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ANALYZING AND SOLVING NON-LINEAR STOCHASTIC DYNAMIC MODELS ON NON-PERIODIC DISCRETE TIME DOMAINS

A Thesis
Presented to
The Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
Of the Requirements for the Degree
Master of Science

By
Gang Cheng

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ANALYZING AND SOLVING NON-LINEAR STOCHASTIC DYNAMIC MODELS ON NON-PERIODIC DISCRETE TIME DOMAINS

Date Recommended 04/03/2013

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Date

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CONTENTS

ANALYZING AND SOLVING NON-LINEAR STOCHASTIC DYNAMIC MODELS ON NON-PERIODIC DISCRETE TIME DOMAINS

Gang Cheng May 2013 44 Pages

Directed by: Dr. Ferhan Atici, Dr. Tom Richmond, and Dr. Di Wu

Department of Mathematics Western Kentucky University

Stochastic dynamic programming is a recursive method for solving sequential or multistage decision problems. It helps economists and mathematicians construct and solve a huge variety of sequential decision making problems in stochastic cases. Research on stochastic dynamic programming is important and meaningful because stochastic dynamic programming reflects the behavior of the decision maker without risk aversion; i.e., decision making under uncertainty. In the solution process, it is extremely difficult to represent the existing or future state precisely since uncertainty is a state of having limited knowledge. Indeed, compared to the deterministic case, which is decision making under certainty, the stochastic case is more realistic and gives more accurate results because the majority of problems in reality inevitably have many unknown parameters. In addition, time scale calculus theory is applicable to any field in which a dynamic process can be described with discrete or continuous models. Many stochastic dynamic models are discrete or continuous, so the results of time scale calculus are directly applicable to them as well. The aim of this thesis is to introduce a general form of a stochastic dynamic sequence problem on complex discrete time domains and to find the optimal sequence which maximizes the sequence problem.

Chapter 1
INTRODUCTION

Time scale calculus is a theory that combines the two approaches of dynamic modeling: difference and differential equations. It is applicable to any field where dynamic processes can be described with discrete or continuous models. Many stochastic dynamic models in economics are discrete or continuous, so the results of time scale calculus are directly applicable to them as well. The goal of this thesis is to analyze and solve some nonlinear stochastic dynamic models on isolated time scales. It is necessary and useful to introduce some basic definitions, theorems and lemmas of time scale calculus to the readers. Also, studying the stochastic dynamic model will involve some functions and formulas about the exponential function and stochastic calculus. Introducing them as background knowledge in advance is important to help readers understand and follow the solution process in the later chapters of this thesis.

1.1. Time Scale Calculus

In this section, we will introduce some definitions and theorems connected to time scale calculus. Many of these basic definitions, theorems, and their proofs can be found in the book by Bohner and Peterson [**1**].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the Cantor set, and the closed interval $[0,1]$ are examples of time scales. On the other hand, the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval $(0,1)$ are not time scales.

Definition 1.1.1. *For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

In this definition we put $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, then t is called *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, we say that t is *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say that t is *left-dense*. Moreover, t is called an *isolated point* when $\rho(t) < t < \sigma(t)$, and it is called a *dense point* when $\rho(t) = t = \sigma(t)$. If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^\kappa = \mathbb{T} - \{\sup \mathbb{T}\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1.1.2. *The* graininess function $\mu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\mu(t) = \sigma(t) - t.$$

If every $t \in \mathbb{T}$ is an isolated point, then the time scale $\mathbb{T}$ is called an *isolated time scale*. For example, if $\mathbb{T} = \mathbb{N}$, then for any $t \in \mathbb{N}$, $\sigma(t) = \inf\{s \in \mathbb{N} : s > t\} = t + 1$ and $\rho(t) = \sup\{s \in \mathbb{N} : s < t\} = t - 1$. Thus every point $t \in \mathbb{N}$ is isolated. In this example the graininess function is $\mu(t) = 1$. The set of integers $\mathbb{Z}$, and $q^{\mathbb{N}} = \{q^n \mid n \in \mathbb{N}\}$ where $q > 1$ are other examples of isolated time scales. Note that for any isolated time scale, the *graininess function* must be strictly positive.

Definition 1.1.3. *For a function $f : \mathbb{T} \to \mathbb{R}$ we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by*

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}.$$

Definition 1.1.4. *If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$, then we define the* delta derivative *of f at t (i.e., $f^\Delta(t)$) to be the number (provided it exists) with the property that given any $\epsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon |\sigma(t) - s| \text{ for all } s \in U.$$

Moreover, we say that f is *delta differentiable* on $\mathbb{T}^\kappa$, if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. If f is differentiable at t, then f is continuous at t. If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{1.1}$$

If t is right-dense, then f is differentiable at t with

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided the limit exists. Note that if $\mathbb{T} = \mathbb{R}$, then we have $f^\Delta(t) = f'(t)$ and if $\mathbb{T} = \mathbb{N}$, then we have $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$.

If f is differentiable at $t \in \mathbb{T}^\kappa$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Assume f, g are differentiable at $t \in \mathbb{T}^\kappa$. Then the sum rule on time scale $\mathbb{T}$ is given by

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

The product rule on time scale $\mathbb{T}$ is given by

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)), \tag{1.2}$$

and if $g(t)g(\sigma(t)) \neq 0$, the quotient rule on time scale $\mathbb{T}$ is given by

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \tag{1.3}$$

Definition 1.1.5. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* rd-continuous *provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (and are finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$.

Definition 1.1.6. *The* Cauchy integral *is defined by*

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for all } r, s \in \mathbb{T},$$

where F is the antiderivative of f.

Theorem 1.1.7. *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T}$ is an isolated time scale, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t) f(t) \qquad \text{if } a < b.$$

(ii) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

As an example, if $\mathbb{T} = \mathbb{Z}$ and $f(t) = 2^t$, then we have

$$\int_a^b 2^t \Delta t = \sum_{t=a}^{b-1} 2^t = 2^b - 2^a.$$

THEOREM 1.1.8. *If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

1.2. The Exponential Function

In this section, we will define a generalized exponential function on time scales and list some important properties of the exponential function. First, we give some basic definitions.

DEFINITION 1.2.1. *The function $p : \mathbb{T} \to \mathbb{R}$ is* regressive *if*

$$1 + \mu(t)p(t) \neq 0 \qquad \textit{for all } t \in \mathbb{T}^\kappa.$$

$\mathcal{R}$ is defined as the set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$.

DEFINITION 1.2.2. *If $p, q \in \mathcal{R}$, the operation "circle plus" addition $\oplus$ is defined by*

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t) \qquad \textit{for all } t \in \mathbb{T}^\kappa.$$

DEFINITION 1.2.3. *If $p \in \mathcal{R}$, the operation "circle minus" $\ominus$ is defined by*

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \qquad \textit{for all } t \in \mathbb{T}^\kappa.$$

DEFINITION 1.2.4. *If $p \in \mathcal{R}$, then the exponential function is defined by*

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \tag{1.4}$$

for $t \in \mathbb{T}$ and $s \in \mathbb{T}^\kappa$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \dfrac{Log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0, \end{cases} \tag{1.5}$$

where Log *denotes the principal logarithm function.*

THEOREM 1.2.5. *If $p \in \mathcal{R}$, then the exponential function $e_p(t, t_0)$ is the unique solution of the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1. \tag{1.6}$$

Here are some examples of $e_p(t, t_0)$ on some various time scales. If $\mathbb{T} = \mathbb{R}$, then $e_p(t, t_0) = e^{p(t-t_0)}$; if $\mathbb{T} = \mathbb{Z}$, then $e_p(t, t_0) = (1+p)^{(t-t_0)}$; if $\mathbb{T} = h\mathbb{Z}$, then $e_p(t, t_0) = (1+ph)^{\frac{(t-t_0)}{h}}$.

DEFINITION 1.2.6. *If $p \in \mathcal{R}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then the dynamic equation*

$$y^{\Delta}(t) = p(t)y(t) + f(t) \tag{1.7}$$

is called regressive.

THEOREM 1.2.7. *(Variation of Constants) Suppose (1.7) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution to the first order dynamic equation on $\mathbb{T}$*

$$y^{\Delta}(t) = p(t)y(t) + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(t)) f(\tau) \Delta\tau.$$

The following two results and relative proofs can be found in the paper [**4**] by Merrell, Ruger and Severs.

THEOREM 1.2.8. *(Variation of Constants for First Order Recurrence Relations) Assume $p(t) \neq 0$, for every $t \in \mathbb{T}^{\kappa}$. Then the unique solution to the IVP*

$$y^{\sigma} - p(t)y = r(t), \; y(t_0) = y_0$$

is given by

$$y(t) = e_{\frac{p-1}{\mu}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{p-1}{\mu}}(t,\sigma(s))\frac{r(s)}{\mu(s)}\Delta s.$$

LEMMA 1.2.9. *The exponential function* $e_{\frac{p-1}{\mu}}(t,t_0)$ *is given by*

$$e_{\frac{p-1}{\mu}}(t,t_0) = \prod_{\tau\in[t_0,t)} p(\tau) \qquad \text{if } t \geq t_0$$
$$e_{\frac{p-1}{\mu}}(t,t_0) = \prod_{\tau\in[t,t_0)} \frac{1}{p(\tau)} \qquad \text{if } t < t_0.$$

Let $0 < p < 1$ be a constant number, and for $t > t_0$, let $t = t_n$ on the time scale $\mathbb{T} = \{t_0, t_1, ..., t_n, t_{n+1}, ...\}$. Also, let n_t be the function of t that counts the number of isolated points on the interval $[t_0, t) \cap \mathbb{T}$. Then by Lemma 1.2.9, the exponential function becomes

$$e_{\frac{p-1}{\mu}}(t,t_0) = \prod_{\tau\in[t_0,t)} p(\tau) = p^{n_t}, \tag{1.8}$$

where the counting function n_t on isolated time scales, is given by

$$n_t(t,s) = \int_s^t \frac{\Delta(\tau)}{\mu(\tau)}. \tag{1.9}$$

1.3. Stochastic Calculus

In this section, we introduce some basic definitions and properties from stochastic calculus. For further reading and for their proofs, we refer the readers to the books by Mikosch [**2**] and Klebaner [**3**].

