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Higher Derivatives of the Hurwitz Zeta Function

Jason Musser
Western Kentucky University, jason.musser@topper.wku.edu

HIGHER DERIVATIVES OF THE HURWITZ ZETA FUNCTION

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Dominic Lanphier, Director of Thesis

Tilak Bhattacharya

Claus Ernst

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Dean, Graduate Studies and Research Date

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Directed by: Dominic Lanphier, Tilak Bhattacharya, Claus Ernst

Department of Mathematics Western Kentucky University

The Riemann zeta function $\zeta(s)$ is one of the most fundamental functions in number theory. Euler demonstrated that $\zeta(s)$ is closely connected to the prime numbers and Riemann gave proofs of the basic analytic properties of the zeta function. Values of the zeta function and its derivatives have been studied by several mathematicians. Apostol in particular gave a computable formula for the values of the derivatives of $\zeta(s)$ at $s = 0$.

The Hurwitz zeta function $\zeta(s, q)$ is a generalization of $\zeta(s)$. We modify Apostol's methods to find values of the derivatives of $\zeta(s, q)$ with respect to s at $s = 0$. As a consequence, we obtain relations among certain important constants, the generalized Stieltjes constants. We also give numerical estimates of several values of the derivatives of $\zeta(s, q)$.

Chapter 1

The Riemann Zeta and Hurwitz Zeta Functions

In this chapter an introduction, definitions of the gamma function, the Riemann zeta function, and the Hurwitz zeta function are given. Various properties of each function including integral representations that give their meromorphic continuation to $\mathbb{C}$ which can be found in [1] and [6] are also shown. This is done in preparation for ultimately discussing the derivatives of the Hurwitz zeta function. Many of these properties can be found in any introductory analytic number theory text.

1.1 Introduction

In 1859 Bernhard Riemann discussed the significance of the Riemann zeta function $\zeta(s)$ in his paper *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (*On the Number of Primes Less Than a Given Magnitude*). He outlined a method whereby one can study prime numbers from the analytic properties of $\zeta(s)$. These ideas culminated in a proof of the celebrated prime number theorem in 1900 by Hadamard and de la Vallée Poussin, which gives an asymptotic formula for the number of primes less than a given value.

In Riemann's original paper he stated a conjecture that is now famously known as the Riemann hypothesis. The critical strip of the Riemann zeta function is the set $s \in \mathbb{C}$ so that $0 < \mathrm{Re}(s) < 1$. From properties of $\zeta(s)$ we know that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$. The analytic continuation of $\zeta(s)$ allows us to define $\zeta(s)$ for any $s \in \mathbb{C}$, $s \neq 1$. It can be shown that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) \geq 1$ and this result implies the Prime Number Theorem. It is known that there are values $\sigma_0 = \frac{1}{2} + it$ so that $\zeta(\sigma_0) = 0$. The Riemann hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on the vertical line at $\mathrm{Re}(s) = \frac{1}{2}$. Therefore, the Riemann hypothesis asserts that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2}$. Although it has not been shown to be true or false, many zeros lie on this line. One of the reasons the Riemann hypothesis is important is because given that the Riemann hypothesis is true, further very powerful deductions can be made about the distribution of prime numbers.

To further study the Riemann zeta function, Adolf Hurwitz defined a more generalized form of the Riemann zeta function refered to as the Hurwitz zeta function $\zeta(s, q)$. Although the relationship to the prime numbers is lost in the generalization, insight can be gained on the Riemann zeta function by studying the Hurwitz zeta function due to relationships between the two.

One of the properties that has been studied is the derivatives of the Riemann zeta function at $s = 0$ [2]. As with $\zeta(s)$, the Hurwitz zeta function has an analytic continuation to all $s \in \mathbb{C}$, $s \neq 1$. Thus $\zeta(s, q)$ is analytic at $s = 0$. The Taylor series at $s = 0$ has a radius of convergence 1. In this thesis we find the derivatives at $s = 0$ of the Hurwitz zeta function by modifying methods used to study $\zeta(s)$. Some numerical results are also found. For all instances that refer to Mathematica the software Mathematica 7.0 was used. Note that in all chapters $f^{(\ell)}(s) = \frac{d^\ell f}{ds^\ell}$ is defined for any meromorphic f.

1.2 The Gamma Function

The gamma function $\Gamma(n)$ is defined as follows from Chapter one of [6] : For $n \in \mathbb{Z}$, $n > 0$,

$$\Gamma(n) = (n-1)!, \ n = 1, 2, \ldots \text{ and } 0! = 1.$$

The gamma function has an integral representation from chapter one of [6] given by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

that converges absolutely for $\mathrm{Re}(s) > 0$ and allows a definition of $\Gamma(s)$ for all $s \in \mathbb{C}$. The integral converges for all complex s with $s \neq 0, -1, -2, \cdots$. The gamma function has simple poles at these points.

Some selected properties of the gamma function follow. Three formulas that are functional equations of the gamma function are given. From integration by parts the recursion formula can be obtained ((1.1.6) in [6])

$$\Gamma(s+1) = s\Gamma(s) \text{ and } \Gamma(n) = (n-1)!,\ n = 1, 2, \dots$$

Euler's reflection formula ((1.2.1) in [6]) states

$$\Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin(\pi n)}.$$

The duplication formula ((1.5.1) in [6]) states

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\Gamma\left(\frac{1}{2}\right)\Gamma(2z)$$

A well known value of the gamma function ((2.14) in [3]) is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{1}$$

1.3 The Riemann Zeta Function

The Riemann zeta function $\zeta(s)$ as found in Chapter twelve of [1] is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\mathrm{Re}(s) > 1$. It has an analytic continuation to all $s = \sigma + it$ except for a pole at $s = 1$ with residue 1 as found in the introduction of [10], and it can be obtained from the integral representation

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}dz.$$

The contour C starts at infinity on the positive real axis, encircles the origin once, excluding the points $\pm 2\pi i, \pm 4\pi i, \ldots$ and returns to where it begins. $\Gamma(s)$ is the gamma function. Also, if $\mathrm{Re}(s) > 1$ then we can express the product of the gamma function and Riemann zeta function as found in the introduction of [10] by

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

In the introduction of [10], Euler found that for $\mathrm{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\forall \text{ primes } p} \frac{1}{1 - p^{-s}}.$$

This shows an interesting relationship between the Riemann zeta function and the prime numbers.

Like the gamma function, the Riemann zeta function obeys certain formulas. For example, the Riemann zeta function as found in Chapter twelve of [1] has a functional equation

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \ \forall s \in \mathbb{C}. \tag{2}$$

Many values of the Riemann zeta function are particularly interesting. Values of the Riemann zeta function have a connection with the Bernoulli numbers. The Bernoulli numbers B_m are the coefficients of the following series expansion ((1.2.10)in [6])

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m t^m}{m!} \tag{3}$$

for $t \in \mathbb{R}$ and $t \neq 0$. For example $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$,

$B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$. If k is an even and positive integer we have ((1.2.11)in [6])

$$\zeta(k) = (-1)^{\frac{k}{2}+1} \frac{B_k (2\pi)^k}{(2k)!}$$

As examples, this equation gives

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

These results were first discovered by Euler.

1.4 The Hurwitz Zeta Function

The Hurwitz zeta function $\zeta(s, q)$ is a generalized form of the Riemann zeta function and as found in Chapter twelve of [1] and is defined as

$$\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}$$

where $\mathrm{Re}(s) > 1$ and $0 < q \leq 1$. Note that $\zeta(s, 1) = \zeta(s)$.

The Hurwitz zeta function also has an analytic continuation for all $s = \sigma + it$ (except for the pole at $s = 1$ with residue 1) as found in chapter twelve of [1] and is given by

$$\zeta(s, q) = \Gamma(1-s) \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^{qz}}{1 - e^z} dz.$$

The contour C loops around the negative real axis by starting at negative infinity on the real axis, encircles the origin once and returns to where it begins.

Similar to the Riemann zeta function, the Hurwitz Zeta function also has a functional equation as found in Chapter twelve of [1] and is given by

$$\zeta\left(1-s,\frac{h}{k}\right)=\frac{2\Gamma(s)}{(2\pi k)^s}\sum_{r=1}^{k}\left[\cos\left(\frac{\pi s}{2}-\frac{2\pi rh}{k}\right)\zeta\left(s,\frac{r}{k}\right)\right],\ \forall s\in\mathbb{C} \tag{4}$$

where h and k are integers and 1≤ h ≤ k. Similar to the Riemann zeta function and the gamma function, the Hurwitz zeta function obeys several identities. The Hurwitz zeta function also has the following integral representation as found in chapter twelve of [1]

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-qx}}{1-e^{-x}}dx$$

where $\mathrm{Re}(s)>1$ and $0<q\leq 1$. Note that, however, the Hurwitz zeta function does not have an Euler product expansion over the prime numbers. Hurwitz discovered a formula which as found in Chapter twelve of [1] is refered to as Hurwitz's formula. Given that $\mathrm{Re}(s)>1$ and $t\in\mathbb{R}$ we have

$$F(t,s)=\sum_{n=1}^{\infty}\frac{e^{2\pi int}}{n^s}$$

where $F(t,s)$ is periodic with period 1 and is called the periodic zeta function. If $\mathrm{Re}(s)>1$ then $F(t,s)$ converges absolutely. Note that $F(1,s)=\zeta(s)$. Hurwitz's formula is

$$\zeta(1-s,q)=\frac{\Gamma(s)}{(2\pi)^s}\left[e^{\frac{-i\pi s}{2}}F(q,s)+e^{\frac{i\pi s}{2}}F(-q,s)\right]$$

where $\mathrm{Re}(s)>1$ and $0<q\leq 1$. Hurwitz's formula is also valid if $q\neq 1$ and $\mathrm{Re}(s)>0$. As with the Riemann zeta function, some values of the Hurwitz zeta function can be obtained explicitly. We have a formula for the values $\zeta(-n,q)$ for

$n = 0, 1, 2, \ldots$ as found in Chapter twelve of [1]. Given

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \ \forall x \in \mathbb{C}, \ n = 0, 1, 2, \ldots,$$

where $B_n(0) = B_n$ and B_k are the Bernoulli numbers previously defined we have

$$\zeta(-n, q) = -\frac{B_{n+1}(q)}{n+1}, \ n = 0, 1, 2, \ldots$$

with $0 < q \leq 1$. The formula for the derivative of $\zeta(s, q)$ with respect to the second arguement q is

$$\frac{\partial}{\partial q}\zeta(s, q) = -s\zeta(s+1, q),$$

where $0 < q \leq 1$ and $s \neq 0, 1$. A multiplication theorem for the second variable of the Hurwitz zeta function can be found in [11]:

$$\zeta(s, kz) = \sum_{n=0}^{\infty} \binom{s+n-1}{n} (1-k)^n z^n \zeta(s+n, z),$$

where $\mathrm{Re}(s) > 1$ and $0 < kz \leq 1$.

Chapter 2

The Stieltjes Constants and Euler's Constant

2.1 The Stieltjes Constants

The generalized Stieltjes constants are denoted as $\gamma_n(q)$ and arise when considering the Laurent expansion of the Hurwitz zeta function. Note that $\gamma_n(q)$ is a function with arguement q not to be confused with Stieltjes constants $\gamma_n = \gamma_n(1)$. From Berndt [4] for $0 < q \leq 1$ and for all complex $s \neq 1$,

$$\zeta(s,q) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{n!} \gamma_n(q)(s-1)^n \right]. \tag{5}$$

Berndt [4] shows the following.

Theorem (Theorem 1 [4]) For $0 < q \leq 1$ and $n = 0, 1, 2, \cdots$

$$\gamma_n(q) = \lim_{m\to\infty} \left(\sum_{k=0}^{m} \frac{\log^n(k+q)}{k+q} - \frac{\log^{n+1}(m+q)}{n+1} \right) \tag{6}$$

It is unknown whether $\gamma_n(q)$ is rational or irrational for any q.

The Stieltjes constants denoted γ_n arise when considering the Laurent expansion of the Riemann zeta function which can be found by taking $q = 1$ in (5), [2].

