Green's Functions of Discrete Fractional Calculus Boundary Value Problems and an Application of Discrete Fractional Calculus to a Pharmacokinetic Model

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GREEN’S FUNCTIONS OF DISCRETE FRACTIONAL CALCULUS
BOUNDARY VALUE PROBLEMS AND AN APPLICATION OF DISCRETE
FRACTIONAL CALCULUS TO A PHARMACOKINETIC MODEL

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GREEN'S FUNCTIONS OF DISCRETE FRACTIONAL CALCULUS
BOUNDARY VALUE PROBLEMS AND AN APPLICATION OF DISCRETE
FRACTIONAL CALCULUS TO A PHARMACOKINETIC MODEL

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Fractional calculus has been used as a research tool in the fields of pharmacology, biology, chemistry, and other areas [3]. The main purpose of this thesis is to calculate Green’s functions of fractional difference equations, and to model problems in pharmacokinetics. We claim that the discrete fractional calculus yields the best prediction performance compared to the continuous fractional calculus in the application of a one-compartmental model of drug concentration.

In Chapter 1, the Gamma function and its properties are discussed to establish a theoretical basis. Additionally, the basics of discrete fractional calculus are discussed using particular examples for further calculations.

In Chapter 2, we use these basic results in the analysis of a linear fractional difference equation. Existence of solutions to this difference equation is then established for both initial conditions (IVP) and two-point boundary conditions (BVP).

In Chapter 3, Green’s functions are introduced and discussed, along with examples. Instead of using Cauchy functions, the technique of finding Green’s functions by a traditional method is demonstrated and used throughout this chapter. The solutions of the BVP play an important role in analysis and construction of the
Green’s functions. Then, Green’s functions for the discrete calculus case are calculated using particular problems, such as boundary value problems, discrete boundary value problems (DBVP) and fractional boundary value problems (FBVP). Finally, we demonstrate how the Green’s functions of the FBVP generalize the existence results of the Green’s functions of DVBP.

In Chapter 4, different compartmental pharmacokinetic models are discussed. This thesis limits discussion to the one-compartmental model. The Mathematica FindFit command and the statistical computational techniques of mean square error (MSE) and cross-validation are discussed. Each of the four models (continuous, continuous fractional, discrete and discrete fractional) is used to compute the MSE numerically with the given data of drug concentration. Then, the best fit and the best model are obtained by inspection of the resulting MSE.

In the last Chapter, the results are summarized, conclusions are drawn, and directions for future work are stated.
Chapter 1

INTRODUCTION

Fractional calculus deals with derivatives and integrals of non-integer order, such as one-half, \( \pi \), or \( e \) order, which are not considered in ordinary calculus. Fractional calculus was first introduced more than 300 years ago. In 1695, L’Hôpital and Leibniz were exchanging ideas about the notation for derivatives with non-integer order.

“Can the derivatives of integer order \( \frac{d^n y}{dx^n} \) be extended to any order: fractional, irrational or complex?” asked Leibniz.

“What would be the result of half-differentiating?” asked L’Hôpital.

Leibniz replied, “It leads to a paradox, from which one day useful consequences will be drawn.”

In the nineteenth century, several notable mathematicians, including Riemann, Liouville, Grünwald and Letnikov worked on the development of fractional calculus theory. In recent years, fractional calculus has been used as a tool to study and model problems in applied mathematics, such as pharmacology, biology, chemistry, and economics [7, 8].

There are three versions of fractional calculus: continuous, discrete, and quantum. In this thesis, we focus on the discrete version.
1.1. The Gamma Function (Γ)

The Gamma function is one of the fundamental functions in mathematics [11]. In this section, we introduce the Gamma function which is an extension of the factorial function to any real number and some properties of the Gamma function, which are crucial in studying the theory of fractional calculus.

**Definition 1.1.1.** Let $\alpha \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$. The Gamma function is defined by

$$
\Gamma(\alpha) = \int_{0}^{\infty} e^{-t}t^{\alpha-1}dt.
$$

If $n \in \mathbb{Z}$, then we have $\Gamma(n) = (n - 1)!$.

**Lemma 1.1.2.** Let $\alpha > 0$. Then

$$
\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).
$$

**Proof.** By using the Definition 1.1.1 of the Gamma function and applying the integration by parts formula, we have

$$
\begin{align*}
\Gamma(\alpha + 1) &= \int_{0}^{\infty} e^{-t}t^{\alpha}dt \\
&= \lim_{b \to \infty} \int_{0}^{b} e^{-t}t^{\alpha}dt \\
&= \lim_{b \to \infty} \left[ -e^{-t}t^{\alpha} \big|_{0}^{b} + \alpha \int_{0}^{b} e^{-t}t^{\alpha-1}dt \right] \\
&= 0 + \alpha \Gamma(\alpha).
\end{align*}
$$

Therefore $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$. □
1.2. Basics of Discrete Fractional Calculus

The theory of discrete calculus is parallel to the theory of ordinary calculus. There are similarities in concepts but differences in methods of calculation. For example, the differential operator (′) is similar to the difference operator (Δ), and the integral operator (∫) is similar to the summation operator (∑). In this section, we introduce basic discrete properties with some examples to prepare the reader for subsequent topics. Some properties are proved, and others only stated.

1.2.1. The Difference Operator (Δ).

Definition 1.2.1. Let \( \mathbb{N}_a = \{a, a+1, a+2, \ldots\} \) and \( y : \mathbb{N}_a \rightarrow \mathbb{R} \). The “difference operator” \( \Delta \) is defined by

\[
\Delta y(t) = y(t + 1) - y(t).
\]

Generally, we define domain on the natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

Higher order differences are defined by composing the difference operator with itself [11]. The second order difference is

\[
\Delta^2 y(t) = \Delta(\Delta y(t))
\]

\[
= \Delta(y(t + 1) - y(t))
\]

\[
= (y(t + 2) - y(t + 1)) + (y(t + 1) - y(t))
\]

\[
= y(t + 2) - 2y(t + 1) - y(t).
\]
The $n^{th}$ order difference is obtained by mathematical induction.

\[ \Delta^n y(t) = y(t + n) - ny(t + n - 1) + \frac{n(n-1)}{2!} y(t + n - 2) + \ldots + (-1)^n y(t) \]

\[ = \sum_{k=0}^{n} (-1)^k \binom{n}{k} y(t + n - k). \]

1.2.2. Falling Factorial Powers.

**Definition 1.2.2.** Let $t > 1$. The falling factorial power $t^{(\nu)}$ [4], read as “$t$ to the $\nu$ falling,” is defined as follows.

(i) If $\nu \in \mathbb{R}$, $t^{(\nu)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \nu)}$.

(ii) If $\nu \in \mathbb{Z}^+$, $t^{(\nu)} = t(t-1)(t-2)\ldots(t-\nu+1)$.

(iii) If $\nu \in \mathbb{Z}^-$, $t^{(\nu)} = \frac{1}{t(t-1)(t-2)\ldots(t-\nu+1)}$.

Note that

\[ t^{(\nu)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \nu)} = \frac{t\Gamma(t)}{\Gamma(t - \nu + 1)} = \frac{t(t-1)\Gamma(t-1)}{\Gamma(t - \nu + 1)} \]

\[ \vdots \]

\[ = \frac{t(t-1)\ldots(t-\nu+1)\Gamma(t-\nu+1)}{\Gamma(t - \nu + 1)} \]

\[ t^{(\nu)} = t(t-1)(t-2)\ldots(t-\nu+1). \]
1.2.3. Forward Difference Operator.

Theorem 1.2.3. (The Difference of a Power [4]) Let $\alpha$ be a constant. Then

$$\Delta t^\alpha = (t + 1)^\alpha - t^\alpha.$$ 

Theorem 1.2.4. ([4]) Assume that the following factorial functions are well defined. Then

(i) $\Delta t^{(\nu)} = \nu t^{(\nu-1)}$, where $\Delta$ is the forward difference operator.
(ii) $(t - \mu)t^{(\mu)} = t^{(\mu + 1)}$, where $\mu \in \mathbb{R}$.
(iii) $\mu^{(\mu)} = \Gamma(\mu + 1)$, where $\mu \in \mathbb{R}$.
(iv) If $t \leq r$, then $t^{(\nu)} \leq r^{(\nu)}$ for any $\nu > r$.
(v) If $0 < \nu < 1$, then $t^{(\alpha \nu)} \leq (t^{(\alpha)})^{(\nu)}$.
(vi) $t^{(\alpha + \beta)} = (t - \beta)^{(\alpha)}t^{(\beta)}$.

Proof. Let $\nu$, $\alpha$ and $\beta$ be positive integers.

(i) By applying Definition 1.2.1, we have

$$\Delta t^{(\nu)} = (t + 1)^{(\nu)} - t^{(\nu)}$$

$$= (t + 1)(t)(t - 1)\ldots(t - \nu + 2) - (t)(t - 1)(t - 2)\ldots(t - \nu + 2)(t - \nu + 1)$$

$$= (t)(t - 1)\ldots(t - \nu + 2)[(t + 1) - (t - \nu + 1)]$$

$$= \nu t^{(\nu - 1)}.$$
(ii) By using Definition 1.2.2, we have

\[
(t - \mu) t^{(\mu)} = (t - \mu) \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \mu)}
\]

\[
= (t - \mu) \frac{\Gamma(t + 1)}{\Gamma(t - \mu + 1)}
\]

\[
= (t - \mu) \frac{\Gamma(t + 1)}{(t - \mu) \Gamma(t - \mu)}
\]

\[
= \frac{\Gamma(t + 1)}{\Gamma(t + 1 - (\mu + 1))}
\]

\[
= t^{(\mu+1)}.
\]

(iii) By using Definition 1.2.2, we have

\[
\mu^{(\mu)} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \mu)}
\]

\[
= \Gamma(\mu + 1) \quad \text{since} \quad \Gamma(1) = 0! = 1.
\]

(iv) Let \( t \leq r \) for any \( \nu > r \).

We will employ Euler’s infinite product as follows.

\[
\Gamma(u) = \frac{1}{u} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^u}{1 + \left(\frac{u}{n}\right)}.
\]

Then we have

\[
t^{(\nu)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \nu)}
\]

\[
= \frac{1}{t + 1} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^{t+1}}{1 + \left(\frac{t + 1}{n}\right)}
\]

\[
= \frac{1}{t + 1 - \nu} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^{t+1-\nu}}{1 + \left(\frac{t + 1 - \nu}{n}\right)}
\]

6
\[
\frac{t + 1 - \nu}{t + 1} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^\nu \frac{1 + \frac{t + 1 - \nu}{n}}{(1 + \frac{t + 1}{n})} \\
= (1 - \frac{\nu}{t + 1}) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^\nu (1 - \frac{\nu}{n + t + 1}) \\
\leq (1 - \frac{\nu}{r + 1}) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^\nu (1 - \frac{\nu}{n + t + 1}) \\
\leq r^{(\nu)}
\]

since \( t \leq r \) implies \( -\nu \leq \frac{\nu}{r + 1} \).

(v) We will employ the log-convexity property to obtain the result.

Let \( f \) be a real-valued function defined on an interval \( \mathbb{I} \subset \mathbb{R} \). Then the log-convexity property is defined by

\[
f[ux + (1 - u)y] \leq uf(x) + (1 - u)f(y) \\
\leq [f(x)]^u[f(y)]^{1-u}
\]

for all \( x, y \in \mathbb{I} \) and all \( u \in [0, 1] \).