A random experiment is an act or process that yields an outcome that cannot be predicted with certainty. A sample space Ω is the set of all possible values of a random experiment. A random variable is a function, which assigns unique numerical values to all possible outcomes of a random experiment under fixed conditions. In fact, a random variable is not a variable but rather a function that maps events to numbers. In mathematical language, the random variable $X = X\{w\}$ is a real-valued function defined on Ω, such that

$$X : \Omega \to \mathbb{R}.$$

We define a random variable $X(t)$ on isolated time scale $\mathbb{T}$ as

$$X(t) : \Omega \times \mathbb{T} \to \mathbb{R}.$$

DEFINITION 1.3.1. *A stochastic process is a collection of random variables $X(t)$.*

DEFINITION 1.3.2. *The expectation of a random variable X is*

$$\mathbb{E}(X) = \sum_{all w \in \Omega} X(w)P(w),$$

where $P(w)$ is the probability function to describe the likelihood of w occurring.

THEOREM 1.3.3. *Assume X and Y are random variables, and a,b are constants. Then*

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y).$$

DEFINITION 1.3.4. *The variance of random variable X is defined by*

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

DEFINITION 1.3.5. *The covariance of two random variable X and Y is defined by*

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

If two random variables X and Y are independent, then $Cov(X,Y) = 0$.

THEOREM 1.3.6. *(Dominated-Convergence Theorem) Suppose* (X_n) *is a sequence of random variables,* X *is a random variable, and* $X_n \to X$*. If* $|X_n| \leq Y$ *for all* n*, where* Y *is a random variable and* $E(Y) < \infty$*, then* $E(X_n) \to E(X)$.

For further reading about this theorem, we refer the readers to the book by Williams [**17**].

Chapter 2

LOG-LINEARIZATION

2.1. Definition of Log-linearization

Log-linearization is a useful tool to help analyze and solve nonlinear dynamic stochastic models. Before introducing the definition of log-linearization, we like to recall some basic definitions and theorems.

Definition 2.1.1. *A function f is called a linear function if it is represented by $f(x) = ax + b$, where a and b are constants.*

For example, $f(x) = 3x + 2$ is a linear function; while $f(x) = \log x - 3$ is not linear.

Theorem 2.1.2. *(Taylor's theorem with one variable)*

If a function f is differentiable through order $n+1$ in an interval I containing c, then for each x in I, there exists z between x and c, such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + ... + \frac{f^n(c)}{n!}(x-c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}$.

For a function that is sufficiently smooth, the higher order derivatives will be small and the function can be well approximated linearly as

$$f(x) \approx f(c) + f'(c)(x-c), \tag{2.1}$$

which is also called the first-order Taylor approximation.

Theorem 2.1.3. *(Taylor's theorem with two variables)*

Let f be of class C^{n+1} in a neighborhood of $p_0 = (x_0, y_0)$. Then with $p = (x, y)$,

$$f(p) = f(p_0) + \frac{(x-x_0)}{1}\frac{\partial f}{\partial x}\Big|_{p_0} + \frac{(y-y_0)}{1}\frac{\partial f}{\partial y}\Big|_{p_0} + \frac{(x-x_0)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{p_0}$$
$$+ \frac{(x-x_0)}{1!}\frac{(y-y_0)}{1!}\frac{\partial^2 f}{\partial x \partial y}\Big|_{p_0} + \frac{(y-y_0)^2}{2!}\frac{\partial^2 f}{\partial y^2}\Big|_{p_0}$$
$$+ \ldots + \frac{(x-x_0)^n}{n!}\frac{\partial^n f}{\partial x^n}\Big|_{p_0} + \frac{(x-x_0)^{n-1}}{(n-1)!}\frac{(y-y_0)}{1!}\frac{\partial^n f}{\partial x^{n-1}\partial y}\Big|_{p_0}$$
$$+ \ldots + \frac{(y-y_0)^n}{n!}\frac{\partial^n f}{\partial y^n}\Big|_{p_0} + R_n(x,y)$$

where $R_n(x,y) = \frac{(x-x_0)^{(n+1)}}{(n+1)!}\frac{\partial^{n+1} f}{\partial x^{n+1}}\Big|_{p^*} + \ldots + \frac{(y-y_0)^{(n+1)}}{(n+1)!}\frac{\partial^{n+1} f}{\partial y^{n+1}}\Big|_{p^*}$ *and* p^* *is a point on the line segment joining* p_0 *and* p.

In fact, Taylor's theorem also applies equally well to functions of more than two variables. For further reading about Theorems (2.1.2) and (2.1.3), we refer the readers to the books by Larson, H. Edwards [**15**] and C. Buck [**16**].

Next, let us make clear what is the log-linearization. Suppose we have a function $f(x) = \log x_t$ with its initial state x_{t_0} when $t = t_0$. According to the first-order Taylor approximation (2.1), we get

$$\log x_t \approx \log x_{t_0} + \frac{1}{x_{t_0}}(x_t - x_{t_0})$$
$$\log x_t - \log x_{t_0} \approx \frac{x_t - x_{t_0}}{x_{t_0}}.$$

If we define $\hat{x}_t \equiv \frac{x_t - x_{t_0}}{x_{t_0}}$, then we have $\hat{x}_t \approx \log x_t - \log x_{t_0}$. Although the function $\log x_t - \log x_{t_0}$ is not linear, it is linear in logarithms, which is the reason why we call it log-linear. In fact, the log-linearization method converts a non-linear function into a function that is linear in logarithms. For further reading about log-linearization, we refer the readers to the papers by C. D. Stein [**20**] and Uhlig [**23**].

2.2. Methods and Examples of Log-linearization

There are several techniques to obtain the log-linearization of a function. We introduce only two of them in this thesis.

Log-linearization: First Approach

This approach is to use a first order Taylor approximation around the initial state to replace the equation with approximation. Then we built up
$\hat{x}_t \equiv \log x_t - \log x_{t_0}$ and $\widehat{f(x_t)} \equiv \dfrac{f(x_t) - f(x_{t_0})}{f(x_{t_0})}$.

$$f(x_t) \approx f(x_{t_0}) + f'(x_{t_0})(x_t - x_{t_0})$$

$$f(x_t) - f(x_{t_0}) \approx f'(x_{t_0})(x_t - x_{t_0}),$$

dividing by $f(x_{t_0})$ on both sides, we get

$$\frac{f(x_t) - f(x_{t_0})}{f(x_{t_0})} \approx \frac{f'(x_{t_0})(x_t - x_{t_0})}{f(x_{t_0})}$$

$$\approx \frac{f'(x_{t_0})x_{t_0}}{f(x_{t_0})} \frac{(x_t - x_{t_0})}{x_{t_0}},$$

that is

$$\widehat{f(x_t)} \approx \frac{f'(x_{t_0})x_{t_0}}{f(x_{t_0})} \hat{x}_t.$$

Example 2.2.1. A product of functions $x_t y_t$:

$$\begin{aligned} x_t y_t &\approx x_{t_0} y_{t_0} + y_{t_0}(x_t - x_{t_0}) + x_{t_0}(y_t - y_{t_0}) \\ &\approx x_{t_0} y_{t_0}\left(1 + \frac{x_t - x_{t_0}}{x_{t_0}} + \frac{y_t - y_{t_0}}{y_{t_0}}\right) \\ &\approx x_{t_0} y_{t_0}(1 + \hat{x}_t + \hat{y}_t). \end{aligned}$$

Example 2.2.2. A Cobb-Douglas production function $y_t = k_t^{\alpha} n_t^{1-\alpha}$, which is an important function of dynamic model in economics:

$$y_t \approx y_{t_0} + \alpha k_{t_0}^{\alpha-1} n_{t_0}^{1-\alpha}(k_t - k_{t_0}) + k_{t_0}^{\alpha}(1-\alpha) n_{t_0}^{-\alpha}(n_t - n_{t_0})$$

$$y_t - y_{t_0} \approx \frac{\alpha k_{t_0}^{\alpha} n_{t_0}^{1-\alpha}(k_t - k_{t_0})}{k_{t_0}} + \frac{(1-\alpha)k_{t_0}^{\alpha} n_{t_0}^{1-\alpha}(n_t - n_{t_0})}{n_{t_0}}$$
$$\approx \frac{\alpha y_{t_0}(k_t - k_{t_0})}{k_{t_0}} + \frac{(1-\alpha)y_{t_0}(n_t - n_{t_0})}{n_{t_0}}.$$

Dividing by y_{t_0} on both sides, we obtain

$$\frac{y_t - y_{t_0}}{y_{t_0}} \approx \alpha \frac{(k_t - k_{t_0})}{k_{t_0}} + (1-\alpha)\frac{n_t - n_{t_0}}{n_{t_0}},$$

that is,

$$\hat{y}_t \approx \alpha \hat{k}_t + (1-\alpha)\hat{n}_t.$$

Log-linearization: Second Approach

Assume that for any variable x_t, we define $X_t \equiv \log x_t$. Then we have

$$\hat{x}_t \approx X_t - X_{t_0}.$$

EXAMPLE 2.2.3. A product of functions $x_t y_t$:

$$x_t y_t = e^{\log x_t} e^{\log y_t}$$
$$= e^{X_t} e^{Y_t}$$
$$= e^{X_t + Y_t}.$$