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n \frac{(-1)^n}{n!}(s-1)^n \\ &= \frac{1}{s-1} + \gamma_0 - \gamma_1(s-1) + \gamma_2 \frac{(s-1)^2}{2!} + \cdots \end{aligned} \tag{7}$$

which has one pole at $s = 1$ with residue 1, where $\gamma_n = \gamma_n(1)$. We have that

$\gamma_n(1) = \gamma_n$, and from [2] the nth Stieltjes constant is expressed as

$$\gamma_n = \lim_{m\to\infty} \left[\left(\sum_{k=1}^{m} \frac{(\log(k))^n}{k} \right) - \frac{(\log(m))^{n+1}}{n+1} \right]. \tag{8}$$

2.2 Euler's Constant

Euler's constant γ_0 as defined in the introduction of [10] is defined by

$$\begin{aligned} \gamma_0 &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) \\ &\approx 0.57721\ldots \end{aligned} \tag{9}$$

As with $\gamma_n(q)$ and γ_n very little is known about the arithmetic properties of γ_0. Euler's constant is the first in the sequence of the Stieltjes constants γ_n. Some properties of Euler's constant follow. We first give some relationships of γ_0 to the derivatives of the gamma function discussed previously. From [2] we have

$$\begin{aligned} \Gamma'(1) &= -\gamma_0 \\ \Gamma''(1) &= \gamma_0^2 + \zeta(2) = \gamma_0^2 + \frac{\pi^2}{6} \\ \Gamma'''(1) &= -\gamma_0^3 - 3\gamma_0\zeta(2) - 2\zeta(3) = -\gamma_0^3 - \gamma_0\frac{\pi^2}{2} - 2\zeta(3) \end{aligned}$$

From Chapter six of [7],

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma_0 + \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+x} \right) \text{ for } x > 0 \text{ and } x \in \mathbb{R}.$$

For example at $x = 1$ we have

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma_0.$$

γ_0 also has an integral representation from chapter twelve of [7] we have

$$\gamma_0 = \int_0^1 \frac{1 - e^{-u} - e^{-\frac{1}{u}}}{u} du$$

The Laurent expansion of $\Gamma(s)\zeta(s)$ ((7) in [2]) involves Euler's constant and the other Stieltjes constants γ_n. It is given by

$$\Gamma(s)\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} c_n (s-1)^n, \tag{10}$$

where

$$c_n = \frac{\Gamma^{(n+1)}(1)}{(n+1)!} \sum_{k=0}^{n} \gamma_k \frac{\Gamma^{(n-k)}(1)}{(n-k)!}. \tag{11}$$

In particular for $n = 0, 1$ and 2 (11) gives

$$\begin{aligned}
c_0 &= \gamma_0, \\
c_1 &= -\frac{\gamma_0^4}{2} + \frac{\gamma_0^2 \gamma_1}{2} - \gamma_0^2 \zeta(2) + \gamma_1 \zeta(2), \\
c_2 &= -\left(\frac{\gamma_0^3}{6} + \frac{2\zeta(2)\gamma_0}{3} + \zeta(3) \right) \left(\frac{\gamma_0^3}{2} + \gamma_0 \zeta(2) - \gamma_1 \gamma_0 + \gamma_2 \right).
\end{aligned}$$

Chapter 3

Higher Derivatives of the Riemann Zeta Function

We review a method to find the values of $\zeta^{(n)}(0)$ for arbitrary n. The value $\zeta(0)$ was found by Riemann, as was $\zeta^{'}(0)$. Ramanujan found $\zeta^{''}(0)$. Apostol [2] found a formula for $\zeta^{(n)}(0)$.

3.1 Series Expansion of the Functional Equation

Using (2), the functional equation of $\zeta(s)$, derivatives of the Riemann zeta function at $s = 0$ were found by Apostol [2]. His method from [2] is as follows. Start with (2) and some manipulation gives an alternate form of the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s).$$

Rewriting in yet another form, by substituting

$$2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right) = e^{\frac{i\pi s}{2} - s\log(2\pi)} + e^{-\frac{i\pi s}{2} - s\log(2\pi)},$$

this gives

$$\zeta(1-s) = \Gamma(s)\zeta(s)\left[e^{-s\left(\log(2\pi) - \frac{\pi i}{2}\right)} + e^{-s\left(\log(2\pi) + \frac{\pi i}{2}\right)}\right]. \tag{12}$$

Taking the kth derivative of both sides by using Leibniz's rule found in Chapter five of [5]

$$[f(z)g(z)]^{(k)} = \sum_{m=0}^{k} \binom{k}{m} f^{(m)}(z)g^{(k-m)}(z)$$

Apostol then shows that for any $k \geq 1$ and for all $s \in \mathbb{C}$ we have

$$
\begin{aligned}
&(-1)^k \zeta^{(k)}(1-s) \\
&= \sum_{m=0}^{k} \binom{k}{m} \left[e^{-s\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2}\right)^{k-m} (-1)^{k-m} \right. \\
&\qquad \left. + e^{s\left(-\log(2\pi)+\frac{\pi i}{2}\right)} \left(-\log(2\pi) + \frac{\pi i}{2}\right)^{k-m} \right] [\Gamma(s)\zeta(s)]^{(m)}. \\
&= e^{-s\log(2\pi)} \sum_{m=0}^{k} \binom{k}{m} \left[e^{-\frac{s\pi i}{2}} \left(\log(2\pi) + \frac{\pi i}{2}\right)^{k-m} (-1)^{k-m} \right. \\
&\qquad \left. + e^{\frac{s\pi i}{2}} \left(-\log(2\pi) + \frac{\pi i}{2}\right)^{k-m} \right] [\Gamma(s)\zeta(s)]^{(m)}.
\end{aligned} \tag{13}
$$

Apostol then rewrites (13) in terms of the sum of its real and imaginary parts.

$$
\begin{aligned}
&(-1)^k \zeta^{(k)}(1-s) \\
&= \sum_{m=0}^{k} \binom{k}{m} \left[\mathrm{Re}\left[\left(-\log(2\pi) - \frac{\pi i}{2}\right)^{k-m} \right] \cos\left(\frac{\pi s}{2}\right) \right. \\
&\qquad \left. + \mathrm{Im}\left[\left(-\log(2\pi) - \frac{\pi i}{2}\right)^{k-m} \right] \sin\left(\frac{\pi s}{2}\right) \right] [\Gamma(s)\zeta(s)]^{(m)}.
\end{aligned} \tag{14}
$$

Where $\mathrm{Re}[z]$ is the real part of z and $\mathrm{Im}[z]$ is the imaginary part of z. Leibniz's rule is then applied to $[\Gamma(s)\zeta(s)]^{(m)}$ in (14) to give

$$
\begin{aligned}
&(-1)^k \zeta^{(k)}(1-s) \\
&= \sum_{m=0}^{k} \sum_{r=0}^{m} \binom{k}{m} \binom{m}{r} \left[\mathrm{Re}\left[\left(-\log(2\pi) - \frac{\pi i}{2}\right)^{k-m} \right] \cos\left(\frac{\pi s}{2}\right) \right. \\
&\qquad \left. + \mathrm{Im}\left[\left(-\log(2\pi) - \frac{\pi i}{2}\right)^{k-m} \right] \sin\left(\frac{\pi s}{2}\right) \right] \Gamma^{(r)}(s)\zeta^{(m-r)}(s).
\end{aligned} \tag{15}
$$

As with (13), for certain sets of integer values of s, (15) can be simplified. Some examples follow. Taking $s = 2n+1$, $n \in \mathbb{N}$ and $k \geq 1$ in (15) then

$$\begin{aligned}(-1)^k \zeta^{(k)}(-2n) &\\ &= \frac{2(-1)^n}{(2\pi)^{2n+1}} \sum_{m=0}^{k} \sum_{r=0}^{m} \left[\binom{k}{m} \binom{m}{r} \right. \\ &\qquad \left. \operatorname{Im} \left[\left(-\log(2\pi) - \frac{\pi i}{2} \right)^{k-m} \right] \Gamma^{(r)}(2n+1) \zeta^{(m-r)}(2n+1) \right].\end{aligned}$$

Taking $s = 2n$, $n \in \mathbb{N}$ and $k \geq 1$ in (15) then

$$\begin{aligned}(-1)^k \zeta^{(k)}(1-2n) &\\ &= \frac{2(-1)^n}{(2\pi)^{2n}} \sum_{m=0}^{k} \sum_{r=0}^{m} \left[\binom{k}{m} \binom{m}{r} \right. \\ &\qquad \left. \operatorname{Re} \left[\left(-\log(2\pi) - \frac{\pi i}{2} \right)^{k-m} \right] \Gamma^{(r)}(2n) \zeta^{(m-r)}(2n) \right].\end{aligned}$$

Now series expansions of the members of (12) can be found. Since $\zeta(1-s)$ is analytic at $s = 1$ the expansion at $s = 1$ follows

$$\zeta(1-s) = \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{(n)}(0)}{n!} (s-1)^n. \tag{16}$$

Looking at the exponential factor in (12), we can expand it in a Taylor series about $s = 1$,

$$\begin{aligned} e^{-s\left(\log(2\pi)+\frac{\pi i}{2}\right)} &= e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} e^{(s-1)\left(-\log(2\pi)-\frac{\pi i}{2}\right)} \\ &= \sum_{n=0}^{\infty} e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2} \right)^n \frac{(-1)^n}{n!} (s-1)^n \end{aligned} \tag{17}$$

Previously in (10) we have the Laurent expansion of the product $\Gamma(s)\zeta(s)$,

$$\Gamma(s)\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} c_n (s-1)^n,$$

where as before

$$c_n = \frac{\Gamma^{(n+1)}(1)}{(n+1)!} \sum_{k=0}^{n} \gamma_k \frac{\Gamma^{(n-k)}(1)}{(n-k)!},$$

and γ_n denote the Stieltjes constants as in (8). Using the expansions of $\Gamma(s)\zeta(s)$ from (10) and $e^{-s\left(\log(2\pi)+\frac{\pi i}{2}\right)}$ from (17), we take the product and have the Laurent expansion

(18)

$$\begin{aligned}
&\Gamma(s)\zeta(s)e^{-s\left(\log(2\pi)+\frac{\pi i}{2}\right)} \\
&\quad = \left(\frac{1}{s-1} + \sum_{n=0}^{\infty} c_n (s-1)^n\right) \\
&\qquad\qquad \left(\sum_{n=0}^{\infty} e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2}\right)^n \frac{(-1)^n}{n!}(s-1)^n\right) \\
&\quad = e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \frac{1}{s-1} \\
&\qquad + \sum_{n=0}^{\infty} \left[e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} \right. \\
&\qquad + \left. \sum_{k=n}^{\infty} c_k e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2}\right)^{n-k} \frac{(-1)^{n-k}}{(n-k)!} \right] (s-1)^n.
\end{aligned}$$

The numbers c_k are defined as in (11). Starting from the functional equation (12) and applying the product of $\Gamma(s)\zeta(s)$ with $e^{-s\left(\log(2\pi)+\frac{\pi i}{2}\right)}$ from (18) above, and equating the coefficients of the $(s-1)^n$ terms for $n \geq 0$, Apostol then shows

$$
\begin{aligned}
&(-1)^n \frac{\zeta^{(n)}(0)}{n!} \\
&\quad = e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} \\
&\qquad + e^{-\left(\log(2\pi)-\frac{\pi i}{2}\right)} \left(\log(2\pi) - \frac{\pi i}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} \\
&\qquad + \sum_{k=0}^{n} c_k \left[e^{-\left(\log(2\pi)+\frac{\pi i}{2}\right)} \left(\log(2\pi) + \frac{\pi i}{2}\right)^{n-k} \frac{(-1)^{n-k}}{(n-k)!} \right. \\
&\qquad\qquad \left. + e^{-\left(\log(2\pi)-\frac{\pi i}{2}\right)} \left(\log(2\pi) - \frac{\pi i}{2}\right)^{n-k} \frac{(-1)^{n-k}}{(n-k)!} \right].
\end{aligned}
$$

After simplifying the above formula Apostol obtains the following Theorem.