Since \( t + 1 - \alpha \nu = t + 1 - \alpha \nu + \nu t - \nu t - \nu + \nu \)

\[
= \nu t + \nu - \alpha \nu + t + 1 - \nu t - \nu \\
= \nu (t + 1 - \alpha) + (1 - \nu)(t + 1), \text{ we have}
\]

\[
f^{(\alpha \nu)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \alpha \nu)} \\
= \frac{\Gamma(t + 1)}{\Gamma[\nu(t + 1 - \alpha) + (1 - \nu)(t + 1)]}.
\]

By applying the log-convexity property of the gamma function, we obtain

\[
\Gamma[\nu x + (1 - \nu)y] \leq [\Gamma(x)]^\nu[\Gamma(y)]^{1-\nu} \quad \text{where} \quad 0 < \nu < 1.
\]
Letting \( x = t + 1 - \alpha \) and \( y = t + 1 \), we have

\[
\Gamma[\nu(t + 1 - \alpha) + (1 - \nu)(t + 1)] \leq \left[\Gamma(t + 1 - \alpha)\right]^{\nu} \left[\Gamma(t + 1)\right]^{1-\nu}
\]

\[
\frac{1}{\Gamma[\nu(t + 1 - \alpha) + (1 - \nu)(t + 1)]} \geq \frac{1}{\Gamma(t + 1)} \geq \left[\Gamma(t + 1 - \alpha)\right]^{\nu} \left[\Gamma(t + 1)\right]^{1-\nu}
\]

Hence \( t^{(\alpha\nu)} \geq (t^{(\alpha)})^{(\nu)}. \)

(vi) By using Definition 1.2.2, we have

\[
t^{(\alpha+\beta)} = \frac{\Gamma(t + 1)}{\Gamma(t - (\alpha + \beta) + 1)}
\]

\[
= \frac{\Gamma(t - \beta + 1)}{\Gamma(t - (\alpha + \beta) + 1)} \cdot \frac{\Gamma(t + 1)}{\Gamma(t - \beta + 1)}
\]

\[
= (t - \beta)^{(\alpha)} t^{(\beta)}.
\]

\[\square\]

**Theorem 1.2.5.** ([11]) Let \( a \) be a constant. Then

\[
\Delta a^t = a^t(a - 1).
\]

1.2.4. **Summation** (\( \Sigma \)). The summation operator in discrete calculus is similar to the integral sign in ordinary calculus [11].
Definition 1.2.6. For a real-valued function \( f(t) \), the indefinite sum (or antiderivative) is written as \( \sum f(t) \). It is any function so that

\[
\Delta(\sum f(t)) = f(t).
\]

Corollary 1.2.7. Let \( F(t) \) be defined on \( \{a, a+1, a+2, \ldots\} \) where \( a \in \mathbb{R} \), \( f(t) \) be an indefinite sum of \( F(t) \), and \( C \) be any constant. Then

\[
\sum F(t) = f(t) + C \text{ where } \Delta C = 0.
\]

Theorem 1.2.8. Let \( \alpha \) and \( C \) be constants. Then

\[
(i) \quad \sum \alpha^t = \frac{\alpha^t}{\alpha - 1} + C, \text{ if } \alpha \neq 1.
\]

\[
(ii) \quad \sum t^{(\alpha)} = \frac{t^{(\alpha+1)}}{\alpha + 1} + C, \text{ if } \alpha \neq -1.
\]

Theorem 1.2.9. (The First Fundamental Theorem of Calculus \([11]\)) Let \( f(t) \) be definite sum of \( F(t) \) over the interval \( [a,b] \) and \( C \) be any constant. Then

\[
\sum_{t=a}^{b} f(t) = F(t)\Big|_{a}^{b+1} = F(b + 1) - F(a) + C \text{ where } \Delta C = 0.
\]

Proof. We need to show that \( \sum_{t=a}^{b} f(t) = F(b + 1) - F(a) \).

By applying \( \Delta \) to \( \sum_{t=a}^{b} f(t) \), we have

\[
\Delta \sum_{t=a}^{b} f(t) = \sum_{t=a}^{b+1} f(t) - \sum_{t=a}^{b} f(t)
\]

\[
= f(b + 1).
\]

Also, applying \( \Delta \) to \( F(b + 1) - F(a) \), we have
$$\Delta(F(b + 1) - F(a)) = \Delta F(b + 1) - \Delta F(a)$$

$$= \Delta F(b + 1)$$

$$= f(b + 1).$$

Then $\Delta \sum_{t=a}^{b} f(t) = \Delta F(b + 1) - F(a)$.

Therefore $\sum_{t=a}^{b} f(t) = F(b + 1) - F(a) + C$ where $\Delta C = 0$. \hfill \square$

Some fundamental properties of $\Delta$, the product rule and the quotient rule are given and proved below.

**Theorem 1.2.10.** *(The Product Rule [11]*)

$$\Delta(f(t)g(t)) = \Delta f(t)g(t) + f(t + 1)\Delta g(t).$$

**Proof.** By using Definition 1.2.1, we have

$$\Delta(f(t)g(t)) = f(t + 1)g(t + 1) - f(t)g(t)$$

$$= f(t + 1)g(t + 1) + f(t + 1)g(t) - f(t + 1)g(t) - f(t)g(t)$$

$$= f(t + 1)[g(t + 1) - g(t)] + g(t)[f(t + 1) - f(t)]$$

$$= \Delta f(t)g(t) + f(t + 1)\Delta g(t).$$ \hfill \square$

**Theorem 1.2.11.** *(The Quotient Rule [11]*)

$$\Delta\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\Delta f(t) - f(t)\Delta g(t)}{g(t)g(t + 1)}.$$

**Proof.** By using Definition 1.2.1, we have

$$\Delta\frac{f(t)}{g(t)} = \frac{f(t + 1)}{g(t + 1)} - \frac{f(t)}{g(t)}$$

10
\[ f(t) = \frac{f(t+1)g(t) - f(t)g(t+1)}{g(t)g(t+1)} \]
\[ = \frac{f(t+1)g(t) + f(t)g(t) - f(t)g(t) - f(t)g(t+1)}{g(t)g(t+1)} \]
\[ = \frac{g(t)f(t) - f(t)g(t)}{g(t)g(t+1)}. \]

\[ g(t)g(t+1) = f(t+1)g(t) + f(t)g(t) - f(t)g(t) - f(t)g(t+1). \]

\[ \square \]

1.2.5. The \( \nu \)th Fractional Sum. Let \( \nu > 0 \) and let \( \sigma(s) = s + 1 \). Define \( \Delta^j = \Delta(\Delta^{j-1}) \), where \( j \) is a nonnegative integer. We consider the following initial value problem (IVP) [4].

\[ \Delta^\nu u(t) = f(t), \text{ where } t = a + 1, a + 2, \ldots, \]
\[ u(a + j - 1) = 0, \text{ where } j = 1, 2, \ldots, n. \]

The solution of the IVP is

\[ \Delta^{-\nu} f(t) = \sum_{s=a}^{t-1} \frac{(t - \sigma(s))^{(\nu-1)}}{(\nu - 1)!} f(s). \] (1.1)

Note that the Cauchy function \( \frac{(t - \sigma(s))^{(\nu-1)}}{(\nu - 1)!} \) vanishes at \( s = t - (\nu - 1), \ldots, t - 1 \).

Equation (1.1) is equivalent to

\[ \Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s-(\nu-1))} f(s). \]

In fact,

\[ \Delta^{-\nu} f(t) = \sum_{s=a}^{t-\nu} \frac{(t - \sigma(s))^{(\nu-1)}}{(\nu - 1)!} f(s) \]
\[ = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s) \]
\[ = \frac{1}{(\nu - 1)!} \sum_{s=a}^{t-\nu} (t - (s + 1))^{(\nu-1)} f(s) \]
\[
\begin{align*}
\sum_{s=a}^{t-\nu} f(s) &= \frac{1}{(\nu - 1)!} \sum_{s=a}^{t-\nu} \frac{\Gamma(t - s)}{\Gamma(t - s - (\nu - 1))} f(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \frac{\Gamma(t - s)}{\Gamma(t - s - (\nu - 1))} f(s).
\end{align*}
\]

We call \( \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s) \) the \( \nu \)th fractional sum of \( f \).

Note that \( f \) is defined for \( s = a, \mod(1) \) and \( \Delta^{-\nu} f \) is defined for \( t = a + \nu, \mod(1) \) ([4]).

1.2.6. Power Rule. By mathematical induction, the following lemma first was obtained by Miller and Ross for only positive integer exponents only. For the proof of the general case where \( \mu \in \mathbb{R}\backslash\{-1, -2, \ldots\} \), we refer the reader to the paper ([4]).

**Lemma 1.2.12.** Let \( \mu \in \mathbb{R}\backslash\{-1, -2, \ldots\} \). Then

\[
\Delta^{-\nu} t(\mu) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} t(\mu + \nu).
\]

1.2.7. Fractional Difference.

**Definition 1.2.13.** Let \( \mu > 0 \) and assume that \( m - 1 < \mu < m \), where \( m \) denotes a positive integer. Set \( \nu = m - \mu \). Then ([4])

\[
\Delta^\mu u(t) = \Delta^{m-\nu} u(t) = \Delta^m (\Delta^{-\nu} u(t)).
\]

**Example 1.2.14.** Let \( \mu = \frac{1}{3} \).

Then considering that \( \frac{1}{3} = 1 - \frac{2}{3} \); we have

\[
\Delta^{\frac{1}{3}} u(t) = \Delta^{1-\frac{2}{3}} u(t) = \Delta^1 (\Delta^{-\frac{2}{3}} u(t)).
\]

**Example 1.2.15.** Find \( \Delta^{\frac{3}{5}} 5 \).
\[
\Delta^{\frac{1}{5}} = \Delta \Delta^{\frac{2}{5}} = 5\Delta(\Delta^{\frac{2}{5}} t^{(0)}) = 5\Delta\left(\frac{\Gamma(1)}{\Gamma\left(\frac{2}{5}\right)} t^{(\frac{1}{5})}\right) = 5.54\Delta t^{(\frac{2}{5})} = 5.54t^{(\frac{2}{5})},
\]

since \(\Gamma\left(\frac{5}{3}\right) = 0.9027\). Note that difference of a constant is not zero as in continuous calculus.

### 1.2.8. Some properties of the fractional sum and difference operator.

The following theorems are commutative type properties of the fractional sum and the difference operator. We will use these to obtain existence results for fractional difference equations in Chapter 2. We refer readers to the paper [4] for the proofs not presented here.

**Theorem 1.2.16.** Let \(f\) be a real-valued function, and let \(\mu, \nu > 0\). Then for all \(t\) such that \(t = \mu + \nu, \text{ mod}(1)\),

\[
\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu + \nu)} f(t) = \Delta^{-\mu}[\Delta^{-\nu} f(t)].
\]

**Proof.** By applying the \(\nu\)th fractional sum of \(f\), we have

\[
\Delta^{-\mu}[\Delta^{-\nu} f(t)] = \Delta^{-\mu}\left[\frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-\nu} \frac{(t - \sigma(s))^{(\nu-1)}}{(\nu-1)!} f(r)\right]
\]

\[
= \frac{1}{\Gamma(\nu)} \Delta^{-\mu} \sum_{r=0}^{t-\nu} \frac{(t - \sigma(s))^{(\nu-1)}}{(\nu-1)!} f(r).
\]

Let \(g(t) = \sum_{r=0}^{t-\nu} \frac{(t - \sigma(s))^{(\nu-1)}}{(\nu-1)!} f(r)\). Then we have

\[
\Delta^{-\mu}[\Delta^{-\nu} f(t)] = \frac{1}{\Gamma(\nu)} \Delta^{-\mu} g(t)
\]

\[
= \frac{1}{\Gamma(\nu)} \frac{1}{\Gamma(\mu)} \sum_{s=\nu}^{t-\mu}(t - \sigma(s))^{(\mu-1)} \sum_{r=0}^{s-\nu}(s - \sigma(r))^{(\nu-1)} f(r)
\]

\[
= \frac{1}{\Gamma(\nu)} \frac{1}{\Gamma(\mu)} \sum_{s=\nu}^{t-\mu} \sum_{r=0}^{s-\nu}(t - \sigma(s))^{(\mu-1)}(s - \sigma(r))^{(\nu-1)} f(r).
\]
Since $x = s - \sigma(r)$, we have $x + 1 = s + 1 - \sigma(r)$.