Let $P_{t_0} = (X_{t_0}, Y_{t_0})$. Then

$$x_t y_t \approx e^{(X_{t_0}+Y_{t_0})} + \left.\frac{\partial (e^{X_t+Y_t})}{\partial X_t}\right|_{P_{t_0}} (X_t - X_{t_0}) + \left.\frac{\partial (e^{X_t+Y_t})}{\partial Y_t}\right|_{P_{t_0}} (Y_t - Y_{t_0})$$
$$\approx e^{(X_{t_0}+Y_{t_0})} + e^{(X_{t_0}+Y_{t_0})}(X_t - X_{t_0}) + e^{(X_{t_0}+Y_{t_0})}(Y_t - Y_{t_0})$$
$$\approx x_{t_0} y_{t_0} + x_{t_0} y_{t_0} \hat{x}_t + x_{t_0} y_{t_0} \hat{y}_t$$
$$\approx x_{t_0} y_{t_0}(1 + \hat{x}_t + \hat{y}_t).$$

EXAMPLE 2.2.4. A Cobb-Douglas production function $y_t = k_t^{\alpha} n_t^{1-\alpha}$:

Firstly, let us start with the left-hand side of the function:

$$y_t = e^{\log y_t}$$

$$
\begin{aligned}
&= e^{Y_t} \\
&\approx e^{Y_{t_0}} + \left.\frac{\partial(e^{Y_t})}{\partial Y_t}\right|_{Y_t = Y_{t_0}} (Y_t - Y_{t_0}) \\
&\approx e^{Y_{t_0}} + e^{Y_{t_0}}(Y_t - Y_{t_0}) \\
&\approx y_{t_0} + y_{t_0}\hat{y}_t \\
&\approx y_{t_0}(1 + \hat{y}_t).
\end{aligned}
$$

Next, we proceed with right-hand side of the function:

$$
\begin{aligned}
k_t^\alpha n_t^{1-\alpha} &= e^{\log(k_t^\alpha n_t^{1-\alpha})} \\
&= e^{\log k_t^\alpha + \log n_t^{1-\alpha}} \\
&= e^{\alpha \log k_t + (1-\alpha)\log n_t} \\
&= e^{\alpha K_t + (1-\alpha) N_t}.
\end{aligned}
$$

Let $P_{t_0} = (K_{t_0}, N_{t_0})$. Then

$$
k_t^\alpha n_t^{1-\alpha} \approx e^{\alpha K_{t_0} + (1-\alpha) N_{t_0}} + \left.\frac{\partial(e^{\alpha K_t + (1-\alpha) N_t})}{\partial K_t}\right|_{P_{t_0}} (K_t - K_{t_0})
$$

$$
\begin{aligned}
k_t^\alpha n_t^{1-\alpha} &\approx e^{\alpha K_{t_0} + (1-\alpha) N_{t_0}} + \left.\frac{\partial(e^{\alpha K_t + (1-\alpha) N_t})}{\partial K_t}\right|_{P_{t_0}} (K_t - K_{t_0}) \\
&\quad + \left.\frac{\partial(e^{\alpha K_t + (1-\alpha) N_t})}{\partial N_t}\right|_{P_{t_0}} (N_t - N_{t_0}) \\
&\approx e^{\alpha K_{t_0} + (1-\alpha) N_{t_0}} + \alpha(e^{\alpha K_{t_0} + (1-\alpha) N_{t_0}})(K_t - K_{t_0}) \\
&\quad + (1-\alpha)(e^{\alpha K_{t_0} + (1-\alpha) N_{t_0}})(N_t - N_{t_0}).
\end{aligned}
$$

At the initial state, we have

$$
e^{\alpha K_{t_0} + (1-\alpha) N_{t_0}} = k_{t_0}^\alpha n_{t_0}^{1-\alpha} = y_{t_0}.
$$

By elementary algebra, we obtain

$$
k_t^\alpha n_t^{1-\alpha} \approx y_{t_0}(1 + \alpha \hat{k}_t + (1-\alpha)\hat{n}_t).
$$

Then we have

$$y_{t_0}(1 + \hat{y}_t) \approx y_{t_0}(1 + \alpha\hat{k}_t + (1 - \alpha)\hat{n}_t)$$

$$1 + \hat{y}_t \approx 1 + \alpha\hat{k}_t + (1 - \alpha)\hat{n}_t$$

$$\hat{y}_t \approx \alpha\hat{k}_t + (1 - \alpha)\hat{n}_t.$$

For further reading and more examples, we refer the readers to the paper by C. D. Stein [**20**].

Chapter 3

RATIONAL EXPECTATIONS ON ISOLATED TIME DOMAINS

3.1. Conditional Expectation

In this section, in order to help the readers understand rational expectations in Section 3.3, we will introduce some basic definitions and important properties of conditional expectation.

Definition 3.1.1. *A partition of a nonempty set* Ω *is a collection* $\{B_i\}_{i\in I}$ *of nonempty mutually disjoint subsets of* Ω *with* $\bigcup_{i\in I} B_i = \Omega$.

The field (algebra) generated by the partition is the collection of all finite unions of B_i's and their complements.

Definition 3.1.2. *Let a random variable* X *take distinct values* $x_1, ..., x_n$, *and the field (algebra)* $\mathcal{F}$ *be generated by a partition* $\{B_1, B_2, ..., B_k\}$ *of* Ω. *Then the* conditional expectation *of* X *given* $\mathcal{F}$ *is defined by*

$$E(X|\mathcal{F}) = \sum_{i=1}^{n} x_i P(X = x_i|\mathcal{F}),$$

where $P(X = x_i|\mathcal{F})$ *is the conditional probability of* X *given* $\mathcal{F}$ *and* $X : \Omega \to \mathbb{R}$.

Basic Properties of Conditional Expectation

(i) Linearity: For random variables X_1, X_2 and constants a_1, a_2,

$$E([a_1X_1 + a_2X_2]|\mathcal{F}) = a_1E(X_1|\mathcal{F}) + a_2E(X_2|\mathcal{F}).$$

(ii) The expectation law:

$$E(X) = E[E(X|\mathcal{F})].$$

(iii) Positivity: If $X \geq 0$, then $E(X|\mathcal{F}) \geq 0$.

(iv) Independence law: If X is independent of $\mathcal{F}$, then $E(X|\mathcal{F}) = E(X)$.

(v) Tower property: If $\mathcal{F}$ and $\mathcal{H}$ are two fields with $\mathcal{F} \subset \mathcal{H}$, then

$$E(E(X|\mathcal{H})|\mathcal{F}) = E(X|\mathcal{F}) \text{ or } E(E(X|\mathcal{F})|\mathcal{H}) = E(X|\mathcal{F}).$$

(vi) Stability: If X is $\mathcal{F}$ measurable, then $E(X|\mathcal{F}) = X$.

(vii) Constants: For any scalar a, $E(a|\mathcal{F}) = a$.

For further reading and for the proofs, we refer the readers to the book by Williams [**17**].

3.2. Martingales

In this section, we will discuss martingales that play an important role in the dynamic functions of Section 3.3. Before introducing the definition of martingales, we define $\sigma - fields$.

Definition 3.2.1. *A family $\mathcal{X}$ of subsets of a set of X is called a $\sigma - field$ (or a $\sigma - algebra$) in case:*

1. $\varnothing, X \in \mathcal{X}$.

2. *If $A \in \mathcal{X}$, then the complement $A^c \in \mathcal{X}$, where $A^c = X \backslash A$.*

3. *If (A_n) is a sequence of sets in $\mathcal{X}$, then the countable union*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{X}.$$

It is easy to prove that each of the following collections $\mathcal{X}$ is a $\sigma - field$.

1. Let X be any set and let $\mathcal{X}$ be the family of all subsets of X.

2. Let $\mathcal{X}$ be the family consisting of precisely two subsets of X, namely $\varnothing$ and X.

3. If $\mathcal{Y}$ and $\mathcal{Z}$ are $\sigma - fields$ of subsets of X, let $\mathcal{X}$ be the intersection of $\mathcal{Y}$ and $\mathcal{Z}$.

More examples and relevant proofs can be found in the book written by G. Bartle [**21**].

Definition 3.2.2. *The collection* $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ *of* $\sigma - fields$ *on* Ω *is called a* filtration *if*

$$\mathcal{F}_m \subset \mathcal{F}_n \text{ for all } 0 \leq m \leq n.$$

Definition 3.2.3. *Let A be a nonempty collection of subsets of X. The intersection of all the* $\sigma - fields$ *containing A is the smallest* $\sigma - field$ *containing A. This smallest* $\sigma - field$ *is called* the $\sigma - field$ generated by A *and denoted by* $\sigma(A)$.

The discrete-time process $X = (X_t, t = 0, 1, ...)$ is said to be *adapted to the filtration* $(\mathcal{F}_t, t = 0, 1, ...)$ if

$$\sigma(X_t) \subset \mathcal{F}_t \quad \text{for all} \quad t = 0, 1, 2, ...$$

where $\sigma(X_t)$ is the $\sigma - field$ generated by the random variable X_t.

Definition 3.2.4. *The stochastic process* $X = (X_n, n = 0, 1, ...)$ *is called a* discrete-time martingale *with respect to the filtration* $(\mathcal{F}_n, n = 0, 1, ...)$, *we write* $(X, (\mathcal{F}_n))$, *if*

(i) for each n, $E|X_n| < \infty$;

(ii) X is adapted to $(\mathcal{F}_n)$;

(iii) for each n, $E(X_{n+1}|\mathcal{F}_n) = X_n$.

In probability theory, a martingale is a mathematical model of a fair game and its knowledge of past events cannot help predict future outcomes to win. In fact, a martingale is a special case of a sequence of random variables. For more relative knowledge and examples of martingales, readers could reference the thesis by Ekiz [**12**] and the book by Williams [**17**].

