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Predictability Time of Chaotic Cosmologies

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PREDICTABILITY TIME OF CHAOTIC COSMOLOGIES

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By

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ABSTRACT

We examine the predictability time scales for a cosmological model from the Einstein field equations coupled to the Klein-Gordon equations for a spin zero scalar field with an interaction potential V(ϕ). The cosmological equations resulting from this coupling are nonlinear in the scale cosmic parameter and scalar field, thus exhibiting characteristics of chaos. The equations can be linearized in the neighborhood of equilibrium points and then diagonalized to yield its Lyapunov exponents. One e-folding time of the system is then found to estimate the predictability time. This time is compared to the Big Rip time theorized by Yurov, Moruno, and Gonzalez-Diaz. The predictability time of the system in the neighborhood of the critical point chosen is found to be smaller than the Big Rip time.

Keywords: Cosmology, Chaos, Nonlinear Dynamics, Big Rip, Predictability Time

Dedicated to my parents and grandparents, who gathered the kindling,
and to Sheila Haynes, who struck the spark.

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CHAPTER 1

GENERAL RELATIVITY

In 1905, Albert Einstein began considering how the laws of physics would change if, as Galileo had postulated, there were no absolute reference frames [1]. That is, if there was no such thing as being absolutely at rest, there is only motion relative to other bodies. Einstein began with the notion that, because the speed of light was a physical constant, light must appear to be moving at the same speed no matter how fast the observer is moving. From this concept grew the theory of special relativity, which describes phenomenon such as time dilation, length contraction, and relative simultaneity. It also presents the famous relation between mass and energy, $E = mc^2$. All of this is accomplished by considering the universe to have four spatial dimensions: the three we are familiar with and one of time.

Special relativity has the weakness of (and derives its name from) only working in special inertial reference frames. This means that special relativity does not accurately describe scenarios in which reference frames are accelerating. Einstein then set out to generalize his theory for all reference frames. He made his first step when he considered relativity in the context of a gravitational field.

Because sitting at rest in a gravitational field is locally identical to acceleration in empty space, a free falling observer, though accelerating, would be experiencing special relativistic effects. This means that being in the presence of a massive body produces similar effects as moving through spacetime. From this observation, it was derived that bodies with mass distort and warp spacetime.

Imagine the Pythagorean Theorem, which describes the distance between two points in flat Cartesian coordinates, $x^2 + y^2 = d^2$. Each of these terms has a coefficient of 1, but it is easy to imagine other coefficients, even variable coefficients, such as $\mathrm{Sin}(x)\, x^2 + x\, y^2 = d^2$. This means the distance between two points depends on where those points are located, thus describing a curved space. Though variable coefficients in the distance equation had been extensively explored by Riemann, Einstein's breakthrough was realizing that this curved distance was to be analyzed in the context of a 4 dimensional spacetime, and that the coefficients depended on the mass and energy present. Thus, mass and energy curve space and time. The final set of equations that Einstein published, the Einstein Field Equations, was [1]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (1)$$

where $R_{\mu\nu}$ the Ricci curvature tensor, R the scalar curvature, and $g_{\mu\nu}$ the metric tensor (all of which are measures of the curvature of space), Λ is the cosmological constant, G is Newton's classical gravitational constant, c the speed of light, and $T_{\mu\nu}$ is the stress-energy tensor (a description of the mass and energy present). Each term is a 4x4 matrix, and so this formula represents 16 equations. Because some of these are not independent, and after being restricted to 4 spacetime dimensions, this reduces to 10 equations. Further simplifications can be made if it is assumed that the universe is both homogeneous (uniformly distributed contents) and isotropic (the same in all directions). This allows the 4 dimensional tensor equation to be presented as

$$\frac{8\pi}{3m_p^2}\left(V(\phi)+\frac{1}{2}(\dot{\phi})^2\right)=\left(\frac{\dot{a}}{a}\right)^2 \qquad (2)$$

where a is the scale factor (the "size" of the universe), m_p is the Planck mass, ϕ is the scalar matter field, and V is the interaction potential for this field. The term $\frac{\dot{a}}{a}$ can also be written as H, the Hubble constant. This equation leaves the source of this distortion, the matter field ϕ, up for substitution.

