5-29-2012

Remainders and Connectedness of Ordered Compactifications

Sinem Ayse Karatas
sinem_karatas@msn.com

REMAINDERS AND CONNECTEDNESS OF ORDERED COMPACTIFICATIONS

A Thesis
Presented to
The Faculty of the Department of Mathematics and Computer Science
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
Of the Requirements for the Degree
Master of Science

By
Sinem Ayse Karatas

May 2012

REMAINDERS AND CONNECTEDNESS OF ORDERED COMPACTIFICATIONS

Date Recommended 04/25/2012

Tom Richmond

Dr. Tom Richmond, Director of Thesis

Ferhan Atici

Dr. Ferhan Atici

John S. Spraker

Dr. John Spraker

Kinchel C Doerner 22-May-2012

Dean, Graduate Studies and Research Date

ACKNOWLEDGMENTS

This thesis would not be possible without the proper guidance, help, and support of many individuals who contributed to the research in one way or another. I would like to express my sincere gratitude to my advisor, Prof. Tom Richmond, for his guidance, patience, enthusiasm, motivation, and support from the beginning. During our studies, he has encouraged me to overcome the difficulties and to continue my pursuit of an academic career. In the future, I hope that I will become as good an advisor to my students as Dr. Richmond has been to me.

Besides my advisor, I am also grateful to the rest of my thesis committee, Prof. Ferhan Atici and Prof. John Spraker, for their constructive criticisms and insightful comments.

I warmly thank my family for their encouragement all my life. None of this would have been possible without their love and unconditional support. My family is an inspiration to me; I am lucky to have them.

I would like to thank my dear friend Nihan Acar. She supports me in many ways and our friendship is greatly valuable to me. I am thankful to my close friend Önder Bağcı for his optimistic perspective and strength.

CONTENTS

TABLE OF FIGURES

REMAINDERS AND CONNECTEDNESS OF ORDERED COMPACTIFICATIONS

Sinem Ayse Karatas May 2012 33 Pages

Directed by: Tom Richmond, Ferhan Atici, and John Spraker

Department of Mathematics and Computer Science Western Kentucky University

The aim of this thesis is to establish the principal properties for the theory of ordered compactifications relating to connectedness and to provide particular examples. The initial idea of this subject is based on the notion of the Stone-Čech compactification. The ordered Stone-Čech compactification $\beta_o X$ of an ordered topological space X is constructed analogously to the Stone-Čech compactification βX of a topological space X, and has similar properties. This technique requires a conceptual understanding of the Stone-Čech compactification and how its product applies to the construction of ordered topological spaces with continuous increasing functions. Chapter 1 introduces background information.

Chapter 2 addresses connectedness and compactification. If (A, B) is a separation of a topological space X, then $\beta(A \cup B) = \beta A \cup \beta B$, but in the ordered setting, $\beta_o(A \cup B)$ need not be $\beta_o A \cup \beta_o B$. We give an additional hypothesis on the separation (A, B) to make $\beta_o(A \cup B) = \beta_o A \cup \beta_o B$. An open question in topology is when is $\beta X - X = X$. We answer the analogous question for ordered compactifications of totally ordered spaces. So, we are concerned with the remainder, that is, the set of added points $\beta_o X - X$. We demonstrate the topological properties by using filters. Moreover, results of lattice theory turn out to be some of the basic tools in our original approach.

In Chapter 3, specific examples and counterexamples are given to illustrate earlier results.

Chapter 1
INTRODUCTION

1.1. Stone-Čech Compactifications

The first chapter is devoted to the progress of Stone-Čech compactifications and constructions. Chronologically, the fundamental study in this direction was originated with the investigation of Tychonoff in 1930 (see [**14**]). He introduced completely regular spaces and stated that a topological space X is embedded in a compact Hausdorff space if and only if X is a completely regular space (see [**4**]). A completely regular Hausdorff space is also known as a Tychonoff space. A topological space is said to be a Hausdorff or T_2 space if for each pair x, y of distinct points of X, there exist disjoint neighborhoods U of x and V of y.

To observe Tychonoff's technique for a (Hausdorff) compactification of a completely regular space, we need to elaborate on the definitions and results of the general construction which we will be using. The relation between compactification and complete regularity is indicated by this general construction.

When we summarize the timeline of the Stone-Čech compactification, Tychonoff's characterization (1930) is the main influence. Then, the properties of M.H. Stone and E. Čech with the filter concept of Henri Cartan (1937) led to the Wallman compactification. The process goes on with the ultrafilter compactifications of Pierre Samuel (1948) and observations about $C^*(X)$ and $C(X)$ of Edwin Hewitt(1948)(see [**15**]).

Definition 1.1.1. *A Hausdorff space X is said to be completely regular if for every closed subset F of X, and for each point x of X not in F, there is a continuous real-valued function f on X such that $f(x) = 0$ and $f(F) = \{1\}$.*

Definition 1.1.2. *If a set A is a subspace of a space X and $f : A \to Y$ is a continuous mapping, then a continuous mapping $g : X \to Y$ such that $g|_A = f$ is called a continuous extension to X of the mapping f.*

Definition 1.1.3. *If X is a topological space then*

$C(X) = \{f : X \to \mathbb{R}|\ f$ is continuous$\}$,

$C^*(X) = \{f : X \to \mathbb{R}|\ f$ is continuous and bounded$\}$, and

for any $f \in C(X)$, $Z(f) = \{x \in X|\ f(x) = 0\}$, the zero set of f.

The family of all zero sets of functions on X will be denoted by $Z(X)$. $Z(X)$ is closed under finite unions and finite intersections.

Definition 1.1.4. *If $A \subseteq X$ and every $f \in C(A)$ has a continuous extension to X, the set A is said to be C-embedded in X. Similarly, if each $f \in C^*(A)$ has a continuous bounded extension to X, the set A is said to be C^*-embedded in X.*

We will see that the Stone-Čech compactification of a space X is a compact Hausdorff space containing X as a dense C^*-embedded subspace. Thus, the concept of C^*-embedding is one of the main tools for studying the Stone-Čech compactification.

Definition 1.1.5. *A compactification of a space X is a compact Hausdorff space αX and an embedding $\alpha : X \to \alpha X$ so that $\alpha(X)$ is dense in αX, i.e., $\overline{\alpha(X)} = \alpha X$ where $\overline{\alpha(X)}$ is the closure of $\alpha(X)$.*

Since X is homeomorphic to $\alpha(X) \subseteq X$, we may think of X as being equal to $\alpha(X)$, so a compactification of X may be viewed as a compact T_2 space αX containing X as a dense subset.

Tychonoff's technique was expanded by Čech who presented the standard notation βX in 1937. The symbol βX is used to denote the Stone-Čech compactification of the space X. The Stone-Čech compactification of X is constructed in such a way that every bounded, real-valued continuous function on X will extend continuously to the compactification (see [**15**]). From now on, all our topological spaces are required to be completely regular Hausdorff spaces.

For each $f \in C^*(X)$, f is bounded. So there exists a closed and bounded interval I_f in $\mathbb{R}$ which contains $f(X)$. We will consider the function $e : X \to \prod_{f \in C^*(X)} I_f \subseteq \mathbb{R}^{C^*(X)}$ defined by $e(x) = \prod_{f \in C^*(X)} f(x)$. The following result is in Chandler's text (see [**4**]).

Proposition 1.1.6. *The function e is a topological embedding of X into $\prod_{f \in C^*(X)} I_f$ (that is, X is homeomorphic to $e(X) \subseteq \prod_{f \in C^*(X)} I_f$) if X is completely regular.*

We consider $\beta X = \overline{e(X)} \subseteq \prod_{f \in C^*(X)} I_f$, so $e(X)$ is dense in βX. Since each $I_f \subseteq \mathbb{R}$ is Hausdorff and an arbitrary product of Hausdorff spaces is Hausdorff, $\prod_{f \in C^*(X)} I_f$ is Hausdorff. Also, the Hausdorff property is hereditary, so $\beta X \subseteq \prod_{f \in C^*(X)} I_f$ is Hausdorff. Tychonoff's theorem (see [**14**]) states that the product of any collection of compact topological spaces is compact. Since each closed and bounded interval I_f in $\mathbb{R}$ is compact, $\prod_{f \in C^*(X)} I_f$ is compact by Tychonoff's theorem. The closed set βX in the compact space $\prod_{f \in C^*(X)} I_f$ is compact. Thus, βX is a compactification of X, and we call this compactification of X the Stone-Čech compactification.

Theorem 1.1.7. *X is completely regular if and only if X is a subspace of a compact T_2 space.*

Proof. If X is completely regular then X is a (dense) subspace of the compact T_2 space βX.

Conversely, let X be a subspace of a compact T_2 space Y. Since Y is compact T_2, Y is a normal space. So, Y is completely regular. Since complete regularity is hereditary, X is also completely regular. $\square$

In view of the following definitions and theorems, we will establish the general techniques for Chapter 2.

Definition 1.1.8. *A collection $\mathcal{F}$ of subsets of a topological space X is called a filter if the following properties hold:*

i) $X \in \mathcal{F}$,

ii) If $A_1, A_2 \in \mathcal{F}$, *then* $A_1 \cap A_2 \in \mathcal{F}$,

iii) If $A \in \mathcal{F}$ *and* $A \subseteq B$, *then* $B \in \mathcal{F}$,

iv) $\varnothing \notin \mathcal{F}$.