EXAMPLE 3.2.5. *If y_t is a random variable and $y_t^{\Delta} = x_t$, where the Δ derivative is with respect to t, then $M(t) = \int x_t \Delta t - y_t$ is a martingale.*

3.3. Rational Expectations

In this section, we will give a brief introduction to rational expectations. We know that expectations are defined as the prediction of future economic events or economists' opinions about future prices, incomes, taxes or other important variables in economics. Many areas of economics such as wage bargaining in the labor market, cost benefit analysis, and exchange rates include expectations.

In this thesis, we will take into account the rational expectations. It is a theory in economics that provides the people or economic agents making future decision with available information and past experience. The theory of rational expectations is not only used in one specific economic field, but it has also been extended to many fields of economics such as finance, labor economics, and industrial organization. For further reading we refer the readers to the books by P. Tucci [**7**] and Broze and Szafarz [**9**].

The following dynamic equation was introduced in the thesis by Ekiz [**12**]:

$$y_t = aE_t[y_t^{\sigma}] + f(t, z_t), \tag{3.1}$$

where $t \in \mathbb{T}$.

LEMMA 3.3.1. *If y_t is a random variable, then*

$$\int E_t[y_t^\sigma]\Delta t = E_t[\int y_t^\sigma \Delta t] + M(t)$$

where $M(t)$ is an arbitrary martingale.

THEOREM 3.3.2. *Let $\mathbb{T}$ be an isolated time scale. Then the solution of the equation (3.1) is given by*

$$y_t = e_{\frac{1-a}{a\mu}}(t,0)M(t) - e_{\frac{1-a}{a\mu}}(t,0)\int e_{\ominus\frac{1-a}{a\mu}}(t,0)\frac{1}{\mu(t)}f(t,z_t)\Delta t,$$

where $t \in \mathbb{T}$ and the $M(t)$ is an arbitrary martingale.

This is just a single equation on isolated time scales. However, many equations may be involved in one system, such as

$$y_1(t) = a_{11}E_t[y^\sigma(t)] + ... + a_{1n}E_t[y^{\sigma^n}(t))] + f_1(t,z_t)$$

$$y_2(t) = a_{21}E_t[y^\sigma(t)] + ... + a_{2n}E_t[y^{\sigma^n}(t)] + f_2(t,z_t)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$y_n(t) = a_{n1}E_t[y^\sigma(t)] + ... + a_{nn}(t)E_t[y^{\sigma^n}(t)] + f_n(t,z_t).$$

This system can be written as an equivalent vector equation,

$$Y_t = AE_t[Y_t^\sigma] + F(t,Z_t) \tag{3.2}$$

where A is an invertible $n \times n$ matrix and

$$Y_t = \begin{bmatrix} y_1(t) \\ y_2(t) \\ . \\ . \\ . \\ y_n(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \qquad F(t, Z_t) = \begin{bmatrix} f_1(t, z_t) \\ f_2(t, z_t) \\ . \\ . \\ . \\ f_n(t, z_t) \end{bmatrix}.$$

THEOREM 3.3.3. *The solution of (3.2) is given as*

$$Y_t = e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M(t) - e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0) \int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t \quad (3.3)$$

where $t \in \mathbb{T}$ and I is the $n \times n$ identity matrix.

For $\mathbb{T} = \mathbb{Z}$, the equation (3.2) will be given by

$$Y_t = AE_t[Y_{t+1}] + F(t, Z_t). \quad (3.4)$$

COROLLARY 3.3.4. *$Y_t = A^{-t}M(t) - A^{-t}\sum A^t F(t, Z_t)$ is the general solution of equation (3.4).*

For the proofs of Lemma (3.3.1), Theorem (3.3.2), (3.3.3) and Corollary (3.3.4) we refer the readers to the thesis by Ekiz [**12**].

Chapter 4

ANALYZING A GENERAL FORM OF STOCHASTIC DYNAMIC MODELS

We will introduce one kind of stochastic sequence problems on isolated time domains in Section 4.1. In Section 4.2, we will introduce Euler equations and transversality conditions for the stochastic sequence problem. After formulating corresponding Euler equations and transversality conditions, we will prove a theorem to justify the sufficiency of the Euler equation and transversality condition.

4.1. Stochastic Dynamic Sequence Problem

Suppose $\mathbb{T}$ is an isolated time domain with $\sup \mathbb{T} = \infty$, and let $\mathbb{T} \bigcap [0, \infty) = [0, \infty)_{\mathbb{T}}$. Then we define the stochastic sequence problem on the isolated time domain $\mathbb{T}$ as

$$(SP) \qquad \sup_{\{C_t, K_t\}_{t=0}^{\infty}} E \sum_{t \in [0,\infty)_{\mathbb{T}}} [e_{\frac{\beta-1}{\mu}}(t,0) F(C_t) \mu_t] \tag{4.1}$$

$$C_t = f(K_{\rho(t)}, K_t, Z_t),$$

$$K_{\rho(0)}, C_0 \in X \text{ given},$$

where

- K_t is the state variable;
- C_t is the optimal control or choice variable;
- Z_t is the random state variable;
- E denotes the mathematical expectation of the objective function F;
- $F(.,.)$ is a strictly increasing, concave, continuous and differentiable

real-valued objective function;

- X is the space of sequences $\{C_t, K_t\}_{t=0}^{\infty}$ that maximize the sequence problem;

- $f : X \to X$ is a strictly concave, continuous and differentiable real-valued function, increasing with respect to $K_{\rho(t)}$ and decreasing with respect to K_t.

Our goal is to find the optimal sequence $\{C_t, K_t\}_{t=0}^{\infty}$ that maximizes the expected utility in the sequence problem.

4.2. Euler Equations and Transversality Conditions

An Euler equation is an intertemporal version of a first-order condition characterizing an optimal choice. There are many methods to obtain Euler equations such as calculus of variations, the Lagrangian method, optimal control theory or dynamic programming. Assuming (SP) (4.1) attains its supremum at $\{C_t^*, K_t^*\}_{t=0}^{\infty}$, we define the Euler equation for our problem as follow:

$$\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t) + E_t[\beta \mu_{(\sigma(t))} F'(C_{\sigma(t)}^*) f_x(K_t^*, K_{\sigma(t)}^*, Z_{\sigma(t)})] = 0. \tag{4.2}$$

The transversality conditions are optimality conditions often used along with Euler equations to characterize the optimal paths of dynamic economic models. If the terminal point is not fixed, there may be several paths satisfying the Euler equation. In our case, the transversality condition enables one to single out the optimal sequence among those satisfying the Euler equation.

We define the transversality condition for our problem (4.1) as:

$$\lim_{T \to \infty} E_0[e_{\frac{\beta-1}{\mu}}(T, 0) \mu_{(T)} F'(C_T^*) f_y(K_{\rho(T)}^*, K_T^*, Z_T) K_T^*] = 0. \tag{4.3}$$

For further reading on Euler equations and transversality conditions, we refer the readers to the thesis by Turhan [**14**] and the paper by Kamihigashi [**22**].

THEOREM 4.2.1. *If the sequence* $\{C_t^*, K_t^*\}_{t=0}^{\infty}$ *satisfies* (4.2) *and* (4.3)*, then it is optimal for the problem* (4.1)*.*

Proof. Assume the sequence $\{C_t^*, K_t^*\}_{t=0}^{\infty}$ satisfies (4.2) and (4.3). It is sufficient to show that the difference between the objective functions in (SP) evaluated at $\{C_t^*, K_t^*\}_{t=0}^{\infty}$ and at $\{C_t, K_t\}_{t=0}^{\infty}$ is nonnegative. Therefore, we start by setting the difference as

$$(SP)^* - (SP) = E \lim_{T\to\infty} \{ \sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t[F(C_t^*) - F(C_t)]\}.$$

According to Theorem (1.3.6), we have

$$\begin{aligned}(SP)^* - (SP) &= E \lim_{T\to\infty} \{ \sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t[F(C_t^*) - F(C_t)]\} \\
&= \lim_{T\to\infty} E\{ \sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t[F(C_t^*) - F(C_t)]\} \\
&\geqslant \lim_{T\to\infty} E\{ \sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t F'(C_t^*)[C_t^* - C_t]\} \\
&\geqslant \lim_{T\to\infty} E\{ \sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t[F'(C_t^*)f_x(K_{\rho(t)}^*, K_t^*, Z_t)\cdot \\
&\qquad (K_{\rho(t)}^* - K_{\rho(t)}) + F'(C_t^*)f_y(K_{\rho(t)}^*, K_t^*, Z_t)(K_t^* - K_t)]\},\end{aligned}$$

$$\begin{aligned}&\lim_{T\to\infty} E\{ \sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t[F'(C_t^*)f_x(K_{\rho(t)}^*, K_t^*, Z_t)(K_{\rho(t)}^* - K_{\rho(t)}) \\
&\quad +F'(C_t^*)f_y(K_{\rho(t)}^*, K_t^*, Z_t)(K_t^* - K_t)]\} \\
&= \lim_{T\to\infty} E\{[e_{\frac{\beta-1}{\mu}}(0,0)\mu_0 F'(C_0^*)f_x(K_{\rho(0)}^*, K_0^*, Z_0)(K_{\rho(0)}^* - K_{\rho(0)}) \\
&\quad +e_{\frac{\beta-1}{\mu}}(\sigma(0),0)\mu_{\sigma(0)}F'(C_{\sigma(0)}^*)f_x(K_0^*, K_{\sigma(0)}^*, Z_{\sigma(0)})(K_0^* - K_0) \\
&\quad +\cdots \\
&\quad +e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*)f_x(K_{\rho(T)}^*, K_T^*, Z_T)(K_{\rho(T)}^* - K_{\rho(T)})]\end{aligned}$$

$$+\sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)(K_t^* - K_t)\}.$$

At the initial state, $K_{\rho(0)}^* = K_{\rho(0)}$, and thus $K_{\rho(0)}^* - K_{\rho(0)} = 0$. That is,