CHAPTER 2

QUANTUM MECHANICS

In 1913, there was mounting experimental data that described the atom as being nuclear and having discrete energy states. All of these observations, however, were unexplained by classical mechanics. Niels Bohr combined several observations and theories into a single description of the atom [2]. He combined the nuclear model from Rutherford's gold foil experiments with Planck and Einstein's quantized energy hypothesis. In Bohr's atom, electrons could only orbit at fixed intervals of angular momentum that are an integer multiple of ħ, or Planck's constant, h, divided by 2π. This model almost predicted the correct emission spectrum of hydrogen, but it did not take relativistic effects into account, and did not hold for more complex atoms.

It was Erwin Schrödinger who made further progress. He was following up on de Broglie's formula for the wavelength of any particle, $h = \lambda p$, where λ is the wavelength of the particle, and p is the particle's momentum. Schrödinger was trying to describe the wave function of electrons and its relation to their energy. He finally developed his time-dependent equation [2]:

$$E\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi(x,t) + V(x)\Psi(x,t) \qquad (3)$$

where E is the total energy of the particle, m is the particle's mass, V is the potential acting on the particle, and Ψ is the wave function. This successfully recreated the energy levels predicted by Bohr's model, but suffered the same problems of being nonrelativistic. In an attempt to create a relativistic wave equation, Schrödinger substituted the relativistic mass-energy relation, $E^2 = m^4c^4+p^2c^2$, into his equation, to give the following [2]:

$$\nabla^2\Psi - \frac{1}{c^2}\ddot{\Psi} = \frac{m^2c^2}{\hbar^2}\Psi \qquad (4)$$

Unfortunately, this contradicted a relativistic modification to the Bohr model, and so Schrödinger discarded this equation. Oscar Klein and Walter Gordon also derived this equation, from which it gains its name, but again found problems in describing electrons. The problem that Klein, Gordon, and Schrödinger all ran into was that the electron is a spin-1/2 particle, while the Klein-Gordon equation describes spin-0 particles.

This version of the equation describes individual particles' waves. If a field is treated in the same manner as the particle, the result is the second quantized Klein-Gordon equation describing the quantum field ϕ:

$$\ddot{\phi} + 3H\dot{\phi} + \Lambda = -V'(\phi) \qquad (5)$$

This equation can be used to describe a scalar field ϕ made of spin-0 quanta.

CHAPTER 3

CHAOS

Chaos, in a mathematical context, refers to the unpredictability of a deterministic system [3]. The criteria for a system to be deemed chaotic are complex, but this category essentially refers to a sensitive dependence on initial conditions. For example, a chaotic model might yield wildly different predictions due to any error in initial measurements. Chaotic solutions do not incorporate randomness. These solutions are deterministic -- given initial data, they yield reproducible results. However, the fact that any error can result in drastically different outcomes means that useful predictions (beyond a certain time scale) are practically impossible.

The "sensitive dependence on initial conditions" exhibited by chaotic systems manifests in the divergence of close phase space trajectories. In order to see how these systems compare with non-chaotic systems, let us define $\Delta(0)$ to be the distance between two phase space trajectories that are arbitrarily close together, or

$$\Delta(0) = y(x + \varepsilon, t = 0) - y(x, t = 0) \tag{6}$$

for ε that is arbitrarily small. As the trajectories develop, that is, as $t \to \infty$, there are three possible scenarios. The first is if the two trajectories do not grow apart, or even get closer together, which is to say

$$|\Delta(t)| \leq |\Delta(0)| \tag{7}$$

This system is both deterministic and predictable. No matter the error in initial conditions, there will be an equivalent (or lower) error in the prediction.