Then $\sigma(x) = \sigma(s) - \sigma(r)$.

We switch the order of summation to obtain

$$\Delta^{-\mu}[\Delta^{-\nu}f(t)] = \frac{1}{\Gamma(\nu)} \frac{1}{\Gamma(\mu)} \sum_{r=0}^{t-\mu} \sum_{s=r+\nu}^{t-\mu} (t - \sigma(s))(\mu^{-1})(s - \sigma(r))(\nu^{-1})f(r)$$

$$= \frac{1}{\Gamma(\nu)} \frac{1}{\Gamma(\mu)} \sum_{r=0}^{t-\mu} \sum_{x=\nu}^{t-\mu} (t - \sigma(r) - \sigma(x))(\mu^{-1})(x)(\nu^{-1})f(r)$$

$$= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-\mu} \left[ \frac{1}{\Gamma(\mu)} \sum_{x=\nu}^{t-\mu} (t - \sigma(r) - \sigma(x))(\mu^{-1})(x)(\nu^{-1})f(r) \right]$$

$$= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-\mu} [\Delta^{-\mu}(t - \sigma(r))\nu^{-1}]f(r).$$

By applying $\Delta^{-\nu}t^{\mu} = \mu^{-\nu}t^{\mu+\nu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \nu)}t^{\mu+\nu}$ then we have

$$\Delta^{-\mu}[\Delta^{-\nu}f(t)] = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-\mu} [(\nu - 1)^{-\mu}(t - \sigma(r))\nu^{-1}]f(r)$$

$$= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-\mu} \left[ \frac{\Gamma(\nu)}{\Gamma(\nu + \mu)}(t - \sigma(r))\nu(\nu-1)\nu^{-1} \right]f(r)$$

$$= \frac{1}{\Gamma(\nu + \mu)} \sum_{r=0}^{t-\mu} (t - \sigma(r))\nu^{\mu+\nu-1}f(r)$$

$$= \Delta^{-(\mu+\nu)}f(t).$$

\[\square\]

**Theorem 1.2.17.** Let $\nu > 0$ and suppose $f$ is defined on $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$.

Then

$$\Delta^{-\nu}\Delta f(t) = \Delta\Delta^{-\nu}f(t) - \frac{(t - a)^{(\nu-1)}}{\Gamma(\nu)}f(a).$$

Note that $\Delta^{-\nu} : \mathbb{N}_a \to \mathbb{N}_{a+\nu}$ where $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$.  

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**Theorem 1.2.18.** Let $\nu \in \mathbb{R}$, integer $p > 0$ and suppose $f$ is defined on $\mathbb{N}_a$. Then

$$\Delta^{-\nu} \Delta^p f(t) = \Delta^p \Delta^{-\nu} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\nu-p+k}}{\Gamma(\nu+k-p+1)} \Delta^k f(a).$$

**Theorem 1.2.19.** Let $p$ be a positive integer and $\nu > p$. Then

$$\Delta^p \Delta^{-\nu} f(t) = \Delta^{-\nu} f(t).$$

We will use these properties to obtain existence results for fractional difference equations in Chapter 2. We refer readers to the paper [4] for the proofs not presented here.
Chapter 2

EXISTENCE RESULTS FOR FRACTIONAL DIFFERENCE EQUATIONS

We use basics of the discrete fractional calculus in the analysis of a linear fractional difference equation. Existence of solutions to this difference equation is then established for both initial conditions (IVP) and two-point boundary conditions (BVP).

2.1. Existence of Solutions of Fractional Initial Value Problem (IVP)

In this section, we introduce a linear fractional difference equation with an initial condition. We then obtain the existence and uniqueness of a solution. Let $f$ be a real-valued function, $a_0 \in \mathbb{R}$, $\nu \in (0, 1]$, and suppose $y(t)$ is defined on $N_{\nu-1}$. We consider the following initial value problem [5]:

\begin{equation}
\Delta^\nu y(t) = f(t + \nu - 1, y(t + \nu - 1)), \text{ where } t = 0, 1, 2, \ldots,
\end{equation}

\begin{equation}
\Delta^{\nu-1} y(t)\big|_{t=0} = a_0.
\end{equation}

Let $t \in N_{\nu-1}$. The unique solution of the IVP (2.1)-(2.2) is given by

\begin{equation}
y(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s + \nu - 1, y(s + \nu - 1)).
\end{equation}

To obtain the solution, we have the following steps.

Applying $\Delta^{-\nu}$ to both sides of (2.1), we have

\begin{equation}
\Delta^\nu y(t) = f(t + \nu - 1, y(t + \nu - 1))
\end{equation}
\[ \Delta^{-\nu} \Delta^{\nu} y(t) = \Delta^{-\nu} f(t + \nu - 1, y(t + \nu - 1)). \]

Applying Theorem 1.2.16 to the right-hand side, we have

\[ \Delta^{-\nu} \Delta^{-1-\nu} y(t) = \Delta^{-\nu} f(t + \nu - 1, y(t + \nu - 1)). \]

Next, we apply Theorem 1.2.17 and \( \Delta^{-1-\nu} y(t)|_{t=0} = y(\nu - 1) = a_0 \) to obtain

\[ \Delta \Delta^{-\nu} \Delta^{-1-\nu} y(t) - \frac{t^{(\nu-1)}}{\Gamma(\nu)} y(\nu - 1) = \Delta^{-\nu} f(t + \nu - 1, y(t + \nu - 1)) \]

\[ y(t) - \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 = \Delta^{-\nu} f(t + \nu - 1, y(t + \nu - 1)) \]

\[ y(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \Delta^{-\nu} f(t + \nu - 1, y(t + \nu - 1)). \]

By applying the \( \nu \)th fractional sum of \( f \), we obtain the solution as

\[ y(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s + \nu - 1, y(s + \nu - 1)). \]

Next, we demonstrate how to solve discrete fractional equations with constant coefficients.

**Example 2.1.1.** Let \( y(t) \) be defined on \( N_{\nu-1} \). We consider the linear fractional difference equation as follows.

\[ \Delta^{\nu} y(t) = \lambda y(t + \nu - 1), \quad \text{where} \quad t = 0, 1, 2, \ldots \quad \tag{2.4} \]

\[ \Delta^{\nu-1} y(t)|_{t=0} = a_0. \quad \tag{2.5} \]

Noting that \( \Delta^{-(1-\nu)} y(t)|_{t=0} = y(\nu - 1) \), and substituting into (2.4)-(2.5),

the IVP can be written as

\[ \Delta^{\nu} y(t) = \lambda y(t + \nu - 1), \quad \text{where} \quad t = 0, 1, 2, \ldots, \]
\[ y(\nu - 1) = a_0. \]

The solution of the IVP is given by

\[ y(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} y(s + \nu - 1). \] (2.6)

Next, we employ the method of successive approximations [5]. Let

\[ y_0(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0. \]

Since \( \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} y_{m-1}(s + \nu - 1) = \Delta^{-\nu} y_{m-1}(t + \nu - 1) \), where \( m = 1, 2, \ldots \), then

\[ y_m(t) = y_0(t) + \frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} y_{m-1}(s + \nu - 1) \]

\[ = y_0(t) + \lambda \Delta^{-\nu} y_{m-1}(t + \nu - 1). \]

We will apply the power rule and mathematical induction to obtain

\[ y_m(t) = a_0 \sum_{i=0}^{m} \frac{\lambda^i}{\Gamma((i+1)\nu)} (t + \nu - 1)^{(i\nu+\nu-1)}. \] (2.7)

In fact, the case \( m = 1 \) is true since

\[ y_1(t) = y_0(t) + \lambda \Delta^{-\nu} y_0(t + \nu - 1) \]

\[ = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \frac{\lambda}{\Gamma(\nu)} \frac{(t + \nu - 1)^{(\nu-1)}}{\Gamma(\nu)} a_0 \]

\[ = \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \frac{\lambda}{\Gamma(2\nu)} (t + \nu - 1)^{(2\nu-1)} a_0 \]

\[ = a_0 \left[ \frac{t^{(\nu-1)}}{\Gamma(\nu)} + \frac{\lambda}{\Gamma(2\nu)} (t + \nu - 1)^{(2\nu-1)} \right]. \]

The case \( m = 2 \) is true since

\[ y_2(t) = y_0(t) + \lambda \Delta^{-\nu} y_1(t + \nu - 1) \]
Suppose the result holds for \( m = k \). Then

\[
y_k(t) = a_0 \sum_{i=0}^{k} \frac{\lambda^i}{\Gamma((i + 1)\nu)} (t + i(\nu - 1))^{(i\nu + \nu - 1)}
\]

\[
= y_0(t) + \lambda \Delta^{-\nu} y_{k-1}(t + \nu - 1).
\]

We need to show that the result holds for \( m = k + 1 \).

\[
y_{k+1}(t) = y_0(t) + \lambda \Delta^{-\nu} y_{k}(t + \nu - 1)
\]

\[
= \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \lambda \Delta^{-\nu} \left[ a_0 \sum_{i=0}^{k} \frac{\lambda^i}{\Gamma((i + 1)\nu)} (t + i\nu - 1 + i(\nu - 1))^{(i\nu + \nu - 1)} \right]
\]

\[
= \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \lambda \left[ a_0 \sum_{i=0}^{k} \frac{\lambda^i}{\Gamma((i + 1)\nu)} \Delta^{-\nu} (t + i\nu - 1 + i(\nu - 1))^{(i\nu + \nu - 1)} \right]
\]

\[
= \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + \lambda \left[ a_0 \sum_{i=0}^{k} \frac{\lambda^i}{\Gamma((i + 1)\nu)} \Delta^{-\nu} (t + (i + 1)(\nu - 1))^{(i\nu + \nu - 1)} \right]
\]

\[
= \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + a_0 \sum_{i=0}^{k} \frac{\lambda^{i+1}}{\Gamma((i + 1)\nu + \nu)} (t + (\nu + 1)(i + 1))^{(i+1)\nu - 1 + \nu}
\]

\[
= \frac{t^{(\nu-1)}}{\Gamma(\nu)} a_0 + a_0 \sum_{u=1}^{k+1} \frac{\lambda^{u}}{\Gamma((u + 1)\nu)} (t + u(\nu - 1))^{(u\nu + \nu - 1)}
\]

where \( u = i + 1 \).

\[
y_{m}(t) = a_0 \sum_{i=0}^{m} \frac{\lambda^i}{\Gamma((i + 1)\nu)} (t + i(\nu - 1))^{(i\nu + \nu - 1)} , \text{ where } m = 0, 1, 2, \ldots
\]

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Generally, we take the limit as $m \to \infty$ to obtain

$$y(t) = a_0 \sum_{i=0}^{\infty} \frac{\lambda_i}{\Gamma((i+1)\nu)}(t + i(\nu - 1))^{(\nu+\nu-1)}. \quad (2.8)$$

Obtaining this solution is crucial to support our initial assumption. We will apply this solution to prove the claim that discrete fractional calculus will give us the best fit and best prediction performance in the application of a one-compartmental model for drug concentration in Chapter 4.