$$\lim_{T\to\infty} E\{\sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t [F'(C_t^*) f_x(K_{\rho(t)}^*, K_t^*, Z_t)(K_{\rho(t)}^* - K_{\rho(t)})$$
$$+F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)(K_t^* - K_t)]\}$$
$$= \lim_{T\to\infty} E\{[e_{\frac{\beta-1}{\mu}}(0,0)\mu_0 F'(C_0^*) f_x(K_{\rho(0)}^*, K_0^*, Z_0)\cdot 0$$
$$+e_{\frac{\beta-1}{\mu}}(\sigma(0),0)\mu_{\sigma(0)} F'(C_{\sigma(0)}^*) f_x(K_0^*, K_{\sigma(0)}^*, Z_{\sigma(0)})(K_0^* - K_0)$$
$$+\cdots$$
$$+e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*) f_x(K_{\rho(T)}^*, K_T^*, Z_T)(K_{\rho(T)}^* - K_{\rho(T)})]$$
$$+\sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)(K_t^* - K_t)\}$$
$$= \lim_{T\to\infty} E\{\sum_{t\in[0,T)_{\mathbb{T}}} \beta\mu_{\sigma(t)} F'(C_{\sigma(t)}^*) f_x(K_t^*, K_{\sigma(t)}^*, Z_{\sigma(t)})(K_t^* - K_t)$$
$$+\sum_{t\in[0,\sigma(T))_{\mathbb{T}}} e_{\frac{\beta-1}{\mu}}(t,0)\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)(K_t^* - K_t)\}$$
$$= \lim_{T\to\infty} E\{\{\sum_{t\in[0,T)_{\mathbb{T}}} [\beta\mu_{\sigma(t)} F'(C_{\sigma(t)}^*) f_x(K_t^*, K_{\sigma(t)}^*, Z_{\sigma(t)})$$
$$+e_{\frac{\beta-1}{\mu}}(t,0)\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)](K_t^* - K_t)\}$$
$$+e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*) f_y(K_{\rho(T)}^*, K_T^*, Z_T)(K_T^* - K_T)\}$$
$$= \lim_{T\to\infty}\{\sum_{t\in[0,T)_{\mathbb{T}}} \{\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)$$
$$+E_t[\beta\mu_{\sigma(t)} F'(C_{\sigma(t)}^*) f_x(K_t^*, K_{\sigma(t)}^*, Z_{\sigma(t)})]\}(K_T^* - K_T)$$
$$+E[e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*) f_y(K_{\rho(T)}^*, K_T^*, Z_T)(K_T^* - K_T)]\}.$$

Let
$$I = \lim_{T\to\infty}\{\sum_{t\in[0,T)_{\mathbb{T}}} \{\mu_t F'(C_t^*) f_y(K_{\rho(t)}^*, K_t^*, Z_t)$$
$$+E_t[\beta\mu_{\sigma(t)} F'(C_{\sigma(t)}^*) f_x(K_t^*, K_{\sigma(t)}^*, Z_{\sigma(t)})]\}(K_T^* - K_T)\}$$
$$+\lim_{T\to\infty} E[e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*) f_y(K_{\rho(T)}^*, K_T^*, Z_T)(K_T^* - K_T)].$$

According to the Euler equation (4.2), we have

$$I = 0 + \lim_{T\to\infty} E[e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*) f_y(K_{\rho(T)}^*, K_T^*, Z_T)(K_T^* - K_T)]$$
$$= \lim_{T\to\infty} E_0[e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C_T^*) f_y(K_{\rho(T)}^*, K_T^*, Z_T)(K_T^* - K_T)].$$

Since $f_y(K^*_{\rho(T)}, K^*_T, Z_T) \leqslant 0$ and by the transversality condition (4.3), we have

$$I \geqslant \lim_{T\to\infty} E_0[e_{\frac{\beta-1}{\mu}}(T,0)\mu_T F'(C^*_T) f_y(K^*_{\rho(T)}, K^*_T, Z_T) K^*_T] = 0.$$

Thus, we have $(SP)^* - (SP) \geqslant 0$, establishing the desired result. $\square$

Chapter 5

THE NEOCLASSICAL GROWTH MODEL

In this chapter, we will introduce the neoclassical growth model on an isolated time domain. Then we will study the social planner's problem and perform all the calculations step by step.

Step 1: Construct the Euler equation and the transversality condition;

Step 2: Find the initial state;

Step 3: Log-linearize the constraints and the Euler equation;

Step 4: Solve for the corresponding recursive equations by matrix operations, instead of the method of undetermined coefficients as recommended by the economic reference document by Uhlig [**23**];

Step 5: Use the techniques for solving rational expectations models on an isolated time domain and reach an optimal solution.

5.1. The Stochastic Growth Model

Suppose $\mathbb{T}$ is an isolated time domain with $\sup\mathbb{T} = \infty$, and let $\mathbb{T}\bigcap[0,\infty) = [0,\infty)_{\mathbb{T}}$. Then the social planner's problem is given by

$$\sup_{\{C_t,K_t\}_{t=0}^{\infty}} E \sum_{t\in[0,\infty)_{\mathbb{T}}} [e_{\frac{\beta-1}{\mu}}(t,0)\frac{C_t^{1-\eta}-1}{1-\eta}\mu_t] \tag{5.1}$$

$$C_t = Z_t K_{\rho(t)}^{\alpha} + (1-\delta)K_{\rho(t)} - K_t$$

$$logZ_t = (1-\psi)logZ_0 + \psi logZ_{\rho(t)} + \varepsilon_t, \qquad \varepsilon_t \sim i.i.d.N(0;\sigma^2)$$

$$K_{\rho(0)}, Z_0, C_0 \text{ are given,}$$

where E is the expectation; C_t is the control variable; K_t is the state variable; Z_t is the random state variable; $0 < \alpha < 1; 0 < \beta < 1; 0 < \psi < 1; \eta \geqslant 1$; $i.i.d$ means independent identically distributed and $\varepsilon_{\sigma(0)}$=0. For further reading about this sequence problem, we refer the readers to the paper by Uhlig [**23**].

5.2. The Social Planner's Problem

In this section, we will study and solve the problem (5.1) in the following five steps.

Step 1: Construct the Euler equation and the transversality condition. According to the formula of the Euler equation (4.2), we have

$$\mu_t C_t^{-\eta}(-1) + E_t[\beta \mu_{\sigma(t)} C_{\sigma(t)}^{-\eta}(\alpha Z_{\sigma(t)} K_t^{\alpha-1} + (1-\delta))] = 0.$$

Since $\mu_t C_t^{-\eta}(-1) = E_t[\mu_t C_t^{-\eta}(-1)]$, the first order condition of (5.1) is

$$E_t\left[\beta \frac{\mu_{\sigma(t)}}{\mu_t}\left(\frac{C_t}{C_{\sigma(t)}}\right)^{\eta} R_{\sigma(t)}\right] = 1, \tag{5.2}$$

where $R_t = \alpha Z_t K_{\rho(t)}^{\alpha-1} + (1-\delta)$.

By applying the formula (4.3), the transversality condition of (5.1) is

$$\lim_{T\to\infty} E_0[e_{\frac{\beta-1}{\mu}}(T,0)\mu_T C_T^{-\eta} K_T] = 0. \tag{5.3}$$

Step 2: Find the initial state.

To find the initial state, we rewrite these necessary conditions:

$$C_t = Z_t K_{\rho(t)}^{\alpha} + (1-\delta)K_{\rho(t)} - K_t \tag{5.4}$$

$$R_t = \alpha Z_t K_{\rho(t)}^{\alpha-1} + (1-\delta) \tag{5.5}$$

$$logZ_t = (1-\psi)logZ_0 + \psi logZ_{\rho(t)} + \varepsilon_t \tag{5.6}$$

where $\varepsilon_t \sim i.i.d.N(0;\sigma^2)$.

At the initial state $t = 0$, the corresponding equations of (5.2), (5.4), (5.5) and (5.6) are:

$$C_0 = Z_0 K_{\rho(0)}^{\alpha} + (1-\delta)K_{\rho(0)} - K_0$$

$$R_0 = \alpha Z_0 K_{\rho(0)}^{\alpha-1} + (1-\delta)$$

$$E_t\left[\beta \frac{\mu_{\sigma(0)}}{\mu_0}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta} R_{\sigma(0)}\right] = 1$$

$$\psi log Z_0 = \psi log Z_{\rho(0)} + \varepsilon_0.$$

Step 3: Log-linearize the constraints and the Euler equation.

We will apply the first approach of log-linearization which was introduced in Chapter 2, and define $\hat{x}_t = \dfrac{X_t - X_0}{X_0}$.