The next scenario is one in which the trajectories separate at a rate which is defined by a polynomial, or

$$|\Delta(t)| = |\Delta(0)| * bt^n \tag{8}$$

for constants b and n, with $n \geq 1$. Because the error of predictions made from these equations grows with time, they are only weakly predictable, but they are still capable of generating useful predictions.

The final possibility is that of a chaotic system, in which arbitrarily close initial conditions separate from each other exponentially:

$$|\Delta(t)| = |\Delta(0)| * e^{\lambda t} \tag{9}$$

Because of this qualitatively different speed of separation, we say that even though these systems are deterministic, they are not predictable. Here λ represents the Lyapunov exponent, which is a measure of how fast the trajectories separate and thus how badly chaotic the system is [3].

Because chaotic systems are less and less useful the farther out they are used to make a prediction, every chaotic system has a time scale on which it can be used for predictions. This time is known as the system's predictability time, t_{pred}. After this predictability time, the error in the prediction causes any information about initial conditions to be lost, and thus predictions beyond this point are useless.

The Lorenz model, originally used to model weather patterns, is a system that is known to be chaotic, but is simple enough to have been solved, and is therefore a useful tool for demonstrating the effects of chaos. The Lorenz equations are:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y \qquad (10)$$
$$\dot{z} = xy - \beta z$$

where σ, ρ, and β are constant. I arbitrarily set σ = -3, ρ = 26.5, and β = 1, and used *Mathematica* to solve these equations for two sets of initial conditions Δx=0.01 apart using the command NDSolve. I then parametrically plotted the two trajectories using ParametricPlot3D and Manipulate. Figure 3.1 demonstrates the exponential divergence of very close initial conditions and the manifestation of a predictability time.

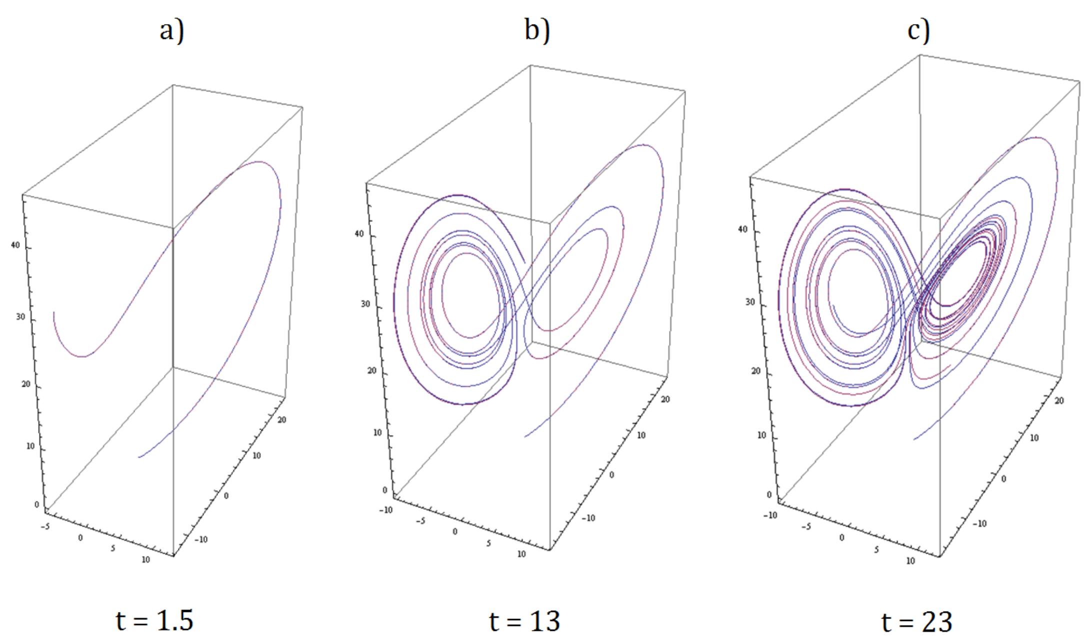

Figure 3.1 - Predictability Time of Lorenz Map

In image **a)**, two trajectories, blue and purple, with $\Delta(0) = 0.01$ are started. In image **b)**, the trajectories can be seen separating slightly, and in image **c)**, the two trajectories visibly separate quickly. It is roughly this point in time, when the trajectories head towards different attractors in the phase space, that represents the predictability time of the system.

