1j} e^{-i s_{1j} t} + c_{2j} e^{i s_{2j} t} - d_{2j} e^{-i s_{2j} t}\right) \right|^2 \, dt \\ &\quad + \int_0^T \mu \left| \sum_{j\in\mathbb{N}} \left(b_1 (c_{1j} e^{i s_{1j} t} - d_{1j} e^{-i s_{1j} t}) + b_2 (c_{2j} e^{i s_{2j} t} - d_{2j} e^{-i s_{2j} t})\right) \right|^2 \, dt. \end{aligned} \tag{4.20}$$

By the generalized Young's Inequality where $|2 a_j b_j| \leq \frac{a_j^2}{\epsilon_2} + \epsilon_1 b_j^2$, for $\epsilon_1, \epsilon_2 \in \mathbb{R}$

$$\begin{aligned} &\geq \int_0^T \rho \sum_{j\in\mathbb{N}} \left(\left(1 - \tfrac{1}{\epsilon_2}\right) |c_{1j} e^{i s_{1j} t} - d_{1j} e^{-i s_{1j} t}|^2 + (1 - \epsilon_2) |c_{2j} e^{i s_{2j} t} - d_{2j} e^{-i s_{2j} t}|^2 \right) \, dt \\ &\quad + \int_0^T \mu \sum_{j\in\mathbb{N}} \left(\left(b_1^2 - \tfrac{|b_1 b_2|}{\epsilon_1}\right) |(c_{1j} e^{i s_{1j} t} - d_{1j} e^{-i s_{1j} t})|^2 \right. \\ &\qquad \left. + (b_2^2 - |b_1 b_2| \epsilon_2) |c_{2j} e^{i s_{2j} t} - d_{2j} e^{-i s_{2j} t}|^2 \right) \, dt. \end{aligned} \tag{4.21}$$

Factoring out our constants, we obtain

$$\begin{aligned} &= \left\{ \rho \left(1 - \tfrac{1}{\epsilon_2}\right) + \mu \left(b_1^2 - \tfrac{\rho}{\mu \epsilon_1}\right) \right\} \int_0^T \sum_{j\in\mathbb{N}} |c_{1j} e^{i s_{1j} t} - d_{1j} e^{-i s_{1j} t}|^2 \, dt \\ &\quad + \left\{ \rho (1 - \epsilon_2) + \mu \left(b_2^2 - \tfrac{\rho}{\mu} \epsilon_1\right) \right\} \int_0^T \sum_{j\in\mathbb{N}} |c_{2j} e^{i s_{2j} t} - d_{2j} e^{-i s_{2j} t}|^2 \, dt. \end{aligned} \tag{4.22}$$

Now choose $\epsilon_1 = 2 + \frac{\mu}{2\rho} \left(\frac{\alpha\mu - \rho\beta}{\beta\gamma\mu}\right)^2 > 0$ and $\epsilon_2 = (1 + \frac{\mu}{\rho} b_1^2) \left(2 + \frac{\mu}{\rho} \left(\frac{\alpha\mu - \rho\beta}{\beta\gamma\mu}\right)^2\right) > 0$ so that $\rho\left(1-\frac{1}{\epsilon_2}\right)+\mu\left(b_1^2-\frac{\rho}{\mu\epsilon_1}\right)>0$ and $\rho\left(1-\epsilon_2\right)+\mu\left(b_2^2-\frac{\rho}{\mu}\epsilon_1\right)>0$.

By the Ingham's theorem, for $T>\frac{2L}{\sigma}=2L\max\left(\zeta_1,\zeta_2\right)$ where $\sigma=\min\left\{\frac{1}{\zeta_1},\frac{1}{\zeta_2}\right\}$, there exists a constant $C(T)$ such that

$$\begin{aligned}\int_0^T\left(\rho|v_t(L,t)|^2+\mu|p_t(L,t)|^2\right)\,dt &\geq C(T)\sum_{j\in\mathbb{N}}\left(|c_{1j}|^2+|d_{1j}|^2+|c_{2j}|^2+|d_{2j}|^2\right)\\ &\asymp \|\varphi^0\|_{\mathrm{H}}^2.\end{aligned} \tag{4.23}$$

□

5 Finite Difference Space-Discretized Piezoelectric Beam Equation

In this section, the goal is to show that the discretized coupled wave equations by the central finite differences lose the positive exact observability result with two boundary observations. We show that the unfiltered solutions of the strongly-coupled system of semi-discretized wave equations, the uniform observability property does not hold as the discretization parameter tends to 0. We show after directly filtering high frequency spurious eigenvalues, that the filtered solutions satisfy an exactly observability inequality.

5.1 Discrete Spectral Analysis and Development of Solutions

We first introduce the semi-discretized version of (4.1) by the same central difference discretization done in section 3.

$$\begin{cases} \rho v_j''(t) - \Delta_h(\alpha v_j - \gamma\beta p_j) = 0 \\ \mu p_j''(t) - \Delta_h(\beta p_j - \gamma\beta v_j) = 0, \\ v_0 = p_0 = 0,\ v_{N+1} = v_N,\ p_{N+1} = p_N \\ (v_j, p_j, v_j', p_j')(0) = (v_0^j, p_0^j, v_1^j, p_1^j) \quad j = 1, 2, \ldots, N, \end{cases} \tag{5.1}$$

where

$$\Delta_h = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & \ldots & 0 & 1 & -1 \end{pmatrix}. \tag{5.2}$$

Putting (5.1) now in its first-order form, we let $\{\phi_j\}_{j=1}^4 \in \mathbb{C}^N$, and let $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3, \phi_4]^{\mathrm{T}}$. Now we rewrite (5.1) as the following

$$\boldsymbol{\phi}' = \tilde{\boldsymbol{A}}\boldsymbol{\phi} = \begin{pmatrix} 0_{2N\times 2N} & I_{2N\times 2N} \\ -A_{2N\times 2N} & 0_{2N\times 2N} \end{pmatrix} \boldsymbol{\phi}, \quad \boldsymbol{\phi}(0) = \boldsymbol{\phi}_0 \tag{5.3}$$

where

$$A_{2N\times 2N} = \begin{pmatrix} -\frac{\alpha}{\rho}\Delta_h & \frac{\gamma\beta}{\rho}\Delta_h \\ \frac{\gamma\beta}{\mu}\Delta_h & -\frac{\beta}{\mu}\Delta_h \end{pmatrix}. \tag{5.4}$$

Next, consider the auxiliary eigenvalue problem

Our eigenvalue problem is as follows:

$$\begin{cases} -\Delta_h \psi_j = \lambda \psi_j, \\ \psi_0 = 0, \quad \psi_{N+1} = \psi_N, \quad j = 1, 2, \ldots, N. \end{cases} \tag{5.5}$$

which we can write as

$$A \begin{pmatrix} \boldsymbol{\psi}^1 \\ \boldsymbol{\psi}^2 \end{pmatrix} = \tilde{\tilde{\lambda}}(h) \begin{pmatrix} \boldsymbol{\psi}^1 \\ \boldsymbol{\psi}^2 \end{pmatrix} \tag{5.6}$$

which is equivalent to solving

$$\begin{cases} \Delta_h \left(\alpha_1 \boldsymbol{\psi}^1 + \gamma\beta \left(\gamma \boldsymbol{\psi}^1 - \boldsymbol{\psi}^2\right)\right) = -\tilde{\tilde{\lambda}}(h)\rho \boldsymbol{\psi}^1 \\ -\beta\Delta_h(\gamma \boldsymbol{\psi}^1 - \boldsymbol{\psi}^2) = -\tilde{\tilde{\lambda}}(h)\mu \boldsymbol{\psi}^2, \\ \psi_0^1 = \psi_0^2 = 0, \ \psi_{N+1}^1 = \psi_N^1, \ \psi_{N+1}^2 = \psi_N^2. \end{cases} \tag{5.7}$$

For $\boldsymbol{u} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4]^{\mathrm{T}}$ and $\tilde{\lambda}(h) = -\tilde{\tilde{\lambda}}(h)$, the eigenvalue problem $\tilde{\boldsymbol{A}}\boldsymbol{u} = \tilde{\tilde{\lambda}}(h)\boldsymbol{u}$

is equivalent to

$$\begin{pmatrix} \frac{\alpha}{\rho}\Delta_h - \tilde{\lambda}^2 I & -\frac{\gamma\beta}{\rho}\Delta_h \\ -\frac{\gamma\beta}{\mu}\Delta_h & \frac{\beta}{\mu}\Delta_h - \tilde{\lambda}^2 I \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \mathbf{0}. \tag{5.8}$$

By (5.5),

$$\begin{pmatrix} \tilde{\lambda}^2 + \frac{\alpha}{\rho}\lambda & -\frac{\gamma\beta}{\rho}\lambda \\ -\frac{\gamma\beta}{\mu}\lambda & \tilde{\lambda}^2 + \frac{\beta}{\mu}\lambda \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \mathbf{0} \tag{5.9}$$

where the characteristic equation is

$$\tilde{\lambda}^4 + \left(\frac{\alpha}{\rho} + \frac{\beta}{\mu}\right)\lambda\tilde{\lambda}^2 + \left(\frac{\alpha\beta}{\mu\rho} - \frac{\gamma^2\beta^2}{\mu\rho}\right)\lambda^2 = 0. \tag{5.10}$$

Following the eigenvalue analysis in Theorem 4.10, the following theorem is immediate:

Theorem 5.12. *For $k, j \in \{1, 2, \ldots, N\}$, the operator $\tilde{\mathcal{A}}$ has eigenvalues*

$$\tilde{\lambda}_{1k}^{\mp}(h) = \mp i\frac{1}{\zeta_1}\sqrt{\lambda_k(h)}, \quad \tilde{\lambda}_{2k}^{\mp}(h) = \mp i\frac{1}{\zeta_2}\sqrt{\lambda_k(h)}, \quad k = 1, 2, \ldots, N, \tag{5.11}$$

and since $\tilde{\lambda}_{1k}^- = -\tilde{\lambda}_{1k}^+, \quad \tilde{\lambda}_{2k}^- = -\tilde{\lambda}_{2k}^+$*, the corresponding eigenfunctions are*

$$\mathbf{\Psi}_k^1(h) = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1k}^+}\boldsymbol{\psi}_k \\ \frac{b_1}{\tilde{\lambda}_{1k}^+}\boldsymbol{\psi}_k \\ \boldsymbol{\psi}_k \\ b_1\boldsymbol{\psi}_k \end{pmatrix}, \ \mathbf{\Psi}_k^{-1}(h) = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1k}^+}\boldsymbol{\psi}_k \\ \frac{b_1}{\tilde{\lambda}_{1k}^+}\boldsymbol{\psi}_k \\ -\boldsymbol{\psi}_k \\ -b_1\boldsymbol{\psi}_k \end{pmatrix},$$

$$\mathbf{\Psi}_k^2(h) = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{2k}^+}\boldsymbol{\psi}_k \\ \frac{b_1}{\tilde{\lambda}_{2k}^+}\boldsymbol{\psi}_k \\ \boldsymbol{\psi}_k \\ b_2\boldsymbol{\psi}_k \end{pmatrix}, \ \mathbf{\Psi}_k^{-2}(h) = \begin{pmatrix} \frac{1}{\tilde{\lambda}_{2j}^+}\boldsymbol{\psi}_k \\ \frac{b_2}{\tilde{\lambda}_{2j}^+}\boldsymbol{\psi}_k \\ -\boldsymbol{\psi}_k \\ -b_2\boldsymbol{\psi}_k \end{pmatrix}. \tag{5.12}$$

For fixed k,

$$\lim_{h\to 0} \sqrt{\tilde{\lambda}_{ik}(h)} = \frac{(2k-1)\pi}{2L\zeta_i}, \quad i = 1,2, \quad k = 1,2,\ldots,N.$$

The solutions to (5.1) *are given by The Fourier series solution of* (5.1) *is given by*

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{pmatrix} = \sum_{|\mu_k|\le\sqrt{\Gamma}} \left[c_{1k} \begin{pmatrix} \boldsymbol{\psi}_k \\ b_1\boldsymbol{\psi}_k \end{pmatrix} e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k} \begin{pmatrix} \boldsymbol{\psi}_k \\ b_2\boldsymbol{\psi}_k \end{pmatrix} e^{e^{\frac{i\mu_k t}{\zeta_2}}} \right]. \tag{5.13}$$

where $\mu_k = \sqrt{\lambda_k}$ *for* $k > 0$ *and* $\mu_{-k} = -\mu_k$*. for the initial data*

$$\varphi^0 = \sum_{j=1}^{N} \begin{pmatrix} \frac{1}{\tilde{\lambda}_{1j}^+}(c_{1j}+d_{1j}) + \frac{1}{\tilde{\lambda}_{2j}^+}(c_{2j}+d_{2j}) \\ \frac{b_1}{\tilde{\lambda}_{1j}^+}(c_{1j}+d_{1j}) + \frac{b_2}{\tilde{\lambda}_{2j}^+}(c_{2j}+d_{2j}) \\ (c_{1j}-d_{1j}) + (c_{2j}-d_{2j}) \\ b_1(c_{1j}-d_{1j}) + b_2(c_{2j}-d_{2j}) \end{pmatrix} \psi_j(h) \tag{5.14}$$

where $\{c_{kj}, d_{kj}, \quad k = 1, 2, \quad j \in \mathbb{N}\}$ *are complex numbers such that*

$$\begin{aligned} \|\varphi^0\|^2_{\mathbb{R}^N} &\asymp \sum_{j=1}^{N} \left(|c_{1j}|^2 + |d_{1j}|^2 + |c_{2j}|^2 + |d_{2j}|^2\right), \text{i.e.} \\ \tilde{C}_1 \ \|\varphi^0\|^2_{\mathbb{R}^N} &\le \sum_{i=1}^{2}\sum_{j=1}^{N} \left(|c_{ij}|^2 + |d_{ij}|^2\right) \le \tilde{C}_2 \ \|\varphi^0\|^2_{\mathbb{R}^N} \end{aligned} \tag{5.15}$$

with two positive constants $\tilde{C}_1, \tilde{C}_2$ *which are independent of the particular choice of* φ^0.

Lemma 5.1 (Conservation of Energy)**.** *For any* $h > 0$ *and* v, p *solutions of* (5.1)*, we have*

$$E_h(t) = E_h(0), \forall t \in [0, T]. \tag{5.16}$$

Proof. Multiply (5.1) by v_j' and p_j' respectively to get

$$\rho v_j'' v_j' - \alpha \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} v_j' + \gamma\beta \frac{p_{j+1} - 2p_j + p_{j-1}}{h^2} v_j' = 0, \tag{5.17}$$

$$\mu p_j'' p_j' - \beta \frac{p_{j+1} - 2p_j + p_{j-1}}{h^2} p_j' + \gamma\beta \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} p_j' = 0. \tag{5.18}$$

Now, sum from 1 to N and combine

$$\begin{aligned} \sum_{j=1}^{N} (\rho v_j'' v_j' + \mu p_j'' p_j') = \tfrac{1}{h^2} \sum_{j=1}^{N} [\alpha(v_{j+1} - 2v_j + v_{j-1}) v_j' + \beta(p_{j+1} - 2p_j + p_{j-1}) p_j' \\ -\gamma\beta[(p_{j+1} - 2p_j + p_{j-1}) v_j + (v_{j+1} - 2v_j + v_{j-1}) p_j]]. \end{aligned} \tag{5.19}$$

We know

$$\sum_{j=1}^{N} (\rho v_j'' v_j' + \mu p_j'' p_j') = \frac{1}{2} \frac{d}{dt} \sum_{j=0}^{N} (\alpha |v_j'|^2 + \beta |p_j'|^2) \tag{5.20}$$

and

$$
\begin{aligned}
&-\sum_{j=1}^{N}[\alpha(v_{j+1}-2v_j+v_{j-1})v_j'+\beta(p_{j+1}-2p_j+p_{j-1})p_j'] \\
&=\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N}(\alpha|v_j-v_{j+1}|^2+\beta|p_j-p_{j+1}|^2)
\end{aligned}
\tag{5.21}
$$

also,

$$
\begin{aligned}
&-\gamma\beta\sum_{j=1}^{N}[(p_{j+1}-2p_j+p_{j-1})v_j'+(v_{j+1}-2v_j+v_{j-1})p_j'] \\
&=-\frac{\gamma\beta}{2}\frac{d}{dt}\sum_{j=1}^{N}(p_{j+1}-2p_j+p_{j-1})(v_{j+1}-2v_j+v_{j-1}).
\end{aligned}
\tag{5.22}
$$

Combining (5.20), (5.21), and (5.22) we get that $\frac{d}{dt}E(t)=0$. □

5.2 Lack of Uniform Observability with Respect to the Discretization Parameter

Lemma 5.2. *For any eigen-pair* $\left[\tilde{\tilde{\lambda}}(h),\begin{pmatrix}\boldsymbol{\psi}^1\\ \boldsymbol{\psi}^2\end{pmatrix}\right]$ *of* $A_{2N\times 2N}$ *in (5.7), the following identities hold:*

$$
\alpha_1\sum_{j=0}^{N}\left|\frac{\psi^1_{j+1}-\psi^1_j}{h}\right|^2+\beta\sum_{j=0}^{N}\left|\frac{(\gamma\psi^1_{j+1}-\psi^2_{j+1})-(\gamma\psi^1_j-\psi^2_j)}{h}\right|^2=\tilde{\tilde{\lambda}}\left(\sum_{j=0}^{N}\rho|\psi^1_j|^2+\mu|\psi^2_j|^2\right), \tag{5.23}
$$

and

$$
\begin{aligned}
&\frac{\rho\tilde{\tilde{\lambda}}(2L-h)}{2h}|\psi^1_N|^2+\frac{\mu\tilde{\tilde{\lambda}}(2L-h)}{2h}|\psi^2_N|^2=\tilde{\tilde{\lambda}}\left[2\rho\left(1-\frac{\rho\tilde{\tilde{\lambda}}(h)h^2}{4\alpha_1}\right)\sum_{j=0}^{N}(\psi^1_j)^2\right. \\
&\quad\left.+2\mu\left(1-\frac{\alpha\mu\tilde{\tilde{\lambda}}h^2}{4\beta\alpha_1}\right)\sum_{j=0}^{N}(\psi^2_j)^2-\frac{\rho\mu\gamma\tilde{\tilde{\lambda}}h^2}{\alpha_1}\sum_{j=0}^{N}\psi^1_j\psi^2_j\right].
\end{aligned}
\tag{5.24}
$$

Proof. Dot-product the first and second equations in (5.7) by $\boldsymbol{\psi}^1_j$ and $\boldsymbol{\psi}^2_j$, respectively, and add these two equations:

$$\frac{-\alpha_1}{h^2}\sum_{j=1}^{N}(\psi^1_{j+1}-2\psi^1_j+\psi^1_{j-1})\psi^1_j$$
$$-\frac{\beta}{h^2}\sum_{j=1}^{N}\left[(\gamma\psi^1_{j+1}-\psi^2_{j+1})-2(\gamma\psi^1_j-\psi^2_j)+(\gamma\psi^1_{j-1}-\psi^2_{j-1})\right](\gamma\psi^1_j-\psi^2_j)$$
$$=\tilde{\tilde{\lambda}}\sum_{j=1}^{N}\left(\rho(\psi^1_j)^2+\mu(\psi^2_j)^2\right).$$