Theorem 1 [2] For $n \geq 0$

$$
\begin{aligned}
&(-1)^n \frac{\zeta^{(n)}(0)}{n!} \\
&\quad = \mathrm{Im}\left[\left(-\log(2\pi) - \frac{\pi i}{2}\right)^{n+1}\right] \frac{1}{\pi(n+1)!} \\
&\qquad + \frac{1}{\pi}\sum_{k=1}^{n-1} c_k \mathrm{Im}\left[\left(-\log(2\pi) - \frac{\pi i}{2}\right)^{n-k}\right] \frac{1}{(n-k)!},
\end{aligned}
\tag{19}
$$

where the numbers c_k are defined as in (11).

This formula can be used to find exact values of $\zeta^{(n)}(0)$ for small n. It can also be used to find good numerical approximations to $\zeta^{(n)}(0)$ for large n. Using (19), $\zeta^{(n)}(0)$ is given below for $0 \leq n \leq 4$. The first two results were obtained by Riemann, the third was obtained first by Ramanujan, and the subsequent formulas were obtained by Apostol.

$$\zeta(0) = -\frac{1}{2}$$

$$\zeta'(0) = -\frac{1}{2}\log(2\pi)$$

$$\begin{aligned} \zeta''(0) &= -\frac{1}{2}\log^2(2\pi) + \frac{\pi^2}{24} - c_1 \\ &= -\frac{1}{2}\log^2(2\pi) + \frac{\pi^2}{24} + \frac{\gamma_0^4}{2} - \frac{\gamma_0^2\gamma_1}{2} + \gamma_0^2\frac{\pi^2}{6} - \gamma_1\frac{\pi^2}{6} \end{aligned} \tag{20}$$

$$\zeta'''(0) = -\frac{1}{2}\log^3(2\pi) + \frac{\pi^2}{8}\log(2\pi) - 3c_1\log(2\pi) + 3c_2$$

$$\begin{aligned} \zeta^{(4)}(0) = &-\frac{1}{2}\log^4(2\pi) + \frac{\pi^2}{4}\log^2(2\pi) - \frac{\pi^4}{160} - 6c_1\log^2(2\pi) + \frac{\pi^2}{2}c_1 \\ &+ 12c_2\log(2\pi) + 12c_3 \end{aligned}$$

Chapter 4

Higher Derivatives of the Hurwitz Zeta Function

As with finding the derivatives of the Riemann zeta function, to find $\zeta^{(n)}(0,q)$ we start with (4), the functional equation for the Hurwitz zeta function. We then expand its members in Taylor and Laurent series about $s=1$. Then we equate coefficients of $(s-1)^n$, resulting in a formula for $\zeta^{(n)}(0,q)$.

4.1 Series Expansion of the Functional Equation

From (5) we have the Laurent expansion of the Hurwitz zeta function about $s=1$,

$$\zeta\left(s,\frac{r}{k}\right)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\gamma_n\left(\frac{r}{k}\right)(s-1)^n\right].$$

$\gamma_n(q)$ are the generalized Stieltjes constants from (6). Taking the Taylor expansion of $\zeta(1-s,\frac{h}{k})$ about $s=1$ we have

$$\zeta\left(1-s,\frac{h}{k}\right)=\sum_{n=0}^{\infty}\frac{\zeta^{(n)}\left(0,\frac{h}{k}\right)(-1)^n(s-1)^n}{n!}. \tag{21}$$

The following Lemma will be used to find the Taylor expansion of $\Gamma(s)$ about $s=1$. This result shows how to obtain the Taylor expansion of $f(x)$ about $x=a$ from the Taylor expansion of $\log(f(x))$ about $x=a$.

Lemma 1[8] Let

$$[a_1,a_2,\ldots,a_n]=\det\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ (n-1) & a_1 & a_2 & \cdots & a_n-1 \\ 0 & (n-2) & a_1 & \cdots & a_n-2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}$$

and let $f(x)$ be a positive function differentiable at $x = a$ so that

$$\log(f(x)) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{n}(x-a)^n.$$

Then

$$f(x) = e^{a_0}\left(1 + \sum_{n=0}^{\infty}\left[a_1, -a_2, ..., (-1)^{n+1}a_n\right]\frac{(x-a)^n}{n!}\right).$$

Proof: Let

$$D_{j,n} = \det\begin{pmatrix} (-1)^{j-1}a_j & (-1)^j a_{j+1} & \cdots & (-1)^{n-1}a_n \\ (n-j) & a_1 & \cdots & (-1)^{n-j-1}a_{n-j} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots 0 & 1 & a_1 \end{pmatrix}.$$

Then

$$\begin{aligned} D_{1,n} &= \left[a_1, -a_2, \ldots, (-1)^{n-1}a_n\right] \\ &= a_1\left[a_1, \ldots, (-1)^{n-2}a_{n-1}\right] - (n-1)D_{2,n} \end{aligned}$$

and for $j < n$,

$$D_{j,n} = (-1)^{j-1}a_j\left[a_1, \ldots, (-1)^{n-j-2}a_{n-j}\right] - (n-j)D_{j+1,n}.$$

Applying these equations repeatedly,

$$\begin{aligned} D_{1,n} &= \left[a_1, -a_2, \ldots, (-1)^{n-1}a_n\right] \\ &= a_1\left[a_1, \ldots, (-1)^{n-2}a_{n-1}\right] - (n-1)D_{2,n} \\ &= a_1\left[a_1, \ldots, (-1)^{n-2}a_{n-1}\right] + (n-1)a_2\left[a_1, \ldots, (-1)^{n-3}a_{n-2}\right] \\ &\qquad - (n-1)(n-2)D_{3,n}. \end{aligned}$$

Iterating, we obtain

$$D_{1,n} = \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!} a_j \left[a_1, \ldots, (-1)^{n-j-1}a_{n-j}\right]. \tag{22}$$

Setting $f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$, it can then be seen that $b_0 = e^{a_0}$ and $b_1 = e^{a_0} a_1$, so assume that

$$b_m = \frac{e^{a_0}}{m!}\left[a_1, -a_2, \ldots, (-1)^{m-1}a_m\right]$$

for each $m < n$. Since $f'(x) = \left(\sum_{n=0}^{\infty} b_n (x-a)^n\right)\left(\sum_{n=1}^{\infty} a_n (x-a)^{n-1}\right)$ then equating terms gives us

$$b_n n = \sum_{j=1}^{n} a_j b_{n-j}.$$

From the induction hypothesis we can rewrite this as

$$b_n n = e^{a_0}\left(\sum_{j=1}^{n} \frac{a_j}{(n-j)!}\left[a_1, \ldots, (-1)^{n-j-1}a_{n-j}\right]\right).$$

From (22),

$$\begin{aligned} b_n n! &= e^{a_0}\left(\sum_{j=1}^{n} a_j \frac{(n-1)!}{(n-j)!}\left[a_1, \ldots, (-1)^{n-j-1}a_{n-j}\right]\right) \\ &= e^{a_0}\left[a_1, \ldots, (-1)^{n-1}a_n\right] \end{aligned}$$

which is the result.

□

From [1],

$$\log(\Gamma(s)) = -\gamma_0(s-1) + \sum_{n=2}^{\infty}(-1)^n\zeta(n)(s-1)^n.$$

The hypothesis of Lemma 1 gives

$$a_0 = 0,$$

$$a_1 = -\gamma_0,$$

$$\vdots$$

$$a_n = n(-1)^n\zeta(n)$$

for n>1. Thus Lemma 1 gives

$$\Gamma(s) = 1 + \sum_{n=1}^{\infty}[-\gamma_0, a_2, ..., (-1)^{2n+1}n\zeta(n)]\frac{(s-1)^n}{n!} \tag{23}$$

where $[-\gamma_0, a_2, ..., (-1)^{2n+1}n\zeta(n)]$ is the determinant as defined in Lemma 1.

Let $C_0, C_1, C_2, \ldots, C_n, \ldots$ be the coefficients of the Taylor expansion of $\cos\left(\frac{\pi s}{2} - \frac{2\pi rh}{k}\right)\frac{2}{(2\pi k)^s}$ about $s = 1$. Thus we have

$$\frac{2}{(2\pi k)^s}\cos\left(\frac{\pi s}{2} - \frac{2\pi rh}{k}\right) = \sum_{n=0}^{\infty} C_n(s-1)^n. \tag{24}$$

Letting $f(s) = \cos\left(\frac{\pi s}{2} - \frac{2\pi r h}{k}\right)\frac{2}{(2\pi k)^s}$ then $C_n = \frac{f^{(n)}(1)}{n!}$ and

$$C_0 = \frac{1}{\pi k}\sin\left(\frac{2\pi h r}{k}\right)$$

$$C_1 = \frac{-1}{2k}\cos\left(\frac{2h\pi r}{k}\right) - \frac{\log(k)}{k\pi}\sin\left(\frac{2h\pi r}{k}\right) - \frac{\log(2\pi)}{k\pi}\sin\left(\frac{2h\pi r}{k}\right)$$

$$\begin{aligned}(25)\qquad C_2 =& \frac{\log^2(k)}{2\pi k}\sin\left(\frac{2\pi h r}{k}\right) + \frac{\log^2(2\pi)}{2\pi k}\sin\left(\frac{2\pi h r}{k}\right) - \frac{\pi}{8k}\sin\left(\frac{2\pi h r}{k}\right)\\ &+\frac{\log(2\pi)\log(k)}{\pi k}\sin\left(\frac{2\pi h r}{k}\right) + \frac{\log(k)}{2k}\cos\left(\frac{2\pi h r}{k}\right)\\ &+\frac{\log(2\pi)}{2k}\cos\left(\frac{2\pi h r}{k}\right).\end{aligned}$$

Note that for every $n = 0, 1, \ldots$, the number C_n depends on h, k and r.

4.2 Higher Derivatives of the Hurwitz Zeta Function

To find $\zeta^{(N)}\left(0, \frac{h}{k}\right)$ we apply the series expansions (21),(23), (24) and (5) to the functional equation (4). Another form of the functional equation (4) is then given by

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\zeta^{(n)}\left(0, \frac{h}{k}\right)(-1)^n(s-1)^n}{n!}\\ (26)\qquad &= \left(1 + \sum_{j=1}^{\infty}\left[-\gamma_0, a_2, ..., (-1)^{2j+1}j\zeta(j)\right]\frac{(s-1)^j}{j!}\right)\\ &\sum_{r=1}^{k}\left[\left(\sum_{\ell=0}^{\infty}C_\ell(s-1)^\ell\right)\left(\frac{1}{s-1} + \sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\gamma_m\left(\frac{r}{k}\right)(s-1)^m\right]\right)\right]\end{aligned}$$

where C_ℓ (defined in (25)) is in terms of h, k and r and $\left[-\gamma_0, a_2, ..., (-1)^{2j+1}j\zeta(j)\right]$

is the determinant found in Lemma 1. Because we are looking for the Nth derivative of $\zeta(s,q)$ at $s=0$ we take only the Nth term of (21) which is

$$\frac{\zeta^{(N)}\left(0,\frac{h}{k}\right)(-1)^N}{N!}(s-1)^N. \tag{27}$$

In the effort to equate (27) with the sum of all coefficients of $(s-1)^N$ from the right hand side of (26), an expression for the sum of these coefficients will now be found. To find such an expression how the coefficients of $(s-1)^N$ arise from the product of the three expansions on the right hand side of (26) must be considered. First the following sum in (26) is considered.

$$\sum_{r=1}^{k}\left[\left(\sum_{\ell=0}^{\infty}C_\ell(s-1)^\ell\right)\left(\frac{1}{s-1}+\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\gamma_m\left(\frac{r}{k}\right)(s-1)^m\right]\right)\right] \tag{28}$$

It is important to now note that for $h,k\in\mathbb{N}$ and $1\leq h<k$,

$$\begin{aligned}\sum_{r=1}^{k}\sin\left(\frac{2\pi hr}{k}\right)&=0,\ \text{and}\\ \sum_{r=1}^{k}\cos\left(\frac{2\pi hr}{k}\right)&=0.\end{aligned} \tag{29}$$