### 2.2. Existence of Solutions of Fractional Boundary Value Problem (FBVP)

In this section, after obtaining the general solution of a linear fractional difference equation with an initial condition, we continue to the boundary value problem for a finite fractional boundary value problem (FBVP) [6].

Let $f : [\nu, \nu + b] \times \mathbb{R} \to \mathbb{R}$ be continuous, $\nu \in (1, 2]$ be a real number, and $b \geq 2$ be an integer. The FBVP is

$$-\Delta^\nu y(t) = f(t + \nu - 1, y(t + \nu - 1)), \quad (2.9)$$

$$y(\nu - 2) = 0, \quad y(\nu + b + 1) = 0, \quad (2.10)$$

where $t = 1, 2, \ldots, b + 1$.

**Lemma 2.2.1.** Let $0 \leq N - 1 < \nu \leq N$ and $C_i \in \mathbb{R}$ for $i = 1, 2, \ldots, N$. Then the homogeneous solution of (2.9) and (2.10) is given by

$$y(t) = C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)} + \ldots + C_N t^{(\nu-N)} \quad \text{where} \quad C_1, C_2, \ldots, C_N \in \mathbb{R}. $$
Then, we analyze and construct the Green’s function $G(t, s)$ for FBVP with the boundary conditions as follows

$$-\Delta^\nu y(t) = h(t + \nu - 1), \; t = 0, 1, 2, ..., b + 1, \; \nu \in (1, 2)$$

$$y(\nu - 2) = 0, \; y(\nu + b + 1) = 0. \quad (2.12)$$

The general solution of the FBVP (2.11)-(2.12) is given by

$$y(t) = -\Delta^{-\nu} h(t + \nu - 1) + C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)}. \quad (2.13)$$

**Theorem 2.2.2.** Define $y(t)$ on $[\nu - 2, \nu + b + 1] \subset N_{\nu-2}$ and let $1 < \nu \leq 2$.

The unique solution of FBVP (2.11)-(2.12) is

$$y(t) = \sum_{s=1}^{b+1} G(t, s) h(s - \nu - 1)$$

where

$$G(t, s) = \frac{1}{\Gamma(\nu)} \left\{ \begin{array}{lr}
\frac{t^{(\nu-1)}(\nu + b + 1 - \sigma(s))^{(\nu-1)}}{(\nu + b + 1)^{(\nu-1)}} - (t - \sigma(s)) & s < t - \nu + 1 \leq b + 1 \\
\frac{t^{(\nu-1)}(\nu + b + 1 - \sigma(s))^{(\nu-1)}}{(\nu + b + 1)^{(\nu-1)}} & t - \nu + 1 \leq s \leq b + 1.
\end{array} \right. \quad (2.14)$$

The proof is given in the paper [6].

**Theorem 2.2.3.** The Green’s function $G(t, s)$ satisfies the condition $G(t, s) > 0$

for $t \in [\nu - 1, \nu + n]_{N_{\nu-1}}, \; s \in [1, b + 1]_{N}$ and

$$\Delta_t G(t, s) > 0 \; \text{for} \; t - \nu + 1 < s \leq b + 1$$

$$\Delta_t G(t, s) < 0 \; \text{for} \; s < t - \nu + 1 \leq b + 1.$$
Lemma 2.2.1, Theorem 2.2.2 and 2.2.3 are given in the paper [6], and are included here for reference. These will be used to obtain further results of Green’s functions.

Further discussion of Green’s functions will be continued in Chapter 3.
The concept of Green’s functions was discovered by the British mathematician George Green in the 1830s and named after him. Green’s functions are primarily used to solve non-homogeneous boundary value problems (BVP). Green’s functions of a non-homogeneous BVP exist when a corresponding homogeneous BVP has only a trivial solution. For the continuous case, let $s \in [a, b]$ and suppose $h$ is continuous on $[a, b]$. We consider the following non-homogeneous BVP [12].

\[ L u = h(t), \]
\[ \alpha u(a) - \beta u'(a) = A, \]
\[ \gamma u(b) + \delta u'(b) = B, \]

where $Lu$ is the non-homogeneous equation, and $A$ and $B$ are constants. Suppose the homogeneous BVP $Lu=0$ has only the trivial solution. The unique solution of the non-homogeneous BVP $Lu=h(t)$ is defined by

\[ x(t) = \int_a^b G(t, s) h(s) ds, \]

where $G(t, s)$ is the Green’s function.

The Green’s function of the BVP

\[ Lu=0, \]
\[ \alpha u(a, s) - \beta u'(a, s) = 0, \]
\[ \gamma u(b, s) + \delta u'(b, s) = -\gamma x(b, s) - \delta x(b, s), \]
is given by
\[ G(t, s) = \begin{cases} 
  u(t, s) & ; \ a \leq t \leq s \leq b \\
  v(t, s) & ; \ a \leq s \leq t \leq b
\end{cases} \]
where \( v(t, s) = u(t, s) + x(t, s) \), \( u(t, s) \) is the solution of BVP and \( x(t, s) \) is the Cauchy function for \( Lu=0 \). We refer the readers to the book [12] for the definition of the Cauchy function and the details of Green’s functions.

### 3.1. Green’s Functions of Boundary Value Problem

Having calculated Green’s functions in the continuous case, in this section we will calculate Green’s functions in the discrete calculus case. In Chapter 2, the existence of solutions of a fractional boundary value problem was established as

\[ -\Delta_\nu y(t) = h(t + \nu - 1), \]
\[ y(\nu - 2) = 0, y(\nu + b + 1) = 0, \]

where \( t = 0, 1, 2, ..., b + 1, \ 1 \leq \nu \leq 2 \).

The unique solution of FBVP will be given by

\[ y(t) = \sum_{s=1}^{b+1} G(t, s) h(s - \nu - 1) \]

where \( G(t, s) \) is the discrete fractional calculus Green’s function introduced in this chapter.

Since we developed the necessary background in Chapter 1, we are able to apply the basics of the discrete calculus to obtain Green’s functions in the discrete fractional calculus. There are a few common methods to find Green’s functions.
As shown in the introduction, we obtain Green’s functions in the continuous case by using a Cauchy function $x(t, s)$. On the other hand, in this thesis, we will find Green’s functions using a traditional method. The following is an example of how to find the Green’s function using a traditional method. We will begin with a boundary value problem in the continuous case since it is easier for readers to follow. Then we will employ the same method for discrete boundary value problems (DBVP) and fractional boundary value problems (FBVP).

**Calculation of Green’s functions $G(t, s)$ in a continuous case by the traditional method is demonstrated as follows.**

We consider the following non-homogeneous boundary value problem (BVP).

**Equation:** $y'' = h(t),$

**Boundary Conditions:** $y(a) = 0, y(b) = 0,$

**Solution:** $y(t) = \int_a^b G(t, s)h(s)ds.$

The Green’s function exists when the homogeneous BVP $y'' = 0, y(a) = 0, y(b) = 0$ has only the trivial solution. Since the homogeneous solution $y(t) = C_1 + C_2 t$ where $C_1 = 0$ and $C_2 = 0$ is trivial, the Green’s function exists.

By taking the integral of both sides, we have

$$\int_a^t y''(s)ds = \int_a^t h(s)ds.$$  

We obtain

$$y'(t) = y'(a) + \int_a^t h(s)ds.$$
By integrating both sides again, we have

\[
\int_a^t y'(t)\,ds = \int_a^t [y'(a) + \int_a^t h(s)\,ds]\,ds,
\]

\[
y(t) = y(a) + y'(a)(t - a) + \int_{\tau=a}^{s=t} \int_{\tau=a}^{s=t} h(\tau)\,d\tau\,ds.
\]

We switch the order of integration to obtain

\[
y(t) = y(a) + y'(a)(t - a) + \int_{\tau=a}^{s=t} \int_{\tau=a}^{s=t} h(\tau)\,d\tau\,ds
\]

\[
= y(a) + y'(a)(t - a) + \int_{\tau=a}^{s=t} \int_{\tau=a}^{s=t} h(\tau)\,d\tau\,ds
\]

\[
= y(a) + y'(a)(t - a) + \int_{\tau=a}^{s=t} h(\tau)(t - \tau)\,d\tau.
\]

We apply the boundary condition \(y(a) = 0, \, y(b) = 0\) to obtain \(y'(a)\).

Then we have

\[
y'(a) = -\frac{1}{(b - a)} \int_a^b h(\tau)(b - \tau)\,d\tau.
\]

It follows that

\[
y(t) = -\frac{1}{(b - a)} \int_a^b h(\tau)(b - \tau)\,d\tau(t - a) + \int_{\tau=a}^{s=t} \int_{\tau=a}^{s=t} h(\tau)(t - \tau)\,d\tau\,ds
\]

We employ \(\int_a^b = \int_a^t + \int_t^b\) to obtain

\[
y(t) = \int_a^t [(\frac{a-t}{b-a})(b - \tau) + (t - \tau)]h(\tau)\,d\tau + \int_t^b [(\frac{a-t}{b-a})(b - \tau)]h(\tau)\,d\tau.
\]

Then the Green’s function is

\[
G(t, s) = \begin{cases} 
\frac{b-t}{b-a}(a-s) & ; s \leq t \\
\frac{a-t}{b-a}(b-s) & ; t \leq s.
\end{cases}
\]
One can easily see that this Green’s function is symmetric, i.e., $G(t, s) = G(s, t)$.

### 3.2. Green’s Functions of Discrete Boundary Value Problem (DBVP)

To obtain Green’s functions in the discrete case, some basics of the discrete fractional calculus in Chapter 1 will be used in this section. We begin with the discrete boundary value problem (DBVP) with boundary conditions. The following examples show how to find the Green’s function $G(t, s)$ of the DBVP.

**Example 3.2.1.** We consider the Green’s function of the following DBVP [11].

*Equation:* $\Delta^2 y(t - 1) = h(t),$

*Boundary Conditions:* $y(a) = 0, y(b + 1) = 0,$

*Solution:* $y(t) = \sum_{s=1}^{b+1} G(t, s) h(s).$

The Green’s function exists when the homogeneous DBVP $\Delta^2 y(t - 1) = 0, y(a) = 0, y(b + 1) = 0$ has only the trivial solution. Since the homogeneous solution $y(t - 1) = C_1 + C_2(t - 1)$ where $C_1 = 0$ and $C_2 = 0$ is trivial, the Green’s function exists.

**Proof.** By applying the summation $\sum_{s=a}^{t}$ to both sides, we have

$$\sum_{s=a}^{t} \Delta^2 y(s - 1) = \sum_{s=a}^{t} h(s).$$

We apply $\Delta y(t) = f(t)$ and the First Fundamental of Calculus to obtain

$$\Delta y(t) = \Delta y(a - 1) + \sum_{s=a}^{t} h(s).$$

By applying the summation $\sum_{s=a}^{t-1}$ to both sides again, we have
\[
\sum_{s=a}^{t-1} \Delta y(s) = \sum_{s=a}^{t-1} [\Delta y(a - 1) + \sum_{s=a}^{t} h(s)],
\]

\[
y(t) = y(a) + \Delta y(a - 1)(t - a) + \sum_{s=a}^{t-1} \sum_{\tau=a}^{t-1} h(\tau).
\]

We switch the order of summation to obtain

\[
y(t) = y(a) + \Delta y(a - 1)(t - a) + \sum_{\tau=a}^{t-1} \sum_{s=\tau}^{t-1} h(\tau)
\]

\[
= y(a) + \Delta y(a - 1)(t - a) + \sum_{\tau=a}^{t-1} h(\tau) \sum_{s=\tau}^{t-1} 1
\]

\[
= y(a) + \Delta y(a - 1)(t - a) + \sum_{\tau=a}^{t-1} h(\tau)(t - \tau).
\]

We apply the boundary condition \(y(a) = 0, y(b + 1) = 0\) to obtain \(\Delta y(a - 1)\).