This implies that

$$\frac{\alpha_1}{h^2}\sum_{j=1}^{N}\left(2(\psi^1_j)^2+2\psi^1_{j+1}\psi^1_j\right)+\frac{\beta}{h^2}\sum_{j=1}^{N}\left[2(\gamma\psi^1_j-\psi^2_j)^2-2(\gamma\psi^1_{j+1}-\psi^2_{j+1})(\gamma\psi^1_j-\psi^2_j)\right]$$
$$+\frac{\alpha_1}{h^2}\left|\psi^1_N\right|^2+\frac{\beta}{h^2}\left|\gamma\psi^1_N-\psi^2_N\right|^2=\tilde{\tilde{\lambda}}\sum_{j=1}^{N}\left(\rho(\psi^1_j)^2+\mu(\psi^2_j)^2\right),$$

and finally, keeping in mind our boundary conditions

$$\frac{\alpha_1}{h^2}\sum_{j=0}^{N}\left((\psi^1_{j+1})^2-2\psi^1_{j+1}\psi^1_j+(\psi^1_j)^2\right)+\frac{\beta}{h^2}\sum_{j=0}^{N}\left[\left(\gamma\psi^1_{j+1}-\psi^2_{j+1}\right)^2\right.$$
$$-2(\gamma\psi^1_{j+1}-\psi^2_{j+1})(\gamma\psi^1_j-\psi^2_j)$$
$$\left.+\left(\gamma\psi^1_j-\psi^2_j\right)^2\right]=\tilde{\tilde{\lambda}}\sum_{j=0}^{N}\left(\rho(\psi^1_j)^2+\mu(\psi^2_j)^2\right).$$

After rewriting, we see this is exactly the claim in (5.23).

To obtain (5.41), first multiply the j^{th} equations in the first and second equations in (5.7) by $j(\psi^1_{j+1}-\psi^1_{j-1})$ and $j(\psi^2_{j+1}-\psi^2_{j-1})$ respectively, and add for $j=1,2,\ldots,N$ to get

$$\sum_{j=0}^{N}\frac{\alpha_1}{h^2}\left(-2(\psi^1_j)^2+2\psi^1_{j+1}\psi^1_j\right)-\frac{\alpha_1}{h^2}(\psi^1_N)^2$$
$$+\frac{\beta}{h^2}\sum_{j=0}^{N}\left[-2(\gamma\psi^1_j-\psi^2_j)^2\right]-\frac{\beta}{h^2}(\gamma\psi^1_N-\psi^2_N)^2$$
$$+\frac{\beta}{h^2}\sum_{j=0}^{N}[2(\gamma\psi^1-\psi^2)_{j+1}(\gamma\psi^1-\psi^2)_j]=\tilde{\tilde{\lambda}}\sum_{j=0}^{N}\left(\rho\psi^1_{j+1}\psi^1_j+\mu\psi^2_{j+1}\psi^2_j\right)$$
$$-\frac{\rho\tilde{\tilde{\lambda}}L}{h}|\psi^1_N|^2-\frac{\mu\tilde{\tilde{\lambda}}L}{h}|\psi^2_N|^2. \tag{5.25}$$

Now multiply the second equation in (5.7) by γ and add it to the first equation in (5.7), and multiply the first equation in (5.7) by $\frac{\gamma\beta}{\alpha}$ and add it to the second equation in (5.7), respectively, to get the following equalities

$$\begin{aligned} -\alpha_1 \sum_{j=0}^{N} \Delta_h \boldsymbol{\psi}^1 &= \tilde{\tilde{\lambda}} \sum_{j=0}^{N} \left(\rho \boldsymbol{\psi}^1 + \gamma\mu \boldsymbol{\psi}^2\right), \\ -\sum_{j=0}^{N} \Delta_h \boldsymbol{\psi}^2 &= \tilde{\tilde{\lambda}} \sum_{j=0}^{N} \left(\tfrac{\rho\gamma}{\alpha_1} \boldsymbol{\psi}^1 + \tfrac{\mu\alpha}{\beta\alpha_1} \boldsymbol{\psi}^2\right). \end{aligned} \tag{5.26}$$

Now multiply the first equation in (5.26) by $\boldsymbol{\psi}_1$, and simplify to obtain

$$\sum_{j=0}^{N} \psi_{j+1}^1 \psi_j^1 = \tfrac{1}{2}|\psi_N^1|^2 + \left(1 - \tfrac{2\rho\tilde{\tilde{\lambda}}h^2}{4\alpha_1}\right) \sum_{j=0}^{N} |\psi_j^1|^2 - \tfrac{\gamma\mu\tilde{\tilde{\lambda}}h^2}{2\alpha_1} \sum_{j=0}^{N} \psi_j^1 \psi_j^2. \tag{5.27}$$

Similarly, multiply the second equation in (5.26) by $\boldsymbol{\psi}_2$, and simplify to obtain

$$\sum_{j=0}^{N} \psi_{j+1}^2 \psi_j^2 = \tfrac{1}{2}|\psi_N^2|^2 + \left(1 - \tfrac{2\alpha\mu h^2}{4\beta\alpha_1}\right) \sum_{j=0}^{N} |\psi_j^2|^2 - \tfrac{\gamma\rho\tilde{\tilde{\lambda}}h^2}{2\alpha_1} \sum_{j=0}^{N} \psi_j^1 \psi_j^2, \tag{5.28}$$

and finally, multiply the second equation in (5.7) by $\frac{\gamma\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2}{\beta}$ to deduce

$$\begin{aligned} \sum_{j=0}^{N} (\gamma\psi_{j+1}^1 - \psi_{j+1}^2)(\gamma\psi_j^2 - \psi_j^2) &= \tfrac{1}{2}|\gamma\psi_N^1 - \psi_N^2|^2 + \sum_{j=0}^{N} (\gamma\psi_j^1 - \psi_j^2)^2 \\ &+ \tfrac{\mu\tilde{\tilde{\lambda}}}{\beta} \sum_{j=0}^{N} (\gamma\psi_j^1 - \psi_j^2)\psi_j^2, \end{aligned} \tag{5.29}$$

Finally, substitute (5.27)-(5.29) in (5.25)

$$\begin{aligned} &\left(\tfrac{\rho\tilde{\tilde{\lambda}}L}{h} - \tfrac{\alpha_1}{h^2}\right)(\psi_N^1)^2 - \tfrac{\beta}{h^2}(\gamma\psi_N^1 - \psi_N^2)^2 + \tfrac{\mu\lambda L}{h}(\psi_N^2)^2 = \tfrac{2\alpha_1}{h^2} \sum_{j=1}^{N} (\psi_j^1)^2 + \tfrac{2\beta}{h^2} \sum_{j=1}^{N} (\gamma\psi_j^1 - \psi_j^2)^2 \\ &+ \left(\tilde{\tilde{\lambda}}\rho - \tfrac{2\alpha_1}{h^2}\right)\left[\tfrac{|\psi_N^1|^2}{2} + \left(1 - \tfrac{\tilde{\tilde{\lambda}}\rho h^2}{2\alpha_1}\right) \sum_{j=1}^{N} (\psi_j^1)^2 - \tfrac{\tilde{\tilde{\lambda}}\mu\gamma h^2}{2\alpha_1} \sum_{j=1}^{N} \psi_j^1 \psi_j^2\right] \\ &+ \mu\tilde{\tilde{\lambda}}\left[\tfrac{|\psi_N^2|^2}{2} + \left(1 - \tfrac{\alpha\mu h^2}{2\beta\alpha_1}\right) \sum_{j=1}^{N} (\psi_j^2)^2 - \tfrac{\tilde{\tilde{\lambda}}\rho\gamma h^2}{2\alpha_1} \sum_{j=1}^{N} \psi_j^1 \psi_j^2\right] \\ &- \tfrac{2\beta}{h^2}\left[\tfrac{|\gamma\psi_N^1 - \psi_N^2|^2}{2} + \sum_{j=1}^{N} \left(\gamma\psi_j^1 - \psi_j^2\right)^2 + \tfrac{\tilde{\tilde{\lambda}}\mu h^2}{2\beta} \sum_{j=1}^{N} (\gamma\psi_j^1 - \psi_j^2)\psi_j^2\right], \end{aligned}$$

and after collecting like-terms and simplifying once more, the result (5.41) is established.

□

Now we are ready to prove the non-uniform observability for the solutions of (5.3).