From (29) for all $\ell\in\mathbb{N}_0$, $h,k\in\mathbb{N}$ and $1\leq h<k$,

$$\sum_{r=1}^{k}C_\ell=0. \tag{30}$$

Now (30) implies that for $h,k\in\mathbb{N}$ and $1\leq h<k$,

$$\sum_{r=1}^{k}\left[\left(\sum_{\ell=0}^{\infty}C_\ell(s-1)^\ell\right)\left(\frac{1}{s-1}\right)\right]=0. \tag{31}$$

Applying (31) it is seen that for $h, k \in \mathbb{N}$ and $1 \leq h < k$ (28) can be expressed as

$$\sum_{r=1}^{k}\left[\left(\sum_{\ell=0}^{\infty} C_\ell (s-1)^\ell\right)\left(\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\gamma_m\left(\frac{r}{k}\right)(s-1)^m\right]\right)\right] \tag{32}$$

Also for $h, k \in \mathbb{N}$ and $1 \leq h < k$ (26) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\zeta^{(n)}\left(0, \frac{h}{k}\right)(-1)^n (s-1)^n}{n!} \\
&= \left(1 + \sum_{j=1}^{\infty}\left[-\gamma_0, a_2, ..., (-1)^{2j+1} j\zeta(j)\right]\frac{(s-1)^j}{j!}\right) \\
&\sum_{r=1}^{k}\left[\left(\sum_{\ell=0}^{\infty} C_\ell (s-1)^\ell\right)\left(\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{m!}\gamma_m\left(\frac{r}{k}\right)(s-1)^m\right]\right)\right].
\end{aligned} \tag{33}$$

Now to ensure that all coefficients of $(s-1)^N$ are included the first $N+1$ terms of both factors found within the sum of (33) must be considered and are given by

$$\sum_{m=0}^{N}\left[\frac{(-1)^m}{m!}\gamma_m\left(\frac{r}{k}\right)(s-1)^m\right] \tag{34}$$

and from (24) the first $N+1$ terms are

$$\sum_{\ell=0}^{N} C_\ell (s-1)^\ell. \tag{35}$$

We also must consider the first $N+1$ terms of (23)

$$1 + \sum_{j=1}^{N}[-\gamma_0, a_2, ..., (-1)^{2j+1} j\zeta(j)]\frac{(s-1)^j}{j!}. \tag{36}$$

To help prevent the expressions from becoming cumbersome the sequence H_m for

$0 \leq m \leq N$ where H_m is the coefficient of $(s-1)^m$ from (34) is defined as

$$H_m = \frac{(-1)^m}{m!}\gamma_m\left(\frac{r}{k}\right). \tag{37}$$

So all coefficients of $(s-1)^n$ from the product of (34) and (35) for $h, k \in \mathbb{N}$, $1 \leq h < k$ and $0 \leq n \leq N$ summed from $r = 1$ to k as is necessary can be expressed as

$$\sum_{r=1}^{k}\left[\sum_{m=0}^{N}\left[H_m \sum_{\ell=0}^{N-m} C_\ell (s-1)^{\ell+m}\right]\right]. \tag{38}$$

As before, to help prevent the expressions from becoming cumbersome the sequence G_j for $j \leq N$ where G_j is the coefficient of $(s-1)^j$ from (36) is defined as

$$\begin{aligned} &G_j = 0, \text{ for } j < 0,\\ &G_0 = 1, \\ &G_j = [-\gamma_0, a_2, ..., (-1)^{2j+1} j\zeta(j)]\frac{1}{j!}, \text{ for } 1 \leq j \leq N. \end{aligned} \tag{39}$$

So (36) can be expressed as

$$\sum_{j=0}^{N} G_j (s-1)^j. \tag{40}$$

Taking the product of (40) and (38) for $h, k \in \mathbb{N}$ and $1 \leq h < k$ gives

$$\left(\sum_{j=0}^{N} G_j (s-1)^j\right)\left(\sum_{r=1}^{k}\left[\sum_{m=0}^{N}\left[H_m \sum_{\ell=0}^{N-m} C_\ell (s-1)^{\ell+m}\right]\right]\right). \tag{41}$$

Now notice that in (41) for the product of any selected terms to result in $(s-1)^N$ then $h + \ell + m = N$. So given some ℓ and m then $h = N - \ell - m$. So the product of

(40) and (38) for $h, k \in \mathbb{N}$ and $1 \le h < k$ with only coefficients of $(s-1)^N$ can be expressed as

$$\sum_{r=1}^{k} \left[\sum_{m=0}^{N} \left[H_m \sum_{\ell=0}^{N-m} C_\ell \, G_{N-\ell-m} \right]\right] (s-1)^N. \tag{42}$$

Equating (27) with (42) for $h, k \in \mathbb{N}$ and $1 \le h < k$ gives

$$\begin{aligned} &\frac{\zeta^{(N)}\left(0, \frac{h}{k}\right)(-1)^N}{N!}(s-1)^N \\ &\qquad = \sum_{r=1}^{k} \left[\sum_{m=0}^{N} \left[H_m \sum_{\ell=0}^{N-m} C_\ell \, G_{N-\ell-m} \right]\right] (s-1)^N. \end{aligned} \tag{43}$$

Solving (43) for $\zeta^{(N)}\left(0, \frac{h}{k}\right)$ for $1 \le h < k$ finally results in

$$\zeta^{(N)}\left(0, \frac{h}{k}\right) = (-1)^N N! \sum_{r=1}^{k} \left[\sum_{m=0}^{N} \left[H_m \sum_{\ell=0}^{N-m} C_\ell \, G_{N-\ell-m} \right]\right]. \tag{44}$$

Substituting in (37) and (39) into (44) and keeping in mind that $G_j = 0$ when $j < 0$ the following theorem is obtained.

Theorem 2 Let $h, k \in \mathbb{N}$ and $1 \le h < k$. Then

$$\begin{aligned} &\zeta^{(N)}\left(0, \frac{h}{k}\right) \\ &\quad = (-1)^N N! \sum_{r=1}^{k} \left[\sum_{m=0}^{N} \left[\frac{(-1)^m}{m!} \gamma_m\left(\frac{r}{k}\right) \right.\right. \\ &\qquad \sum_{\ell=0}^{N-m} C_\ell \, [-\gamma_0, a_2, ..., (-1)^{2(N-\ell-m)+1}(N-\ell-m)\zeta((N-\ell-m))] \\ &\qquad\qquad \left.\left. \frac{(s-1)^{(N-\ell-m)}}{(N-\ell-m)!} \chi_{\mathbb{N}_0}(N-\ell-m) \right]\right] \end{aligned} \tag{45}$$

where γ_n and $\gamma_n\left(\frac{h}{k}\right)$ are the Stieltjes constants and generalized Stieltjes constants defined in (8) and (6) respectively, and $\chi_{\mathbb{N}_0}(x)$ is the characteristic function defined by

$$\chi_{\mathbb{N}_0}(x) = \begin{cases} 1 & : x \in \mathbb{N} \cup \{0\} \\ 0 & : x \notin \mathbb{N} \cup \{0\} \end{cases}. \tag{46}$$

Note that the inclusion of $\chi_{\mathbb{N}_0}(x)$ is necessary due to the condition $G_j = 0$ for $j < 0$. Theorem 2 can give exact results for small N and good estimates for large N.

4.3 Main Results

Another representation of $\zeta^{(N)}\left(0, \frac{h}{k}\right)$ for $N = 0$ is now found by using the discussion from section 4.2. Starting with (38) for $h, k \in \mathbb{N}$ and $1 \leq h < k$,

$$\begin{aligned} &\sum_{r=1}^{k}\left[\sum_{m=0}^{N}\left[H_m \sum_{\ell=0}^{N-m} C_\ell (s-1)^{\ell+m}\right]\right] \\ &\qquad = \sum_{r=1}^{k}\left[C_0 \gamma_0\left(\frac{r}{k}\right)\right]. \end{aligned} \tag{47}$$

From (36),

$$1 + \sum_{m=1}^{N}[-\gamma_0, a_2, ..., (-1)^{2m+1} m\zeta(m)]\frac{(s-1)^m}{m!} = 1. \tag{48}$$

This gives

$$\zeta\left(0, \frac{h}{k}\right) = \sum_{r=1}^{k}\left[C_0 \gamma_0\left(\frac{r}{k}\right)\right] \tag{49}$$

and so the following Theorem is obtained.

Theorem 3 If $h, k \in \mathbb{N}$ and $1 \leq h < k$ then

$$\zeta\left(0, \frac{h}{k}\right) = \frac{1}{k\pi} \sum_{r=1}^{k} \left[\sin\left(\frac{2h\pi r}{k}\right) \gamma_0\left(\frac{r}{k}\right)\right]. \tag{50}$$

where $\gamma_0\left(\frac{r}{k}\right)$ are the generalized Stieltjes constants defined in (6).

Using (44) we find $\zeta^{(n)}(0, q)$ for $n = 1$ to $n = 3$.

Theorem 4 If $q = \frac{h}{k}$, $h, k \in \mathbb{N}$ and $1 \leq h < k$ then

(51)

$$\zeta'(0, q) = \sum_{r=1}^{k} \left(-C_1\gamma_0\left(\frac{r}{k}\right) + C_0\left[\gamma_0\gamma_0\left(\frac{r}{k}\right) + \gamma_1\left(\frac{r}{k}\right)\right]\right)$$

$$\zeta''(0, q) = \sum_{r=1}^{k} \left(2C_2\gamma_0\left(\frac{r}{k}\right) + C_1\left[-2\gamma_0\gamma_0\left(\frac{r}{k}\right) - 2\gamma_1\left(\frac{r}{k}\right)\right]\right.$$
$$\left. + C_0\left[\gamma_0^2\gamma_0\left(\frac{r}{k}\right) - 2\zeta(2)\gamma_0\left(\frac{r}{k}\right) + 2\gamma_0\gamma_1\left(\frac{r}{k}\right) + \gamma_2\left(\frac{r}{k}\right)\right]\right)$$

$$\zeta^{(3)}(0, q) = \sum_{r=1}^{k} \left(-6C_3\gamma_0\left(\frac{r}{k}\right) + C_2\left[6\gamma_0\gamma_0\left(\frac{r}{k}\right) + 6\gamma_1\left(\frac{r}{k}\right)\right]\right.$$
$$+ C_1\left[6\zeta(2)\gamma_0\left(\frac{r}{k}\right) - 3\gamma_0^2\gamma_0\left(\frac{r}{k}\right) - 6\gamma_0\gamma_1\left(\frac{r}{k}\right) - 3\gamma_2\left(\frac{r}{k}\right)\right]$$
$$+ C_0\left[-6\gamma_0\zeta(2)\gamma_0\left(\frac{r}{k}\right) + 6\zeta(3)\gamma_0\left(\frac{r}{k}\right) - 6\zeta(2)\gamma_1\left(\frac{r}{k}\right) + \gamma_0^3\gamma_0\left(\frac{r}{k}\right)\right.$$
$$\left.\left. + 3\gamma_0^2\gamma_1\left(\frac{r}{k}\right) + 3\gamma_0\gamma_2\left(\frac{r}{k}\right) + \gamma_3\left(\frac{r}{k}\right)\right]\right)$$

where γ_0 and $\gamma_n\left(\frac{h}{k}\right)$ are Euler's constant and the generalized Stieltjes constants defined in (9) and (6) respectively.

Due to $\zeta^{(n)}(0, q)$ becoming cumbersome for $n > 3$, the values $\zeta^{(n)}(0, q)$ for

$n = 4$ to $n = 9$ are located in the Appendix.

4.4 Applications to the Stieltjes Constants

The formula discovered for $\zeta\left(0, \frac{h}{k}\right)$ will be applied to the following Theorem.

Theorem 5 [12] For $h, k \in \mathbb{N}$ and $1 \leq h < k$

$$\zeta\left(0, \frac{h}{k}\right) = \frac{1}{2} - \frac{h}{k}. \tag{52}$$

Equating (50) and (52) for $h, k \in \mathbb{N}$ and $1 \leq h < k$ gives

$$\begin{aligned} \frac{1}{2} - \frac{h}{k} &= \sum_{r=1}^{k} \left[C_0 \gamma_0 \left(\frac{r}{k}\right)\right] \\ &= \frac{1}{k\pi} \sum_{r=1}^{k} \left[\sin\left(\frac{2h\pi r}{k}\right) \gamma_0 \left(\frac{r}{k}\right)\right]. \end{aligned} \tag{53}$$

After some manipulation of (53) for $h, k \in \mathbb{N}$ and $1 \leq h < k$,

$$\pi(k - 2h) = \sum_{r=1}^{k} \left[\sin\left(\frac{2h\pi r}{k}\right) \gamma_0 \left(\frac{r}{k}\right)\right], \tag{54}$$

and from (54) a representation of π in terms of $\gamma_0\left(\frac{r}{k}\right)$ follows.