Then we have

\[
\Delta y(a - 1) = -\frac{1}{(b + 1 - a)} \sum_{\tau=a}^{b+1} h(\tau)(b + 1 - \tau).
\]

We employ \(\sum_{a}^{b+1} = \sum_{a}^{t-1} + \sum_{t}^{b+1}\) to obtain

\[
y(t) = \sum_{a}^{t-1} \left[ \frac{(a - \tau)(b + 1 - t)}{b + 1 - a} \right] h(\tau) + \sum_{t}^{b+1} \left[ \frac{(a - t)(b + 1 - \tau)}{b + 1 - a} \right] h(\tau).
\]

Then the Green’s function is

\[
G(t, s) = \begin{cases} 
\frac{(a - s)(b + 1 - t)}{b + 1 - a} & ; s \leq t - 1 \\
\frac{(a - t)(b + 1 - s)}{b + 1 - a} & ; t \leq s.
\end{cases}
\]

It is clear that this Green’s function is symmetric, i.e., \(G(t, s) = G(s, t).\)
Next, we introduce discrete boundary value problems (DBVP) with two-point boundary value conditions. This are called Sturm-Liouville difference equations. We consider the following equation from the book [11].

Equation: \(-\Delta^2 y(t - 1) = h(t)\), where \(t = 1, 2, ..., b + 1\),

Boundary Conditions: \(\alpha y(0) - \beta \Delta y(0) = 0\),

\(\gamma y(b + 1) + \delta \Delta y(b + 1) = 0\),

where \(\alpha, \beta, \gamma, \delta\) are constant satisfying \(\alpha^2 + \beta^2 \neq 0\) and \(\gamma^2 + \delta^2 \neq 0\).

**Example 3.2.2.** *Find the Green’s function of the Sturm-Liouville problem.*

The Green’s function exists when the homogeneous DBVP \(-\Delta^2 y(t - 1) = 0\) has only the trivial solution. Since the homogeneous solution \(y(t - 1) = C_1 + C_2(t - 1)\) where \(C_1 = 0\) and \(C_2 = 0\) is trivial, the Green’s function exists.

As the calculation is similar to the previous example, we follow the same method to obtain the results of follows.

\[
y(t) = y(1) + \Delta y(0)(t - 1) + \sum_{\tau=1}^{t-1} h(\tau)(t - \tau), \quad (3.1)
\]

\[
\Delta y(t) = \Delta y(0) - \sum_{s=1}^{t} h(s). \quad (3.2)
\]

We find \(y(1)\) by using the condition \(\alpha y(0) - \beta \Delta y(0) = 0\).

Then we have

\[
y(1) = \left(\frac{\alpha + \beta}{\rho}\right) \Delta y(0). \quad (3.3)
\]
Next, we find \( \Delta y(0) \) by using the condition \( \gamma y(b + 1) + \delta \Delta y(b + 1) = 0 \).

We define \( \rho = \gamma \beta + \gamma \alpha (b + 1) + \alpha \delta \).

Then

\[
\Delta y(0) = \frac{\alpha}{\rho} \left[ \gamma \sum_{\tau=1}^{b} h(\tau)(b + 1 + \tau) + \delta \sum_{s=1}^{b+1} h(s) \right].
\] (3.4)

We substitute \( \Delta y(0) \) into (3.2) to obtain

\[
y(1) = \left( \frac{\alpha + \beta}{\rho} \right) \left[ \gamma \sum_{\tau=1}^{b} h(\tau)(b + 1 + \tau) + \delta \sum_{s=1}^{b+1} h(s) \right].
\] (3.5)

We substitute \( y(1) \) and \( \Delta y(0) \) into (3.1).

Then we have

\[
y(t) = \left( \frac{\alpha + \beta}{\rho} \right) \left[ \gamma \sum_{\tau=1}^{b} h(\tau)(b + 1 + \tau) + \delta \sum_{s=1}^{b+1} h(s) \right] \\
+ \frac{\alpha}{\rho} \left[ \gamma \sum_{\tau=1}^{b} h(\tau)(b + 1 + \tau) + \delta \sum_{s=1}^{b+1} h(s) \right] (t - 1) - \sum_{\tau=1}^{t-1} h(\tau) (t - \tau) \\
= \left( \frac{\alpha t + \beta}{\rho} \right) \left[ \gamma \sum_{\tau=1}^{b} h(\tau)(b + 1 + \tau) + \delta \sum_{s=1}^{b+1} h(s) \right] - \sum_{\tau=1}^{t-1} h(\tau) (t - \tau) \\
= \sum_{\tau=1}^{t-1} \left( \frac{\alpha t + \beta}{\rho} \right) \gamma (b + 1 + \tau) h(\tau) + \sum_{s=1}^{t-1} \left( \frac{\alpha t + \beta}{\rho} \right) \delta h(s) - \sum_{\tau=1}^{t-1} h(\tau) (t - \tau) \\
+ \sum_{\tau=1}^{b+1} \left( \frac{\alpha t + \beta}{\rho} \right) \gamma (b + 1 + \tau) h(\tau) + \sum_{s=1}^{b+1} \left( \frac{\alpha t + \beta}{\rho} \right) \delta h(s).
\]

We obtain

\[
y(t) = \sum_{\tau=1}^{t-1} \left( \frac{\alpha t + \beta}{\rho} \right) \gamma (b + 1 + \tau) - (t - \tau) h(\tau) + \sum_{s=1}^{t-1} \left( \frac{\alpha t + \beta}{\rho} \right) \delta h(s) \\
+ \sum_{\tau=1}^{b+1} \left( \frac{\alpha t + \beta}{\rho} \right) \gamma (b + 1 + \tau) h(\tau) + \sum_{s=1}^{b+1} \left( \frac{\alpha t + \beta}{\rho} \right) \delta h(s).
\]

Then the Green’s function is
\[
G(t, s) = \begin{cases} 
\frac{(\alpha s + \beta)[\gamma(b + 1) + \delta - \gamma t]}{\rho}; & s \leq t - 1 \\
\frac{(\alpha t + \beta)[\gamma(b + 1) + \delta - \gamma s]}{\rho}; & t \leq s
\end{cases}
\]

where \( \rho = \gamma\beta + \gamma\alpha(b + 1) + \alpha\delta \).

We also observe that this Green’s function is symmetric, i.e., \( G(t, s) = G(s, t) \).

3.3. Green’s Functions of Fractional Boundary Value Problem (FBVP)

In the previous section, we introduced and obtained the Green’s functions of discrete boundary value problems (DBVP) using the traditional method. In this section, we consider linear finite fractional difference equations (or fractional boundary value problem, FBVP) with boundary conditions and with two-point boundary conditions. To obtain Green’s functions of FBVP, readers are asked to recall the existence of solutions of FBVP from Chapter 2. Moreover, the general solution of FBVP (2.11) and (2.12) plays an important role in this section.

**Example 3.3.1.** We employ the FBVP (2.11)-(2.12) and the solution in section (2.2) to find the following Green’s function [6].

Equation: \(-\Delta^\nu y(t) = h(t + \nu + 1)\),

Boundary Conditions: \(y(\nu - 2) = 0\) and \(y(b + \nu - 1) = 0\),

Solution: \(y(t) = \sum_{s=1}^{b+1} G(t, s)h(s + \nu - 1)\).
The Green’s function exists when the homogeneous FBVP $-\Delta^{\nu} y(t) = 0$, $y(\nu - 2) = 0$ and $y(b + \nu - 1) = 0$ has only the trivial solution. Since the homogeneous solution $y(t) = C_1 t^{(\nu - 1)} + C_2 t^{(\nu - 2)}$ where $C_1 = 0$ and $C_2 = 0$ is trivial, the Green’s function exists.

The general solution of FBVP is given by

$$y(t) = -\Delta^{\nu} h(t + \nu - 1) + C_1 t^{(\nu - 1)} + C_2 t^{(\nu - 2)}.$$ 

We find $C_2$ by using the boundary condition $y(\nu - 2) = 0$.

We substitute $t = \nu - 2$ into (2.13) to obtain

$$y(\nu - 2) = -\Delta^{\nu} h(t + \nu - 1)|_{t=\nu-2} + C_1 (\nu - 2)^{(\nu - 1)} + C_2 (\nu - 2)^{(\nu - 2)}.$$ 

We apply $\Delta^{\nu} y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu - 1)} f(s)$.

It follows that

$$0 = \frac{-1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu - 1)} h(s + \nu - 1)|_{t=\nu-2} + C_1 (0) + C_2 \Gamma(\nu - 1),$$

$$0 = C_2 \Gamma(\nu - 1).$$

This implies $C_2 = 0$.

We do the same process using the boundary condition $y(b + \nu - 1) = 0$.

We substitute $t = b + \nu - 1$ into (2.13).

We have

$$y(b + \nu - 1) = -\Delta^{\nu} h(t + \nu - 1)|_{t=b+\nu-1} + C_1 (b + \nu - 1)^{(\nu - 1)} + C_2 (b + \nu - 1)^{(\nu - 2)},$$

$$0 = \frac{-1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu - 1)} h(s + \nu - 1)|_{t=b+\nu-1} + C_1 (b + \nu - 1)^{(\nu - 1)} + 0,$$
\begin{equation}
0 = \frac{-1}{\Gamma(\nu)} \sum_{s=1}^{b-1} (b + \nu - 1 - \sigma(s))^{(\nu-1)} h(s + \nu - 1) + C_1(b + \nu - 1)^{(\nu-1)}.
\end{equation}

This implies

\begin{equation}
C_1 = \frac{1}{(b + \nu - 1)^{(\nu-1)}} \frac{1}{\Gamma(\nu)} \sum_{s=1}^{b-1} (b + \nu - 1 - \sigma(s))^{(\nu-1)} h(s + \nu - 1).
\end{equation}

We substitute \( C_1 \) and \( C_2 \) into (2.13).

Then we have

\[ y(t) = \frac{-1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu-1)} h(s + \nu - 1) + \frac{1}{(b + \nu - 1)^{(\nu-1)}} \frac{1}{\Gamma(\nu)} \sum_{s=1}^{b-1} (b + \nu - 1 - \sigma(s))^{(\nu-1)} h(s + \nu - 1) t^{(\nu-1)}.
\]

We employ \( \sum_{1}^{b-1} = \sum_{1}^{t-\nu+1} + \sum_{t-\nu+1}^{b-1} \) and \( \sigma(s) = s + 1 \).

We obtain

\[ y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} \left[ \frac{t^{(\nu-1)}(b + \nu - 1 - \sigma(s))^{(\nu-1)}}{(b + \nu - 1)^{(\nu-1)}} - (t - \sigma(s))^{(\nu-1)} \right] h(s + \nu - 1) + \frac{1}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{b-1} \frac{t^{(\nu-1)}(b + \nu - 1 - \sigma(s))^{(\nu-1)}}{(b + \nu - 1)^{(\nu-1)}} h(s + \nu - 1).
\]

Then the Green’s function is

\[ G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} 
\frac{t^{(\nu-1)}(b + \nu - 1 - \sigma(s))^{(\nu-1)}}{(b + \nu - 1)^{(\nu-1)}} - (t - \sigma(s))^{(\nu-1)} & ; s \leq t - \nu + 1 \\
\frac{t^{(\nu-1)}(b + \nu - 1 - \sigma(s))^{(\nu-1)}}{(b + \nu - 1)^{(\nu-1)}} & ; t - \nu + 2 < s.
\end{cases}
\]

Next, we introduce a fractional boundary value problem (FBVP) with two-point boundary value conditions as follows.