Theorem 5.13. *For any $T > 0$, as $h \to 0$*

$$\sup_{\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{pmatrix} \text{ solves (5.1)}} \frac{E_h(0)}{\int_0^T \left(\rho \left|\boldsymbol{v}_N'\right|^2 + \mu \left|\boldsymbol{p}_N'\right|^2\right) \, dt} \to \infty. \tag{5.30}$$

Proof. Consider the solution $\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\psi}_N \\ b_k \boldsymbol{\psi}_N \end{pmatrix}$ corresponding to the eigenvalue $\frac{\lambda_N(h)}{(\zeta_k)^2}$ where $k = 1, 2$. By the normalization $h \sum_{j=0}^{N} (\psi_{N,j})^2 = 1$, (5.41) yields

$$\begin{aligned} &\tilde{\tilde{\lambda}}_N(h) \tfrac{(2L-h)}{2h} \left(\rho + \mu (b_k)^2\right) \left|\psi_{N,N}\right|^2 = \tfrac{\tilde{\tilde{\lambda}}_N(h)}{h} \Big[2\rho \left(1 - \tfrac{\rho \lambda_N(h) h^2}{4\alpha_1 (\zeta_k)^2}\right) \\ &\quad + 2\mu (b_k)^2 \left(1 - \tfrac{\alpha \mu \lambda_N(h) h^2}{4\beta \alpha_1 (\zeta_k)^2}\right) - \tfrac{\rho \mu \gamma \lambda_N(h) h^2}{\alpha_1 (\zeta_k)^2} b_k\Big] \end{aligned} \tag{5.31}$$

and therefore

$$\begin{aligned} &\tilde{\tilde{\lambda}}_N(h) \tfrac{(2L-h)}{2} \left|\psi_{N,N}\right|^2 \\ &\quad = E_h(0) \Big[2\rho \left(\tfrac{4\alpha_1 (\zeta_k)^2 - \rho \lambda_N(h) h^2}{4\alpha_1 (\zeta_k)^2}\right) + 2\mu (b_k)^2 \left(1 - \tfrac{\alpha \mu \lambda_N(h) h^2}{4\beta \alpha_1 (\zeta_k)^2}\right) - \tfrac{\rho \mu \gamma \lambda_N(h) h^2}{\alpha_1 (\zeta_k)^2} b_k\Big]. \end{aligned} \tag{5.32}$$

Thus

$$E_h(0) = \frac{\frac{(2L-h)}{2} \tilde{\tilde{\lambda}}_N(h) \left|\psi_{N,N}\right|^2}{\left[2\rho \left(\frac{4\alpha_1 (\zeta_k)^2 - \rho \lambda_N(h) h^2}{4\alpha_1 (\zeta_k)^2}\right) + 2\mu (b_k)^2 \left(1 - \frac{\alpha \mu \lambda_N(h) h^2}{4\beta \alpha_1 (\zeta_k)^2}\right) - \frac{\rho \mu \gamma \lambda_N(h) h^2}{\alpha_1 (\zeta_k)^2} b_k\right]}, \tag{5.33}$$

and

$$\begin{aligned} &\frac{E_h(0)}{\int_0^T \left(\rho \left|v_N'\right|^2 + \mu \left|p_N'\right|^2\right) \, dt} \\ &\quad = \frac{2L-h}{2T(\rho + \mu b_k^2) \left[2\rho \left(\frac{4\alpha_1 (\zeta_k)^2 - \rho \lambda_N(h) h^2}{4\alpha_1 (\zeta_k)^2}\right) + 2\mu (b_k)^2 \left(1 - \frac{\alpha \mu \lambda_N(h) h^2}{4\beta \alpha_1 (\zeta_k)^2}\right) - \frac{\rho \mu \gamma \lambda_N(h) h^2}{\alpha_1 (\zeta_k)^2} b_k\right]}. \end{aligned} \tag{5.34}$$

By (3.24), recall as $h \to 0$, $\lambda_N(h)h^2 \to 4$, and this reduces (5.32) to

$$\frac{E_h(0)}{\int_0^T \left(\rho\left|v_N'\right|^2 + \mu\left|p_N'\right|^2\right)\,dt} \to \frac{L}{T(\rho+\mu b_k^2)\left[2\rho\left(\frac{\alpha_1(\zeta_k)^2-\rho}{\alpha_1(\zeta_k)^2}\right)+2\mu(b_k)^2\left(1-\frac{\alpha\mu}{\beta\alpha_1(\zeta_k)^2}\right)-\frac{4\rho\mu\gamma}{\alpha_1(\zeta_k)^2}b_k\right]}. \tag{5.35}$$

Now, utilizing (4.6) and factoring out $\frac{2\mu}{\beta(b_k\gamma\mu+\rho)}$ in the denominator of (5.35) yields

$$\begin{aligned}\frac{E_h(0)}{\int_0^T \left(\rho\left|v_N'\right|^2 + \mu\left|p_N'\right|^2\right)\,dt} &\to \frac{L}{T(\rho+\mu b_k^2)\left[2\mu(b_k)^2\left(1-\frac{\alpha\mu}{\beta\alpha_1(\zeta_k)^2}\right)-\frac{2\rho\mu\gamma}{\alpha_1(\zeta_k)^2}b_k\right]}\\ &\to \frac{\beta(b_k\gamma\mu+\rho)}{2\mu b_k}\frac{L}{T(\rho+\mu b_k^2)[b_k(\beta(b_k\gamma\mu+\rho)-\alpha\mu)-\rho\gamma\beta]}.\end{aligned} \tag{5.36}$$

Now, distributing terms in the denominator,

$$\begin{aligned}\frac{E_h(0)}{\int_0^T \left(\rho\left|v_N'\right|^2 + \mu\left|p_N'\right|^2\right)\,dt} &\to \frac{\beta(b_k\gamma\mu+\rho)}{2\mu b_k}\frac{L}{T(\rho+\mu b_k^2)[\beta\gamma\mu(b_k)^2+\rho\gamma b_k-\alpha\mu b_k-\rho\gamma\beta]}\\ &\to \frac{\beta(b_k\gamma\mu+\rho)}{2\mu b_k}\frac{L}{T(\rho+\mu b_k^2)[\gamma\beta\mu(b_k)^2-(\alpha\mu-\rho\beta)b_k-\rho\gamma\beta]}\\ &\to \infty\end{aligned} \tag{5.37}$$

where $\frac{\beta(b_k\gamma\mu+\rho)}{2\mu b_k} \neq 0$, and $\beta\gamma\mu(b_k)^2 - (\alpha\mu - \beta\rho)b_k - \rho\gamma\beta = 0$ by (4.6). □

5.3 Uniform Observability of Filtered Solutions with Respect to the Discretization Parameter

Lemma 5.3. *Equipartition of Energy: For any $h > 0$ and v, p solutions of* (5.1)*, the following identity holds:*

$$\begin{aligned}-h\sum_{j=1}^{N}\int_0^T(\rho|v_j'|^2+\mu|p_j'|^2)\,dt &= \alpha_1 h\sum_{j=1}^{N}\int_0^T\left|\frac{v_j-v_{j+1}}{h}\right|^2\\ +\beta h\sum_{j=1}^{N}\int_0^T&\left|\frac{(\gamma v_j-p_j)-(\gamma v_{j+1}-p_{j+1})}{h}\right|^2\,dt+Y_h(t)=0\end{aligned} \tag{5.38}$$

with

$$Y_h(t) = h\sum_{j=1}^{N}(\rho v_j' v_j + \mu p_j' p_j). \tag{5.39}$$

Proof: First, rewrite (5.1) as the following with $\alpha = \alpha_1 + \gamma^2\beta$ to get

$$\begin{aligned} &\rho v_j'' - \alpha_1 \tfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} - \\ &\gamma\beta\left(\gamma\tfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} - \tfrac{p_{j+1}-2p_j+p_{j-1}}{h^2}\right) = 0, \end{aligned} \tag{5.40}$$

$$\mu p_j'' + \beta\left[\gamma\tfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} - \tfrac{p_{j+1}-2p_j+p_{j-1}}{h^2}\right] = 0. \tag{5.41}$$

Now, multiply (5.40) and (5.41) by v_j and p_j respectively and combine to get

$$\begin{aligned} &\textstyle\sum_{j=1}^{N}\int_0^T(\rho v_j'' v_j + \mu p_j'' p_j)\ dt - \alpha_1 \sum_{j=1}^{N}\int_0^T \tfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} v_j\ dt \\ &-\beta\textstyle\sum_{j=1}^{N}\int_0^T \tfrac{(\gamma v_{j+1}-p_{j+1})+(\gamma v_{j-1}-p_{j-1})-2(\gamma v_j-p_j)}{h^2}(\gamma v_j - p_j)\ dt. \end{aligned} \tag{5.42}$$

Now,

$$\rho\sum_{j=1}^{N}\int_0^T v_j'' v_j\ dt = -\rho\sum_{j=1}^{N}\int_0^T |v_j'|^2\ dt + \rho\sum_{j=1}^{N} v_j' v_j|_0^T \tag{5.43}$$

via integration by parts with $u = v_j$ and $dv = v_j''\ dt$. Similarly,

$$\mu\sum_{j=1}^{N}\int_0^T p_j'' p_j\ dt = -\mu\sum_{j=1}^{N}\int_0^T |p_j'|^2\ dt + \mu\sum_{j=1}^{N} p_j' p_j|_0^T. \tag{5.44}$$

We also know from [11]

$$\sum_{j=1}^{N}(u_{j+1} + u_{j-1} - 2u_j)u_j = -\sum_{j=0}^{N}|u_j - u_{j+1}|^2. \tag{5.45}$$

So we can write

$$\alpha_1 \sum_{j=1}^{N} (v_{j+1} + v_{j-1} - 2v_j)v_j = -\alpha_1 \sum_{j=0}^{N} |v_j - v_{j+1}|^2 \tag{5.46}$$

and

$$\begin{aligned} &\beta \sum_{j=1}^{N} \left[(\gamma v_{j+1} - p_{j+1}) + (\gamma v_{j-1} - p_{j-1}) - 2(\gamma v_j - p_j)\right](\gamma v_j - p_j) \\ &= -\beta \sum_{j=0}^{N} |(\gamma v_j - p_j) - (\gamma v_{j+1} - p_{j+1})|^2. \end{aligned} \tag{5.47}$$

Combining these last 5 equalities, the claim holds. □

Lemma 5.4. *For any eigenvectors v and p with eigenvalue λ of* (5.7)*, the following identity holds:*

$$\alpha_1 \sum_{j=0}^{N} \left|\frac{v_{j+1} - v_j}{h}\right|^2 + \beta \sum_{j=0}^{N} \left|\frac{(\gamma v_{j+1} - p_{j+1}) - (\gamma v_j - p_j)}{h}\right|^2 = \tilde{\lambda} \sum_{j=0}^{N} (\rho v_j^2 + \mu p_j^2). \tag{5.48}$$