Thus, by induction, we conclude that filters are closed under finite intersections. Using the filter definition, we also define a z-filter on a space X as a subfamily $\mathcal{Z}$ of $Z(X)$ with the same properties given in Definition 1.1.9. We now provide some further explanation about filters. A collection $\mathscr{B}$ of subsets of X is a filter base if every intersection of finitely many elements of $\mathscr{B}$ is nonempty. If $\mathscr{B}$ is a filter base, $[\mathscr{B}] = \{F : \exists B \in \mathscr{B}, B \subseteq F\}$ is a filter called the filter generated by the base $\mathscr{B}$.

Definition 1.1.9. *Let $\mathcal{U}$ be a filter on X. $\mathcal{U}$ is called an ultrafilter if for all $A \subset X$ either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$ where A^c denotes the complement of A in X.*

The following theorem about ultrafilters, which follows from the definition and a standard application of Zorn's lemma indicates the main property of ultrafilters (see [**5**]).

Theorem 1.1.10. *(The Ultrafilter Theorem)*

Let $\mathcal{F}$ be a filter on X. There is an ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subset \mathcal{U}$.

Example 1.1.11. *Consider $X = \mathbb{Q}$, the set of rational numbers, and the filter $\mathcal{F} = [\{(\pi - \varepsilon, \pi + \varepsilon) : \varepsilon > 0\}]$ on $\mathbb{Q}$. Since $\mathcal{H} = [\{(\pi, \pi + \varepsilon) : \epsilon > 0\}]$ is a filter which contains $\mathcal{F}$, $\mathcal{F}$ is not an ultrafilter on $\mathbb{Q}$. In addition, the filter $\mathcal{J} = [\{(\pi, \pi + \varepsilon) : \epsilon > 0\} \cap \{\frac{a}{2^n} : a \in \mathbb{Z}, n \in \mathbb{N}\}]$ on $\mathbb{Q}$ satisfies $\mathcal{H} \subseteq \mathcal{J}$, but $\mathcal{J}$ is still not an ultrafilter.*

We will exhibit the following definition and then demonstrate the theorem for convergence in terms of filters.

Definition 1.1.12. *We say a filter $\mathcal{F}$ converges to x, denoted $\mathcal{F} \to x$, if for any neighborhood N_x of x, $N_x \in \mathcal{F}$.*

Theorem 1.1.13. *Let X be a topological space. X is Hausdorff if and only if every filter has at most one limit.*

Proof. A topological space is a Hausdorff space if for each pair x, y of distinct points of X, there exist disjoint neighborhoods U of x and V of y. Thus, no filter contains both U and V, so a filter may converge to either x or y but not both. Then each filter has at most one limit.

Conversely, assume that every neighborhood N_x of x and every neighborhood N_y of y are not disjoint. Thus, $N_x \cap N_y$ is a basis for a filter which has two limit points x and y. By the assumption, every filter has at most one limit point, so, this is a contradiction, and thus X is Hausdorff. $\square$

A topological space is said to be sequentially compact if and only if every sequence has a convergent subsequence. Sequential compactness and compactness are equivalent for metric spaces. But, filters work in the opposite way, so finding a convergent subspace corresponds to expanding a filter to a larger filter which converges to some point x.

This leads us to the following proposition which will allow us to characterize convergence in a completely regular space in terms of z-ultrafilters (see [**15**]) .

PROPOSITION 1.1.14. *A topological space X is compact if and only if every ultrafilter converges.*

DEFINITION 1.1.15. *Subsets A and B of a space X are said to be completely separated in X if there exists an $f \in C^*(X)$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.*

We shall present the following equivalent properties of the Stone-Čech compactifications of a completely regular Hausdorff space X.

THEOREM 1.1.16. *Every space X has a compactification βX, which has the following equivalent properties:*

i) X is C^*-embedded in βX, and βX is the only compactification of X having this property.

ii) Every continuous function from X into a compact Hausdorff space Y has a continuous extension from βX to Y.

iii) Disjoint zero sets of X have disjoint closures in βX.

iv) If Z_1 and Z_2 are zero sets in X, then $cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 = cl_{\beta X}(Z_1 \cap Z_2)$.

v) Completely separated sets in X have disjoint closures in βX.

vi) Every point in βX is the limit of a unique z-ultrafilter on X.

vii) βX is maximal in the set of compactifications of X partially ordered by $\alpha X \geq \gamma X$ if and only if there is a continuous function $f : \alpha X \to \gamma X$ such that $f|_X = id_X$.

The equivalency of the properties is shown in Walker's text (see [**15**]).

The following property of the Stone-Čech compactification will help us to describe the ordered case and determine the characteristic properties of our general construction.

PROPOSITION 1.1.17. *A subspace A of X is C^*-embedded in X if and only if $\beta A = cl_{\beta X} A$.*

PROOF. If A is C^*-embedded in X, each $f \in C^*(A)$ has a continuous extension to X and $X \subseteq \beta X$, so the set A is C^*-embedded in βX by Theorem 1.1.16(i). Now $cl_{\beta X} A$ is compact Hausdorff and contains A as a dense subspace, so it is a compactification of A. Any $f \in C^*(A)$ can be extended to $\widehat{f} \in C^*(\beta X)$, and $\widehat{f}|_{cl_{\beta X} A}$ is an extension of f to $cl_{\beta X} A$. Thus, by Theorem 1.1.16(i), $cl_{\beta X} A = \beta A$.

Conversely, suppose $\beta A = cl_{\beta X} A$. By Theorem 1.1.16(i), A is C^*-embedded in βA, and by the Tietze extention theorem (see [**11**]), $cl_{\beta X} A$ is C^*-embedded in the normal space βX. Thus, any $f \in C^*(A)$ can be extended to $\beta A = cl_{\beta X} A$ and then to βX, and the restriction of this extention to X gives an extention of f to X. Thus, A is C^*-embedded in X. □

Further information about the extention principle and the properties of the Stone-Čech compactification can also be found in L. Gillman and M. Jerison's text, *Rings of Continuous Functions* (see [**6**]) and P. Jackson's doctoral thesis, *Iterated Remainders in Compactifications* (see [**8**]).

1.2. Ordered Topological Spaces

An early development of the concept of order (or partial order) in mathematics is given in C.S. Pierce's work in 1880. The later systematic study of ordered set theory was elaborated by Dedekind, Hausdorff and Emmy Noether. We first define a concept of ordered sets, then observe some technical results.

Definition 1.2.1. *A partial order is a binary relation* $\leq$ *over a set* X *which is reflexive, transitive and antisymmetric so that it has the following properties:*

i) *If* $x \in X$, *then* $x \leq x$;

ii) *If* $x, y, z \in X$, $x \leq y$, *and* $y \leq z$, *then* $x \leq z$;

iii) *If* $x, y \in X$ *and both* $x \leq y$ *and* $y \leq x$, *then* $x = y$.

A total order on X *is a partial order which is decisive, that is:*

iv) *If* $x, y \in X$, *then either* $x \leq y$ *or* $y \leq x$.

A set with a partial order is said to be a partially ordered set or poset (see [**11**]*). A set* $\{(\leftarrow, x) : x \in X\}$ *shows the elements with less than a point* x.

Definition 1.2.2. *A subset* A *of a partially ordered set* $(X, \leq)$ *is said to be decreasing if* $x \leq a$ *and* $a \in A$ *imply* $x \in A$.

The decreasing hull of A, denoted $d(A)$, is the smallest decreasing set containing A. Thus, a point x belongs to $d(A)$ if and only if there is a point $a \in A$ such that $x \leq a$. Dually, we can also define the increasing hull of A as the smallest increasing set $i(A)$ containing a given subset $A \subset X$. We observe that A is an increasing set if and only if $A = i(A)$ and A is an decreasing set if and only if $A = d(A)$. We now describe increasing and decreasing functions.

Definition 1.2.3. *Let A and B be partially ordered sets. A function f from A to B is said to be an increasing function or order-preserving function if $a, b \in A$ and $a \leq b$ imply $f(a) \leq f(b)$. A function f from A to B is said to be a decreasing function or order-reversing function if $a, b \in A$ and $a \leq b$ imply $f(a) \geq f(b)$.*

Definition 1.2.4. *Let X be a partially ordered set. A subset A of X is convex if $x, y \in A$ and $x \leq z \leq y$ imply $z \in A$.*

We note that A is convex if and only if $A = i(A) \cap d(A)$.

Definition 1.2.5. *A partially ordered topological space X is said to be locally convex if at every point the set of convex neighborhoods is a base for the neighborhood system of that point (see* [**11**]*).*

Local convexity is a common compatibility condition between the topology and order on a partially ordered topological spaces. A partially ordered topological space $(X, \tau, \leq)$ is T_2-ordered if $x \nleq y$ in X implies there exist disjoint neighborhoods, which need not be open, U of x and V of y, with U being an increasing set and V being a decreasing set. We only consider T_2-ordered, locally convex, partially ordered topological spaces, with special attention to totally ordered spaces.

Definition 1.2.6. *Let X be an ordered topological space. X is a totally ordered space if X is T_2-ordered, the order on X is total, and the topology on X is locally convex (see* [**1**]*).*

Also, Kent and Richmond noted that if the neighborhood filter at each point has a filter base of convex sets, then this topology is locally convex (see [**9**]).

Chapter 2

A COMPARISON OF βX *and* $\beta_o X$

In this chapter, we introduce the concept of an ordered version $\beta_o X$ of the Stone-Čech compactification of a totally ordered topological space X. Nachbin proved that an ordered topological space has an ordered compactification if and only if it is completely regular ordered. We focus on ordered versions of connectedness to determine which versions of ordered separations (A, B) of X satisfy $\beta_o(A \cup B) = \beta_o A \cup \beta_o B$ where $\beta_o X$ is the ordered Stone-Čech compactification of X.