A common approximation of the predictability time of a system is one e-folding time, or the time it takes for the separation between paths to increase by a factor of e. This is found by inverting the Lyapunov exponent of the system:

$$t_{\text{pred}} = \frac{1}{\lambda} \tag{11}$$

CHAPTER 4

COUPLING AND BIG RIP PREDICTION

Andrei Linde, at Stanford University, used both the reduced general relativity equation and the Klein-Gordon equation to create a model for the behavior of the Universe [4]. He coupled the equations, using the Klein-Gordon equation to describe the matter field within the fabric of spacetime as described by Einstein. After coupling the equations, Linde obtained the following cosmological model:

$$b\ddot{a} + \frac{b(\ddot{a}^2 + 1)}{2a} + \frac{a\dot{\phi}}{8} - \frac{a\mathrm{V}(\phi)}{4} = 0 \tag{12}$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{a}}{a} + V'(\phi) = 0 \tag{13}$$

Here $b = m_p^2/16\pi$.

The pressure and density of the Universe behave according to an equation of state similar to that which governs an ideal gas. Imagine the Ideal Gas Law:

$$pV = nRT \tag{14}$$

where p is pressure, V is volume, n is the number of moles of gas, R is the ideal gas constant, and T is the temperature. This can be rewritten as:

$$p = \frac{nRT}{V} \tag{15}$$

n/V, moles per cubic meter, is the density of the gas. Assuming the temperature remains constant, this can be rewritten as

$$p = w\rho \tag{16}$$

where *w* = RT and ρ is the density.

In a cosmological context, this *w* constant describes the proportionality between the expansive force and the energy density of the Universe. For *w* = 1/3, we say that the Universe is filled with radiation and ultra-relativistic matter. For *w* = 0, a nonrelativistic dust fills the universe. Observation of the redshifts of supernovae (indicating expansion) places the value of *w* very close to -1 [5]. If *w* = -1, we get the equation of state for dark energy, which causes the universe to expand exponentially. If this value is at all below -1 it gives the equation of state for phantom energy, and a Big Rip scenario occurs. The Big Rip refers to a time when the scale factor *a* of the universe goes to infinity in a finite time. The fabric of spacetime fractalizes, breaking apart at the subatomic level.

Later, Yurov, Moruno, and Gonzalez-Diaz analyzed Linde's cosmology [6]. Using a custom potential function *V(ϕ)*, they determined when the Big Rip would occur. The potential used in the Yurov et. al. analysis was

$$V(\phi) = \frac{1}{2}(|w| + 1)c^2 e^{-3i(\phi-\phi_0)\sqrt{|w|-1}} \tag{17}$$

where *c* does not equal the speed of light, but is a constant of value

$$c = \frac{6{a_0}^{-\frac{3}{2}(w-1)}(1 + {a_0}^{3w}H(w-1) + w)}{w-1} \tag{18}$$

The Big Rip time derived by Yurov et. al. was

$$t_{\text{rip}} = t_0 + \frac{w - 1}{9(1 + {a_0}^{3w} H(w - 1) + w)(|w| - 1)} \tag{19}$$

CHAPTER 5

PREDICTABILITY TIME ANALYSIS

The Linde cosmology is badly nonlinear, and attempts to solve it both analytically and numerically are unsuccessful. These traits suggest that this is a chaotic system. As with any chaotic system, an associated predictability time would exist. Because Yurov et. al. used the Linde cosmology to make a prediction about the Big Rip time of this model, the question must be asked: does this Big Rip prediction fall within the predictability time of the model?