For the equation (5.4), we will log-linearize explicitly. Our idea is to log-linearize the parts of the equations separately and then combine them. Firstly, we log-linearize C_t:

$$C_t \approx C_0(1 + \hat{c}_t).$$

Secondly, log-linearizing $Z_t K_{\rho(t)}^{\alpha}$, we have

$$\begin{aligned} Z_t K_{\rho(t)}^{\alpha} &\approx Z_0 K_{\rho(0)}^{\alpha} + K_{\rho(0)}^{\alpha}(Z_t - Z_0) + Z_0 \alpha K_{\rho(0)}^{\alpha-1}(K_{\rho(t)} - K_{\rho(0)}) \\ &\approx Z_0 K_{\rho(0)}^{\alpha} + Z_0 K_{\rho(0)}^{\alpha}\frac{(Z_t - Z_0)}{Z_0} + Z_0 \alpha K_{\rho(0)}^{\alpha}\frac{(K_{\rho(t)} - K_{\rho(0)})}{K_{\rho(0)}} \\ &\approx Z_0 K_{\rho(0)}^{\alpha}(1 + \hat{z}_t + \alpha \hat{k_{\rho(t)}}). \end{aligned}$$

Thirdly, by log-linearizing $(1-\delta)K_{\rho(t)} - K_t$, we obtain

$$(1-\delta)K_{\rho(t)} - K_t \approx (1-\delta)K_{\rho(0)}(1 + \hat{k_{\rho(t)}}) - K_0(1 + \hat{k}_t).$$

Finally, combining these equations, we have

$$\begin{aligned} C_t &\approx C_0 + C_0 \hat{c}_t \\ &\approx Z_0 K_{\rho(0)}^{\alpha}(1 + \hat{z}_t + \alpha \hat{k_{\rho(t)}}) + (1-\delta)K_{\rho(0)}(1 + \hat{k_{\rho(t)}}) - K_0(1 + \hat{k}_t) \end{aligned}$$

$$\approx Z_0 K_{\rho(0)}^{\alpha} + (1-\delta)K_{\rho(0)} - K_0 + Z_0 K_{\rho(0)}^{\alpha}(\hat{z}_t + \alpha \hat{k_{\rho(t)}}) + (1-\delta)K_{\rho(0)}\hat{k_{\rho(t)}} - K_0 \hat{k}_t$$

$$\approx C_0 + Z_0 K_{\rho(0)}^{\alpha}(\hat{z}_t + \alpha \hat{k_{\rho(t)}}) + (1-\delta)K_{\rho(0)}\hat{k_{\rho(t)}} - K_0 \hat{k}_t.$$

By elementary algebra, we obtain

$$C_0 \hat{c}_t \approx Z_0 K_{\rho(0)}^{\alpha}(\hat{z}_t + \alpha \hat{k_{\rho(t)}}) + (1-\delta)K_{\rho(0)}\hat{k_{\rho(t)}} - K_0 \hat{k}_t$$

$$\approx Z_0 K_{\rho(0)}^{\alpha}\hat{z}_t + (\alpha Z_0 K_{\rho(0)}^{\alpha-1} + (1-\delta))K_{\rho(0)}\hat{k_{\rho(t)}} - K_0 \hat{k}_t.$$

In Step 2, we have $R_0 = \alpha Z_0 K_{\rho(0)}^{\alpha-1} + (1-\delta)$, and thus

$$C_0 \hat{c}_t \approx Z_0 K_{\rho(0)}^{\alpha}\hat{z}_t + R_0 K_{\rho(0)}\hat{k_{\rho(t)}} - K_0 \hat{k}_t.$$

Dividing by C_0 on both sides, we obtain

$$\hat{c}_t \approx \frac{Z_0 K_{\rho(0)}^{\alpha}}{C_0}\hat{z}_t + \frac{R_0 K_{\rho(0)}}{C_0}\hat{k_{\rho(t)}} - \frac{K_0}{C_0}\hat{k}_t. \tag{5.7}$$

For the equation (5.5), log-linearizing on both sides, we have

$$R_t \approx R_0(1 + \hat{r}_t)$$

$$\approx \alpha Z_0 K_{\rho(0)}^{\alpha-1} + \alpha K_{\rho(0)}^{\alpha-1}(Z_t - Z_0) + \alpha Z_0(\alpha-1)K_{\rho(0)}^{\alpha-2}(K_{\rho(t)} - K_{\rho(0)}) + (1-\delta)$$

$$\approx R_0 + \alpha K_{\rho(0)}^{\alpha-1}(Z_t - Z_0) + \alpha Z_0(\alpha-1)K_{\rho(0)}^{\alpha-2}(K_{\rho(t)} - K_{\rho(0)}).$$

By elementary algebra, we obtain

$$R_0 \hat{r}_t \approx \alpha K_{\rho(0)}^{\alpha-1}(Z_t - Z_0) + \alpha Z_0(\alpha-1)K_{\rho(0)}^{\alpha-2}(K_{\rho(t)} - K_{\rho(0)})$$

$$\approx \alpha K_{\rho(0)}^{\alpha-1} Z_0 \frac{(Z_t - Z_0)}{Z_0} + \alpha Z_0(\alpha-1)K_{\rho(0)}^{\alpha-1}\frac{(K_{\rho(t)} - K_{\rho(0)})}{K_{\rho(0)}}$$

$$\approx \alpha K_{\rho(0)}^{\alpha-1} Z_0(\hat{z}_t + (\alpha-1)\hat{k_{\rho(t)}})$$

$$\approx (R_0 - (1-\delta))(\hat{z}_t + (\alpha-1)\hat{k_{\rho(t)}}).$$

Dividing by R_0 on both sides, we have

$$\hat{r}_t \approx \frac{(R_0 - (1-\delta))}{R_0}(\hat{z}_t + (\alpha-1)\hat{k_{\rho(t)}}). \tag{5.8}$$

For the equation (5.2), we log-linearize the term $\dfrac{\mu_{\sigma(t)}}{\mu_t}\left(\dfrac{C_t}{C_{\sigma(t)}}\right)^{\eta} R_{\sigma(t)}$ first.

$$\frac{\mu_{\sigma(t)}}{\mu_t}\left(\frac{C_t}{C_{\sigma(t)}}\right)^{\eta} R_{\sigma(t)} \approx \frac{\mu_{\sigma(0)}}{\mu_0}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta} R_{\sigma(0)} + \frac{1}{\mu_0}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta} R_{\sigma(0)}(\mu_{\sigma(t)} - \mu_{\sigma(0)})$$
$$-\frac{\mu_{\sigma(0)}}{\mu_0^2}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta} R_{\sigma(0)}(\mu_t - \mu_0)$$
$$+\frac{\mu_{\sigma(0)}}{\mu_0}\eta\left(\frac{C_0^{\eta-1}}{C_{\sigma(0)}^{\eta}}\right) R_{\sigma(0)}(C_t - C_0)$$
$$+\frac{\mu_{\sigma(0)}}{\mu_0}(-\eta)\left(\frac{C_0^{\eta}}{C_{\sigma(0)}^{\eta+1}}\right) R_{\sigma(0)}(C_{\sigma(t)} - C_{\sigma(0)})$$
$$+\frac{\mu_{\sigma(0)}}{\mu_0}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta}(R_{\sigma(t)} - R_{\sigma(0)})$$
$$\approx \frac{\mu_{\sigma(0)}}{\mu_0}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta} R_{\sigma(0)}(1 + \eta\hat{c}_t - \eta\hat{c_{\sigma(t)}} + \hat{r_{\sigma(t)}} + \hat{\mu_{\sigma(t)}} - \hat{\mu}_t).$$

Since $E_t\left[\beta\frac{\mu_{\sigma(0)}}{\mu_0}\left(\frac{C_0}{C_{\sigma(0)}}\right)^{\eta} R_{\sigma(0)}\right] = 1$, we have

$$E_t[\eta\hat{c}_t - \eta\hat{c_{\sigma(t)}} + \hat{r_{\sigma(t)}} + \hat{\mu_{\sigma(t)}} - \hat{\mu}_t] = 0. \tag{5.9}$$

If we rewrite the equation (5.6) at $\sigma(t)$, then we have

$$logZ_{\sigma(t)} = (1-\psi)logZ_0 + \psi logZ_t + \varepsilon_{\sigma(t)}.$$

After log-linearizing on both sides, we have

$$log(Z_{\sigma(0)}) + \frac{1}{Z_{\sigma(0)}}(Z_{\sigma(t)} - Z_{\sigma(0)}) \approx (1-\psi)logZ_0 + \psi logZ_0 + \frac{\psi}{Z_0}(Z_t - Z_0) + \varepsilon_{\sigma(t)}$$
$$log(Z_{\sigma(0)}) + \hat{z_{\sigma(t)}} \approx (1-\psi)logZ_0 + \psi logZ_0 + \psi\hat{z}_t + \varepsilon_{\sigma(t)}.$$

Since $log(Z_{\sigma(0)}) = (1-\psi)logZ_0 + \psi logZ_0 + \varepsilon_{\sigma(0)}$, we have

$$\hat{z_{\sigma(t)}} \approx \psi\hat{z}_t + \varepsilon_{\sigma(t)} - \varepsilon_{\sigma(0)}.$$

Adding expectation on both sides, with $\varepsilon_{\sigma(0)}$=0, we have

$$E_t(\hat{z_{\sigma(t)}}) \approx \psi\hat{z}_t. \tag{5.10}$$

Step 4: Solve for the corresponding recursive equations by matrix operations, instead of the method of undetermined coefficients as recommended in the paper by Uhlig [**23**].