2.1. Ordered Compactifications of Totally Ordered Spaces

We first recall the definition of a completely regular ordered space and present the concept of ordered compactification of totally ordered spaces. Ordered compactifications of totally ordered spaces were described as a compactification with an order by Blatter (see [**3**]), Kent and Richmond (see [**9**]), and Bezhanishivili and Morandi (see [**1**]).

Definition 2.1.1. *Let X be a partially ordered topological space. X is a completely regular ordered space if the following conditions are satisfied:*

i) For each $x, y \in X$ with $x \nleq y$, there exists a continuous order-preserving function $f : X \to [0,1]$ such that $f(x) > f(y)$.

ii) For each $x \in X$ and each closed subset F of X with $x \notin F$, there exist a continuous order-preserving $f : X \to [0,1]$ and a continuous order-reversing $g : X \to [0,1]$ such that $f(x) = 1 = g(x)$ and $F \subseteq f^{-1}(0) \cup g^{-1}(0)$.

$\beta_o X$ is constructed analogously to the product construction for βX given in Chapter 1, using the set $C^{*\uparrow}(X)$ of continuous, bounded, increasing real valued functions on X as the index set. The fact that X is a completely regular ordered space implies the evaluation map e is an embedding and order isomorphism. Furthermore, analogous to βX, the extension property holds in $\beta_o X$. So, given any continuous increasing function $f : X \to Y$ of X into a compact T_2-ordered space Y, the function f extends uniquely to a continuous increasing function $\widehat{f} : \beta_o X \to Y$. By the construction of $\beta_o X$ as a subspace of $\prod_{f \in C^{*\uparrow}(X)} \mathbb{R}$, $\beta_o X$ essentially contains a copy of each $f \in C^{*\uparrow}(X)$. The projection function $\pi_f : \beta_o X \to \mathbb{R}$ of $\beta_o X \subseteq \prod_{f \in C^{*\uparrow}(X)} \mathbb{R}$ onto its f^{th} coordinate is a continuous increasing extension of $f \in C^{*\uparrow}(X)$.

Let us consider the ordered compactification $\beta_o\mathbb{N}$ of the natural numbers. Given two sequences $A = \{2, 4, 6, ...\}$ and $B = \{1, 3, 5, ...\}$, each must have a limit point in $\beta_o\mathbb{N}$. If ∞_1 is a limit point of A and ∞_2 is a limit point of B, the intersection of any neighborhood U of ∞_1 and any neighborhood of V of ∞_2 cannot be an empty set by the convexity. So it is not a Hausdorff space if $\infty_1 \neq \infty_2$. Therefore, the 1-point compactification of the natural numbers is the only ordered compactification of $\mathbb{N}$, so $\beta_o\mathbb{N} = \mathbb{N} \cup \{\infty\}$.

An open question in topology is which spaces X are homeomorphic to $\beta X - X$. This question has been known since the 1980s from the work of Stannett (see [**12**] and [**13**]) and other students of C.J. Knight at the University of Sheffield (see [**7**] and [**8**]). We start with totally ordered spaces and ask when X is topologically and ordered equivalent to $\beta_o X - X$. First, we give the lemmas related to the construction of $\beta_o X$, then we recall that a complete lattice is defined as a poset in which every subset has a least upper bound and a greatest lower bound (see [**2**]).

Lemma 2.1.2. *Let $f : Y \to Z$ be a continuous function and S be a dense subset of Y for which $f|_S$ is a homeomorphism. If Y is a Hausdorff space, then $f(Y - S) \subset Z - f(S)$ (see Theorem 1.6 in* [**4**]*).*

Lemma 2.1.3. *Every compact totally ordered topological space is a complete lattice.*

Proof. If X is a finite totally ordered space, then X has to be a complete lattice. So, suppose compact totally ordered topological space X is infinite and not a complete lattice. If some set $S \subset X$ has no upper bound, then the open cover $C = \{(\leftarrow, x) : x \in S\} \cup \{(b, \rightarrow) : b$ is an upper bound of $S\}$ of X contains no $(b, \rightarrow)$ type sets and it has only $(\leftarrow, x)$ for $x \in S$. If C has a finite subcover $(\leftarrow, x_i)$ for $i = 1, ..., n$, then $\bigcup_{i=1}^{n}(\leftarrow, x_i) = (\leftarrow, x_j)$ where $x_j = max\ x_i$ which does not cover $x_j = max\ x_i$. If S has upper bounds but no least one, any finite subcover would contain only finitely many $(b_i, \rightarrow)$ for $i = 1, ..., n$. So, $\bigcup(b_i, \rightarrow) = (b_j, \rightarrow)$ where $b_j = min\ b_i$ for $i = 1, ..., n$ does not include b_j and b_j is not covered by any $(\leftarrow, x)$ for $x \in S$. This contradicts the assumption that X is a compact totally ordered topological space. The dual argument covers the case of S having no greatest lower bound. □

Theorem 2.1.4. *Let X be a totally ordered space. If $f : X \to \beta_o X - X$ is a homeomorphism and order isomorphism, then there exists an increasing extension $\widehat{f} : \beta_o X \to \beta_o X$ and $\widehat{f}$ has no fixed points.*

Proof. f has an extension $\widehat{f}$ by the ordered version of Theorem 1.1.16(ii). By Lemma 2.1.3, since X is a totally ordered space, $\beta_o X$ has a largest element $x \in X$ and a smallest element $y \in X$. Also, since X is mapping to points not in X, $\widehat{f}(x) < x$ and $\widehat{f}(y) > y$. Applying Lemma 2.1.2 with $Y = Z = \beta_o X$ and $S = X$ we have $\widehat{f}(\beta_o X - X) \subseteq \beta_o X - \widehat{f}(X)$. Since the image $\widehat{f}(X)$ is same as the image $f(X)$,

$\widehat{f}(X) = \beta_o X - X$. Therefore, $\widehat{f}(\beta_o X - X) \subseteq \beta_o X - \widehat{f}(X) = \beta_o X - (\beta_o X - X) = X$. Thus, for each $z \in \beta_o X$, $\widehat{f}(z) \neq z$, so the extension $\widehat{f}$ has no fixed point. $\square$

If f and g are continuous functions on $\mathbb{R}$, the set $Y = \{x \in \mathbb{R} : f(x) \geq g(x)\}$ is closed. It is sufficient to show that $\mathbb{R} - Y$ is open. Since $\mathbb{R}$ is T_2, for any $x \in \mathbb{R}$ with $f(x) < g(x)$ there exist disjoint convex neighborhoods U of $f(x)$ and V of $g(x)$. Then $f^{-1}(U) \cap g^{-1}(V)$ is a neighborhood of x contained in $\mathbb{R} - Y$ on which g is strictly larger than f. So $\mathbb{R} - Y$ is open and Y is closed in $\mathbb{R}$. The same proof shows that this property holds for functions f and g on any locally convex totally ordered topological space. Now if f and $\widehat{f}$ are as in Theorem 2.1.4, and $g(x) = x$, then we see that $C = \{x \in X : \widehat{f}(x) > x\}$ and $D = \{x \in X : \widehat{f}(x) < x\}$ form a separation of X, since $\widehat{f}$ has no fixed points, and $\widehat{C} = \{x \in \beta_o X : \widehat{f}(x) > x\}$ and $\widehat{D} = \{x \in \beta_o X : \widehat{f}(x) < x\}$ form a separation of $\beta_o X$. Furthermore, we have the following lemma about these sets C, D, $\widehat{C}$ and $\widehat{D}$.

Lemma 2.1.5. *$cl_{\beta_o X} C = \widehat{C}$ and $cl_{\beta_o X} D = \widehat{D}$.*

Proof. $\widehat{C}$ is closed and contains C, so $cl_{\beta_o X} C \subseteq \widehat{C}$. If $x \in \widehat{C}$, $x \notin cl_{\beta_o X} D$ since $cl_{\beta_o X} D \subseteq \widehat{D}$. But since $\beta_0 X = cl_{\beta_o X}(C \cup D) = cl_{\beta_o X} C \cup cl_{\beta_o X} D$, $x \notin cl_{\beta_o X} D$ implies $x \in \beta_o X - cl_{\beta_o X} D \subseteq cl_{\beta_o X} C$. Thus, $\widehat{C} \subseteq cl_{\beta_o X} C$, so $\widehat{C} = cl_{\beta_o X} C$. Similarly, $\widehat{D} = cl_{\beta_o X} D$. $\square$

We first consider the case of a topological space, without order. The notation $\uplus$ represents the disjoint union.