Because the system is nonlinear and cannot be solved directly, it must be linearized before it can be analyzed. This is accomplished by taking the linear terms of a Taylor series expansion of the equations near critical points of the system. The Taylor series of a function f is:

$$f(x) = \sum_{n=0}^{\infty} \frac{\partial^n f(x_0)}{\partial x^n} \frac{(x - x_0)^n}{n!} \tag{20}$$

where x_0 is the critical point. In linearization, only the first two terms are kept. For example, if this formula is applied to a single equation with one variable, the equation becomes

$$f(x) \approx f(x_0) + m\,(x - x_0) \tag{21}$$

where m is the slope of the function at the critical point, a form easily recognized as the equation for a line. For a system of multiple multivariable equations, the derivative coefficient in the Taylor series becomes the Jacobian matrix of the system.

The eigenvalues of the Jacobian matrix analyzed at a critical point yields the Lyapunov exponents of the system around that point. Inverting the largest of these will give the predictability time for the system. In order to accomplish these calculations, *Mathematica* v. 7.0 was used.

First, the Yurov et. al. cosmology had to be reconstructed. The two second order differential equations of the Linde model were reduced to four first order differential equations:

$$\begin{gathered}
\dot{a} = x \\
\dot{\phi} = y \\
\dot{x} = \frac{aV(\phi)}{4b} - \frac{x^2+1}{2a} - \frac{ay}{8b} \\
\dot{y} = -\frac{3yx}{a} - V'(\phi)
\end{gathered} \tag{22}$$

Difficulties arose in using the exact potential function present in their cosmological model, so a series expansion of the potential about ϕ to the third term was found using the Series command:

$$V(\phi) \approx \frac{1}{2} c^2 e^{3i\phi_0\sqrt{-1+|w|}}(1+|w|) - \frac{3}{2} i c^2 e^{3i\phi_0\sqrt{-1+|w|}}\sqrt{-1+|w|}(1+|w|)\phi - \frac{9}{4}(c^2 e^{3i\phi_0\sqrt{-1+|w|}}(-1+|w|)(1+|w|))\phi^2 \quad (23)$$

$$V'(\phi) \approx -\frac{3}{2} i c^2 e^{3i\phi_0\sqrt{-1+|w|}}\sqrt{-1+|w|}(1+|w|) - \frac{9}{2} c^2 e^{3i\phi_0\sqrt{-1+|w|}}(-1+|w|)(1+|w|)\phi \quad (24)$$

This potential function was then used in conjunction with the Linde model.

Because of the nonlinear nature of the system, the eigenvalues would have to be found at a critical point of the system. These were found by setting each $\dot{a}, \dot{\phi}, \dot{x}, \dot{y}$ to be zero and solving using the Solve command.

$$\dot{a} = \dot{\phi} = \dot{x} = \dot{y} = 0 \quad (25)$$

This yielded two critical points:

$$y \to 0, x \to 0, \phi \to -\frac{i\sqrt{-1+|w|}}{-3+3|w|}$$
$$a \to -\frac{\sqrt{b} e^{-\frac{3}{2} i\phi_0\sqrt{-1+|w|}}\sqrt{-8+8|w|}}{c\sqrt{-1+|w|^2}} \quad (26)$$

$$y \to 0, x \to 0, \phi \to -\frac{i\sqrt{-1+\mid w \mid}}{-3+3\mid w \mid}$$

$$a \to \frac{\sqrt{b}e^{-\frac{3}{2}i\phi_0\sqrt{-1+\mid w \mid}}\sqrt{-8+8\mid w \mid}}{c\sqrt{-1+\mid w \mid}^2} \tag{27}$$

Then the Jacobian matrix **J** of these four variables was constructed.

$$\mathbf{J}=\begin{pmatrix} \frac{\partial \dot{a}}{\partial a} & \frac{\partial \dot{a}}{\partial \phi} & \frac{\partial \dot{a}}{\partial x} & \frac{\partial \dot{a}}{\partial y} \\ \frac{\partial \dot{\phi}}{\partial a} & \frac{\partial \dot{\phi}}{\partial \phi} & \frac{\partial \dot{\phi}}{\partial x} & \frac{\partial \dot{\phi}}{\partial y} \\ \frac{\partial \dot{x}}{\partial a} & \frac{\partial \dot{x}}{\partial \phi} & \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial a} & \frac{\partial \dot{y}}{\partial \phi} & \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} \tag{28}$$