Equation: \(-\Delta^\nu y(t) = h(t + \nu + 1)\), where \( t = 1, 2, \ldots \) and \( 1 < \nu \leq 2 \),

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Boundary Conditions: \( \alpha y(\nu - 2) - \beta \Delta y(\nu - 2) = 0, \)
\[ \gamma y(b + \nu - 1) + \delta \Delta y(b + \nu - 1) = 0. \]

This is called the Sturm-Liouville difference equation. The Sturm-Liouville problem has to satisfy the condition \( \alpha^2 + \beta^2 \neq 0 \) and \( \gamma^2 + \delta^2 \neq 0 \). The solution is given by
\[
y(t) = \sum_{s=1}^{b+1} G(t, s) h(s + \nu - 1).
\]

**Example 3.3.2.** Find the Green’s function of the above Sturm-Liouville problem.

The calculation is similar to the case involving boundary value conditions. The general solution of FBVP is given by
\[
y(t) = -\Delta^{-\nu} h(t + \nu - 1) + C_1 t^{(\nu - 1)} + C_2 t^{(\nu - 2)}. \]

Since the homogeneous solution \( y(t) = C_1 t^{(\nu - 1)} + C_2 t^{(\nu - 2)} \) where \( C_1 = 0 \) and \( C_2 = 0 \) is trivial, the Green’s function exists.

We find \( C_1 \) by using the boundary condition \( \alpha y(\nu - 2) - \beta \Delta y(\nu - 2) = 0. \)

Then, we substitute \( t = \nu - 2 \) into (2.13) to obtain
\[
y(\nu - 2) = -\Delta^{\nu} h(t + \nu - 1)|_{t=\nu-2} + C_1(\nu - 2)^{(\nu - 1)} + C_2(\nu - 2)^{(\nu - 2)}
\]
\[
= -\frac{1}{\Gamma(\nu)} \sum_{s=1}^{\nu-\nu} (t - \sigma(s))^{(\nu - 1)} h(s + \nu - 1)|_{t=\nu-2} + C_1(0) + C_2 \Gamma(\nu - 1)
\]
\[
= C_2 \Gamma(\nu - 1).
\]

Multiplying both sides by \( \alpha \), we have
\[
\alpha y(\nu - 2) = \alpha C_2 \Gamma(\nu - 1).
\]

(3.6)
Applying $\Delta$ to both sides of (2.13), we have

$$\Delta y(t) = \Delta[-\Delta^{-\nu}h(t + \nu - 1) + C_1 t^{(\nu-1)} + C_2 t^{(\nu-2)}]$$

$$= -\Delta\Delta^{-\nu}h(t + \nu - 1) + \Delta C_1 t^{(\nu-1)} + \Delta C_2 t^{(\nu-2)}$$

$$= \Delta[-\frac{1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu-1)} h(s + \nu - 1)] + C_1 (\nu - 1) t^{(\nu-2)} + C_2 (\nu - 2) t^{(\nu-3)}$$

$$= -\frac{1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (\nu - 1)(t - \sigma(s))^{(\nu-2)} h(s + \nu - 1) + (t + 1 - \sigma(t)) h(t + \nu - 1)$$

$$+ C_1 (\nu - 1) t^{(\nu-2)} + C_2 (\nu - 2) t^{(\nu-3)}.$$

We employ $\Delta \sum_{a}^{t-1} f(t, s) = \sum_{a}^{t-1} f(t, s) + f(t + 1, t)$ and $\sigma(t) = t + 1$.

We obtain

$$\Delta y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (\nu - 1)(t - \sigma(s))^{(\nu-2)} h(s + \nu - 1) + C_1 (\nu - 1) t^{(\nu-2)} + C_2 (\nu - 2) t^{(\nu-3)}. \quad (3.7)$$

We substitute $t = \nu - 2$ into (3.7), and we then have

$$\Delta y(\nu - 2) = -\frac{1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (\nu - 1)(t - \sigma(s))^{(\nu-2)} h(s + \nu - 1) \big|_{t=\nu-2} + C_1 (\nu - 1)(\nu - 2)^{(\nu-2)}$$

$$+ C_2 (\nu - 2)(\nu - 2)^{(\nu-3)}$$

$$= C_1 (\nu - 1)(\nu - 2)^{(\nu-2)} + C_2 (\nu - 2)(\nu - 2)^{(\nu-3)}.$$

Multiplying both sides by $\beta$, we obtain

$$\beta \Delta y(\nu - 2) = \beta C_1 (\nu - 1)(\nu - 2)^{(\nu-2)} + \beta C_2 (\nu - 2)(\nu - 2)^{(\nu-3)}. \quad (3.8)$$

We solve (3.6) and (3.8) by using the boundary condition $\alpha y(\nu - 2) - \beta \Delta y(\nu - 2) = 0$.

We obtain
We do the same process using the boundary condition \( \gamma y(b + \nu - 1) + \delta \Delta y(b + \nu - 1) = 0 \).

Substituting \( t = b + \nu - 1 \) into (2.13), we obtain

\[
y(b + \nu - 1) = -\Delta' h(t + \nu - 1)_{|t=b+\nu-1} + C_1(b + \nu - 1)^{(\nu-1)} + C_2(b + \nu - 1)^{(\nu-2)}
\]

Multiplying both sides by \( \gamma \), we have

\[
\gamma y(b + \nu - 1) = -\frac{\gamma}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu-1)} h(s + \nu - 1)_{|s=b+\nu-1} + \gamma C_1(b + \nu - 1)^{(\nu-1)} + \gamma C_2(b + \nu - 1)^{(\nu-2)}.
\]

Substituting \( t = b + \nu - 1 \) into (3.7), we have

\[
\Delta y(b + \nu - 1) = -\frac{1}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu-2)} h(s + \nu - 1)_{|(t=b+\nu-1)} + C_1(\nu - 1)(b + \nu - 1)^{(\nu-2)} + C_2(\nu - 2)(b + \nu - 1)^{(\nu-3)}.
\]

Multiplying both sides by \( \delta \), we have

\[
\delta \Delta y(b + \nu - 1) = -\frac{\delta}{\Gamma(\nu)} \sum_{s=1}^{t-\nu} (t - \sigma(s))^{(\nu-2)} h(s + \nu - 1)_{|(t=b+\nu-1)} + \delta C_1(\nu - 1)(b + \nu - 1)^{(\nu-2)} + \delta C_2(\nu - 2)(b + \nu - 1)^{(\nu-3)}.
\]
We solve (3.9) and (3.10) by using the boundary condition \( \gamma y(b + \nu - 1) + \delta \Delta y(b + \nu - 1) = 0 \) and \( C_1 = \frac{\alpha - \beta(\nu - 2)}{\beta(\nu - 1)} C_2 \).

Then we have

\[
C_2 = \beta \left( \frac{\nu - 1}{M} \right) \frac{1}{\gamma(\nu)} \sum_{s=1}^{b+1} \left[ \gamma(b + \nu - 1 - \sigma(s))^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - \sigma(s))^{(\nu-2)} \right] h(s + \nu - 1),
\]

where

\[
M = \gamma(\alpha - \beta(\nu - 2))(b + \nu - 1)^{(\nu-1)} + [\delta(\alpha - \beta(\nu - 2))(\nu - 1) + \gamma\beta(\nu - 1)](b + \nu - 1)^{(\nu-2)} + \beta(\nu - 1)\delta(\nu - 2)(b + \nu - 1)^{(\nu-3)},
\]

and

\[
C_1 = \frac{(\alpha - \beta(\nu - 2))}{M} \frac{1}{\Gamma(\nu)} \sum_{s=1}^{b+1} \left[ \gamma(b + \nu - 1 - \sigma(s))^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - \sigma(s))^{(\nu-2)} \right] h(s + \nu - 1).
\]

We substitute \( C_1 \) and \( C_2 \) into (2.13). Then we have

\[
y(t) = \frac{-1}{\Gamma(\nu)} \sum_{s=1}^{t - \nu}(t - \sigma(s))^{(\nu-1)} h(s + \nu - 1) + \frac{\alpha - \beta(\nu - 2)}{M} \frac{1}{\Gamma(\nu)} \sum_{s=1}^{b+1} \left[ \gamma(b + \nu - 1 - \sigma(s))^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - \sigma(s))^{(\nu-2)} \right] h(s + \nu - 1) t^{(\nu-1)} + \frac{\beta(\nu - 1)}{M} \frac{1}{\gamma(\nu)} \sum_{s=1}^{b+1} \left[ \gamma(b + \nu - 1 - \sigma(s))^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - \sigma(s))^{(\nu-2)} \right] h(s + \nu - 1) t^{\nu-2}.
\]

We employ \( \sum_{1}^{t - \nu} = \sum_{1}^{b+1} + \sum_{t - \nu + 1}^{b+1} \) and \( \sigma(s) = s + 1 = u \).

Then we obtain

\[
y(t) = \frac{1}{\Gamma(\nu)} \sum_{u=2}^{t-\nu+1} \left[ (t^{(\nu-1)} \frac{\alpha - \beta(\nu - 2)}{M} + t^{(\nu-2)} \beta \frac{(\nu - 1)}{M}) (\gamma(b + \nu - 1 - u)^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - u)^{(\nu-2)}) + \frac{1}{\Gamma(\nu)} \sum_{u=t-\nu+2}^{b+2} \left[ (t^{(\nu-1)} \frac{\alpha - \beta(\nu - 2)}{M} + t^{(\nu-2)} \beta \frac{(\nu - 1)}{M}) (\gamma(b + \nu - 1 - u)^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - u)^{(\nu-2)}) \right].
\]
Then the Green’s function is

\[
G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} \\
\frac{\phi \varphi}{M} - \left( t - \sigma(s) \right)^{(\nu-1)} & ; u \leq t - \nu + 1 \\
\frac{\phi \varphi}{M} & ; t - \nu + 2 \leq u \\
\end{cases}
\]

where

\[
\phi = t^{(\nu-1)}(\alpha - \beta(\nu - 2) + t^{(\nu-2)}\beta(\nu - 1)),
\]

\[
\varphi = \gamma(b + \nu - 1 - u)^{(\nu-1)} + \delta(\nu - 1)(b + \nu - 1 - u)^{(\nu-2)}, \text{ and}
\]

\[
M = \gamma((\alpha - \beta(\nu - 2))(b + \nu - 1)^{(\nu-1)} + \left[ \delta(\alpha - \beta(\nu - 2))(\nu - 1) + \gamma\beta(\nu - 1) \right]
\]

\[
(b + \nu - 1)^{(\nu-2)} + \beta(\nu - 1)\delta(\nu - 2)(b + \nu - 1)^{(\nu-3)}.
\]

### 3.4. Generalization

In this section, we demonstrate how our Green’s function of discrete fractional boundary value problems (FBVP) in Section 3.3 generalizes the existence results in Section 3.2. According to the techniques for the Sturm-Liouville problem

\[-\Delta^\nu y(t - 1) = h(t + \nu + 1), \text{ where } t = 1, 2, \ldots \text{ and } 1 < \nu \leq 2,\]

\[\alpha y(\nu - 2) - \beta \Delta y(\nu - 2) = 0,\]

\[\gamma y(b + \nu - 1) + \delta \Delta y(b + \nu - 1) = 0,\]

we obtain the discrete boundary value problems (DBVP) as in Example 3.2.2 if \(\nu = 2\).