Theorem 2.1.6. *If (A, B) is a separation of a topological space X, then $\beta(A \cup B) = \beta A \uplus \beta B = cl_{\beta X} A \uplus cl_{\beta X} B$, that is, $cl_{\beta X} A \cap cl_{\beta X} B = \varnothing$, $cl_{\beta X} A = \beta A$ and $cl_{\beta X} B = \beta B$.*

Proof. Any point in $\beta(A \cup B)$ is a limit of points in $X = A \uplus B$. Since X is disconnected, the limit points must be a limit point of A or a limit point of B, that is, $\beta(A \cup B) = \beta A \uplus \beta B$. More formally, by Theorem 1.1.16(vi), every point of βX is the limit of a unique z-ultrafilter $\mathcal{F}$. If $\mathcal{F} \to a \in A$, then $\mathcal{F}$ is a z-ultrafilter on A (and not on B), and similarly for $B \in \mathcal{F}$. It follows that $\beta(A \cup B) = \beta A \cup \beta B$. By Theorem 1.1.16(i), A is C^*-embedded in βA. Then it follows that $cl_{\beta X} A = \beta A$ by Proposition 1.1.17. Similarly, $cl_{\beta X} B = \beta B$. Since, A and B are completely separated in X, they have disjoint closures in βX by Theorem 1.1.16(v). Therefore, $cl_{\beta X} A \cap cl_{\beta X} B = \varnothing$. Thus, we have $\beta X = \beta(A \cup B) = \beta A \uplus \beta B = cl_{\beta X} A \uplus cl_{\beta X} B$. □

Since sequences are defined by countable sets, they cannot be used to describe the general situation, but considering them will give some insight. In a compact space, every sequence has a convergent subsequence. By Proposition 1.1.14, βX contains a unique limit for every ultrafilter. If (x_n) is a sequence in $\beta X = \beta(A \cup B)$, it must have a convergent subsequence $(x_{n_i}) \to x_0 \in cl_{\beta X} A \uplus cl_{\beta X} B = cl_{\beta X}(A \cup B) = cl_{\beta X} X = \beta X$. We show that x_0 is not in both $cl_{\beta X} A$ and $cl_{\beta X} B$. Since $x_0 \in cl_{\beta X} A \uplus cl_{\beta X} B$, $x_0 \in cl_{\beta X} A$ or $x_0 \in cl_{\beta X} B$. We suppose $x_0 \in cl_{\beta X} A$ and since $cl_{\beta X} B$ is closed and by Theorem 2.1.6, $cl_{\beta X} A \cap cl_{\beta X} B = \varnothing$, $cl_{\beta X} A = X - cl_{\beta X} B$ is an open neighborhood of x_0. Then (x_{n_i}) is eventually in $cl_{\beta X} A \cap X = A$. So, a subsequence in X converging to $x_0 \in cl_{\beta X} A$ is a subsequence eventually in A. It follows that every subsequence in A is a subsequence in X. So, saying that every sequence in B has a convergent subsequence is equivalent to the property that every sequence in X with a converging subsequence $(x_{n_i}) \to x_0 \in cl_{\beta X} B$ has a converging subsequence in B.

Additionally, if (A, B) is a separation of a topological space X, there exists a continuous function $s : X \to \{0, 1\}$ defined by $s(A) = \{0\}$ and $s(B) = \{1\}$. By Theorem

1.1.16 (ii), the continuous function s from X into a compact T_2 space $\{0,1\}$ has a continuous extension $\widehat{s}$ from βX to $\{0,1\}$. A net is a generalization of a sequence, and it is a function with range the topological spaces. If $z \in cl_{\beta X} A$, there exists a net a_λ in A such that $a_\lambda \to z$, and then $s(a_\lambda) \to s(z)$. Because the image of the net a_λ is equal to 0 under a continuous function s, we have $s(z) = 0$. If $z \in cl_{\beta X} B$, then similarly, $s(z) = 1$. Therefore, this is another way to show that $cl_{\beta X} A$ and $cl_{\beta X} B$ are disjoint sets in βX.

Now we observe this statement in the setting of ordered compactifications of totally ordered spaces and note that the ordered version of the condition $cl_{\beta X} A \cap cl_{\beta X} B = \varnothing$, which was part of the conclusion in Theorem 2.1.6, must be part of the hypothesis.

THEOREM 2.1.7. *If (A, B) is a separation of a locally convex, totally ordered topological space X and $cl_{\beta_o X} A \cap cl_{\beta_o X} B = \varnothing$, then $\beta_o(A \cup B) = \beta_o A \uplus \beta_o B = cl_{\beta_o X} A \uplus cl_{\beta_o X} B$. In particular, $\beta_o A = cl_{\beta_o X} A$ and $\beta_o B = cl_{\beta_o X} B$.*

PROOF. In $\beta_o X$, every convex closed ultrafilter converges by the filter construction of $\beta_o X$ in [**9**]. Let $\beta_o X$ be the set of all convex closed ultrafilters on X and $\dot{x}$ be the set of all supersets of a point x in a totally ordered space X. Then the function $e : X \to \beta_o X$ defined by $e(x) = \dot{x}$ is an embedding of X into $\beta_o X$. If $\mathcal{F}$ is a convex ultrafilter on X which is not of form $\dot{x}$, then $\mathcal{F}$ does not converge. So we need a limit for $\mathcal{F}$ in $\beta_o X$. Given any closed convex ultrafilter $\mathcal{F}$ on X, $\mathcal{F}$ must converge in $\beta_o X$. So $\mathcal{F}$ converges to a point x in $cl_{\beta_o X} A$ or $cl_{\beta_o X} B$. Suppose $x \in cl_{\beta_o X} A$. We claim that there is a neighborhood N_x of x with $N_x \subseteq cl_{\beta_o X} A$. Otherwise, every convex neighborhood of x intersects $cl_{\beta_o X} B = \beta_o X - cl_{\beta_o X} A$. $\beta_o X$ has a base of convex neighborhoods, so $x \in cl_{\beta_o X} B$. This contradicts the fact that x belongs to $cl_{\beta_o X} A$ and $cl_{\beta_o X} A \cap cl_{\beta_o X} B = \varnothing$.

By the definition (Definition 1.1.12) of filter convergence, the convex neighborhood N_x of x belongs to $\mathcal{F}$. Now $\mathcal{F}|_{N_x} = [\{F \cap N_x : F \in \mathcal{F}\}]$ is a convex filter. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{F} \supseteq \mathcal{F}|_{N_x}$. On the other hand, for each $G \in \mathcal{F}|_{N_x}$, $F \cap N_x \subseteq G$ for some $F \in \mathcal{F}$, and since $N_x \in \mathcal{F}$, $F \cap N_x \in \mathcal{F}$ and thus $G \in \mathcal{F}$. This shows $\mathcal{F}|_{N_x} \subseteq \mathcal{F}$. It follows that $\mathcal{F}|_{N_x} = \mathcal{F}$.

So, the arbitrary convex ultrafilter $\mathcal{F}$ on X converging to a point $x \in A$ really turns out to be a convex ultrafilter on A. We note that every closed convex ultrafilter on A is also a closed convex ultrafilter on X (see [**9**]). Then $\beta_o A$ generated by the closed convex ultrafilters on A is equal to $cl_{\beta_o X} A$ generated by the closed convex ultrafilters on X converging to a point of $cl_{\beta_o X} A$. Therefore, $\beta_o A = cl_{\beta_o X} A$. Similarly, if $\mathcal{F}$ converges to a point x in $cl_{\beta_o X} B$, we can find that $\beta_o B = cl_{\beta_o X} B$. Then we finally have $\beta_o (A \cup B) = \beta_o A \uplus \beta_o B = cl_{\beta_o X} A \uplus cl_{\beta_o X} B$, as desired. □

Returning to the question of when $\beta_o X - X = X$, for a totally ordered space X, suppose $f : X \to \beta_o X - X$ is a homeomorphism and order isomorphism as in Theorem 2.1.4. Our specific sets $C = \{x \in X : \widehat{f}(x) > x\}$ with $\widehat{C} = \{x \in \beta_o X : \widehat{f}(x) > x\}$ and $D = \{x \in X : \widehat{f}(x) < x\}$ with $\widehat{D} = \{x \in \beta_o X : \widehat{f}(x) < x\}$ satisfy the hypothesis of Theorem 2.1.7. Now, we obtain the following result.

THEOREM 2.1.8. *If X is a totally ordered space, then $\beta_o X - X \not\approx X$.*

PROOF. 1: Suppose X is totally ordered space and $f : X \to \beta_o X - X$ is a homeomorphism and order isomorphism. The smallest and largest element of $\beta_o X$ are elements of X and X is dense. If the smallest element a of $\beta_o X$ is an element of $\beta_o X - X$, there is a decreasing net in X converging to a, so X has no smallest element. But $X \approx \beta_o X - X$ which had a smallest element.

With $C, D, \widehat{C}, \widehat{D}$ as defined, by Theorem 2.1.7, $f : X \to \beta_o X - X$ is $f : C \cup D \to \beta_o (C \cup D) - (C \cup D) = \beta_o C \uplus \beta_o D - (C \cup D) = \beta_o C - C \uplus \beta_o D - D$ and $f|_C$ is a homeomorphism and order isomorphism from C to $\beta_o C - C$. $f|_C$ is onto because for every $z \in \beta_o C - C \subseteq \beta_o X - X$, there is a value $x \in X$ for which $z = f(x)$. If $x \in D$, then $f(x) < x$ and $\widehat{f}(f(x)) < \widehat{f}(x) = f(x)$, so $z = f(x) \in \widehat{D}$, contrary to $z \in \beta_o C = \widehat{C}$. One to one, continuity and order isomorphism follow for any restriction. Now $C \approx \beta_o C - C$; so the largest element b of $\beta_o C$ is in C, so $f(b) < b$ contrary to $b \in C$. □

With the help of the following Knaster-Tarski fixed theorem (see [**2**]), we give our second proof that $\beta_o X - X \neq X$ if X is a totally ordered space.

Theorem 2.1.9. *(Knatser-Tarski Fixed Point Theorem)*

Let L be a complete lattice and $F : L \to L$ be an order-preserving function. Then the function F has greatest and least fixed points. The set of fixed points of F in L is also a complete lattice.

Proof. 2 of Theorem 2.1.8: Suppose $\beta_o X - X \approx X$. By Theorem 2.1.3, we know that the ordered Stone-Čech compactification $\beta_o X$ of X is a complete lattice. Also, by Theorem 2.1.4, there exists an increasing extension $\widehat{f} : \beta_o X \to \beta_o X$ and $\widehat{f}$ has no fixed points. But, this contradicts the Knaster-Tarski fixed theorem. So, $\beta_o X - X \not\approx X$. □

We saw that the condition $cl_{\beta X} A \cap cl_{\beta X} B = \varnothing$ in Theorem 2.1.6 always happened, so it is not needed as a hypothesis in the topological case. But the condition $cl_{\beta_o X} A \cap cl_{\beta_o X} B = \varnothing$ in Theorem 2.1.7, does not automatically happen.