The eigenvalues of this matrix were found using the Eigenvalues command about the critical point in (27). This point was chosen of the two because the value of the scale factor is positive, a condition likely exhibited by the universe.

$$\lambda_1 = -\frac{\sqrt{c^2e^{3i\phi_0\sqrt{-1+\mid w \mid}} + c^2e^{3i\phi_0\sqrt{-1+\mid w \mid}}\mid w \mid}}{2\sqrt{2}\sqrt{b}} \tag{29}$$

$$\lambda_2 = \frac{\sqrt{c^2e^{3i\phi_0\sqrt{-1+\mid w \mid}} + c^2e^{3i\phi_0\sqrt{-1+\mid w \mid}}\mid w \mid}}{2\sqrt{2}\sqrt{b}} \tag{30}$$

$$\lambda_3 = -\frac{3\sqrt{-c^2e^{3i\phi_0\sqrt{-1+\mid w \mid}} + c^2e^{3i\phi_0\sqrt{-1+\mid w \mid}}\mid w \mid}^2}{\sqrt{2}} \tag{31}$$

$$\lambda_4 = \frac{3\sqrt{-c^2 e^{3i\phi_0\sqrt{-1+|w|}} + c^2 e^{3i\phi_0\sqrt{-1+|w|}}|w|^2}}{\sqrt{2}} \tag{32}$$

These eigenvalues represent the Lyapunov exponents of the system, the largest of which being the leading Lyapunov exponent. It is difficult to ascertain the order of these values, but (32) was chosen because it has the largest non-negative coefficient. This value was then inverted, yielding one e-folding time of the system, thus estimating the predictability time of the cosmology:

$$t_{\text{pred}} = \frac{a_0^{\frac{3}{2}(-1+w)}(-1+w)}{9\sqrt{2}(1+a_0^{3w}H(-1+w)+w)\sqrt{e^{3i\phi_0\sqrt{-1+|w|}}(-1+|w|^2)}} - t_0 \tag{33}$$

This predictability time was compared to the Yurov et. al. Big Rip time. Each of these values were then scaled such that they depended only upon the pressure/density proportionality constant, *w*.

$$t_{\text{pred}} = \frac{w-1}{18\sqrt{2}w\sqrt{|w|^2-1}} \tag{34}$$

$$t_{\text{rip}} = \frac{w-1}{18w(|w|-1)} \tag{35}$$

Since observations place *w* to be very close to -1, and because the Big Rip scenario only occurs when *w* is less than -1, a plot of the Big Rip time and the predictability time was made on a time vs. *w* graph using the Plot command, and the plots were qualitatively analyzed as *w* approached -1 from the left.