Substituting \(\nu = 2\), we have

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\[-\Delta^2 y(t-1) = h(t), \text{ where } t = 1, 2, \ldots, b + 1,\]

\[\alpha y(0) - \beta \Delta y(0) = 0,\]

\[\gamma y(b + 1) + \delta \Delta y(b + 1) = 0.\]

Note that we consider only a part of the Green’s function to show how to obtain the generalization.

We substitute \(\nu = 2\) into the condition \(u \leq t - \nu + 1:\)

\[(t^{(\nu-1)} \frac{(\alpha - \beta (\nu - 2)}{M} + t^{(\nu-2)} \frac{(\nu - 1)}{M})(\gamma (b + \nu - 1 - u)^{(\nu-1)} + \delta (\nu - 1)(b + \nu - 1 - u)^{(\nu-2)}) - (t - \sigma(s))^{(\nu-1)}).\]

For \(u \leq t - \nu + 1\), we then have

\[
\frac{1}{M} (\alpha t + \beta)(\gamma (b + 1) + \delta - \gamma u) - M(t - u)
\]

\[
= \frac{1}{M} (\alpha t + \beta)(\gamma (b + 1) + \delta - \gamma u) - (\gamma \beta + \gamma \alpha (b + 1) + \alpha \delta)(t - u)
\]

\[
= \frac{1}{M} (\alpha t + \beta)(\gamma (b + 1) + \delta) - \alpha t(\alpha u + \beta)
\]

\[
= \frac{1}{M} (\alpha t + \beta)(\gamma (b + 1) + \delta - \gamma t)
\]

where \(M = \gamma \beta + \gamma \alpha (b + 1) + \alpha \delta\).

We substitute \(\nu = 2\) into the condition \(t - \nu + 2 \leq u:\)

\[(t^{(\nu-1)} \frac{(\alpha - \beta (\nu - 2)}{M} + t^{(\nu-2)} \frac{(\nu - 1)}{M})(\gamma (b + \nu - 1 - u)^{(\nu-1)} + \delta (\nu - 1)(b + \nu - 1 - u)^{(\nu-2)})).\]

For \(t - \nu + 2 \leq u\), we then have

\[
\frac{1}{M} (t \alpha + \beta)(\gamma (b + 2 - 1 - u) + \delta) = \frac{1}{M} (\alpha t + \beta)(\gamma (b + 1) + \delta - \gamma u).
\]
We obtain the Green’s function of DBVP as

\[
G(t, s) = \begin{cases} 
\frac{(\alpha u + \beta)(\gamma(b + 1) + \delta - \gamma t)}{M} & ; u \leq t - \nu + 1 \\
\frac{(\alpha t + \beta)(\gamma(b + 1) + \delta - \gamma u)}{M} & ; t - \nu + 2 \leq u.
\end{cases}
\]

This is equivalent to

\[
G(t, s) = \begin{cases} 
\frac{(\alpha s + \beta)[\gamma(b + 1) + \delta - \gamma t]}{\rho} & ; s \leq t - 1 \\
\frac{(\alpha t + \beta)[\gamma(b + 1) + \delta - \gamma s]}{\rho} & ; t \leq s
\end{cases}
\]

where \( u = s + 1 \) and \( M = \rho = \gamma\beta + \gamma\alpha(b + 1) + \alpha\delta \).

Moreover, if \( \nu = 2, \alpha = 1, \beta = 0, \gamma = 1, \) and \( \delta = 0 \), we obtain the discrete boundary value problem (DBVP) as in Example 3.2.1;

\[
\Delta^2 y(t - 1) = h(t),
\]

\[
y(a) = 0, \quad y(b + 2) = 0.
\]

We substitute \( \nu = 2, \alpha = 1, \beta = 0, \gamma = 1, \) and \( \delta = 0 \) into both conditions.

Then we have

\[
\frac{s(b + 1 - t)}{b + 1} ; \quad s \leq t - 1,
\]

\[
\frac{t(b + 1 - s)}{b + 1} ; \quad t \leq s.
\]

Then, we obtain the Green’s function of DBVP as
\[ G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{(a - s)(b + 1 - t)}{b + 1 - a} & ; s \leq t - 1 \\ \frac{(a - t)(b + 1 - s)}{b + 1 - a} & ; t \leq s. \end{cases} \]

Note that \( a = 0 \) but we keep \( a \) instead of 0 to see the comparison.
Chapter 4

AN APPLICATION OF DISCRETE FRACTIONAL CALCULUS TO A ONE COMPARTMENTAL MODEL

Pharmacokinetics is simply defined as the body’s response to a drug or the study of the time-course data of drug absorption, distribution, metabolism and elimination. The idea of a pharmacokinetic model is a correlation between the body and drug concentrations as the drug works through tissues and fluid. Since measurement of drug concentration is not always practical, applying mathematical principles to various processes is the simpler way. To apply mathematical principles, a model of the body needs to be used. The compartmental model is a basic type of model in pharmacokinetics. Compartmental models are categorized by the number of compartments, which represent a group of similar tissues or fluids. Also, these models can be used to predict the time course of drug concentration in the body. There are three models: one-compartment, two-compartment, and multi-compartment models. The one compartment model, which includes organs (e.g., heart, liver, and kidneys) and blood(plasma), is called a “central compartment” or “highly blood-perfused compartment.” The second compartment that includes tissues (e.g., fat and muscle) and cerebrospinal fluid and is called a “peripheral compartment” or “less blood-perfused compartment.” The one-compartment model in Figure 4.0.1 assumes that the drug is distributed into the compartment instantaneously and leaves as elimination. On the other hand, in the two-compartment model in Figure 4.0.2, the drug is distributed at a slower rate since it moves between central and peripheral compartments [8].
In this thesis, we focus on the one-compartmental model with the discrete fractional calculus. Comparison to the continuous fractional calculus result of Amera Almusharff [1] proves the claim that discrete fractional calculus gives us the best fit and the best model. There are two methods, mean squared error (MSE) and cross validation, used to obtain the result. The mean squared error (MSE) determines the best fit and the cross validation method determines the best model.

**Figure 4.0.1.** One-compartmental model

**Figure 4.0.2.** Two-compartmental model

### 4.1. Best Fit for One Compartmental Model

Using the one-compartmental model with a rapid intravenous injection of a drug that distributes to the body and is eliminated, the rate of loss of the drug from
the body is given by

\[ \Delta y(t) = -ky(t), \quad \text{where} \quad t = 0, 1, 2, \ldots \quad (4.1) \]

\[ y(0) = a_0 \quad (4.2) \]

where \( y \) is the concentration of the drug in the body at time \( t \) after injection,

\( k \) is constant apparent first-order elimination rate of the drug, and

\( a_0 \) is the initial concentration at \( t = 0 \).

The negative sign means drug is eliminated from the body [1].

The solution of the IVP of (4.1)-(4.2) defined by

\[ y(t) = a_0(1 - k)^t \quad (4.3) \]

is called as a **discrete model**.

By using fractional difference equations, we aim to get the solution of the initial value problem (4.1). According to the existence and solutions results for an IVP in Section 2.1, we consider the following IVP with an initial condition:

\[ \Delta^\nu y(t) = \lambda y(t + \nu - 1). \]

\[ y(\nu - 1) = a_0, \]

where \( \lambda \) is constant, \( t = 0, 1, 2, \ldots \), and \( 0 < \nu \leq 1 \).

There exists a solution such that

\[ y(t) = a_0 \sum_{i=0}^{\infty} \frac{\lambda^i}{\Gamma((i + 1)\nu)} (t + i(\nu - 1))^{(i\nu + \nu - 1)}. \]
We define $\lambda = -k$ where $k$ is a constant and $\nu = 1$ for an initial condition to get the following problem.

$$\Delta^\nu y(t) = -ky(t + \nu - 1). \quad (4.4)$$

$$y(0) = a_0. \quad (4.5)$$

The solution of the IVP of (4.4)-(4.5) defined by

$$y(t) = a_0 \sum_{i=0}^{\infty} \frac{(-k)^i}{\Gamma((i + 1)v)(t + i(v - 1))^{(iv+v-1)}} \quad (4.6)$$

is called as a discrete fractional model [6].

Note that in model (4.3) and (4.6), there are the same parameters, $a_0$ and $k$.

We aim to estimate the initial parameters $a_0$ and $k$ and use these parameters for both models. For estimation, we use the Mathematica FindFit command. Then we substitute parameters into each model to determine the square of residual error (SQR). The square of residual error or residual sum of squares (RSS) is a statistical computational technique. It is the sum of squares of residuals. A residual error is the difference between observed data ($Y_i$) and predicted data ($\hat{Y}_i$) by an estimation model and is given by $e_i = Y_i - \hat{Y}_i$. The SQR is defined by

$$SQR = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2.$$

We use SQR to determine the fit of each model. A smaller SQR indicates a better fitting model. For more precise results, we use the mean squared error (MSE) which is given by

$$MSE = \frac{\sum_{i=1}^{n} e_i^2}{n} = \frac{SQR}{n}.$$
The best fit has the least mean squared error [2].

<table>
<thead>
<tr>
<th>Time(min)</th>
<th>Concentration(mg/L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>920</td>
</tr>
<tr>
<td>20</td>
<td>800</td>
</tr>
<tr>
<td>30</td>
<td>750</td>
</tr>
<tr>
<td>40</td>
<td>630</td>
</tr>
<tr>
<td>50</td>
<td>610</td>
</tr>
<tr>
<td>60</td>
<td>530</td>
</tr>
<tr>
<td>70</td>
<td>520</td>
</tr>
<tr>
<td>90</td>
<td>380</td>
</tr>
<tr>
<td>110</td>
<td>350</td>
</tr>
<tr>
<td>150</td>
<td>200</td>
</tr>
</tbody>
</table>

Table 4.1.1. Data of drug concentration for various time points of intravenous bolus dosing

Table 4.1.1 shows data of drug concentration versus time points of intravenous bolus dosing, taken from Gabrielsson [7]. For comparison, the continuous model, the continuous fractional model, the discrete model and the discrete fractional model are given below.

Continuous : \[ y(t) = a_0 e^{-kt} \]

Continuous Fractional : \[ y(t) = a_0 t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-kt^\alpha)^n}{\Gamma(\alpha(n+1))}, \text{ where } 0 < \alpha \leq 1 \]

Discrete : \[ y(t) = a_0 (1 - k)^t \]

Discrete Fractional : \[ y(t) = a_0 \sum_{i=0}^{\infty} \frac{(-k)^i}{\Gamma((i+1)\nu)}(t + i(\nu - 1))^{(i\nu+\nu-1)}, \text{ where } 0 < \nu \leq 1. \]
For the continuous models, the parameters \((a_0, K)\) and the MSE value 432.738 are given by Gabrielsson [7] and the least MSE value 426.202 at \(\alpha = 0.98\) of the continuous fractional model is obtained by Amera Almusharff and Ngoc Nguyen [2]. Note that at \(\alpha = 0.98\), the data points are closer to the curve than other \(\alpha\) values. Based on the MSE values in Table 4.1.2, we conclude that the fractional model gives better fitting to the given data compared to the traditional model.