Example 2.1.10. *Consider the subsets of* $\mathbb{R}$, $A = \{[10n, 10n+4]_{n\in\mathbb{Z}}\}$ *and* $B = \{[10n+5, 10n+9]_{n\in\mathbb{Z}}\}$ *as shown in Figure 2.1.*

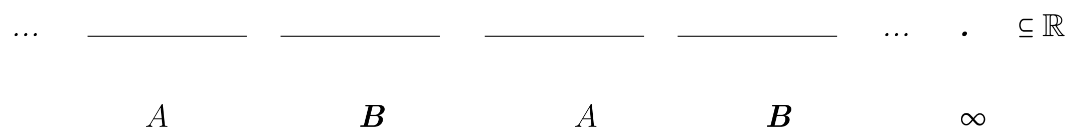

Figure 2.1. Accumulating on the end

Since $X = A \cup B$ *is convex, both the clopen sets* A *and* B *must accumulate to* ∞*. So,* $\infty \in cl_{\beta_o X} A \cap cl_{\beta_o X} B$ *even though* A, B *is a separation of the locally convex totally ordered space* X*. Thus the hypothesis* $cl_{\beta_o X} A \cap cl_{\beta_o X} B = \varnothing$ *is needed in Theorem 2.1.7. Similarly, there are examples where the problem accumulates in middle instead of on the end as shown in Figure 2.2.*

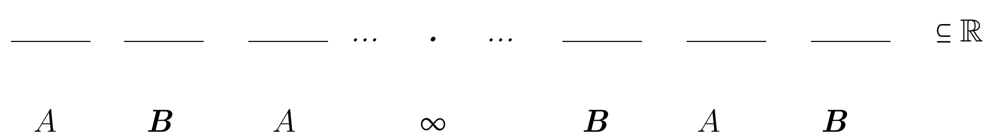

Figure 2.2. Accumulating in middle

2.2. Ordered Versions of Connectedness

The ordered Stone-Čech compactification $\beta_o X$ of X is constructed as a subspace of $\prod_{f \in C^{*\uparrow}(X)} I_f$. By the discussion following Theorem 2.2.1, we see that every $f \in C^{*\uparrow}(X)$ has an extension from X to $\beta_0 X$. So, $\beta_o X$ is the unique ordered compactification in which X is $C^{*\uparrow}$-embedded. We note that in the topological case, any compactification αX can be constructed using the product construction and αX is a subspace of $\prod_{f \in \mathscr{C}} I_f$ where $\mathscr{C} = \{f \in C^*(X)$ which can be extended to $\alpha X\}$ (see [**4**]). If X is an ordered topological space, then $C^{*\uparrow}(X) = \{f : X \to \mathbb{R} |\ f$ is continuous, increasing and bounded $\}$ and if each $f \in C^{*\uparrow}(A)$ has a continuous, increasing and bounded extension to X, the set A is said to be $C^{*\uparrow}$-embedded in X. Thus, any ordered compactification $\alpha_o X$ can be constructed using the corresponding $C^{*\uparrow}$ result. That is, any ordered compactification $\alpha_o X$ is a subspace of $\prod_{f\in\mathscr{C}^{\uparrow}} I_f$ where $\mathscr{C}^{\uparrow} = \{f \in C^{*\uparrow}(X)$ which can be extended to $\alpha_o X\}$. In this section, we show that for any separation (A,B) of a locally convex, totally ordered topological space X, A is $C^{*\uparrow}$-embedded in X.

In the topological case, (A,B) form a separation of X if and only if there is a continuous function $f : A \cup B \to \{0,1\}$ with $f(A) = \{0\}$ and $f(B) = \{1\}$. If $g : A \to [-M, M]$ is continuous for any $M \in \mathbb{R}$, then $\widehat{g}$ defined by

$$\widehat{g}(x) = \begin{cases} g(x) & if \quad x \in A \\ M+1 & if \quad x \in B \end{cases}$$

is a continuous extension, so A is C^{*}-embedded in $X = A \cup B$. But this approach fails for increasing functions on partially ordered topological spaces. For totally ordered spaces, it holds.

Theorem 2.2.1. *If (A,B) is a separation of a T_2-ordered locally convex, totally ordered space X, then A and B are $C^{*\uparrow}$-embedded in X.*

Proof. A is $C^{*\uparrow}$-embedded in X means that each $g \in C^{*\uparrow}(A)$ can be extended to $\widehat{g} \in C^{*\uparrow}(X)$. Let $g : A \to [-M, M] \subseteq \mathbb{R}$ be in $C^{*\uparrow}(A)$, so g is continuous, increasing and bounded. We extend g on A to $\widehat{g}$ on $X = A \cup B$ by

$$\widehat{g}(b) = \begin{cases} \sup\{g(\{(\leftarrow, b) \cap A\})\} & if \quad (\leftarrow, b) \cap A \neq \varnothing \\ -M-1 & if \quad (\leftarrow, b) \cap A = \varnothing \end{cases}$$

for any $b \in B$ and $\widehat{g}(a) = g(a)$ for any $a \in A$.

i) We show that $\widehat{g}$ is an increasing function.

Case 1: $a_1 \leq a_2$ in A.

Since g is an increasing function, we have $g(a_1) \leq g(a_2)$. By the definition of $\widehat{g}$, $g(a_1) = \widehat{g}(a_1)$ and $\widehat{g}(a_2) = g(a_2)$, so we have $\widehat{g}(a_1) \leq \widehat{g}(a_2)$.

Case 2: $b_1 \leq b_2$ in B.

If $b_1 \le b_2$, then $(\leftarrow, b_1) \cap A \subseteq (\leftarrow, b_2) \cap A$, so $g(\{(\leftarrow, b_1) \cap A\}) \subseteq g(\{(\leftarrow, b_2) \cap A\})$. Thus, $\sup\{g(\{(\leftarrow, b_1) \cap A\})\} \le \sup\{g(\{(\leftarrow, b_2) \cap A\})\}$, and thus $\widehat{g}(b_1) \le \widehat{g}(b_2)$.

<u>Case 3:</u> $a \le b$ for $a \in A$ and $b \in B$.

If $a \le b$, then $a \in (\leftarrow, b) \cap A$. So $g(a) \in g(\{(\leftarrow, b) \cap A\})$. Since $g(a) = \widehat{g}(a)$, we have $\widehat{g}(a) \le \sup\{g(\{(\leftarrow, b) \cap A\})\} = \widehat{g}(b)$.

<u>Case 4:</u> $b \le a$ for $a \in A$ and $b \in B$.

Since $\widehat{g}(b) = \sup\{g(\{(\leftarrow, b) \cap A\})\}$, each element of $\{(\leftarrow, b) \cap A\}$ is less than $a \in A$, so $g(c) < g(a)$ for each $c \in \{(\leftarrow, b) \cap A\}$. Thus, $\sup\{g(\{(\leftarrow, b) \cap A\})\} \le g(a) = \widehat{g}(a)$. Then $\widehat{g}(b) \le \widehat{g}(a)$ and $\widehat{g}$ is increasing.

ii) We now show that the increasing function $\widehat{g}$ is continuous. By the definition of continuity, we know that $\widehat{g} : X \to \mathbb{R}$ is continuous if and only if for each $x \in X$ and each neighborhood V of $\widehat{g}(x)$, there is a neighborhood U of x such that $\widehat{g}(U) \subseteq V$.

<u>Case 1:</u> $x \in A$.

Given any neighborhood V of $\widehat{g}(a) = g(a)$ for $a \in A$, since g is continuous, there is a neighborhood U in A of a with $g(U) \subseteq V$. Since A is an open set in X, U is a neighborhood of a in X, and $g(U) = \widehat{g}(U) \subseteq V$.

<u>Case 2:</u> $x = b \in B$.

Let V be a neighborhood of $\widehat{g}(b)$. Since B is an open and locally convex subset of X, there is a convex neighborhood N in X of b with $N \subseteq B$. For $z \in N$, either $[z, b] \subseteq N$ or $[b, z] \subseteq N$, so $(\leftarrow, z] \cap A = (\leftarrow, b] \cap A$, so $\widehat{g}(z) = \widehat{g}(b)$. Thus, $\widehat{g}(N) = \widehat{g}(b) \subseteq V$.

So, by <u>Case 1</u> and <u>Case 2</u>, $\widehat{g}$ is continuous.

Therefore, by (i) and (ii), $\widehat{g} \in C^{*\uparrow}(A)$ and A is $C^{*\uparrow}$-embedded in X. Similarly, B is $C^{*\uparrow}$-embedded in X. $\square$

The proof of Theorem 2.2.1 does not hold in a partial ordered setting, because given any $g \in C^{*\uparrow}(A)$ we cannot extend g to $\widehat{g}$ on X by $\widehat{g}(a) = g(a)$ for any $a \in A$ and $\widehat{g}(b) = \sup\{g(\{a \in A : a < b\})\}$. A counterexample is given in the following.

EXAMPLE 2.2.2. *Consider the subsets,* $A = \{(x, |x|) : x \in [-1,1] - \{0\}\} \cup \{(0,1)\}$ *and* $B = [-1,1] \times \{2\}$ *of* $\mathbb{R}^2$ *as shown in Figure 2.3. The order on* $X = A \cup B$ *is defined by* $(a,b) \leq (c,d) \Leftrightarrow a = c$ *and* $b \leq d$.