Theorem 6 for $h, k \in \mathbb{N}$, $2h \neq k$ and $1 \leq h < k$

$$\pi = \frac{1}{(k - 2h)} \sum_{r=1}^{k} \left[\sin\left(\frac{2h\pi r}{k}\right) \gamma_0 \left(\frac{r}{k}\right)\right]. \tag{55}$$

Examined earlier in (20) from Apostol [2],

$$\zeta'(0) = -\frac{1}{2} \log(2\pi). \tag{56}$$

Theorem 7 [9] For $h, k \in \mathbb{N}$ and $1 \leq h < k$

$$\zeta'\left(0, \frac{h}{k}\right) = \log\left(\Gamma\left(\frac{h}{k}\right)\right) + \zeta'(0). \tag{57}$$

Applying (51) and (56) to (57) for $h, k \in \mathbb{N}$ and $1 \leq h < k$ gives

$$\sum_{r=1}^{k}\left(-C_1\gamma_0\left(\frac{r}{k}\right) + C_0\left[\gamma_0\gamma_0\left(\frac{r}{k}\right) + \gamma_1\left(\frac{r}{k}\right)\right]\right) = \log\left(\Gamma\left(\frac{h}{k}\right)\right) - \frac{1}{2}\log(2\pi)$$

Substituting in the the values for C_0 and C_1 results in the following Theorem.

Theorem 8 For $h, k \in \mathbb{N}$ and $1 \leq h < k$

(58)

$$\log\left(\Gamma\left(\frac{h}{k}\right)\right) - \frac{1}{2}\log(2\pi)$$
$$= \sum_{r=1}^{k}\left[\left(\frac{1}{2k}\cos\left(\frac{2h\pi r}{k}\right) + \frac{\log(k)}{k\pi}\sin\left(\frac{2h\pi r}{k}\right) + \frac{\log(2\pi)}{k\pi}\sin\left(\frac{2h\pi r}{k}\right)\right)\gamma_0\left(\frac{r}{k}\right) + \left(\frac{1}{\pi k}\sin\left(\frac{2\pi hr}{k}\right)\right)\left[\gamma_0\gamma_0\left(\frac{r}{k}\right) + \gamma_1\left(\frac{r}{k}\right)\right]\right].$$

4.5 Examples

We now give some specific examples of (55). Let $h = 1$ and $k = 3$ in (55). After simplifying,

$$\frac{\pi}{\sqrt{3}} = \gamma_0\left(\frac{1}{3}\right) - \gamma_0\left(\frac{2}{3}\right). \tag{59}$$

Let $h = 1$ and $k = 4$ in (55). After simplifying,

$$\pi = \gamma_0\left(\frac{1}{4}\right) - \gamma_0\left(\frac{3}{4}\right). \tag{60}$$

Let $h = 1$ and $k = 5$ in (55). After simplifying,

$$\frac{4\pi}{\sqrt{3}} = \gamma_0\left(\frac{1}{6}\right) - \gamma_0\left(\frac{2}{3}\right) + \gamma_0\left(\frac{1}{3}\right) - \gamma_0\left(\frac{5}{6}\right). \tag{61}$$

Applying (59) to (61) gives

$$\pi\sqrt{3} = \gamma_0\left(\frac{1}{6}\right) - \gamma_0\left(\frac{5}{6}\right). \tag{62}$$

From the examples given it is seen that many equations involving differences of the generalized Stieltjes constants at specific values can be obtained from (55).

Some specific examples of (58) are now given. Let $h = 1$ and $k = 2$ in (58) to give

$$4\log\left(\Gamma\left(\frac{1}{2}\right)\right) - 2\log(2\pi) = -\gamma_0\left(\frac{1}{2}\right) + \gamma_0(1). \tag{63}$$

To further simplify recall from section 2.1 that $\gamma_0(1) = \gamma_0$; applying this as well as (1) in (63) gives

$$\gamma_0 = 2\log(\pi) - 2\log(2\pi) + \gamma_0\left(\frac{1}{2}\right). \tag{64}$$

After some manipulation of (64), a new definition of Euler's constant is found

Theorem 9

$$\gamma_0 = -2\log(2) + \gamma_0\left(\frac{1}{2}\right). \tag{65}$$

Letting $h = 2$ and $k = 4$ in (58) gives

$$\log\left(\Gamma\left(\frac{1}{2}\right)\right) - \frac{1}{2}\log(2\pi) = -\frac{1}{8}\gamma_0\left(\frac{1}{4}\right) + \frac{1}{8}\gamma_0\left(\frac{1}{2}\right) - \frac{1}{8}\gamma_0\left(\frac{3}{4}\right) + \frac{\gamma_0}{8}. \tag{66}$$

Applying (1) and (65), after some manipulation (66) simplifies to

$$2\log(2) = -\gamma_0\left(\frac{1}{4}\right) - \gamma_0\left(\frac{3}{4}\right) + 2\gamma_0. \tag{67}$$

This results in yet another definition of Euler's constant.

Theorem 10

$$\gamma_0 = \frac{1}{2}\gamma_0\left(\frac{1}{4}\right) + \frac{1}{2}\gamma_0\left(\frac{3}{4}\right) + \log(2). \tag{68}$$

Chapter 5

Numerical Results

5.1 Numerical Results

The exact arithmetic values of $\zeta^{(n)}(0, \frac{h}{k})$ become quite cumbersome as n gets bigger. However, we can use the previous methods to find arbitrarily close numerical approximations to the values. Note that Mathematica has Euler's constant, the Stieltjes constants, the generalized Stieltjes constants, and the Riemann zeta function as built in functions.

Using Mathematica, numerical approximations of $\zeta^{(n)}(0, \frac{h}{k})$ were found for some values of h and k where $1 \leq h \leq k$, with $h, k \in \mathbb{Z}$ and $1 \leq n \leq 10$. When finding these approximate values, Mathematica called the built in functions for Euler's constant, the Stieltjes constants, the generalized Stieltjes constants, and the Riemann zeta function. The numerical approximation of the whole formula is then found exactly to seventeen digits. The Mathematica code can be found in the Appendix.

Table 1

n	$\zeta^{(n)}(0, \frac{1}{3})$
1	$4.2264691715694405 \times 10^{-16} i$
2	$-4.440892098500626 \times 10^{-16} + 1.6762294942867273 \times 10^{-16} i$
3	$8.881784197001252 \times 10^{-16} - 2.6979048583040365 \times 10^{-15} i$
4	$-3.8465099820174876 \times 10^{-15} i$
5	$9.237055564881302 \times 10^{-14} + 3.275255055919095 \times 10^{-15} i$
6	$-4.547473508864641 \times 10^{-13} - 7.100212261109741 \times 10^{-14} i$
7	$-1.4510114453388763 \times 10^{-13} i$
8	$2.9103830456733704 \times 10^{-11} - 9.364394113320239 \times 10^{-12} i$
9	$-1.4551915228366852 \times 10^{-10} + 1.8874858745357452 \times 10^{-11} i$
10	$3.259629011154175 \times 10^{-9} - 6.873448021623356 \times 10^{-10} i$

Table 2

n	$\zeta^{(n)}(0, \frac{2}{3})$
1	$2.306320638957115 \times 10^{-16}i$
2	$2.7755575615628914 \times 10^{-16} + 2.630468980525967 \times 10^{-16}i$
3	$1.1102230246251565 \times 10^{-16} - 4.097875495084035 \times 10^{-16}i$
4	$-3.552713678800501 \times 10^{-15} - 2.3007272342993096 \times 10^{-15}i$
5	$7.460698725481052 \times 10^{-14} - 2.0358987852929752 \times 10^{-14}i$
6	$-1.1368683772161603 \times 10^{-13} - 4.8321273195262077 \times 10^{-14}i$
7	$-1.6058265828178264 \times 10^{-12} + 9.167379874024828 \times 10^{-13}i$
8	$1.0913936421275139 \times 10^{-11} + 2.6391278625809323 \times 10^{-12}i$
9	$-2.9103830456733704 \times 10^{-11} + 2.1304990822027376 \times 10^{-11}i$
10	$1.1641532182693481 \times 10^{-10} - 7.92535571903653 \times 10^{-11}i$

Table 3

n	$\zeta^{(n)}(0, \frac{1}{5})$
1	$5.768715489750805 \times 10^{-18}i$
2	$-4.969555567157345 \times 10^{-16}i$
3	$-5.329070518200751 \times 10^{-15} - 8.320504346120992 \times 10^{-16}i$
4	$7.105427357601002 \times 10^{-15} - 9.311479177989934 \times 10^{-16}i$
5	$-8.526512829121202 \times 10^{-14} - 1.7284877673211857 \times 10^{-14}i$
6	$-4.547473508864641 \times 10^{-13} - 5.753604229990332 \times 10^{-13}i$
7	$6.821210263296962 \times 10^{-12} - 6.920456426606928 \times 10^{-13}i$
8	$7.275957614183426 \times 10^{-11} + 6.050033245953069 \times 10^{-11}i$
9	$-1.5861587598919868 \times 10^{-9} - 2.453695314525703 \times 10^{-10}i$
10	$-4.656612873077393 \times 10^{-10} - 2.0238011511888144 \times 10^{-10}i$

Table 4

n	$\zeta^{(n)}(0, \frac{2}{5})$
1	$-3.117616086394899 \times 10^{-17}i$
2	$-4.996003610813204 \times 10^{-16} + 2.6791567644318653 \times 10^{-17}i$
3	$-6.661338147750939 \times 10^{-16} - 1.0644519590978504 \times 10^{-15}i$
4	$-3.552713678800501 \times 10^{-15} + 6.907647467961181 \times 10^{-17}i$
5	$2.4868995751603507 \times 10^{-14} + 5.246660065772101 \times 10^{-14}i$
6	$-9.094947017729282 \times 10^{-13} - 1.336253835834523 \times 10^{-13}i$
7	$4.490630090003833 \times 10^{-12} - 1.5405220999062644 \times 10^{-12}i$
8	$1.0913936421275139 \times 10^{-11} + 3.4366169352245977 \times 10^{-11}i$
9	$-3.346940502524376 \times 10^{-10} - 1.3357457990390365 \times 10^{-10}i$
10	$-7.814189671200447 \times 10^{-10}i$

Table 5

n	$\zeta^{(n)}(0, \frac{3}{5})$
1	$5.551115123125783 \times 10^{-17} + 2.583938206769259 \times 10^{-19}i$
2	$-1.942890293094024 \times 10^{-16} + 6.304469719721828 \times 10^{-17}i$
3	$-3.4891031739627734 \times 10^{-16}i$
4	$7.105427357601002 \times 10^{-15} - 2.9083210926198754 \times 10^{-16}i$
5	$-1.0658141036401503 \times 10^{-14} + 1.425381789358119 \times 10^{-14}i$
6	$-2.5579538487363607 \times 10^{-13} - 1.9880846900258315 \times 10^{-13}i$
7	$1.4850343177386094 \times 10^{-12} - 9.66102315929392 \times 10^{-13}i$
8	$9.094947017729282 \times 10^{-12} + 1.8796293209404037 \times 10^{-11}i$
9	$-2.3646862246096134 \times 10^{-11} - 6.700024429508597 \times 10^{-12}i$
10	$-4.656612873077393 \times 10^{-10} - 1.1056950366385455 \times 10^{-9}i$