According to (5.7) and (5.10), we have

$$E_t[\hat{c_{\sigma(t)}}] = \frac{Z_0 K_{\rho(0)}^{\alpha}}{C_0} E_t[\hat{z_{\sigma(t)}}] + \frac{R_0 K_{\rho(0)}}{C_0}\hat{k}_t - \frac{K_0}{C_0} E_t[\hat{k_{\sigma(t)}}]$$

$$= \frac{Z_0 K_{\rho(0)}^{\alpha}}{C_0}\psi\hat{z}_t + \frac{R_0 K_{\rho(0)}}{C_0}\hat{k}_t - \frac{K_0}{C_0} E_t[\hat{k_{\sigma(t)}}].$$

Rewriting this equation, we obtain

$$\frac{K_0}{C_0} E_t[\hat{k_{\sigma(t)}}] + E_t[\hat{c_{\sigma(t)}}] = \frac{R_0 K_{\rho(0)}}{C_0}\hat{k}_t + \frac{Z_0 K_{\rho(0)}^{\alpha}}{C_0}\psi\hat{z}_t. \tag{5.11}$$

According to (5.8), (5.9) and (5.10), we have

$$E_t[\hat{r_{\sigma(t)}} - \frac{(R_0 - (1-\delta))}{R_0}\hat{z_{\sigma(t)}}] = \frac{(R_0 - (1-\delta))}{R_0}(\alpha - 1)\hat{k}_t$$

$$E_t[\hat{r_{\sigma(t)}}] = -E_t[\eta\hat{c}_t - \eta\hat{c_{\sigma(t)}} + \hat{\mu_{\sigma(t)}} - \hat{\mu}_t]$$

$$E_t(\hat{z_{\sigma(t)}}) = \psi\hat{z}_t.$$

By elementary algebra, we get

$$\eta E_t[\hat{c_{\sigma(t)}}] = \frac{(R_0 - (1-\delta))}{R_0}(\alpha - 1)\hat{k}_t + \eta\hat{c}_t + \frac{(R_0 - (1-\delta))}{R_0}\psi\hat{z}_t + E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t]. \tag{5.12}$$

We could rewrite (5.10), (5.11) and (5.12) in matrix form:

$$C E_t \begin{bmatrix} \hat{k_{\sigma(t)}} \\ \hat{c_{\sigma(t)}} \\ \hat{z_{\sigma(t)}} \end{bmatrix} = B \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} 0 \\ E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ 0 \end{bmatrix},$$

where

$$C = \begin{bmatrix} \frac{K_0}{C_0} & 1 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \dfrac{R_0 K_{\rho(0)}}{C_0} & 0 & \dfrac{Z_0 K_{\rho(0)}^{\alpha}}{C_0}\psi \\ \dfrac{(R_0 - (1-\delta))}{R_0}(\alpha - 1) & \eta & \dfrac{(R_0 - (1-\delta))}{R_0}\psi \\ 0 & 0 & \psi \end{bmatrix}.$$

Next, we will give some specific numbers from the paper by Uhlig [**23**] to replace letters in matrix B and C. Let $\beta = 0.99$, $\psi = 0.95$, $\alpha = 0.36$, $\eta = 1$, $\delta = 0.025$, $R_0 = 1.01$, $\dfrac{K_0}{C_0} = 13.73$, $\dfrac{Z_0 K_{\rho(0)}^{\alpha}}{C_0} = 1.335$ and $K_{\rho(0)} = K_0$. Then we have

$$\begin{bmatrix} \hat{k_t} \\ \hat{c_t} \\ \hat{z_t} \end{bmatrix} = A \begin{bmatrix} \hat{k_{\sigma(t)}} \\ \hat{c_{\sigma(t)}} \\ \hat{z_{\sigma(t)}} \end{bmatrix} + \begin{bmatrix} 0 \\ E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu_t}] \\ 0 \end{bmatrix}, \tag{5.13}$$

where

$$A = \begin{bmatrix} 0.9901 & 0.0721 & -0.0963 \\ 0.022 & 1.0016 & -0.0368 \\ 0 & 0 & 1.0526 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1.0116 & -0.0728 & 0.09 \\ -0.0222 & 1 & 0.0329 \\ 0 & 0 & 0.95 \end{bmatrix}.$$

Step 5: Use the techniques for solving rational expectations models on an isolated time domain and reach an optimal solution.

Let $Y_t = \begin{bmatrix} \hat{k_t} \\ \hat{c_t} \\ \hat{z_t} \end{bmatrix}$, $M_t = \begin{bmatrix} M1(t) \\ M2(t) \\ M3(t) \end{bmatrix}$, $F(t, Z_t) = \begin{bmatrix} 0 \\ E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu_t}] \\ 0 \end{bmatrix}$.

By Theorem (3.3), the solution of (5.13) is given as

$$Y_t = e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)M_t - e_{(I-A)A^{-1}\frac{1}{\mu}}(t,0)\int e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0)\frac{1}{\mu(t)}F(t,Z_t)\Delta t$$

where $M1(t), M2(t), M3(t)$ are arbitrary martingales.

According to the thesis by Ekiz [**12**], we have

$$e_{(A^{-1}-I)\frac{1}{\mu}}(t,0) = \prod\nolimits_{s\in[0,t)} A^{-1} = (A^{-1})^{n_t}$$

$$e_{\ominus(I-A)A^{-1}\frac{1}{\mu}}(t,0) = \prod_{s\in[0,t)} A = A^{n_t},$$

where $n_t(t,0) := \int_0^t \frac{\Delta(\tau)}{\mu(\tau)}$ is a counting function for any isolated time scale $\mathbb{T}$.

Thus, $Y_t = (A^{-1})^{n_t} M_t - (A^{-1})^{n_t} \sum_{t\in[0,T)_{\mathbb{T}}} A^{n_t} F(t, Z_t)$.

Next, we calculate $(A^{-1})^{n_t}$ and A^{n_t} by using the Putzer algorithm. For further reading about this theorem, we refer the readers to the book by Kelley and C. Peterson [**24**].

THEOREM 5.2.1. *(Putzer Algorithm) Let A be a $n \times n$-matrix. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the* eigenvalues *of A, then*

$$A^t = \sum_{i=0}^{n-1} c_{i+1}(t) P_i,$$

where $t \in \mathbb{Z}$, and $c_i(t), (i = 1, 2, ..., n)$ are chosen to satisfy the system

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \\ . \\ . \\ . \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 1 & \lambda_2 & 0 & ... & 0 \\ 0 & 1 & \lambda_3 & ... & 0 \\ . & & & . & \\ . & . & & & \\ . & . & & & \\ 0 & ... & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ . \\ . \\ . \\ c_n(t) \end{bmatrix}, \begin{bmatrix} c_1(0) \\ c_2(0) \\ . \\ . \\ . \\ c_n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

and the P_i are defined by

$$P_0 = I$$

$$P_i = (A - \lambda_i I) P_{i-1}, \qquad (1 \le i \le n).$$

In our case, A^{-1} is a 3×3-matrix, so

$$(A^{-1})^t = c_1(t) P_0 + c_2(t) P_1 + c_3(t) P_2.$$

Next, before calculating the $(A^{-1})^{n_t}$, we need to compute P_0, P_1, P_2, $c_1(t)$, $c_2(t)$, $c_3(t)$ respectively.

$$A^{-1} = \begin{bmatrix} 1.0116 & -0.0728 & 0.09 \\ -0.0222 & 1 & 0.0329 \\ 0 & 0 & 0.95 \end{bmatrix}.$$

.

The matrix A^{-1} has the eigenvalues $\lambda_1 = 0.95, \lambda_2 = 1.0462, \lambda_3 = 0.9654$.

$$P_0 = I$$

$$P_1 = \begin{bmatrix} 0.0616 & -0.0728 & 0.09 \\ -0.0222 & 0.05 & 0.0329 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -0.0005 & -0.0011 & -0.0055 \\ -0.0003 & -0.0007 & -0.0035 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \\ c_3(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}, \quad \begin{bmatrix} c_1(0) \\ c_2(0) \\ c_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solution of the initial value problem

$$c_1(t+1) = \lambda_1 c_1(t), \quad c_1(0) = 1$$

is given by

$$c_1(t) = \lambda_1^t.$$

The solution of the initial value problem

$$c_2(t+1) = c_1(t) + \lambda_2 c_2(t), \quad c_2(0) = 0$$

is given by

$$c_2(t) = \frac{\lambda_1^t - \lambda_2^t}{\lambda_1 - \lambda_2}.$$

The solution of the initial value problem

$$c_3(t+1) = c_2(t) + \lambda_3 c_3(t), \quad c_3(0) = 0$$

is given by

$$c_3(t) = \frac{\lambda_3^t - \lambda_1^t}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)} - \frac{\lambda_3^t - \lambda_2^t}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)},$$

$$c_1(t)P_0 = \begin{bmatrix} \lambda_1^t & 0 & 0 \\ 0 & \lambda_1^t & 0 \\ 0 & 0 & \lambda_1^t \end{bmatrix},$$

$$c_2(t)P_1 = \begin{bmatrix} -0.6403(\lambda_1^t - \lambda_2^t) & 0.7568(\lambda_1^t - \lambda_2^t) & -0.9356(\lambda_1^t - \lambda_2^t) \\ 0.2308(\lambda_1^t - \lambda_2^t) & -0.5198(\lambda_1^t - \lambda_2^t) & -0.342(\lambda_1^t - \lambda_2^t) \\ 0 & 0 & 0 \end{bmatrix},$$

$$c_3(t)P_2 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix},$$

where

$$p_{11} = 0.3333(\lambda_3^t - \lambda_1^t) + 0.641(\lambda_3^t - \lambda_2^t)$$

$$p_{12} = 0.7333(\lambda_3^t - \lambda_1^t) + 0.141(\lambda_3^t - \lambda_2^t)$$

$$p_{13} = 3.6667(\lambda_3^t - \lambda_1^t) + 0.7051(\lambda_3^t - \lambda_2^t)$$

$$p_{21} = 0.2(\lambda_3^t - \lambda_1^t) + 0.0385(\lambda_3^t - \lambda_2^t)$$

$$p_{22} = 0.4667(\lambda_3^t - \lambda_1^t) + 0.0897(\lambda_3^t - \lambda_2^t)$$

$$p_{23} = 2.3333(\lambda_3^t - \lambda_1^t) + 0.4487(\lambda_3^t - \lambda_2^t)$$

$$p_{31} = 0, \quad p_{32} = 0, \quad p_{33} = 0.$$

In our case, the counting number n_t is an integer. So n_t could replace t and we have

$$(A^{-1})^{n_t} = c_1(n_t)P_0 + c_2(n_t)P_1 + c_3(n_t)P_2.$$

Thus we obtain $(A^{-1})^{n_t}$ as

$$(A^{-1})^{n_t} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where

$$a_{11} = 0.0264\lambda_1^{n_t} - 0.0007\lambda_2^{n_t} + 0.9743\lambda_3^{n_t}$$

$$a_{12} = 0.0235\lambda_1^{n_t} - 0.8978\lambda_2^{n_t} + 0.8743\lambda_3^{n_t}$$

$$a_{13} = -4.6023\lambda_1^{n_t} + 0.2305\lambda_2^{n_t} + 4.3718\lambda_3^{n_t}$$

$$a_{21} = 0.0308\lambda_1^{n_t} - 0.2693\lambda_2^{n_t} + 0.2385\lambda_3^{n_t}$$

$$a_{22} = 0.0135\lambda_1^{n_t} + 0.4301\lambda_2^{n_t} + 0.5564\lambda_3^{n_t}$$

$$a_{23} = -2.6753\lambda_1^{n_t} - 0.1067\lambda_2^{n_t} + 2.782\lambda_3^{n_t}$$

$$a_{31} = 0, \quad a_{32} = 0, \quad a_{33} = \lambda_1^{n_t}.$$

Next, we compute A^{n_t} with a similar process.

$$A^t = c_1'(t)P_0' + c_2'(t)P_1' + c_3'(t)P_2'.$$

$$A = \begin{bmatrix} 0.9901 & 0.0721 & -0.0963 \\ 0.022 & 1.0016 & -0.0368 \\ 0 & 0 & 1.0526 \end{bmatrix}.$$

.