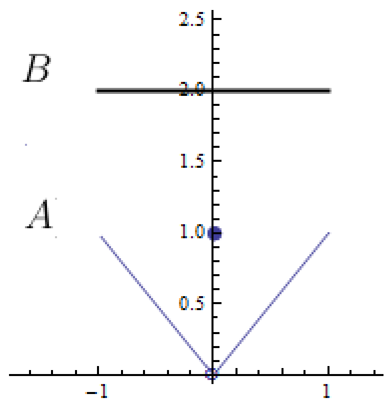

Figure 2.3. A Counterexample for Partial Orders

We try to extend $g \in C^{*\uparrow}(A)$ *defined by* $g((x,y)) = y$ *to* B *by* $\widehat{g}(b) = \sup\{g((\leftarrow, b) \cap A)\}$ *for any* $b \in B$. *For* $(\frac{1}{n}, 2) \in B$, *we have* $\widehat{g}((\frac{1}{n}, 2)) = \frac{1}{n}$. *Now* $\lim_{n\to\infty} \widehat{g}((\frac{1}{n}, 2)) = \lim_{n\to\infty} \frac{1}{n} = 0$. *On the other hand,* $\widehat{g}(\lim_{n\to\infty}(\frac{1}{n}, 2)) = \widehat{g}((0,2)) = 1$. *Therefore, since* $\lim_{n\to\infty} \widehat{g}((\frac{1}{n}, 2)) \neq \widehat{g}(\lim_{n\to\infty}(\frac{1}{n}, 2))$, $\widehat{g}$ *is not continuous. So, the proof of Theorem 2.2.1 fails for partial orders.*

For totally ordered spaces, $\sup\{g(\{a \in A : a < b\})\} = \sup\{g(\{a \in A : a \ngeq b\})\}$, but these may be different in partially ordered spaces. If A, B is a separation of a partially ordered space X, we now ask whether $g \in C^{*\uparrow}(A)$ can be extended to $\tilde{g}$ in B defined by $\tilde{g}(a) = g(a)$ for any $a \in A$ and $\tilde{g}(b) = \sup\{g(\{a \in A : a \ngeq b\})\}$ for any $b \in B$. The first three cases seem to show that $\tilde{g}$ may be increasing.

<u>Case 1:</u> $a_1 \leq a_2$ for $a_1, a_2 \in A$.

Since g is an increasing function, $g(a_1) \le g(a_2)$. By the definition of $\tilde{g}$, it follows that $\tilde{g}(a_1) \le \tilde{g}(a_2)$.

Case 2: $b_1 \le b_2$ for $b_1, b_2 \in B$.

If the points are above b_2, then they are above b_1. Equivalently, we say that if the points are not above b_1, then they are not above b_2 as shown in *Figure 2.4*

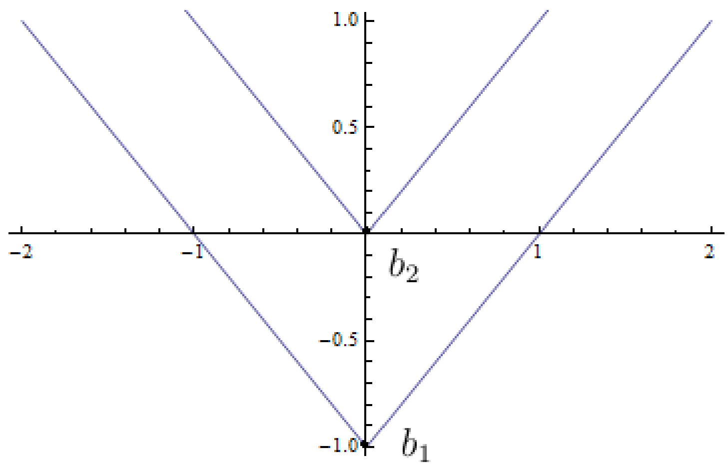

Figure 2.4. An Increasing Extension in Partial Orders

Then, since $\{a \in A : a \ngeq b_1\} \subseteq \{a \in A : a \ngeq b_2\}$, we have $\sup\{g(\{a \in A : a \ngeq b_1\})\} \le \sup\{g(\{a \in A : a \ngeq b_2\})\}$. By the definition of $\tilde{g}$, $\tilde{g}(b_1) \le \tilde{g}(b_2)$.

Case 3: $a_1 \le b_1$ for $a_1 \in A$ and $b_1 \in B$.

Since we have $a_1 \ngeq b_1$, $a_1 \in \{a \in A : a \ngeq b_1\}$, so $g(a_1) \le \sup\{g(\{a \in A : a \ngeq b_1\})\} = \tilde{g}(b_1)$. By the definition of $\tilde{g}$, we know that $g(a_1) = \tilde{g}(a_1)$ for $a_1 \in A$. So, $\tilde{g}(a_1) \le \tilde{g}(b_1)$.

However, the fourth case fails in general.

Case 4: $b \le a_1$ for $a_1 \in A$ and $b \in B$.

Suppose A, B is the separation of $X = A \cup B \subseteq \mathbb{R}^2$ shown in *Figure 2.5*. Give X the order $(x, y) \le (z, w)$ if and only if $x \le z$ and $y \le w$. Take $g \in C^{*\uparrow}(A)$ to be $g((x, y)) = y$. Consider the points b, a_1 and a_b as shown in *Figure 2.5*.

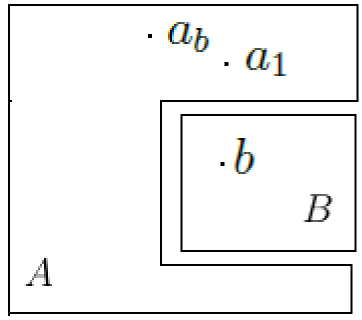

Figure 2.5. A Non-increasing Extension in Partial Orders

Since $a_b \ngeq b$, we have $\tilde{g}(b) = \sup\{g(\{a \in A : a \ngeq b\})\} \geq g(a_b) > g(a_1) = \tilde{g}(a_1)$. Thus, it shows that $\tilde{g}$ is not increasing.

If we had defined an ordered separation of a partially ordered topological space to be $X = A \cup B$ where there exists a continuous, increasing and onto function $f : X \to \{0,1\}$ with $f(A) = \{0\}$ and $f(B) = \{1\}$, then A would be $C^{*\uparrow}$-embedded in X. If $g \in C^{*\uparrow}(A)$ and $g : \mathbb{R} \to [-M, M]$, then $\tilde{g}(a) = g(a)$ for $a \in A$ and $\tilde{g}(b) = M + 1$ defines a continuous increasing bounded extension of g. However, since $A = f^{-1}(\{0\}) = f^{-1}((\leftarrow, \frac{1}{2}))$ is a decreasing set and $B = f^{-1}(\{1\}) = f^{-1}((\frac{1}{2}, \to))$ is an increasing set, this definition requires that we separate X into one decreasing set A and one increasing set B, which is too restrictive to be widely applied.

We now continue with a lemma and two important properties.

Lemma 2.2.3. *Every compact T_2-ordered space is a normally ordered space (see* [**11**]*).*

Proposition 2.2.4. *Let A, B be a separation of a locally convex, T_2-ordered, partially ordered space X. If A is $C^{*\uparrow}$-embedded in X, then A is $C^{*\uparrow}$-embedded in $\beta_o X$.*

Proof. By Lemma 2.2.3, since $\beta_o X$ is a compact T_2-ordered space, then $\beta_o X$ is normally ordered. If A is $C^{*\uparrow}$-embedded in $\beta_o A$, then by the ordered version of Tietze extension theorem (see [**11**]), $\beta_o A$ is $C^{*\uparrow}$-embedded in the normally ordered space $\beta_o X$. Thus, any $g \in C^{*\uparrow}(A)$ can be extended to $\widehat{g}$ on X and then to $\beta_o X$. It follows that A is $C^{*\uparrow}$-embedded in $\beta_o X$. $\square$

The following result is the ordered version of Proposition 1.1.17. The proof is similar.

THEOREM 2.2.5. *A subspace A of an ordered topological space X is $C^{*\uparrow}$-embedded in X if and only if $\beta_o A = cl_{\beta_o X} A$.*

Chapter 3

EXAMPLES

3.1. Examples For Ordered Separations of Topological Spaces

In this chapter, we study specific examples to illustrate the properties we proved. We have shown that if (A, B) is a separation of a locally convex, T_2-ordered and partially ordered topological space X, then A and B need not be $C^{*\uparrow}$-embedded in the ordered Stone-Čech compactification of X. Determining the continuous and increasing extensions in the setting of interlaced separations can be hard to visualize. So we developed some other basic examples to emphasize the problems which arise when the order is not total. In Theorem 2.2.1, we saw that if (A, B) is a separation of a T_2-ordered locally convex, totally ordered space X, then A and B are $C^{*\uparrow}$-embedded in X.

We now see in Example 3.1.1 that A need not be $C^{*\uparrow}$-embedded if we do not assume a T_2-ordered, locally convex and totally ordered topological space.