Table 6

n	$\zeta^{(n)}(0, \frac{4}{5})$
1	$2.7755575615628914 \times 10^{-17} + 2.19601536972525 \times 10^{-17}i$
2	$8.326672684688674 \times 10^{-17} + 6.997550424484177 \times 10^{-18}i$
3	$-4.996003610813204 \times 10^{-16} + 3.039449679968326 \times 10^{-16}i$
4	$3.552713678800501 \times 10^{-15} + 1.8151465605831884 \times 10^{-16}i$
5	$3.730349362740526 \times 10^{-14} + 6.898693286259209 \times 10^{-15}i$
6	$-1.2789769243681803 \times 10^{-13} - 1.3081121664866243 \times 10^{-13}i$
7	$-1.7053025658242404 \times 10^{-13} - 1.889418269235247 \times 10^{-13}i$
8	$1.2732925824820995 \times 10^{-11} + 1.1722182325405858 \times 10^{-11}i$
9	$1.6370904631912708 \times 10^{-11} + 1.0812300131441648 \times 10^{-11}i$
10	$-4.656612873077393 \times 10^{-10} - 4.904970909585042 \times 10^{-10}i$

Table 7

n	$\zeta^{(n)}(0, \frac{1}{7})$
1	$2.220446049250313 \times 10^{-16} + 3.447948119621746 \times 10^{-16}i$
2	$-5.310562783649649 \times 10^{-16}i$
3	$-1.509903313490213 \times 10^{-14} - 1.728034150206612 \times 10^{-16}i$
4	$4.973799150320701 \times 10^{-14} - 2.935748283720043 \times 10^{-14}i$
5	$-1.4210854715202004 \times 10^{-13} - 2.893482140288973 \times 10^{-13}i$
6	$-1.8189894035458565 \times 10^{-12} + 1.176019890776122 \times 10^{-12}i$
7	$2.2737367544323206 \times 10^{-11} + 6.455862611225116 \times 10^{-12}i$
8	$6.701380668799425 \times 10^{-12}i$
9	$-2.3283064365386963 \times 10^{-10} - 3.841041498364184 \times 10^{-10}i$
10	$1.4901161193847656 \times 10^{-8} - 1.921157700638319 \times 10^{-9}i$

Table 8

n	$\zeta^{(n)}(0, \frac{2}{7})$
1	$-1.249000902703301 \times 10^{-16} + 2.4257402815210997 \times 10^{-16}i$
2	$-1.1102230246251565 \times 10^{-15} - 1.2512816566128184 \times 10^{-16}i$
3	$-5.551115123125783 \times 10^{-15} + 1.3210383482137698 \times 10^{-15}i$
4	$3.197442310920451 \times 10^{-14} + 5.3947068768718835 \times 10^{-18}i$
5	$-2.2737367544323206 \times 10^{-13} - 1.1911173243647815 \times 10^{-13}i$
6	$-9.094947017729282 \times 10^{-13} + 2.759131420861941 \times 10^{-13}i$
7	$1.261923898709938 \times 10^{-11} - 1.647645372122514 \times 10^{-12}i$
8	$-2.1827872842550278 \times 10^{-11} - 1.502389321193309 \times 10^{-11}i$
9	$-6.402842700481415 \times 10^{-10} - 2.8996981488407486 \times 10^{-10}i$
10	$2.7939677238464355 \times 10^{-9} - 2.083446763169931 \times 10^{-10}i$

Table 9

n	$\zeta^{(n)}(0, \frac{3}{7})$
1	$-1.6306400674181987 \times 10^{-16} + 1.4217610889545903 \times 10^{-16}i$
2	$-4.996003610813204 \times 10^{-16} + 4.97725435466079 \times 10^{-17}i$
3	$-1.4432899320127035 \times 10^{-15} + 1.704145654832255 \times 10^{-15}i$
4	$1.0658141036401503 \times 10^{-14} - 3.634880373817201 \times 10^{-15}i$
5	$-6.394884621840902 \times 10^{-14} - 3.712690498617261 \times 10^{-14}i$
6	$1.7882852796503897 \times 10^{-13}i$
7	$4.632738637155853 \times 10^{-12} - 2.0493845762943288 \times 10^{-13}i$
8	$-4.18367562815547 \times 10^{-11} + 4.3584911423912 \times 10^{-12}i$
9	$8.731149137020111 \times 10^{-11} - 1.0316808083777059 \times 10^{-10}i$
10	$3.259629011154175 \times 10^{-9} + 4.2732796821797003 \times 10^{-10}i$

Table 10

n	$\zeta^{(n)}(0, \frac{4}{7})$
1	$-1.6653345369377348 \times 10^{-16} + 1.4217664591957426 \times 10^{-16}i$
2	$5.551115123125783 \times 10^{-17} + 1.8703862477713105 \times 10^{-16}i$
3	$-1.8283985436795547 \times 10^{-15} + 8.023015894602931 \times 10^{-16}i$
4	$1.7763568394002505 \times 10^{-15} + 4.7984848202987706 \times 10^{-15}i$
5	$7.105427357601002 \times 10^{-15} - 6.011510936163684 \times 10^{-14}i$
6	$-4.547473508864641 \times 10^{-13} - 1.141866784652263 \times 10^{-13}i$
7	$3.2396307858562068 \times 10^{-12} - 1.5435850905808775 \times 10^{-14}i$
8	$8.185452315956354 \times 10^{-12} + 8.777968132755605 \times 10^{-12}i$
9	$-1.0913936421275139 \times 10^{-10} - 2.010157288032626 \times 10^{-11}i$
10	$-1.96045243289112 \times 10^{-10}i$

Table 11

n	$\zeta^{(n)}(0, \frac{5}{7})$
1	$-8.673617379884035 \times 10^{-19} + 1.0485075349202332 \times 10^{-16} i$
2	$-3.0531133177191805 \times 10^{-16} - 1.1498516572713503 \times 10^{-17} i$
3	$-9.992007221626409 \times 10^{-16} + 1.84139061278034 \times 10^{-16} i$
4	$-3.552713678800501 \times 10^{-15} + 7.830520717308488 \times 10^{-16} i$
5	$1.0658141036401503 \times 10^{-14} - 5.52727538770601 \times 10^{-14} i$
6	$-1.7053025658242404 \times 10^{-13} - 1.0931297624607645 \times 10^{-13} i$
7	$1.2860823517257813 \times 10^{-12} + 6.713349080504979 \times 10^{-13} i$
8	$-1.0913936421275139 \times 10^{-11} + 6.8585243227620045 \times 10^{-12} i$
9	$-9.458744898438454 \times 10^{-11} - 3.446224947407351 \times 10^{-11} i$
10	$1.862645149230957 \times 10^{-9} - 1.3667212395382754 \times 10^{-10} i$

Table 12

n	$\zeta^{(n)}(0, \frac{6}{7})$
1	$-1.3877787807814457 \times 10^{-17} + 7.469226141801285 \times 10^{-17} i$
2	$-1.1102230246251565 \times 10^{-16} + 1.2305676846476443 \times 10^{-16} i$
3	$-1.5681900222830336 \times 10^{-15} + 3.2500791074104106 \times 10^{-16} i$
4	$2.6645352591003757 \times 10^{-15} + 7.77550754584439 \times 10^{-16} i$
5	$-3.552713678800501 \times 10^{-14} - 3.0038798606065904 \times 10^{-14} i$
6	$-1.9895196601282805 \times 10^{-13} - 1.2459700626252083 \times 10^{-13} i$
7	$1.907807245515869 \times 10^{-12} + 2.0740951327175377 \times 10^{-13} i$
8	$1.8189894035458565 \times 10^{-12} + 2.6194240392882737 \times 10^{-12} i$
9	$-7.275957614183426 \times 10^{-11} - 1.9853382113892475 \times 10^{-11} i$
10	$2.3283064365386963 \times 10^{-10} - 6.761162948063146 \times 10^{-11} i$

Table 13

n	$\zeta^{(n)}(0, \frac{1}{9})$
1	$1.9961526040076327 \times 10^{-16} i$
2	$-1.9984014443252818 \times 10^{-15} + 9.496528091546033 \times 10^{-16} i$
3	$3.1086244689504383 \times 10^{-15} - 9.745014226820748 \times 10^{-15} i$
4	$1.4210854715202004 \times 10^{-13} - 2.2497909648003716 \times 10^{-14} i$
5	$-5.968558980384842 \times 10^{-13} + 1.1546882995957668 \times 10^{-13} i$
6	$1.3642420526593924 \times 10^{-12} - 1.106591371527016 \times 10^{-12} i$
7	$3.9108272176235914 \times 10^{-11} + 7.544324379753964 \times 10^{-12} i$
8	$-2.473825588822365 \times 10^{-10} + 5.390948359299618 \times 10^{-11} i$
9	$4.94765117764473 \times 10^{-9} - 4.799695349785388 \times 10^{-10} i$
10	$-2.3283064365386963 \times 10^{-9} + 8.242227786672504 \times 10^{-9} i$

Table 14

n	$\zeta^{(n)}(0, \frac{2}{9})$
1	$5.551115123125783 \times 10^{-17} + 9.029044586144469 \times 10^{-17} i$
2	$6.106226635438361 \times 10^{-16} + 7.314450198858921 \times 10^{-16} i$
3	$8.881784197001252 \times 10^{-16} - 5.149673495723847 \times 10^{-15} i$
4	$-7.105427357601002 \times 10^{-15} - 2.2540937869067317 \times 10^{-15} i$
5	$-2.788880237858393 \times 10^{-13} + 8.445346960374863 \times 10^{-15} i$
6	$9.094947017729282 \times 10^{-13} - 4.919911070293448 \times 10^{-13} i$
7	$1.8189894035458565 \times 10^{-11} + 3.872926580839716 \times 10^{-12} i$
8	$-9.822542779147625 \times 10^{-11} + 2.364188892620111 \times 10^{-11} i$
9	$8.731149137020111 \times 10^{-10} - 4.258634965955523 \times 10^{-10} i$
10	$8.149072527885437 \times 10^{-10} - 8.921470210643557 \times 10^{-10} i$

Table 15

n	$\zeta^{(n)}(0, \frac{4}{9})$
1	$1.0900891914791774 \times 10^{-16} i$
2	$-1.249000902703301 \times 10^{-16} + 4.54336897748052 \times 10^{-16} i$
3	$1.3322676295501878 \times 10^{-15} - 1.078871889854253 \times 10^{-15} i$
4	$1.2434497875801753 \times 10^{-14} - 6.9147966697775334 \times 10^{-15} i$
5	$-1.7741363933510002 \times 10^{-13} + 1.5833509409594492 \times 10^{-14} i$
6	$1.7053025658242404 \times 10^{-13} - 1.8839728766691192 \times 10^{-13} i$
7	$5.258016244624741 \times 10^{-12} - 1.4073055925379515 \times 10^{-12} i$
8	$-5.4569682106375694 \times 10^{-11} + 1.363334032484136 \times 10^{-11} i$
9	$-1.0186340659856796 \times 10^{-10} + 5.54116500580523 \times 10^{-11} i$
10	$2.5320332497358322 \times 10^{-9} - 1.9004532644162623 \times 10^{-10} i$

Table 16

n	$\zeta^{(n)}(0, \frac{5}{9})$
1	$-2.7755575615628914 \times 10^{-17} + 1.2022710607086387 \times 10^{-16} i$
2	$5.898059818321144 \times 10^{-17} + 2.911037766268097 \times 10^{-16} i$
3	$4.3021142204224816 \times 10^{-16} - 5.225281964199046 \times 10^{-17} i$
4	$-8.881784197001252 \times 10^{-15} - 4.325469507301302 \times 10^{-15} i$
5	$-3.419486915845482 \times 10^{-14} + 1.1057172168665047 \times 10^{-15} i$
6	$-2.8421709430404007 \times 10^{-13} - 3.004755751581185 \times 10^{-14} i$
7	$8.029132914089132 \times 10^{-13} + 7.350043480855267 \times 10^{-13} i$
8	$-2.637534635141492 \times 10^{-11} + 8.603733935015266 \times 10^{-12} i$
9	$-2.473825588822365 \times 10^{-10} + 1.2237892916576366 \times 10^{-11} i$
10	$1.4260876923799515 \times 10^{-9} - 9.243286149966664 \times 10^{-12} i$