<table>
<thead>
<tr>
<th>Model</th>
<th>(a_0)</th>
<th>(K)</th>
<th>SQR</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>1002.16</td>
<td>0.0102561</td>
<td>4327.377823</td>
<td>432.7377</td>
</tr>
<tr>
<td>Continuous Fractional</td>
<td>1070.00</td>
<td>0.0104340</td>
<td>4262.016023</td>
<td>426.2016</td>
</tr>
</tbody>
</table>

Table 4.1.2. Data analysis of drug concentration by the continuous models

Using the *Mathematica* FindFit command, we estimated parameters of the discrete model as \(a_0 = 1002.16\) and \(K = 0.0102037\). To obtain the predicted drug concentration of the discrete fractional model, we used the same parameters. We fit the initial parameter \(a_0 = 1002.16\) as a fixed parameter to obtain the \(k\) values by a guessing method with different \(\nu\)-values to determine SQR and MSE.

We used *Mathematica* to determine square of residual error (SQR) and the mean square error (MSE) for each model. In Table 4.1.3, at \(\nu = 0.98\), the data points are closer to the curve or the discrete fractional model has the smaller MSE than other \(\alpha\) values. We then obtained the discrete MSE to be 432.7377 and the discrete fractional MSE to be 424.2536 as shown in Table 4.1.4. It is shown that the discrete fractional model has the least MSE of 424.2536. We then conclude that discrete fractional model gives us the better fit compared to the discrete model.
Table 4.1.3. Data analysis of drug concentration by the discrete fractional models for each $\nu$-values

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>761.6698</td>
</tr>
<tr>
<td>0.98</td>
<td>424.2536</td>
</tr>
<tr>
<td>0.97</td>
<td>774.1716</td>
</tr>
<tr>
<td>0.96</td>
<td>1772.6716</td>
</tr>
<tr>
<td>0.95</td>
<td>3381.7067</td>
</tr>
<tr>
<td>0.94</td>
<td>5563.9444</td>
</tr>
<tr>
<td>0.93</td>
<td>8282.7822</td>
</tr>
<tr>
<td>0.92</td>
<td>11794.9365</td>
</tr>
<tr>
<td>0.91</td>
<td>11502.3692</td>
</tr>
<tr>
<td>0.90</td>
<td>15187.6329</td>
</tr>
</tbody>
</table>

Table 4.1.4. Data analysis of drug concentration by the discrete models

<table>
<thead>
<tr>
<th>Model</th>
<th>$a_0$</th>
<th>$K$</th>
<th>SQR</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete</td>
<td>1002.16</td>
<td>0.0102037</td>
<td>4327.377828</td>
<td>432.7377</td>
</tr>
<tr>
<td>Discrete Fractional</td>
<td>1073.00</td>
<td>0.0104340</td>
<td>4242.536025</td>
<td>424.2536</td>
</tr>
</tbody>
</table>

Figure 4.1.1. Data fitting the discrete models
To see the significance of different MSE, the graphs of both the discrete and the discrete fractional models are provided in Figure 4.1.1 to support it. Note that both curves are overlapping.

The discrete model and the discrete fractional model for predicted drug concentration is given below.

\[
y(t) = 1002.16(0.989796)^t.
\]

\[
y(t) = 1073 \sum_{i=0}^{\infty} \left( \frac{-0.010434}{i!} \right) \frac{1}{\Gamma((i+1)\nu)} (t + i(\nu - 1))^{(\nu+1-\nu)} \text{ where } \nu = 0.98.
\]

Tables 4.1.2 and 4.1.4 show that the fractional model gives better fitting compared to the traditional model. In Table 4.1.5, we then compare the continuous fractional model when \( \alpha = 0.98 \) and the discrete fractional model when \( \nu = 0.98 \). Based on MSE, we see that the discrete fractional model has the least MSE. We conclude that the discrete fractional model is the better fit.

<table>
<thead>
<tr>
<th>Model</th>
<th>( a_0 )</th>
<th>( K )</th>
<th>SQR</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>1002.16</td>
<td>0.0102561</td>
<td>4327.377823</td>
<td>432.7377</td>
</tr>
<tr>
<td>Continuous Fractional</td>
<td>1070.00</td>
<td>0.0104340</td>
<td>4262.016023</td>
<td>426.2016</td>
</tr>
<tr>
<td>Discrete</td>
<td>1002.16</td>
<td>0.0102037</td>
<td>4327.377828</td>
<td>432.7377</td>
</tr>
<tr>
<td>Discrete Fractional</td>
<td>1073.00</td>
<td>0.0104340</td>
<td>4242.536025</td>
<td><strong>424.2536</strong></td>
</tr>
</tbody>
</table>

**Table 4.1.5.** Data analysis of drug concentration by MSE
In Figure 4.1.2, since MSEs of each model are close to each other, all curves are overlapping.

4.2. Best Prediction for One Compartmental Model by Cross Validation

In the previous section, we obtained that the discrete fractional model gives us the best fit (Table 4.1.4) with particular data. For any given data, can we predict that the discrete fractional model will yield the best fit as well? To address this issue, we use the cross validation method to determine the best model. Cross validation is a statistical technique for estimating the performance of a predictive model. There are a few kinds of cross validation. We consider k-fold cross validation, which is a common type. The data set is divided into k folds (subsets). One of the k folds is chosen as the validation (test) set and the other k-1 folds are used as a training set. Then the cross validation process is repeated k times (folds). The advantage of this method is that every observation is used both for validation (exactly once) and training [3].
In this thesis, there are 10 independent observations $Y_1, Y_2, \ldots, Y_{10}$. We chose the $Y_i$ observation where $i = 1, 2, \ldots, 10$ as the validation set and fit the model for the training set (remaining data) to determine the parameters. Then we calculate the predicted value ($\hat{Y}$) and residual which correspond to observed value ($Y$) in the test set by using those parameters. We repeat the same process for each observation and calculate SQR. Then we use MSE to determine the best prediction.

For the data in Table 4.2.1, we used the following steps to calculate MSE using cross validation. Note that we choose the continuous data id #1 as an example for demonstration. Then we repeat the same process for all models.

Step 1. Choose data id # 1 as the validation set and leave the rest (id # 2-10) as the training set.

Step 2. Determine parameters $a_0$ and $K$ by using Mathematica FindFit command.

Step 3. Substitute parameters into four models to get a predicted value $\hat{Y}=891.949$.

Step 4. Find SQR by using a predicted value $\hat{Y}=891.949$ and determine MSE.

Step 5. Repeat the same process for other id#s.
<table>
<thead>
<tr>
<th>id</th>
<th>Time</th>
<th>Continuous</th>
<th>Continuous Fractional</th>
<th>Discrete</th>
<th>Discrete Fractional</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>MSE</td>
<td>Y</td>
<td>MSE</td>
<td>Y</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>891.949</td>
<td>424.31150</td>
<td>909.347</td>
<td>419.13419</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>822.008</td>
<td>409.39451</td>
<td>817.111</td>
<td>408.22310</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>734.196</td>
<td>416.99470</td>
<td>728.690</td>
<td>392.76857</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>669.720</td>
<td>311.21553</td>
<td>663.965</td>
<td>339.53786</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>598.819</td>
<td>424.90615</td>
<td>594.784</td>
<td>406.71100</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
<td>543.311</td>
<td>419.42932</td>
<td>540.242</td>
<td>417.54254</td>
</tr>
<tr>
<td>7</td>
<td>70</td>
<td>483.344</td>
<td>335.81668</td>
<td>486.325</td>
<td>326.21627</td>
</tr>
<tr>
<td>8</td>
<td>90</td>
<td>402.425</td>
<td>401.56036</td>
<td>402.544</td>
<td>380.24875</td>
</tr>
<tr>
<td>9</td>
<td>110</td>
<td>317.309</td>
<td>368.84914</td>
<td>330.269</td>
<td>388.94753</td>
</tr>
<tr>
<td>10</td>
<td>150</td>
<td>219.374</td>
<td>414.39863</td>
<td>226.193</td>
<td>359.70112</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>392.68750</td>
<td>383.90309</td>
<td>382.35177</td>
<td>382.35177</td>
</tr>
</tbody>
</table>

Table 4.2.1. Data analysis of drug concentration by cross validation

Comparing the MSE of the continuous models, the continuous fractional model has the smaller MSE at 383.90309. We obtain that the fractional model is the better continuous model. For the MSE of the discrete and discrete fractional models, the better model is the fractional model, with a smaller MSE of 382.35177. The last comparison is between the fractional models. Base on MSE of 383.90309 and 382.35177, it is obvious that the discrete fractional model has the lower MSE. We obtain that discrete fractional model is the best model for prediction. This supports our claim that the discrete fractional calculus gives us the best fit and the best model.
In this thesis, linear fractional difference equations were considered. We analyzed the existence of Green’s functions as solutions of difference equations with discrete boundary value problems and with fractional boundary value problems. We employed the traditional method to obtain the Green’s functions instead of using the Cauchy functions. Then, we demonstrated that the Green’s function of fractional boundary conditions (FBVP) generalized to the existence results of the Green’s functions of discrete boundary conditions (DBVP) in Section 3.2.

Recently, the discrete fractional calculus has been employed as a tool to analyze real world problems. For example, determining accurate levels of drug concentration in the blood stream is impractical in clinical study so mathematical principles are applied. In this thesis, we applied the discrete fractional calculus to model problems in a field of pharmacology known as pharmacokinetics. A one-compartmental model where the correlation of drug concentration and time was considered. We established that the discrete fractional calculus model yields both the best fit and the highest prediction performance, compared with the continuous fractional calculus results.

In the future, existence of solutions of fractional difference equations with higher order, such as $2 < \nu \leq 3$, or nonlinear fractional difference equations and applications of fractional calculus to a compartmental model with two or more compartments will be considered. To work with these compartments, we will have a system of two or more equations instead of a single equation. Last, using more data
points will allow us to determine the percentage improvement of MSE in the model prediction.
Using the one-compartmental model with a rapid intravenous injection of a drug that distributes to the body and is eliminated, the continuous model, the continuous fractional model, the discrete model, and the discrete fractional model of the rate of loss of the drug from the body are given below.

Let $\alpha = 0.98$ and $\nu = 0.98$, and suppose $y$ is the concentration of the drug in the body at time $t$ after injection, $k$ is constant apparent first-order elimination rate of the drug, and $a_0$ is the initial concentration at $t = 0$.

The negative sign means drug is eliminated from the body.

The linear continuous differential equation with an initial condition is given by

**Equation:** $y' = -ky,$

**Initial Condition:** $y(0) = a_0$,

**Solution:** $y(t) = a_0 e^{-kt}$.

The solution is called a **continuous model** [7].

The linear continuous fractional difference equation with an initial condition is given by

**Equation:** $D^{\alpha} y = -ky,$
Initial Condition: \[ y(0) = a_0, \]

Solution: \[
y(t) = a_0 t^{\alpha-1} E_{\alpha,\alpha}(-kt) \\
= a_0 t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-kt)^n}{\Gamma(\alpha(n+1))},
\]
where \( E_{\alpha,\alpha}(-kt) = \sum_{n=0}^{\infty} \frac{(-kt)^n}{\Gamma(\alpha(n+1))}, \alpha,\alpha > 0. \)

The solution is called a **continuous fractional model** [1].

The linear discrete difference equation with an initial condition is given by

Equation: \[ \Delta y(t) = -ky(t), \]

Initial Condition: \[ y(0) = a_0, \]

Solution: \[ y(t) = a_0 (1-k)^t. \]

The solution is called a **discrete model**.

The linear discrete fractional difference equation with an initial condition is given by

Equation: \[ \Delta^v y(t) = -ky(t+v-1), \]

Initial Condition: \[ y(0) = a_0, \]

Solution: \[ y(t) = a_0 \sum_{i=0}^{\infty} \frac{(-k)^i}{\Gamma((i+1)v)} (t+i(v-1))^{(iv+v-1)}. \]

The solution is called a **discrete fractional model** [6].
BIBLIOGRAPHY