Example 3.1.1. *On $\mathbb{R}^2$ with the usual topology and the order $(x, y) \leq (z, w)$ if and only if $x \leq z$ and $y \leq w$, let $a_n = (2n, \frac{-1}{2n})$ for $n \in \mathbb{N}$, $b_n = (2n+1, \frac{-1}{2n+1})$ for $n \in \mathbb{N}$, $a_{-2} = (-2, 1)$, $a_0 = (0, 1)$, and $b_{-1} = (-1, 1)$. Add a point a_∞ as the limit of $\{a_n\}_{n=1}^{\infty}$ and a point b_∞ as the limit of $\{b_n\}_{n=1}^{\infty}$, with $a_n < a_\infty$ for each $n \in \mathbb{N} \cup \{-2, 0\}$, $b_n < b_\infty$ for each $n \in \mathbb{N} \cup \{-1\}$, and $a_{-2} < b_{-1} < a_0$. Put $A = \{a_n\}_{n=1}^{\infty} \cup \{a_{-2}, a_0, a_\infty\}$, $B = \{b_n\}_{n=1}^{\infty} \cup \{b_{-1}, b_\infty\}$, and $X = A \cup B$, as shown in Figure 3.1.*

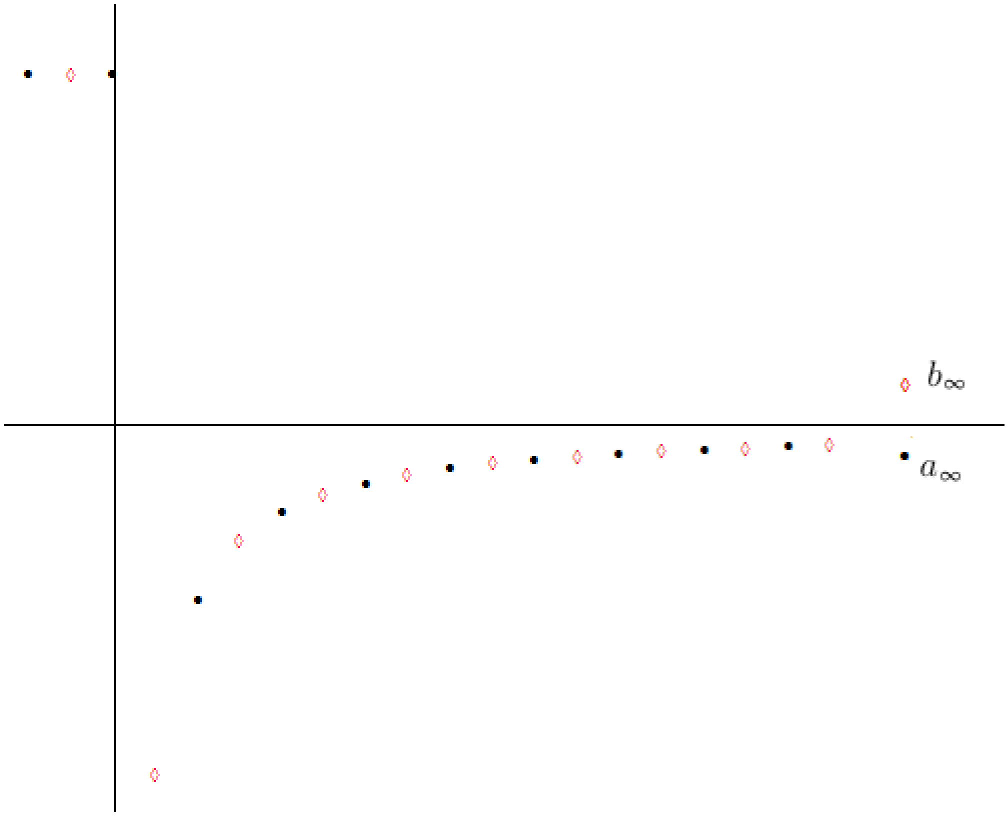

Figure 3.1. An Example on $\mathbb{R}^2$ for the Totally Ordered Spaces

Since we added points a_∞ *to* A *and* b_∞ *to* B *so that* $a_n \to a_\infty$ *where* $a_n = \{(2n, \frac{-1}{2n})\}_{n\in\mathbb{N}}$ *and* $b_n \to b_\infty$ *where* $b_n = \{(2n+1, \frac{-1}{2n+1})\}_{n\in\mathbb{N}}$*, the basic neighborhoods of* a_∞ *are* $\{a_\infty\} \cup \{a_n\}_{n\geq k}$ *and the basic neighborhoods of* b_∞ *are* $\{b_\infty\} \cup \{b_n\}_{n\geq k}$ *for* $k \in \mathbb{N}$*. We note that* A *and* B *are closed, because each set contains its limit point. See Figure 3.1. The set* Y *is* T_2*-ordered if and only if any sequences* x_λ, y_λ *in* Y *satisfies the following. If* $x_\lambda \leq y_\lambda$ *with* $x_\lambda \to x$ *and* $y_\lambda \to y$ *for any* $x, y \in Y$*, then* $x \leq y$*. We have* $a_n < b_n < a_{n+1} < b_{n+1}$ *and if we take the limit of these terms as* n *goes to infinity, we get* $a_\infty \leq b_\infty \leq a_\infty$*. This gives a contradiction that* $a_\infty = b_\infty$*, so* X *is not a* T_2*-ordered space. Basic neighborhoods of* a_∞ *contain* a_k *and* a_{k+1} *but not* b_k*, even though* $a_k < b_k < a_{k+1}$*. So,* X *is not a locally convex space.*

We now show that A is not $C^{*\uparrow}$-embedded in $A \cup B$. We define $f : A \to \mathbb{R}$ by $f(x,y) = y$ and $f(a_\infty) = 0$, so $f \in C^{*\uparrow}(A)$. By Figure 3.1, we have $a_n < b_n < a_{n+1}$. If f is extended to $\widehat{f}$ on $A \cup B$, we get $\widehat{f}(a_n) < \widehat{f}(b_n) < \widehat{f}(a_{n+1})$. So, when we take the limit of $\widehat{f}$ as n goes to infinity, we find $\widehat{f}(a_\infty) \le \widehat{f}(b_\infty) \le \widehat{f}(a_\infty)$. Since $\widehat{f}(a_\infty) = 0$, we have $\widehat{f}(b_\infty) = 0$. Recall $a_{-2} = (-2,1)$, $a_0 = (0,1)$ and $b_{-1} = (-1,1)$. Since $\widehat{f}(a_{-2}) \le \widehat{f}(b_{-1}) \le \widehat{f}(a_0)$ and $\widehat{f}(a_{-2}) = \widehat{f}(a_0) = 1$, then we have $\widehat{f}(b_{-1}) = 1$. We also know that $\widehat{f}(b_{-1}) \le \widehat{f}(b_\infty)$. But, since $\widehat{f}(b_{-1}) = 1$ and $\widehat{f}(b_\infty) = 0$, this gives a contradiction that $1 \le 0$. Thus, we cannot extend f to $\widehat{f}$ on X and A is not $C^{*\uparrow}$-embedded in X.

Now we continue with another example for the partially ordered case. This one is locally convex and T_2-ordered.

EXAMPLE 3.1.2. *On $\mathbb{R}^2$ with the usual topology, consider the subsets*

$A_1 = [-1,0) \times \{1\}$, $A_2 = (0,1] \times \{-1\}$ and $B = [-1,1] \times \{0\}$ with $(x,y) \le (z,w)$ if and only if $x = z$ and $y \le w$, as shown in Figure 3.2. Let $A = A_1 \cup A_2$.

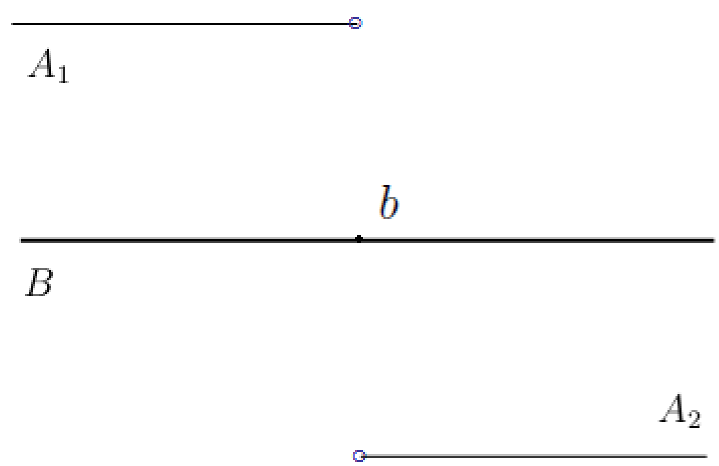

Figure 3.2. An Example on $\mathbb{R}^2$ for the Partially Ordered Spaces

We consider $f : A_1 \cup A_2 \to \{0,1\}$ with $f(A_1) = \{0\}$ and $f(A_2) = \{1\}$. Since there is no order relation between A_1 and A_2, the two figures shown in Figure 3.3 and Figure 3.4 are *topologically and order equivalent, and we see that $f \in C^{*\uparrow}(A)$. But, including the set B does not allow the positions of A_1 and A_2 to change. Referring to Figure 3.2, for $(\frac{1}{n}, -1) \in A_2$ and $(\frac{1}{n}, 0) \in B$, we have $(\frac{1}{n}, 0) > (\frac{1}{n}, -1)$. If we extend f to $\widehat{f}$ on $X = A \cup B$, then $\widehat{f}((\frac{1}{n}, 0)) \geq \widehat{f}((\frac{1}{n}, -1)) = 1$. So, for $b = (0,0)$, taking the limit implies that $\widehat{f}((0,0)) = \widehat{f}(b) \geq 1$.*

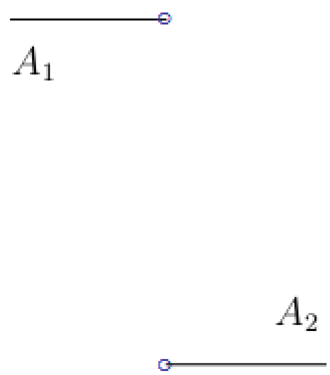

Figure 3.3. First Case without Order

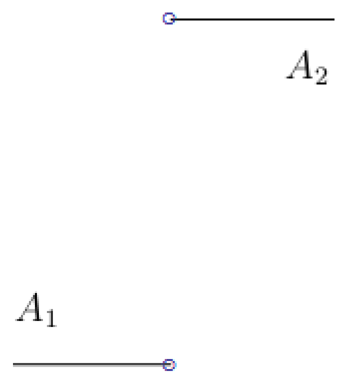

Figure 3.4. Second Case without Order

Similarly, by Figure 3.2, for $(\frac{-1}{n}, 1) \in A_1$ and $(\frac{-1}{n}, 0) \in B$, we have $(\frac{-1}{n}, 1) > (\frac{-1}{n}, 0)$. So, $\widehat{f}((\frac{-1}{n}, 1)) = 0 \geq \widehat{f}((\frac{-1}{n}, 0))$. Thus, as n goes to infinity, for $b = (0,0)$, we get $0 \geq \widehat{f}((0,0)) = \widehat{f}(b)$. In this case, it implies that $0 \geq \widehat{f}(b) \geq 1$ which is a contradiction.