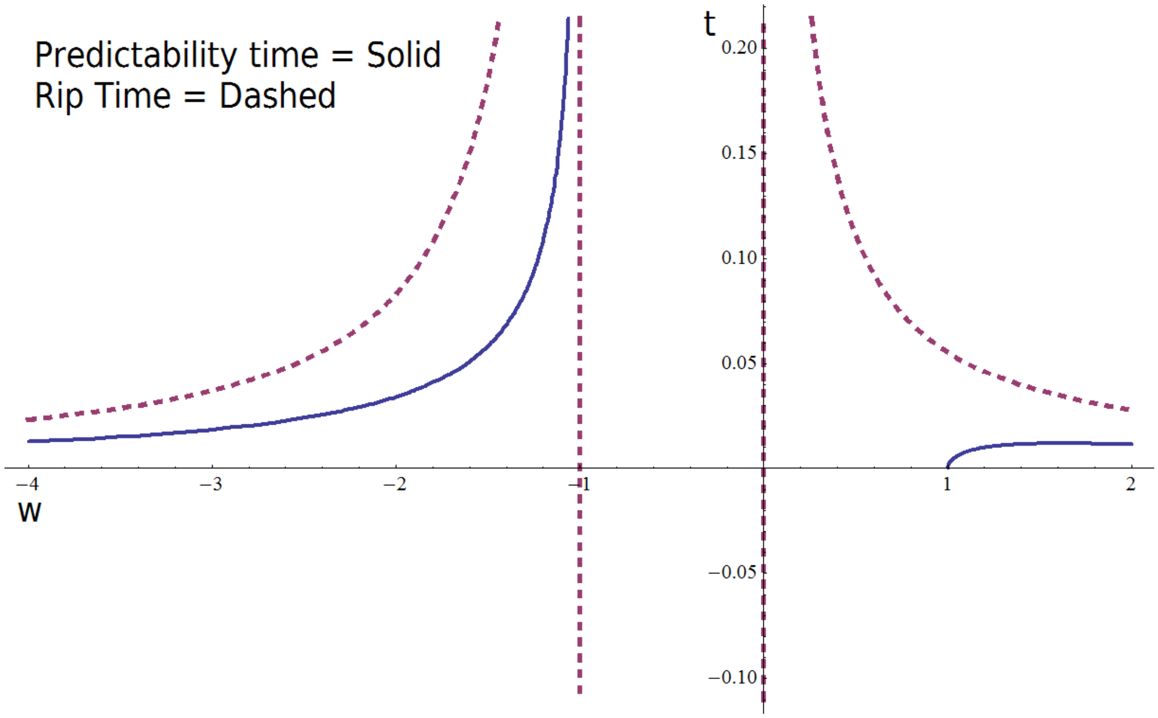

Figure 5.1 - Predictability and Big Rip time comparison

It was then seen that the Rip time approached infinity faster than the predictability time. This implies that, at any $w < -1$, the predictability time occurs before the Rip time.

CHAPTER 6

CONCLUSION

It has been demonstrated how general relativity and quantum mechanics can be coupled into a system of two second order equations describing the scale factor of the Universe and the spin-0 scalar matter field contained within. This system has solutions which exhibit behaviors indicative of chaos, and as such the system has an associated predictability time.

Through matrix linearization, the Lyapunov exponents of the system were found in the neighborhood of a critical point, and the leading Lyapunov exponent was inverted to estimate the predictability time. This was then qualitatively compared to the Big Rip time predicted by Yurov, et. al. as w approached -1 from the left.

Should $w < -1$, a Big Rip scenario is likely. However, the rip time found by Yurov et. al. lies outside of the predictability time of the cosmological model and is therefore not likely indicative of the true time at which this phenomenon would occur.

Future work can be done by determining the predictability times using the same methods here on the remaining Lyapunov exponents and critical point. This method can be refined by finding closer approximations to the potential function $V(\phi)$. The chaotic nature of this model can be further explored through the analysis of Poincaré sections of this model.

References

1. Hartle, J. B. *Gravity: An Introduction to Einstein's General Relativity.* s.l. : Addison-Wesley, 2003.

2. Zettili, N. *Quantum Mechanics: Concepts and Applications, 2nd ed.* s.l. : Wiley, 2009.

3. Taylor, J. *Classical Mechanics.* s.l. : University Science Books, 2005.

4. Linde, A. Inflationary Cosmology. *arXiv.* 2007. [Cited: April 17, 2011.] http://arxiv.org/abs/0705.0164.

5. Kuwalski, M. et. al. Improved Cosmological Constraints from New, Old and Combined Supernova Datasets. *arXiv.* 2008. [Cited: April 17, 2011.] http://arxiv.org/abs/0804.4142.

6. Yurov, A. V., Moruno, P. M. and Gonzalez-Diaz, P. F. New "Bigs" in Cosmology. *arXiv.* 2008. [Cited: April 17, 2011.] http://arxiv.org/abs/astro-ph/0606529.