Proof: Rewrite (5.7) as previously done and multiply by v_j and p_j respectively to get

$$\begin{aligned} &-\tfrac{\alpha_1}{h^2} \sum_{j=1}^{N} (v_{j+1} - 2v_j + v_{j-1})v_j \\ &-\tfrac{\beta}{h^2} \sum_{j=1}^{N} [(\gamma v_{j+1} - p_{j+1}) + (\gamma v_{j-1} - p_{j-1}) \\ &-2(\gamma v_j - p_j)](\gamma v_j - p_j)] = \tilde{\lambda} \sum_{j=0}^{N} (\rho v_j^2 + \mu p_j^2) \end{aligned} \tag{5.49}$$

$$\begin{aligned} &\implies -\tfrac{\alpha_1}{h^2} \sum_{j=1}^{N} (v_{j+1}v_j - 2v_j^2 + v_{j-1}v_j)\ dt \\ &-\tfrac{\beta}{h^2} \sum_{j=1}^{N} [(\gamma v_{j+1} - p_{j+1})(\gamma v_j - p_j) + (\gamma v_{j-1} - p_{j-1})(\gamma v_j - p_j) \\ &\quad -2(\gamma v_j - p_j)^2] \\ &= \tilde{\lambda} \sum_{j=0}^{N} (\rho v_j^2 + \mu p_j^2) \end{aligned} \tag{5.50}$$

$$\Longrightarrow \frac{\alpha_1}{h^2}\sum_{j=1}^{N}(2v_j^2-2v_{j+1}v_j)+\frac{\beta}{h^2}\sum_{j=1}^{N}(2(\gamma v_j-p_j)^2-2(\gamma v_{j+1}-p_{j+1})(\gamma v_j-p_j))$$
$$+\frac{\alpha_1}{h^2}v_N^2+\frac{\beta}{h^2}(\gamma v_N-p_N)^2=\tilde{\lambda}\sum_{j=0}^{N}(\rho v_j^2+\mu p_j^2) \tag{5.51}$$

which yields

$$\frac{\alpha_1}{h^2}\sum_{j=0}^{N}(v_{j+1}^2-2v_{j+1}+v_j^2)$$
$$+\frac{\beta}{h^2}\sum_{j=0}^{N}[(\gamma v_{j+1}-p_{j+1})^2-2(\gamma v_{j+1}-p_{j+1})(\gamma v_j-p_j)+(\gamma v_j-p_j)^2] \tag{5.52}$$
$$=\tilde{\lambda}\sum_{j=0}^{N}(\rho v_j^2+\mu p_j^2)$$

which is exactly the claim □

Lemma 5.5. *For any $h>0$, the solution $\boldsymbol{w}$ of (5.1) satisfies the following*

$$\frac{h}{2}\sum_{j=0}^{N}\left[\int_0^T\left(\rho v_j' v_{j+1}'+\mu p_j' p_{j+1}'\right)+\alpha_1\left|\frac{v_{j+1}-v_j}{h}\right|^2\right.$$
$$\left.+\beta\left|\frac{(\gamma v-p)_{j+1}-(\gamma v-p)_j}{h}\right|^2\right]\,dt+X_h(t)\big|_0^T=\frac{L}{2}\int_0^T(\rho|v_N'|^2+\mu|p_N'|^2)\,dt \tag{5.53}$$

where $X_h(t)=h\sum_{j=1}^{N}\left(\rho v_j' j\frac{v_{j+1}-v_{j-1}}{2}+\mu p_j' j\frac{p_{j+1}-p_{j-1}}{2}\right).$

Proof. Multiply the first and second equations in (5.1) by the multipliers $j\frac{v_{j+1}-v_{j-1}}{2}$ and $j\frac{p_{j+1}-p_{j-1}}{2}$, respectively, add these two equations, and take the sum and integrate by parts from 0 to T. The left hand side is

$$\sum_{j=1}^{N}\int_0^T j\left(\rho v_j''\frac{v_{j+1}-v_{j-1}}{2}+\mu p_j''\frac{p_{j+1}-p_{j-1}}{2}\right)\,dt$$
$$=\sum_{j=1}^{N}\rho v_j' j\frac{v_{j+1}-v_{j-1}}{2}+\mu p_j' j\frac{p_{j+1}-p_{j-1}}{2}\bigg|_0^T-\sum_{j=1}^{N}\int_0^T j\left(\rho v_j'\frac{v_{j+1}'-v_{j-1}'}{2}+\mu p_j'\frac{p_{j+1}'-p_{j-1}'}{2}\right)\,dt$$
$$=\sum_{j=1}^{N}\frac{1}{h}X_h(t)\bigg|_0^T-\frac{1}{2}\sum_{j=1}^{N}\int_0^T\left(j\rho v_j' v_{j+1}'-\rho(j+1)v_j' v_{j+1}'-\mu p_j' p_{j+1}'-\mu(j+1)p_j' p_{j+1}'\right.$$
$$\left.-\rho\frac{N+1}{2}v_N' v_{N+1}'-\mu\frac{N+1}{2}p_N' p_{N+1}'\right)\,dt$$
$$=\sum_{j=1}^{N}\frac{1}{h}X_h(t)\bigg|_0^T+\frac{1}{2}\sum_{j=1}^{N}\int_0^T\left(\rho v_j' v_{j+1}'+\mu p_j' p_{j+1}'\right)\,dt+\int_0^L\left(-\rho\frac{N+1}{2}|v_N'|^2-\mu\frac{N+1}{2}|p_N'|^2\right)\,dt.$$

The right hand side is

$$\sum_{j=1}^{N}\int_0^T \Big[\alpha_1 \left(\tfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} j \tfrac{v_{j+1}-v_{j-1}}{2}\right)$$
$$+\beta\left(\tfrac{(\gamma v-p)_{j+1}-2(\gamma v-p)_j+(\gamma v-p)_{j-1}}{h^2} j \tfrac{(\gamma v-p)_{j+1}-(\gamma v-p)_{j-1}}{2}\right)\Big]\ dt$$
$$= \tfrac{1}{2h^2}\sum_{j=0}^{N}\int_0^T \Big[\alpha_1(jv_{j+1}^2 - jv_{j-1}^2 - 2jv_{j+1}v_j + 2jv_{j-1}v_j)$$
$$+\ \beta\left(j(\gamma v-p)_{j+1}^2 - j(\gamma v-p)_{j-1}^2 - 2j(\gamma v-p)_{j+1}(\gamma v-p)_j + 2j(\gamma v-p)_{j-1}(\gamma v-p)_j\right)\Big]\ dt$$

$$= \tfrac{1}{2h^2}\sum_{j=0}^{N}\int_0^T \Big[\alpha_1(-2v_j^2 + 2v_jv_{j+1}) + \beta\left(-2(\gamma v-p)_j^2 + 2(\gamma v-p)_j(\gamma v-p)_{j+1}\right)\Big]\ dt$$
$$-\tfrac{1}{2h^2}\int_0^T \left(\alpha_1|v_N|^2 + \beta|\gamma v_N - p_N|^2\right)\ dt.$$
$$= -\tfrac{1}{2}\sum_{j=0}^{N}\int_0^T \left[\alpha_1\left|\tfrac{v_{j+1}-v_j}{h}\right|^2 + \beta\left|\tfrac{(\gamma v-p)_{j+1}-(\gamma v-p)_j}{h}\right|^2\right]\ dt.$$

Therefore, merging these two equations yield (5.53). □

Lemma 5.6. *For any $h>0$, the solution $\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \end{pmatrix}$ of (5.1) satisfies the following*

$$-h\sum_{j=1}^{N}\int_0^T \left(\rho|v_j'|^2 + \mu|p_j'|^2\right) + h\sum_{j=0}^{N}\left[\alpha_1\left|\frac{v_{j+1}-v_j}{h}\right|^2 + \beta\left|\frac{(\gamma v-p)_{j+1}-(\gamma v-p)_j}{h}\right|^2\right]\ dt + \mathrm{Y}_h(t)\Big|_0^T = 0 \qquad (5.54)$$

where $\mathrm{Y}_h(t) = h\sum_{j=1}^{N}\left(\rho v_j v_j' + \mu p_j p_j'\right)$.