The matrix A has the eigenvalues $\lambda_a = 1.0526, \lambda_b = 1.0361, \lambda_c = 0.9557$.

$$P_0' = I$$

$$P_1' = \begin{bmatrix} -0.0625 & 0.0721 & -0.0965 \\ 0.022 & -0.051 & -0.0368 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_2' = \begin{bmatrix} 0.0045 & -0.007 & 0.0018 \\ -0.0021 & 0.0033 & -0.0009 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c_1'(t)P_0' = \begin{bmatrix} \lambda_a^t & 0 & 0 \\ 0 & \lambda_a^t & 0 \\ 0 & 0 & \lambda_a^t \end{bmatrix}$$

$$c_2'(t)P_1' = \begin{bmatrix} -3.7879(\lambda_a^t - \lambda_b^t) & 4.3697(\lambda_a^t - \lambda_b^t) & -5.8485(\lambda_a^t - \lambda_b^t) \\ 1.3333(\lambda_a^t - \lambda_b^t) & -3.0909(\lambda_a^t - \lambda_b^t) & -2.2303(\lambda_a^t - \lambda_b^t) \\ 0 & 0 & 0 \end{bmatrix}$$

$$c_3'(t)P_2' = \begin{bmatrix} p_{11}' & p_{12}' & p_{13}' \\ p_{21}' & p_{22}' & p_{23}' \\ p_{31}' & p_{32}' & p_{33}' \end{bmatrix}$$

where

$$p_{11}' = -2.8125(\lambda_c^t - \lambda_a^t) + 3.4615(\lambda_c^t - \lambda_b^t)$$

$$p_{12}' = 4.375(\lambda_c^t - \lambda_a^t) - 5.3846(\lambda_c^t - \lambda_b^t)$$

$$p_{13}' = -1.125(\lambda_c^t - \lambda_a^t) + 1.3846(\lambda_c^t - \lambda_b^t)$$

$$p_{21}' = 1.3125(\lambda_c^t - \lambda_a^t) - 1.6154(\lambda_c^t - \lambda_b^t)$$

$$p'_{22} = -2.0625(\lambda_c^t - \lambda_a^t) + 2.5385(\lambda_c^t - \lambda_b^t)$$

$$p'_{23} = 0.5625(\lambda_c^t - \lambda_a^t) - 0.6923(\lambda_c^t - \lambda_b^t)$$

$$p'_{31} = 0, \quad p'_{32} = 0, \quad p'_{33} = 0.$$

In our case,

$$A^{n_t} = c'_1(n_t)P'_0 + c'_2(n_t)P'_1 + c'_3(n_t)P'_2.$$

Thus we obtain A^{n_t} as

$$A^{n_t} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where

$$b_{11} = 0.0246\lambda_a^{n_t} + 0.3264\lambda_b^{n_t} + 0.649\lambda_c^{n_t}$$

$$b_{12} = -0.0053\lambda_a^{n_t} + 1.0149\lambda_b^{n_t} - 1.0096\lambda_c^{n_t}$$

$$b_{13} = -4.7235\lambda_a^{n_t} + 4.4639\lambda_b^{n_t} + 0.2596\lambda_c^{n_t}$$

$$b_{21} = 0.0208\lambda_a^{n_t} + 0.2821\lambda_b^{n_t} - 0.3029\lambda_c^{n_t}$$

$$b_{22} = -0.0284\lambda_a^{n_t} + 0.5524\lambda_b^{n_t} + 0.476\lambda_c^{n_t}$$

$$b_{23} = -2.7928\lambda_a^{n_t} + 2.9226\lambda_b^{n_t} - 0.1298\lambda_c^{n_t}$$

$$b_{31} = 0, \quad b_{32} = 0, \quad b_{33} = \lambda_a^{n_t}.$$

$$\sum_{t \in [0,T)_{\mathbb{T}}} A^{n_t} F(t, Z_t) = \sum_{t \in [0,T)_{\mathbb{T}}} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 0 \\ E_t[\mu_{\hat{\sigma(t)}} - \hat{\mu}_t] \\ 0 \end{bmatrix}$$

$$= \sum_{t \in [0,T)_{\mathbb{T}}} \begin{bmatrix} b_{12} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ b_{22} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{t \in [0,T)_{\mathbb{T}}} b_{12} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ \sum_{t \in [0,T)_{\mathbb{T}}} b_{22} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ 0 \end{bmatrix}.$$

So we have

$$Y_t = (A^{-1})^{n_t} M_t - (A^{-1})^{n_t} \sum_{t \in [0,T)_{\mathbb{T}}} A^{n_t} F(t, Z_t)$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} M1(t) \\ M2(t) \\ M3(t) \end{bmatrix}$$

$$- \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sum_{t \in [0,T)_{\mathbb{T}}} b_{12} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ \sum_{t \in [0,T)_{\mathbb{T}}} b_{22} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\ 0 \end{bmatrix}.$$

Therefore, the solution of the equation (5.13) is given by

$$\begin{aligned}
\hat{k}_t &= a_{11} M_1(t) + a_{12} M_2(t) + a_{13} M_3(t) - a_{11} \sum_{t \in [0,T)_{\mathbb{T}}} b_{12} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\
&\quad - a_{12} \sum_{t \in [0,T)_{\mathbb{T}}} b_{22} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\
\hat{c}_t &= a_{21} M_1(t) + a_{22} M_2(t) + a_{23} M_3(t) - a_{21} \sum_{t \in [0,T)_{\mathbb{T}}} b_{12} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\
&\quad - a_{22} \sum_{t \in [0,T)_{\mathbb{T}}} b_{22} E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\
\hat{z}_t &= a_{33} M_3(t).
\end{aligned}$$

Finally, the optimal sequence $\{C_t, K_t\}_{t=0}^{\infty}$ that maximizes the expected utility in the sequence problem (5.1) is

$$
\begin{aligned}
K_t &= K_0(1+\hat{k}_t) \\
&= K_0(1 + a_{11}M_1(t) + a_{12}M_2(t) + a_{13}M_3(t) - a_{11}\sum_{t\in[0,T)_{\mathbb{T}}} b_{12}E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t] \\
&\quad -a_{12}\sum_{t\in[0,T)_{\mathbb{T}}} b_{22}E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t]),
\end{aligned}
$$

$$
\begin{aligned}
C_t &= C_0(1+\hat{c}_t) \\
&= C_0(1 + a_{21}M_1(t) + a_{22}M_2(t) + a_{23}M_3(t) - a_{21}\sum_{t\in[0,T)_{\mathbb{T}}} b_{12}E_t[\hat{\mu_{\sigma(t)}} - \hat{\mu}_t]).
\end{aligned}
$$

As a conclusion, if we choose such a sequence $\{C_t, K_t\}_{t=0}^{\infty}$, the sequence problem (5.1) would attain its maximization. In addition, when $\mathbb{T} = \mathbb{Z}$, $\hat{\mu_{\sigma(t)}} = \hat{\mu}_t$, we have

$$
\begin{aligned}
K_t &= K_0(1 + a_{11}M_1(t) + a_{12}M_2(t) + a_{13}M_3(t)), \\
C_t &= C_0(1 + a_{21}M_1(t) + a_{22}M_2(t) + a_{23}M_3(t)),
\end{aligned}
$$

which means that we could choose such a sequence $\{C_t, K_t\}_{t=0}^{\infty}$ to maximize the sequence problem (5.1) in this special case.

Chapter 6
CONCLUSION AND FUTURE WORK

Stochastic dynamic programming is one of the methods included in the theory of stochastic dynamic optimization. Not only does it reflect the behavior of decision making under uncertainty, but it also offers more accurate results in reality than the deterministic case does. Many economists and mathematicians solve a lot of sequential decision making problems in stochastic cases with its help. In addition, time scale calculus theory is suitable to be applied to many discrete or continuous stochastic dynamic models. Research on stochastic dynamic models on time scales is important and meaningful. There are three findings in our study: At first, we formulate Euler equations and transversality conditions of one general form of stochastic dynamic sequence problems on isolated time domains and prove a theorem about the sufficiency of Euler equation and transversality condition. Moreover, instead of the method of undetermined coefficients introduced in the paper by Uhlig [**23**], we solve the corresponding recursive equation just by matrix operations. Furthermore, we explicitly analyze and solve one case of a stochastic growth model and find the optimal sequence maximizing this sequence problem (5.1).

For future work, we will figure out that if the constraint condition $C_t = f(K_{\rho(t)}, K_t, Z_t)$ in Section 4.1 is substituted with $C_t = f(K_t, K_{\sigma(t)}, Z_t)$, what the corresponding Euler equation and the transversality condition of stochastic dynamic models are. Also, we would like to apply the same technique represented in this thesis to solve stochastic dynamic models with some other objective functions, such as $F(C_t) = ln(C_t)$.

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