Table 17

n	$\zeta^{(n)}(0, \frac{7}{9})$
1	$2.0816681711721685 \times 10^{-17} + 5.892460226217698 \times 10^{-17}i$
2	$-1.0755285551056204 \times 10^{-16} + 1.3893752547208237 \times 10^{-16}i$
3	$-1.1102230246251565 \times 10^{-16} - 2.965481516283304 \times 10^{-16}i$
4	$5.329070518200751 \times 10^{-15} - 3.919612465887069 \times 10^{-15}i$
5	$-4.1744385725905886 \times 10^{-14} + 2.9644032797468814 \times 10^{-15}i$
6	$-2.842170943040401 \times 10^{-14} - 9.102602785875876 \times 10^{-14}i$
7	$1.5347723092418164 \times 10^{-12} - 1.1260116005598962 \times 10^{-13}i$
8	$-1.2732925824820995 \times 10^{-11} + 3.460942064039666 \times 10^{-12}i$
9	$-5.4569682106375694 \times 10^{-11} + 9.336213936843651 \times 10^{-12}i$
10	$1.6152625903487206 \times 10^{-9} + 1.8691559294561302 \times 10^{-10}i$

Table 18

n	$\zeta^{(n)}(0, \frac{8}{9})$
1	$-1.3877787807814457 \times 10^{-17} + 5.666090853093189 \times 10^{-17}i$
2	$-3.8163916471489756 \times 10^{-17} + 2.0322919115269383 \times 10^{-16}i$
3	$-6.106226635438361 \times 10^{-16} - 2.808791337135555 \times 10^{-17}i$
4	$8.881784197001252 \times 10^{-16} - 1.409645494743955 \times 10^{-15}i$
5	$-1.7319479184152442 \times 10^{-14} + 6.3841247152569525 \times 10^{-15}i$
6	$-8.526512829121202 \times 10^{-14} - 1.484502177810413 \times 10^{-13}i$
7	$8.242295734817162 \times 10^{-13} - 4.723389247055592 \times 10^{-13}i$
8	$-5.9117155615240335 \times 10^{-12} + 4.855115096263487 \times 10^{-12}i$
9	$-1.418811734765768 \times 10^{-10} + 1.0900765343637892 \times 10^{-11}i$
10	$6.83940015733242 \times 10^{-10} - 7.229230983687623 \times 10^{-11}i$

APPENDIX

Mathematica code used to find $\zeta^{(n)}(0, \frac{h}{k})$ follows.

```
(*Initialize*)
n=1;
export=1;  (* If export=1 then will export final form
of solution without sum to file TestingExport.tex *)

generalize=1; (*If generalize=1 then Some coefficients
will be left as a sequence value C0,C1,...Cn*)

Approximate=0;  (*To approximate Zeta[i] and Stieltjes
constants set =1*)
(*
h=1; (*Do not initialize if desiring general soln*)
k=1; (*Make sure 1<=h<=k and h,k are integers *)
*)

q=h/k; (*Second arguement for Hurwitz zeta function*)

(*Start Code For Gamma Expansion Coefficient Matrix*)
A0=0;
A1=-gamma[0];(*To approximate Zeta[i] and Stieltjes constants set =1*)

Do[Evaluate[Symbol[StringJoin["A",ToString[i]]]]
=i*(-1)i*Zeta[i],{i,2,n+1}];
For[i=2,i<=n+1,i++,
 Evaluate[Symbol[StringJoin["MatriXa",ToString[i]]]]
 =ConstantArray[0,{i,i}];
 For[row=1, row<=i,row++,For[column=1,column<=i,column++,
   If[NameQ["TempMatrix"],Remove[TempMatrix]];
   TempMatrix=Symbol[StringJoin["MatriXa",ToString[i]]];
   If[column==row-1,Remove[Evaluate[StringJoin["MatriXa"
   ,ToString[i]]]];
    Evaluate[Symbol[StringJoin["MatriXa",ToString[i]]]]
    =ReplacePart[TempMatrix,{row,column}->i-column];
    ];
   If[column>=row,
    Remove[Evaluate[StringJoin["MatriXa",ToString[i]]]];
    Evaluate[Symbol[StringJoin["MatriXa",ToString[i]]]]
```

```
    =ReplacePart[TempMatrix,{row,column}->Evaluate[Symbol
    [StringJoin["A",ToString[Evaluate[column-row+1]]]]]];
    ];]];]
For[i=2,i<=n+1,i++,Evaluate[Symbol[StringJoin["DetMatriXa"
,ToString[i]]]]=1/i! Det[Symbol[StringJoin["MatriXa"
,ToString[i]]]]];
TempMatrix=ConstantArray[0,n+3];
GammaExpanCoefficientMatrix=ReplacePart[TempMatrix,2->1];
TempMatrix=GammaExpanCoefficientMatrix;
GammaExpanCoefficientMatrix=ReplacePart[
TempMatrix,3->Evaluate[Symbol["A1"]]];

For[i=2,i<=n+3,i++,
 TempMatrix=GammaExpanCoefficientMatrix;
 GammaExpanCoefficientMatrix=ReplacePart[TempMatrix
 ,i+2->Symbol[StringJoin["DetMatriXa",ToString[i]]]];
 ]

 (*End Code For Gamma Expansion Coefficient Matrix*)

(*Start Code For Laurent Expansion of Hurwitz Zeta
Coefficient Matrix*)

HurwitzZetaLaurentExpanCoefficientMatrix=ConstantArray[1,n+2];
For[i=2,i<=n+2,i++,
 TempMatrix=HurwitzZetaLaurentExpanCoefficientMatrix;
 HurwitzZetaLaurentExpanCoefficientMatrix=
  ReplacePart[TempMatrix,i->(-1)i-2/(i-2)!*Y[i-2]];
 ]

 (*End Code For Laurent Expansion of Hurwitz Zeta
 Coefficient Matrix*)

(*Begin Code For Taylor Expansion of Cosine and Fraction
Coefficient Matrix*)

If[generalize==1,
 CosineFractionExpanCoefficientMatrix=ConstantArray[0,n+3];
 For[i=2,i<=n+2,i++,
  TempMatrix=CosineFractionExpanCoefficientMatrix;
```

```
  CosineFractionExpanCoefficientMatrix=ReplacePart[
  TempMatrix,i->C[i-2]];
  ]]
If[generalize==0,
 f=1/(2p k)s (Power[e,i ((p*s)/2-2p*r*q)]+Power[
 e,-i((p*s)/2-2p*r*q)]);
 CosineFractionExpanCoefficientMatrix=ConstantArray[0,n+3];
 For[i=2,i<=n+3,i++,
  TempMatrix=CosineFractionExpanCoefficientMatrix;
  CosineFractionExpanCoefficientMatrix=ReplacePart[TempMatrix
  ,i->Simplify[SeriesCoefficient[f,{s,1,i-2}],Element[{k,h}
  ,Integers]]];
  ]]

(*End Code For Taylor Expansion of Cosine and Fraction
Coefficient Matrix*)

(*Begin Code For Product of Cosine and Hurwitz Zeta
Coefficient Matrix*)

CosFracHurwitzZetaCoefficientMatrix=ConstantArray[0,n+2];
TempMatrix=CosFracHurwitzZetaCoefficientMatrix;
CosFracHurwitzZetaCoefficientMatrix=ReplacePart[TempMatrix,1->1];
For[j=1,j<=n+2,j++,
  TempMatrix=CosFracHurwitzZetaCoefficientMatrix;
  CosFracHurwitzZetaCoefficientMatrix=
   ReplacePart[TempMatrix,j->Part[TempMatrix,j]+Part[
   HurwitzZetaLaurentExpanCoefficientMatrix,1]*Part[
   CosineFractionExpanCoefficientMatrix,j+1]];
];
For[i=2,i<=n+2,i++,
  For[j=2,i+j<=n+4,j++,
    TempMatrix=CosFracHurwitzZetaCoefficientMatrix;
    CosFracHurwitzZetaCoefficientMatrix=
     ReplacePart[TempMatrix,i+j-2->Part[TempMatrix,i+j-2]
     +Part[HurwitzZetaLaurentExpanCoefficientMatrix,i]
     *Part[CosineFractionExpanCoefficientMatrix,j]];
];];
If[generalize==0,
 For[i=1,i<=n+2,i++,
```

```
  TempMatrix=CosFracHurwitzZetaCoefficientMatrix;
  CosFracHurwitzZetaCoefficientMatrix=ReplacePart[TempMatrix
  ,i-> Sum[Part[TempMatrix,i],{r,1,k}]];
  ]]
If[generalize==0,
 For[i=1,i<=n+2,i++,
  TempMatrix=CosFracHurwitzZetaCoefficientMatrix;
  CosFracHurwitzZetaCoefficientMatrix=ReplacePart[
  TempMatrix,i->Simplify[
Sum[Part[TempMatrix,i],{r,1,k}]
,Element[{k,h},Integers]]];
  ]]

(*End Code For Product of Cosine and Hurwitz Zeta Coefficient Matrix*)

(*Begin Code For Final Solution*)

NthDerivativeSolution=0;
For[i=1,i<=n+2,i++,
  TempSolution=NthDerivativeSolution;
  NthDerivativeSolution=TempSolution+Part[
  CosFracHurwitzZetaCoefficientMatrix,i]
  *Part[GammaExpanCoefficientMatrix,n+4-i];
  ];
If[generalize==1,
  TempSolution=NthDerivativeSolution;
  NthDerivativeSolution=TempSolution*n!*Power[-1,n];
  TempSolution=NthDerivativeSolution;
  NthDerivativeSolution=Expand[TempSolution];
  For[i=0,i<=n,i++,
   TempSolution=NthDerivativeSolution;
   NthDerivativeSolution=Collect[TempSolution,Y[i]];
   ];
  NewSoln=0;
  For[i=0,i<=n,i++,
   TempSoln=NewSoln;
   NewSoln=TempSoln+Y[i]*Coefficient[NthDerivativeSolution,Y[i]];
   ];
  NthDerivativeSolution=NewSoln;
  TempSolution=NthDerivativeSolution;
```

```
 NthDerivativeSolution=Expand[TempSolution];
 For[i=0,i<=n,i++,
  TempSolution=NthDerivativeSolution;
  NthDerivativeSolution=Collect[TempSolution,C[i]];
  ]
  Output=NthDerivativeSolution;
 For[i=0,i<=n,i++,
  TempSolution=Output;
  Output=Replace[TempSolution,Y[i]-> Subscript[g,i][r/k],-1];
  TempSolution=Output;
  Output=Replace[TempSolution,C[i]-> Subscript[C,i],-1] ;];];
If[generalize==1,Output]
If[generalize==1,TempSolution=Output;
 Output=Replace[TempSolution,{g[0] ->Subscript[g,0]},-1];
 TraditionalForm[Output]
 If[export==1,
 Export["TestingExport.tex",TeXForm[Output],"TeX"]
 ]]
If[generalize==0,
 If[Approximate?1,z[i_]=Zeta[i]; Y[b_]=StieltjesGamma[b,q];
 g[c_]=StieltjesGamma[c];
  TempSolution=NthDerivativeSolution;
  NthDerivativeSolution=N[TempSolution*n!*Power[-1,n]],
  TempSolution=NthDerivativeSolution;
  NthDerivativeSolution=TempSolution*n!*Power[-1,n]
  ]]

  (*End Code For Final Solution*)
```

Using (44) and the Mathematica code above we find $\zeta^{(n)}(0,q)$ for $n=4$ to $n=9$, $q=\frac{h}{k}$ where h and k are integers and $1\leq h\leq k$ as is specified by (4). This code was checked for $\zeta^{(n)}(0,q)$ for $n=1$ to $n=3$ against the computed formulas in (51) and found to be correct.