Proof. Multiply the first and second equations in (5.1) by the multipliers v_j and p_j, respectively, add these two equations, and take the sum and integrate by parts from

0 to T:

$$\begin{aligned}
0 = &\sum_{j=1}^{N}\int_0^T\left(\rho v_j'' v_j + \mu p_j'' p_j\right)\,dt - \alpha_1\sum_{j=1}^{N}\int_0^T \frac{v_{j+1}-2v_j+v_{j-1}}{h^2} v_j\,dt\\
&-\beta\sum_{j=1}^{N}\int_0^T \frac{(\gamma v-p)_{j+1}-2(\gamma v-p)_j+(\gamma v-p)_{j-1}}{h^2}(\gamma v-p)_j\,dt\\
&= \frac{1}{h}\sum_{j=1}^{N} Y_h(t)\Big|_0^T - \sum_{j=1}^{N}\int_0^T\left(\rho|v_j'|^2+\mu|p_j'|^2\right)\,dt\\
&\quad +\alpha_1\sum_{j=0}^{N}\left|\frac{v_j-v_{j+1}}{h}\right|^2+\beta\sum_{j=0}^{N}\left|\frac{(\gamma v-p)_j-(\gamma v-p)_{j+1}}{h}\right|^2.
\end{aligned}$$

Therefore, (5.54) follows. □

Lemma 5.7. *For any $h>0$, $t\in[0,T]$ and $\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{p}\end{pmatrix}$ solution of (5.1) in which $\Gamma=\frac{\gamma}{h^2}$ be the largest eigenvalue of $\{\Gamma_k\}$. It follows that*

$$|Z_h(t)| \le \sqrt{\tilde{M}}E_h(0) \tag{5.55}$$

where $\tilde{M} = \max\left\{\frac{2\mu}{\beta}, \frac{\rho}{\alpha_1}+\frac{2\gamma^2\mu}{\alpha_1}\right\}\left(L^2-\frac{M\Gamma h^2}{16}\right)+\frac{1}{\lambda_1}\left(\frac{M^2\Gamma^2}{64}+\frac{M\Gamma}{8}\right),$

$$\begin{aligned}
Z_h(t) &= X_h(t)-\frac{M\Gamma}{8}Y_h(t)\\
&= h\sum_{j=1}^{N}\left(\rho v_j' j\frac{v_{j+1}-v_{j-1}}{2}+\mu p_j' j\frac{p_{j+1}-p_{j-1}}{2}\right)-\frac{M\Gamma}{8}\sum_{j=1}^{N}\left(\rho v_j v_j'+\mu p_j p_j'\right).
\end{aligned} \tag{5.56}$$

and $M := \frac{2\max\left(\frac{2\gamma^2\mu+\rho}{\alpha_1},\frac{2\mu}{\beta}\right)}{\min\left(\frac{1}{\zeta_1^2},\frac{1}{\zeta_2^2}\right)}.$

Note that, for a piezoelectric beam, $\left\{\frac{2\gamma^2\mu+\rho}{\alpha_1}, \frac{2\mu}{\beta}\right\} \ll 1$ and $\left(\frac{1}{\zeta_1^2}, \frac{1}{\zeta_2^2}\right) \gg 1$.

Proof.

$$
\begin{aligned}
|Z_h(t)| &\le h\left[\sum_{j=1}^{N}\rho|v_j'|^2\right]^{1/2}\left[\sum_{j=1}^{N}\rho\left|j\tfrac{v_{j+1}-v_{j-1}}{2}-\tfrac{M\Gamma}{8}v_j\right|^2\right]^{1/2}\\
&\quad +h\left[\sum_{j=1}^{N}\mu|p_j'|^2\right]^{1/2}\left[\sum_{j=1}^{N}\mu\left|j\tfrac{p_{j+1}-p_{j-1}}{2}-\tfrac{M\Gamma}{8}p_j\right|^2\right]^{1/2}\\
&\le h\left[\sum_{j=1}^{N}\rho|v_j'|^2\right]^{1/2}\left[\sum_{j=1}^{N}\rho\left|j\tfrac{v_{j+1}-v_{j-1}}{2}-\tfrac{M\Gamma}{8}v_j\right|^2+\mu\left|j\tfrac{p_{j+1}-p_{j-1}}{2}-\tfrac{M\Gamma}{8}p_j\right|^2\right]^{1/2}\\
&\quad +h\left[\sum_{j=1}^{N}\mu|p_j'|^2\right]^{1/2}\left[\sum_{j=1}^{N}\rho\left|j\tfrac{v_{j+1}-v_{j-1}}{2}-\tfrac{M\Gamma}{8}v_j\right|^2+\mu\left|j\tfrac{p_{j+1}-p_{j-1}}{2}-\tfrac{M\Gamma}{8}p_j\right|^2\right]^{1/2}.
\end{aligned}
\tag{5.57}
$$

On the other hand, since $j < \frac{L}{h}$

$$
\begin{aligned}
&\rho h\left[\sum_{j=1}^{N}\left|j\tfrac{v_{j+1}-v_{j-1}}{2}-\tfrac{M\Gamma}{8}v_j\right|^2\right]\\
&\quad =\rho h\sum_{j=1}^{N}\left[\tfrac{j^2}{4}\left|v_{j+1}-v_{j-1}\right|^2+\tfrac{M^2\Gamma^2}{64}|v_j|^2-\tfrac{M\Gamma}{8}j(v_{j+1}-v_{j-1})v_j\right].
\end{aligned}
\tag{5.58}
$$

By the Triangle Inequality,

$$
\begin{aligned}
&\rho h\left[\sum_{j=1}^{N}\left|j\tfrac{v_{j+1}-v_{j-1}}{2}-\tfrac{M\Gamma}{8}v_j\right|^2\right]\\
&\quad \le\rho h\sum_{j=1}^{N}\left[\tfrac{j^2}{2}\left|v_{j+1}-v_j\right|^2+\tfrac{j^2}{2}\left|v_j-v_{j-1}\right|^2+\tfrac{M^2\Gamma^2}{64}|v_j|^2-\tfrac{M\Gamma}{8}j(v_{j+1}-v_{j-1})v_j\right].
\end{aligned}
\tag{5.59}
$$

Substituting $\frac{L}{h}$ for j (Since $j \le N$)and rearranging,

$$
\begin{aligned}
&\rho h\left[\sum_{j=1}^{N}\left|j\tfrac{v_{j+1}-v_{j-1}}{2}-\tfrac{M\Gamma}{8}v_j\right|^2\right]\\
&\quad \le\rho h L^2\sum_{j=0}^{N}\left[\left|\tfrac{v_{j+1}-v_j}{h}\right|^2+\tfrac{M^2\Gamma^2}{64}|v_j|^2+\tfrac{M\Gamma}{8}v_{j+1}v_j-\tfrac{M\Gamma}{8}|v_N|^2\right]\\
&\quad \le\rho h L^2\sum_{j=0}^{N}\left|\tfrac{v_{j+1}-v_j}{h}\right|^2+\rho h\left(\tfrac{M^2\Gamma^2}{64}+\tfrac{M\Gamma}{8}\right)\sum_{j=0}^{N}|v_j|^2\\
&\qquad -\rho h\tfrac{M\Gamma}{16}\sum_{j=0}^{N}\left(2|v_j|^2-v_{j+1}v_j\right)-\tfrac{ML\Gamma}{8}\rho|v_N|^2.
\end{aligned}
\tag{5.60}
$$

Keeping in mind that $\Gamma < 4$,

$$\begin{aligned}
&\leq \rho h L^2 \sum_{j=0}^{N} \left| \tfrac{v_{j+1}-v_j}{h} \right|^2 + \rho h \left(\tfrac{M^2\Gamma^2}{64} + \tfrac{M\Gamma}{8} \right) \sum_{j=0}^{N} |v_j|^2 \\
&\quad -\rho h \tfrac{M\Gamma}{16} \sum_{j=0}^{N} |v_{j+1} - v_j|^2 + \tfrac{M\Gamma h}{16} |v_{N+1}|^2 - \tfrac{M\Gamma}{8} \rho |v_N|^2 \\
&\leq \tfrac{\rho}{\alpha_1} \left(L^2 - \tfrac{M\Gamma h^2}{16} \right) h \sum_{j=0}^{N} \alpha_1 \left| \tfrac{v_{j+1}-v_j}{h} \right|^2 + \rho h \left(\tfrac{M^2\Gamma^2}{64} + \tfrac{M\Gamma}{8} \right) \sum_{j=0}^{N} |v_j|^2 \\
&\quad + \left(\tfrac{M\Gamma h}{16} - \tfrac{M\Gamma}{8} \right) \rho |v_N|^2.
\end{aligned} \tag{5.61}$$

Analogously,

$$\begin{aligned}
&\mu h \left[\sum_{j=1}^{N} \left| j \tfrac{p_{j+1}-p_{j-1}}{2} - \tfrac{M\Gamma}{8} p_j \right|^2 \right] \\
&\quad \leq \mu \left(L^2 - \tfrac{M\Gamma h^2}{16} \right) h \sum_{j=0}^{N} \left| \tfrac{p_{j+1}-p_j}{h} \right|^2 + \mu h \left(\tfrac{M^2\Gamma^2}{64} + \tfrac{M\Gamma}{8} \right) \sum_{j=0}^{N} |p_j|^2 \\
&\qquad + \left(\tfrac{M\Gamma h}{16} - \tfrac{M\Gamma}{8} \right) \mu |p_N|^2,
\end{aligned} \tag{5.62}$$

and since $\frac{p_{j+1}-p_j}{h} = \frac{p_{j+1}-p_j}{h} + \gamma \frac{v_{j+1}-v_j}{h} - \gamma \frac{v_{j+1}-v_j}{h}$,

$$\begin{aligned}
&\mu h \left[\sum_{j=1}^{N} \left| j \tfrac{p_{j+1}-p_{j-1}}{2} - \tfrac{M\Gamma}{8} p_j \right|^2 \right] \\
&\quad \leq 2\mu \left(L^2 - \tfrac{M\Gamma h^2}{16} \right) h \sum_{j=0}^{N} \left[\left| \tfrac{(\gamma v - p)_{j+1} - (\gamma v - p)_j}{h} \right|^2 + \gamma^2 \left| \tfrac{v_{j+1}-v_j}{h} \right|^2 \right] \\
&\qquad + \mu h \left(\tfrac{M^2\Gamma^2}{64} + \tfrac{M\Gamma}{8} \right) \sum_{j=0}^{N} |p_j|^2 + \left(\tfrac{M\Gamma h}{16} - \tfrac{M\Gamma}{8} \right) \mu |p_N|^2 \\
&\quad = \tfrac{2\mu}{\beta} \left(L^2 - \tfrac{M\Gamma h^2}{16} \right) h \sum_{j=0}^{N} \beta \left[\left| \tfrac{(\gamma v - p)_{j+1} - (\gamma v - p)_j}{h} \right|^2 \right] \\
&\qquad + \tfrac{2\gamma^2 \mu}{\alpha_1} \left(L^2 - \tfrac{M\Gamma h^2}{16} \right) h \sum_{j=0}^{N} \left[\alpha_1 \left| \tfrac{v_{j+1}-v_j}{h} \right|^2 \right] \\
&\qquad + \mu h \left(\tfrac{M^2\Gamma^2}{64} + \tfrac{M\Gamma}{8} \right) \sum_{j=0}^{N} |p_j|^2 + \left(\tfrac{M\Gamma h}{16} - \tfrac{M\Gamma}{8} \right) \mu |p_N|^2.
\end{aligned} \tag{5.63}$$