$A = A_1 \cup A_2$ and B is a separation of a partially ordered, locally convex topological space X, but since f cannot be extended to $\widehat{f}$ on X, A is not $C^{*\uparrow}$-embedded in X.

In Theorem 2.2.5, we showed that A is $C^{*\uparrow}$-embedded in X if and only if $\beta_o A = cl_{\beta_o X} A$. With A and X as in Example 3.1.2, A was not $C^{*\uparrow}$-embedded in X, so $\beta_o A \neq cl_{\beta_o X} A$. We will show this directly. The order $(x,y) \leq (z,w)$ if and only if $x = z$ and $y \leq w$, when restricted to A, gives that no point of A is above or below any other point of A, so the order on A is equality, or the trivial order. Thus, $\beta_o A = \beta A = \beta(A_1 \cup A_2) = \beta A_1 \cup \beta A_2$ by Theorem 2.1.6. Also by Theorem 2.1.6,

$\beta(A \cup B) = \beta A \cup \beta B = \beta A_1 \cup \beta A_2 \cup B$. Now $\beta(A \cup B)$ can be ordered by $\alpha_2 < (0,0) < \alpha_1$ for any $\alpha_1 \in \beta A_1 - A_1$ and any $\alpha_2 \in \beta A_2 - A_2$, giving an ordered compactification of $A \cup B$. This ordered compactification is $\beta_o X$, since $\beta_o X$ is the largest ordered compactification of X, and no compactification can be larger than βX. Now $cl_{\beta_o X} A = \beta A_1 \cup \beta A_2$ with the subspace order from $\beta_o X$, so that $\alpha_1 < \alpha_2$ for $\alpha_1 \in \beta A_1 - A_1$ and $\alpha_2 \in \beta A_2 - A_2$. We saw $\beta_o A$ was also equal to $\beta A_1 \cup \beta A_2$, but since there was no order on A, $\beta_o A$ has no order. This shows that, as sets, $\beta_o A = cl_{\beta_o X} A$, but as ordered topological spaces, they are not equal.

Chapter 4
CONCLUSION AND FUTURE WORK

The Stone-Čech compactification has been studied in various topological spaces since 1937. In this thesis, we expanded this to the concept of an ordered version of the Stone-Čech compactification of totally ordered topological spaces using filter ideas. For totally ordered spaces, we obtained the ordered compactifications by filters of closed convex set. We applied this technique to separations and we developed the properties of ordered versions of connectedness. Using this approach, we determined which versions of ordered separations (A, B) of X satisfy $\beta_o(A \cup B) = \beta_o A \cup \beta_o B$. Since an open question in topology is which spaces X are homeomorphic to $\beta X - X$, we applied our results to get an answer to the analogous question in the setting of ordered compactifications of totally ordered spaces. We showed that no totally ordered space X satisfies $\beta_o X - X = X$. We also used this construction with the continuous increasing functions and we visualized examples in the setting of ordered spaces.

For future work, we would like to use the idea that we presented for the ordered version of the Stone-Čech compactification of the totally ordered topological spaces to answer when $\beta_o X$ is a complete lattice and when $\beta_o X$ has the interval topology. Another open question is whether X can be homeomorphic and order isomorphic to $\beta_o X - X$ if X is a partially ordered topological space whose order is not total.

APPENDIX

The following Mathematica codes are used to compute and plot the graphs.

Figure 2.3

a = Plot[$Abs[x]$, $\{x, -3, 3\}$]

b = ListPlot[{{0, 1}}, PlotStyle → PointSize[.02], PlotRange → {−1, 3}]

c = ListPlot[{{0, 0}}, PlotMarkers → "[EmptyCircle]", PlotRange → {−1, 3}]

d = Graphics [{$Thick$, $Line$[{{−1, 2}, {0, 2}, {1, 2}}]}]

Show [a,b,c,d]

Figure 2.4

a = Plot[$Abs[x] - 1$, $\{x, -2, 2\}$]

b = Plot[$Abs[x]$, $\{x, -2, 2\}$]

Show[a,b]

Figure 3.1

ListPlot[{{{−2, 1}, {0, 1}, {2, −(1/2)}, {4, −(1/4)}, {6, −(1/6)}, {8, −(1/8)},
{10, −(1/10)}, {12, −(1/12)}, {14, −(1/14)}, {16, −(1/16)}, {18, −(1/18)},
{20, −(1/20)}, {22, −(1/22)}, {24, −(1/24)}}, {{−1, 1}, {1, −1}, {3, −(1/3)},
{5, −(1/5)}, {7, −(1/7)}, {9, −(1/9)}, {11, −(1/11)}, {13, −(1/13)}, {15, −(1/15)},
{17, −(1/17)}, {19, −(1/19)}, {21, −(1/21)}, {23, −(1/23)}, {25, −(1/25)}},
{{29, −.15}}, {{29, .15}}} , PlotStyle → {Black, Red, Black, Red}, DataRange
→ {−3, 30}, PlotRange → {−1.2, 1.2}, Ticks → {$None$, $None$}, PlotMarkers
→ {"Bullet", "Diamond", "Bullet a_∞", "Diamond b_∞"}]

Figure 3.2

a = Graphics[$\{Thick, Line[\{\{-1,0\},\{0,0\},\{1,0\}\}]\}$]

b = Graphics[$\{Thin, Line[\{\{-1,1\},\{0,1\}\}]\}$]

c = ListPlot[$\{\{0,1\}\}$, PlotMarkers → "[EmptyCircle]", PlotRange → $\{-1.5, 1.5\}$]

d = ListPlot[$\{\{0,-1\}\}$, PlotMarkers → "[EmptyCircle]", PlotRange → $\{-1.5, 1.5\}$]

e = Graphics[$\{Thin, Line[\{\{0,-1\},\{1,-1\}\}]\}$]

Show[a,b,c,d,e]

Figure 3.3

a = Graphics[$\{Thin, Line[\{\{-1,1\},\{0,1\}\}]\}$]

b = ListPlot[$\{\{0,1\}\}$, PlotMarkers → "[EmptyCircle]", PlotRange $\{-1, 1.2\}$]

c = ListPlot[$\{\{0,0\}\}$, PlotMarkers → "[EmptyCircle]", PlotRange → $\{-1.2, 0.2\}$]

d = Graphics[$\{Thin, Line[\{\{0,0\},\{1,0\}\}]\}$]

Show [a,b,c,d]

Figure 3.4

a = Graphics[$\{Thin, Line[\{\{0,1\},\{1,1\}\}]\}$]

b = ListPlot[$\{\{0,1\}\}$, PlotMarkers → "[EmptyCircle]", PlotRange $\{-1, 1.2\}$]

c = ListPlot[$\{\{0,0\}\}$, PlotMarkers → "[EmptyCircle]", PlotRange → $\{-1.2, 0.2\}$]

d = Graphics[$\{Thin, Line[\{\{-1,0\},\{0,0\}\}]\}$]

Show [a,b,c,d]

BIBLIOGRAPHY

[1] G. Bezhanishivili and P.J. Morandi, *Order-Compactification of Totally Ordered Spaces:Revisited*, Order 28, No. 3, 577-592, 2011.
[2] G. Birkhoff, *Lattice Theory*, American Mathematical Society, Vol. 25, 1993.
[3] J. Blatter, *Order Compactifications of Totally Ordered Topological Spaces*, J. Approximation Theory, Vol. 13, 56–65 1975.
[4] R.E. Chandler, *Hausdorff Compactifications, Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker Inc., NY, 1976.
[5] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.
[6] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand Mathematical Studies, Princeton NJ, 1960.
[7] W. Hussak, *Iteratively Defined Generalisations of Locally Compact and Discrete Topological Spaces*, Doctoral Thesis, Sheffield, 1983.
[8] P.P. Jackson, *Iterated Remainders In Compactifications*, Doctoral Thesis, Sheffield, 1980.
[9] D.C. Kent and T.A. Richmond, *Ordered Compactification of Totally Ordered Spaces*, Internat. J. Math. Math. Sci., Vol. 11 No. 4, 683-694, 1988.
[10] J.R. Munkres, *Topology*, Prentice Hall, Inc. Upper Saddle River, NJ, 1975.
[11] L. Nachbin, *Topology and Order*, Van Nostrand Mathematical Studies, No. 4, Princeton NJ, 1965.
[12] M. P. Stannett, *Internal Topology*, Doctoral Thesis, Sheffield, 1986.
[13] M. P. Stannett, *The Stone-Čech Compactification of the Rational World*, Glasgow Math. J. 30, 181–188, 1988.
[14] A. Tychonoff, *Über die topologische Erweiterung von Räume*, Math. Ann., 102, 544-561, 1930.
[15] R.C. Walker, *The Stone-Čech Compactification*, Springer-Verlag, Berlin, 1974.