Now (5.58) and (5.63) are merged to get

$$
\begin{aligned}
&\rho h\left[\sum_{j=1}^{N}\left|j\frac{v_{j+1}-v_{j-1}}{2}-\frac{M\Gamma}{8}v_j\right|^2+\mu h\sum_{j=1}^{N}\left|j\frac{p_{j+1}-p_{j-1}}{2}-\frac{M\Gamma}{8}p_j\right|^2\right]\\
&\quad\le \frac{2\mu}{\beta}\left(L^2-\frac{M\Gamma h^2}{16}\right)h\sum_{j=0}^{N}\beta\left[\left|\frac{(\gamma v-p)_{j+1}-(\gamma v-p)_j}{h}\right|^2\right]\\
&\qquad+\left(\frac{\rho}{\alpha_1}+\frac{2\gamma^2\mu}{\alpha_1}\right)\left(L^2-\frac{M\Gamma h^2}{16}\right)h\sum_{j=0}^{N}\left[\alpha_1\left|\frac{v_{j+1}-v_j}{h}\right|^2\right]\\
&\qquad+h\left(\frac{M^2\Gamma^2}{64}+\frac{M\Gamma}{8}\right)\sum_{j=0}^{N}(\rho|v_j|^2+\mu|p_j|^2)+\left(\frac{M\Gamma h}{16}-\frac{M\Gamma}{8}\right)(\rho|v_N|^2+\mu|p_N|^2)\\
&\quad\le\left[\frac{2\mu}{\beta}\left(L^2-\frac{M\Gamma h^2}{16}\right)+\frac{1}{\lambda_1}\left(\frac{M^2\Gamma^2}{64}+\frac{M\Gamma}{8}\right)\right]h\sum_{j=0}^{N}\beta\left[\left|\frac{(\gamma v-p)_{j+1}-(\gamma v-p)_j}{h}\right|^2\right]\\
&\qquad+\left[\left(\frac{\rho}{\alpha_1}+\frac{2\gamma^2\mu}{\alpha_1}\right)\left(L^2-\frac{M\Gamma h^2}{16}\right)+\frac{1}{\lambda_1}\left(\frac{M^2\Gamma^2}{64}+\frac{M\Gamma}{8}\right)\right]h\sum_{j=0}^{N}\left[\alpha_1\left|\frac{v_{j+1}-v_j}{h}\right|^2\right]\\
&\quad\le\left[\max\left\{\frac{2\mu}{\beta},\frac{\rho}{\alpha_1}+\frac{2\gamma^2\mu}{\alpha_1}\right\}\left(L^2-\frac{M\Gamma h^2}{16}\right)+\frac{1}{\lambda_1}\left(\frac{M^2\Gamma^2}{64}+\frac{M\Gamma}{8}\right)\right]\cdot\\
&\qquad h\sum_{j=0}^{N}\left[\alpha_1\left|\frac{v_{j+1}-v_j}{h}\right|^2+\beta\left|\frac{(\gamma v-p)_{j+1}-(\gamma v-p)_j}{h}\right|^2\right].
\end{aligned}
\tag{5.64}
$$

Combining (5.57)-(5.64), and by the Young's inequality, we deduce (5.55). □

Now we can state the main theorem of the paper which is the discrete version of (4.12) in Theorem 4.11 for the filtered solutions as the following:

Theorem 5.14. *Assume that* $0<\Gamma<4$. *Then, there exists* $T(\Gamma,h)=$ *such that for all* $T>T(\Gamma,h)=\frac{2\sqrt{\tilde M}}{1-\frac{\Gamma}{4}}=\frac{2L\sqrt{\max\left\{\frac{2\mu}{\beta},\frac{\rho}{\alpha_1}+\frac{2\gamma^2\mu}{\alpha_1}\right\}\left(1-\frac{M\Gamma h^2}{16L^2}\right)+\frac{1}{\lambda_1 L^2}\left(\frac{M^2\Gamma^2}{64}+\frac{\Gamma}{8}\right)}}{1-\frac{\Gamma}{4}}$ *there exists*

$$C(T,\Gamma,h)=\frac{2}{L}\left[T\left(1-\frac{\Gamma}{4}\right)-2\sqrt{\tilde M}\right]$$

such that

$$C(T,\Gamma,h)E_h(0)\le\int_0^T\left(\rho\left|v_N'\right|^2+\mu\left|p_N'\right|^2\right)\,dt \tag{5.65}$$

holds for every solution of (5.3) in the class $C_h(\Gamma)$, *uniformly as* $h\to 0$.

Proof. Consider (5.53) with the energy conservation in mind to simplify it to

$$\begin{aligned} &TE_h(0) + \frac{h}{2}\sum_{j=0}^{N}\int_0^T \left[\rho\left(v'_j v'_{j+1} - |v'_j|^2\right) + \mu(p'_j p'_{j+1} - |p'_j|^2)\right]\, dt + X_h(t)\big|_0^T \\ &\quad = \frac{L}{2}\int_0^T \left(\rho|v'_N|^2 + \mu|p'_N|^2\right)\, dt \end{aligned} \tag{5.66}$$

By (5.13),

$$\begin{pmatrix} \boldsymbol{v}' \\ \boldsymbol{p}' \end{pmatrix} = i \sum_{|\mu_k|\le\sqrt{\Gamma}} \left[c_{1k}\frac{\mu_k}{\zeta_1}\begin{pmatrix} \boldsymbol{\psi}_k \\ b_1\boldsymbol{\psi}_k \end{pmatrix} e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k}\frac{\mu_k}{\zeta_2}\begin{pmatrix} \boldsymbol{\psi}_k \\ b_2\boldsymbol{\psi}_k \end{pmatrix} e^{\frac{i\mu_k t}{\zeta_2}} \right]$$

and therefore,

$$\begin{aligned} &\rho\sum_{j=0}^{N}|v'_j - v'_{j+1}|^2 = \rho\sum_{j=0}^{N}\left|\sum_{|\mu_k|\le\sqrt{\Gamma}}\left[c_{1k}\frac{\mu_k}{\zeta_1}e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k}\frac{\mu_k}{\zeta_2}e^{\frac{i\mu_k t}{\zeta_2}}\right](\psi_{k,j} - \psi_{k,j+1})\right|^2 \\ &\quad = \rho\sum_{j=0}^{N}\sum_{|\mu_k|\le\sqrt{\Gamma}}\left|\frac{c_{1k}}{\zeta_1}e^{\frac{i\mu_k t}{\zeta_1}} + \frac{c_{2k}}{\zeta_2}e^{\frac{i\mu_k t}{\zeta_2}}\right|^2 |\mu_k|^2|\psi_{k,j} - \psi_{k,j+1}|^2. \end{aligned} \tag{5.67}$$

Analogously,

$$\begin{aligned} &\mu\sum_{j=0}^{N}|p'_j - p'_{j+1}|^2 = \mu\sum_{j=0}^{N}|p'_j - p'_{j+1} - \gamma(v'_j - v'_{j+1}) + \gamma(v'_j - v'_{j+1})|^2 \\ &\le 2\mu\sum_{j=0}^{N}|(\gamma v' - p')_j - (\gamma v' - p')_{j+1}|^2 + 2\gamma^2\mu\sum_{j=0}^{N}|(v'_j - v'_{j+1})|^2 \\ &= 2\gamma^2\mu\sum_{j=0}^{N}\sum_{|\mu_k|\le\sqrt{\Gamma}}\left|\frac{c_{1k}}{\zeta_1}e^{\frac{i\mu_k t}{\zeta_1}} + \frac{c_{2k}}{\zeta_2}e^{\frac{i\mu_k t}{\zeta_2}}\right|^2 |\mu_k|^2|\psi_{k,j} - \psi_{k,j+1}|^2 \\ &\quad + 2\mu\sum_{j=0}^{N}\sum_{|\mu_k|\le\sqrt{\Gamma}}\left|\frac{c_{1k}b_1(\gamma - b_1)}{b_1\zeta_1}e^{\frac{i\mu_k t}{\zeta_1}} + \frac{c_{2k}b_2(\gamma - b_2)}{b_2\zeta_2}e^{\frac{i\mu_k t}{\zeta_2}}\right|^2 |\mu_k|^2|\psi_{k,j} - \psi_{k,j+1}|^2 \end{aligned} \tag{5.68}$$

Adding (5.67) and (5.68) and using that $\mu_k^2 h^2 < \Gamma$ yields

$$
\begin{aligned}
&\sum_{j=0}^{N} \rho|v'_j - v'_{j+1}|^2 + \mu|p'_j - p'_{j+1}|^2 \\
&\le \tfrac{(2\gamma^2\mu+\rho)}{\alpha_1} \sum_{j=0}^{N} \sum_{|\mu_k|\le\sqrt{\Gamma}} \alpha_1 \left| \tfrac{c_{1k}}{\zeta_1} e^{\frac{i\mu_k t}{\zeta_1}} + \tfrac{c_{2k}}{\zeta_2} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 u_k^2 |\psi_{k,j} - \psi_{k,j+1}|^2 \\
&\quad + \tfrac{2\mu}{\beta} \sum_{j=0}^{N} \sum_{|\mu_k|\le\sqrt{\Gamma}} \beta \left| \tfrac{c_{1k} b_1(\gamma - b_1)}{b_1\zeta_1} e^{\frac{i\mu_k t}{\zeta_1}} + \tfrac{c_{2k} b_2(\gamma - b_2)}{b_2\zeta_2} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \mu_k^2 |\psi_{k,j} - \psi_{k,j+1}|^2 \\
&\le \max\left(\tfrac{2\gamma^2\mu+\rho}{\alpha_1}, \tfrac{2\mu}{\beta}\right) \left\{ \sum_{j=0}^{N} \sum_{|\mu_k|\le\sqrt{\Gamma}} \alpha_1 \left| \tfrac{c_{1k}}{\zeta_1} e^{\frac{i\mu_k t}{\zeta_1}} + \tfrac{c_{2k}}{\zeta_2} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 u_k^2 |\psi_{k,j} - \psi_{k,j+1}|^2 \right. \\
&\quad \left. + \sum_{|\mu_k|\le\sqrt{\Gamma}} \beta \left| \tfrac{c_{1k} b_1(\gamma - b_1)}{b_1\zeta_1} e^{\frac{i\mu_k t}{\zeta_1}} + \tfrac{c_{2k} b_2(\gamma - b_2)}{b_2\zeta_2} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \mu_k^2 |\psi_{k,j} - \psi_{k,j+1}|^2 \right\} \\
&\le \Gamma \max\left(\tfrac{2\gamma^2\mu+\rho}{\alpha_1}, \tfrac{2\mu}{\beta}\right) \left\{ \sum_{j=0}^{N} \sum_{|\mu_k|\le\sqrt{\Gamma}} \rho \left| c_{1k} e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \tilde{\tilde{\lambda}} |\psi_{k,j}|^2 \right. \\
&\quad \left. + \sum_{|\mu_k|\le\sqrt{\Gamma}} \mu \left| c_{1k} b_1 e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k} b_2 e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \tilde{\tilde{\lambda}} |\psi_{k,j}|^2 \right\}
\end{aligned} \tag{5.69}
$$

Here we use the fact that $\tilde{\tilde{\lambda}}_{k,j} = \frac{\mu_k^2}{\zeta_j^2}$ for $j = 1, 2$.

$$
\begin{aligned}
&\sum_{j=0}^{N} \rho|v'_j - v'_{j+1}|^2 + \mu|p'_j - p'_{j+1}|^2 \\
&\le \Gamma \max\left(\tfrac{2\gamma^2\mu+\rho}{\alpha_1}, \tfrac{2\mu}{\beta}\right) \left\{ \sum_{j=0}^{N} \sum_{|\mu_k|\le\sqrt{\Gamma}} \rho \left| c_{1k} e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \tilde{\tilde{\lambda}} |\psi_{k,j}|^2 \right. \\
&\quad \left. + \sum_{|\mu_k|\le\sqrt{\Gamma}} \mu \left| c_{1k} b_1 e^{\frac{i\mu_k t}{\zeta_1}} + c_{2k} b_2 e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \tilde{\tilde{\lambda}} |\psi_{k,j}|^2 \right\} \\
&\le \Gamma M \left\{ \sum_{j=0}^{N} \sum_{|\mu_k|\le\sqrt{\Gamma}} \rho \left| \tfrac{c_{1k}}{\zeta_1} e^{\frac{i\mu_k t}{\zeta_1}} + \tfrac{c_{2k}}{\zeta_2} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \mu_k^2 |\psi_{k,j}|^2 \right. \\
&\quad \left. + \sum_{|\mu_k|\le\sqrt{\Gamma}} \mu \left| \tfrac{c_{1k} b_1}{\zeta_1} e^{\frac{i\mu_k t}{\zeta_1}} + \tfrac{c_{2k} b_2}{\zeta_2} e^{\frac{i\mu_k t}{\zeta_2}} \right|^2 \mu_k^2 |\psi_{k,j}|^2 \right\} \\
&< \Gamma \left(\sum_{j=0}^{N} \rho|v'_j|^2 + \mu|p'_j|^2 \right)
\end{aligned} \tag{5.70}
$$

where M is defined in Lemma 5.7. Thus,

$$\begin{aligned}
&\sum_{j=0}^{N}\int_0^T \left[\rho\left(v_j'v_{j+1}' - |v_j'|^2\right) + \mu(p_j'p_{j+1}' - |p_j'|^2)\right] \\
&\qquad = -\tfrac{1}{2}\sum_{j=0}^{N}\rho|v_j' - v_{j+1}'|^2 + \mu|p_j' - p_{j+1}'|^2 + \tfrac{1}{2}\left(\rho|v_N'|^2 + \mu|p_N'|^2\right) \\
&\qquad \geq -\tfrac{\Gamma}{2}\sum_{j=0}^{N}\rho|v_j'|^2 + \mu|p_j'|^2 + \tfrac{1}{2}\left(\rho|v_N'|^2 + \mu|p_N'|^2\right)
\end{aligned} \tag{5.71}$$

and

$$\begin{aligned}
&TE_h(0) - \Gamma\tfrac{h}{4}\left(\sum_{j=1}^{N}\rho|v_j'|^2 + \sum_{j=1}^{N}\mu|p_j'|^2\right) + \tfrac{1}{2}\left(\rho|v_N'|^2 + \mu|p_N'|^2\right) + X_h(t)|_0^T \\
&\leq \tfrac{L}{2}\int_0^T \left(\rho|v_N'|^2 + \mu|p_N'|^2\right)\, dt.
\end{aligned} \tag{5.72}$$

Next, rewrite (5.54)

$$h\sum_{j=1}^{N}\int_0^T \left(\rho|v_j'|^2 + \mu|p_j'|^2\right) = TE_h(0) + \frac{1}{2}\,Y_h(t)|_0^T = 0 \tag{5.73}$$

together with (5.72) to obtain

$$\begin{aligned}
T\left[1 - \tfrac{\Gamma}{4}\right]E_h(0) + Z_h(t)|_0^T \quad &\leq \tfrac{2L-h}{4}\int_0^T \left(\rho|v_N'|^2 + \mu|p_N'|^2\right)\, dt \\
&\leq \tfrac{L}{2}\int_0^T \left(\rho|v_N'|^2 + \mu|p_N'|^2\right)\, dt
\end{aligned} \tag{5.74}$$

where Z_h is defined in (5.56). By substituting (5.55) into (5.74) we deduce that

$$\tfrac{L}{2}\int_0^T \left(\rho|v_N'|^2 + \mu|p_N'|^2\right)\, dt \geq \left[T\left(1 - \tfrac{\Gamma}{4}\right) - 2\sqrt{\tilde{M}}\right]E_h(0) \tag{5.75}$$

Taking into account that $\Gamma = \frac{\Gamma}{h^2}$ in $C_h(\Gamma)$ for (5.1), we have obtained the observability inequality (5.76) provided that

$$T > \frac{2\sqrt{\tilde{M}}}{1-\frac{\Gamma}{4}} = \frac{2L\sqrt{\max\left\{\frac{2\mu}{\beta}, \frac{\rho}{\alpha_1} + \frac{2\gamma^2\mu}{\alpha_1}\right\}\left(1 - \frac{M\Gamma h^2}{16L^2}\right) + \frac{1}{\lambda_1 L^2}\left(\frac{M^2\Gamma^2}{64} + \frac{\Gamma}{8}\right)}}{1-\frac{\Gamma}{4}}, \tag{5.76}$$

□

Note that by (4.6)

$$\zeta_1^2 = \tfrac{\rho+\gamma^2\mu}{\alpha_1} + O\left(\tfrac{2\gamma^2\mu^2}{\alpha_1\beta}\right), \quad \zeta_2^2 = \tfrac{\mu}{\beta} + O\left(\tfrac{2\gamma^2\mu^2}{\alpha_1\beta}\right) \tag{5.77}$$

since $\frac{2\gamma^2\mu^2}{\alpha_1\beta}$ is very small in comparison to $\frac{\rho+2\gamma^2\mu}{\alpha_1}$ and $\frac{\mu}{\beta}$.

6 Conclusion and Future work

In this project, we show that the strongly-coupled model of a piezoelectric beam retaining magnetic effects, as in (4.1), satisfies an observability inequality with an optimal observation time. As the system is discretized by finite differences, we lose the positive result for the observability inequality (4.12) as expected, due to the gap between consecutive eigenvalues tending towards zero as the discretization parameter goes to zero. Working off of what is done in [2] for a single wave equation, we first mimic the necessary proofs for proving the negative result for the observability inequality via discrete multipliers. Note that this differs from the work done in [23] since the wave equations are strongly coupled. In fact, to the best of our knowledge, no result has been reported in the literature for strongly coupled wave equations with non-identical wave speeds. Secondly, we directly filter the spurious high-frequency eigenvalues for both branches in order to obtain the observability inequality (5.65) using discrete multipliers. It is important to recognize that this observation time is sub-optimal.

It is also important to note that we only adopted the finite difference discretization for the spacial variable, and for future work, we look to apply other numerical methods such as the finite element method and spectral methods. There will likely be a greater amount of computations than for the finite differences, but we will also likely obtain a slightly better estimate for the observation time. The benefits of using finite differences can be seen in [11, 15].

Additionally, in Section 5, we did not apply the non-harmonic Fourier series method for the discretized model as was done for the continuous model, which provides another opening for future work. Using this method will likely result in the most optimal observation time for the model (5.1).

Another possible direction is applying the indirect filtering technique by adopting the methods in [29] to prove that the closed-loop system is exponentially stable by the boundary feedback.

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