(69)

$$\zeta^{(4)}(0,q)=\sum_{r=1}^{k}\Big(24C_4\gamma_0\left(\frac{r}{k}\right)+C_3\left[-24\gamma_0\gamma_0\left(\frac{r}{k}\right)-24\gamma_1\left(\frac{r}{k}\right)\right]$$
$$+C_2\left[-24\zeta(2)\gamma_0\left(\frac{r}{k}\right)+12\gamma_0^2\gamma_0\left(\frac{r}{k}\right)+24\gamma_0\gamma_1\left(\frac{r}{k}\right)+12\gamma_2\left(\frac{r}{k}\right)\right]$$

$$+C_1\left[24\gamma_0\zeta(2)\gamma_0\left(\frac{r}{k}\right)-24\zeta(3)\gamma_0\left(\frac{r}{k}\right)+24\zeta(2)\gamma_1\left(\frac{r}{k}\right)-4\gamma_0^3\gamma_0\left(\frac{r}{k}\right)\right.$$
$$\left.-12\gamma_0^2\gamma_1\left(\frac{r}{k}\right)-12\gamma_0\gamma_2\left(\frac{r}{k}\right)-4\gamma_3\left(\frac{r}{k}\right)\right]$$
$$+C_0\left[-12\gamma_0^2\zeta(2)\gamma_0\left(\frac{r}{k}\right)+24\gamma_0\zeta(3)\gamma_0\left(\frac{r}{k}\right)-24\gamma_0\zeta(2)\gamma_1\left(\frac{r}{k}\right)\right.$$
$$+12\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)-24\zeta(4)\gamma_0\left(\frac{r}{k}\right)+24\zeta(3)\gamma_1\left(\frac{r}{k}\right)-12\zeta(2)\gamma_2\left(\frac{r}{k}\right)$$
$$\left.\left.+\gamma_0^4\gamma_0\left(\frac{r}{k}\right)+4\gamma_0^3\gamma_1\left(\frac{r}{k}\right)+6\gamma_0^2\gamma_2\left(\frac{r}{k}\right)+4\gamma_0\gamma_3\left(\frac{r}{k}\right)+\gamma_4\left(\frac{r}{k}\right)\right]\right)$$

$$\zeta^{(5)}(0,q)=\sum_{r=1}^{k}\left(-120C_5\gamma_0\left(\frac{r}{k}\right)+C_4\left[120\gamma_0\gamma_0\left(\frac{r}{k}\right)+120\gamma_1\left(\frac{r}{k}\right)\right]\right.$$
$$+C_3\left[120\zeta(2)\gamma_0\left(\frac{r}{k}\right)-60\gamma_0^2\gamma_0\left(\frac{r}{k}\right)-120\gamma_0\gamma_1\left(\frac{r}{k}\right)-60\gamma_2\left(\frac{r}{k}\right)\right]$$
$$+C_2\left[-120\gamma_0\zeta(2)\gamma_0\left(\frac{r}{k}\right)+120\zeta(3)\gamma_0\left(\frac{r}{k}\right)-120\zeta(2)\gamma_1\left(\frac{r}{k}\right)+20\gamma_0^3\gamma_0\left(\frac{r}{k}\right)\right.$$
$$\left.+60\gamma_0^2\gamma_1\left(\frac{r}{k}\right)+60\gamma_0\gamma_2\left(\frac{r}{k}\right)+20\gamma_3\left(\frac{r}{k}\right)\right]$$
$$+C_1\left[60\gamma_0^2\zeta(2)\gamma_0\left(\frac{r}{k}\right)-120\gamma_0\zeta(3)\gamma_0\left(\frac{r}{k}\right)+120\gamma_0\zeta(2)\gamma_1\left(\frac{r}{k}\right)\right.$$
$$-60\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)+120\zeta(4)\gamma_0\left(\frac{r}{k}\right)-120\zeta(3)\gamma_1\left(\frac{r}{k}\right)+60\zeta(2)\gamma_2\left(\frac{r}{k}\right)$$
$$\left.-5\gamma_0^4\gamma_0\left(\frac{r}{k}\right)-20\gamma_0^3\gamma_1\left(\frac{r}{k}\right)-30\gamma_0^2\gamma_2\left(\frac{r}{k}\right)-20\gamma_0\gamma_3\left(\frac{r}{k}\right)-5\gamma_4\left(\frac{r}{k}\right)\right]$$
$$+C_0\left[-20\gamma_0^3\zeta(2)\gamma_0\left(\frac{r}{k}\right)+60\gamma_0^2\zeta(3)\gamma_0\left(\frac{r}{k}\right)-60\gamma_0^2\zeta(2)\gamma_1\left(\frac{r}{k}\right)\right.$$
$$+60\gamma_0\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)-120\gamma_0\zeta(4)\gamma_0\left(\frac{r}{k}\right)+120\gamma_0\zeta(3)\gamma_1\left(\frac{r}{k}\right)$$
$$-60\gamma_0\zeta(2)\gamma_2\left(\frac{r}{k}\right)-120\zeta(2)\zeta(3)\gamma_0\left(\frac{r}{k}\right)+120\zeta(5)\gamma_0\left(\frac{r}{k}\right)+60\zeta(2)^2\gamma_1\left(\frac{r}{k}\right)$$
$$-120\zeta(4)\gamma_1\left(\frac{r}{k}\right)+60\zeta(3)\gamma_2\left(\frac{r}{k}\right)-20\zeta(2)\gamma_3\left(\frac{r}{k}\right)+\gamma_0^5\gamma_0\left(\frac{r}{k}\right)$$
$$\left.\left.+5\gamma_0^4\gamma_1\left(\frac{r}{k}\right)+10\gamma_0^3\gamma_2\left(\frac{r}{k}\right)+10\gamma_0^2\gamma_3\left(\frac{r}{k}\right)+5\gamma_0\gamma_4\left(\frac{r}{k}\right)+\gamma_5\left(\frac{r}{k}\right)\right]\right)$$

$$\zeta^{(6)}(0,q)=\sum_{r=1}^{k}\left(720C_6\gamma_0\left(\frac{r}{k}\right)+C_5\left[-720\gamma_0\gamma_0\left(\frac{r}{k}\right)-720\gamma_1\left(\frac{r}{k}\right)\right]\right.$$
$$+C_4\left[360\gamma_0\left(\frac{r}{k}\right)\gamma_0^2+720\gamma_1\left(\frac{r}{k}\right)\gamma_0-720\zeta(2)\gamma_0\left(\frac{r}{k}\right)+360\gamma_2\left(\frac{r}{k}\right)\right]$$
$$+C_3\left[-120\gamma_0\left(\frac{r}{k}\right)\gamma_0^3-360\gamma_1\left(\frac{r}{k}\right)\gamma_0^2+720\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0-360\gamma_2\left(\frac{r}{k}\right)\gamma_0\right.$$
$$\left.-720\zeta(3)\gamma_0\left(\frac{r}{k}\right)+720\zeta(2)\gamma_1\left(\frac{r}{k}\right)-120\gamma_3\left(\frac{r}{k}\right)\right]$$
$$+C_2\left[30\gamma_0\left(\frac{r}{k}\right)\gamma_0^4+120\gamma_1\left(\frac{r}{k}\right)\gamma_0^3-360\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2+180\gamma_2\left(\frac{r}{k}\right)\gamma_0^2\right.$$
$$+720\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0-720\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0+120\gamma_3\left(\frac{r}{k}\right)\gamma_0+360\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)$$
$$\left.-720\zeta(4)\gamma_0\left(\frac{r}{k}\right)+720\zeta(3)\gamma_1\left(\frac{r}{k}\right)-360\zeta(2)\gamma_2\left(\frac{r}{k}\right)+30\gamma_4\left(\frac{r}{k}\right)\right]$$
$$+C_1\left[-6\gamma_0\left(\frac{r}{k}\right)\gamma_0^5-30\gamma_1\left(\frac{r}{k}\right)\gamma_0^4+120\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^3-60\gamma_2\left(\frac{r}{k}\right)\gamma_0^3\right.$$
$$-360\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2+360\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0^2-60\gamma_3\left(\frac{r}{k}\right)\gamma_0^2$$
$$-360\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)\gamma_0+720\zeta(4)\gamma_0\left(\frac{r}{k}\right)\gamma_0-720\zeta(3)\gamma_1\left(\frac{r}{k}\right)\gamma_0$$

$$+360\zeta(2)\gamma_2\left(\frac{r}{k}\right)\gamma_0 - 30\gamma_4\left(\frac{r}{k}\right)\gamma_0 + 720\zeta(2)\zeta(3)\gamma_0\left(\frac{r}{k}\right) - 720\zeta(5)\gamma_0\left(\frac{r}{k}\right)$$
$$-360\zeta(2)^2\gamma_1\left(\frac{r}{k}\right) + 720\zeta(4)\gamma_1\left(\frac{r}{k}\right) - 360\zeta(3)\gamma_2\left(\frac{r}{k}\right) + 120\zeta(2)\gamma_3\left(\frac{r}{k}\right)$$
$$-6\gamma_5\left(\frac{r}{k}\right)\Big]$$
$$+C_0\Big[\gamma_0\left(\frac{r}{k}\right)\gamma_0^6 + 6\gamma_1\left(\frac{r}{k}\right)\gamma_0^5 - 30\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^4 + 15\gamma_2\left(\frac{r}{k}\right)\gamma_0^4$$
$$+120\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0^3 - 120\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0^3 + 20\gamma_3\left(\frac{r}{k}\right)\gamma_0^3$$
$$+180\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)\gamma_0^2 - 360\zeta(4)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2 + 360\zeta(3)\gamma_1\left(\frac{r}{k}\right)\gamma_0^2$$
$$-180\zeta(2)\gamma_2\left(\frac{r}{k}\right)\gamma_0^2 + 15\gamma_4\left(\frac{r}{k}\right)\gamma_0^2 - 720\zeta(2)\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0$$
$$+720\zeta(5)\gamma_0\left(\frac{r}{k}\right)\gamma_0 + 360\zeta(2)^2\gamma_1\left(\frac{r}{k}\right)\gamma_0 - 720\zeta(4)\gamma_1\left(\frac{r}{k}\right)\gamma_0$$
$$+360\zeta(3)\gamma_2\left(\frac{r}{k}\right)\gamma_0 - 120\zeta(2)\gamma_3\left(\frac{r}{k}\right)\gamma_0 + 6\gamma_5\left(\frac{r}{k}\right)\gamma_0 - 120\zeta(2)^3\gamma_0\left(\frac{r}{k}\right)$$
$$+360\zeta(3)^2\gamma_0\left(\frac{r}{k}\right) + 720\zeta(2)\zeta(4)\gamma_0\left(\frac{r}{k}\right) - 720\zeta(6)\gamma_0\left(\frac{r}{k}\right)$$
$$-720\zeta(2)\zeta(3)\gamma_1\left(\frac{r}{k}\right) + 720\zeta(5)\gamma_1\left(\frac{r}{k}\right) + 180\zeta(2)^2\gamma_2\left(\frac{r}{k}\right) - 360\zeta(4)\gamma_2\left(\frac{r}{k}\right)$$
$$+120\zeta(3)\gamma_3\left(\frac{r}{k}\right) - 30\zeta(2)\gamma_4\left(\frac{r}{k}\right) + \gamma_6\left(\frac{r}{k}\right)\Big]\Big)$$

$$\zeta^{(7)}(0,q) = \sum_{r=1}^{k}\Big(-5040C_7\gamma_0\left(\frac{r}{k}\right) + C_6\Big[5040\gamma_0\gamma_0\left(\frac{r}{k}\right) + 5040\gamma_1\left(\frac{r}{k}\right)\Big]$$
$$+C_5\Big[-2520\gamma_0\left(\frac{r}{k}\right)\gamma_0^2 - 5040\gamma_1\left(\frac{r}{k}\right)\gamma_0 + 5040\zeta(2)\gamma_0\left(\frac{r}{k}\right) - 2520\gamma_2\left(\frac{r}{k}\right)\Big]$$
$$+C_4\Big(840\gamma_0\left(\frac{r}{k}\right)\gamma_0^3 + 2520\gamma_1\left(\frac{r}{k}\right)\gamma_0^2 - 5040\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0$$
$$+2520\gamma_2\left(\frac{r}{k}\right)\gamma_0 + 5040\zeta(3)\gamma_0\left(\frac{r}{k}\right) - 5040\zeta(2)\gamma_1\left(\frac{r}{k}\right) + 840\gamma_3\left(\frac{r}{k}\right)\Big]$$
$$+C_3\Big[-210\gamma_0\left(\frac{r}{k}\right)\gamma_0^4 - 840\gamma_1\left(\frac{r}{k}\right)\gamma_0^3 + 2520\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2$$
$$-1260\gamma_2\left(\frac{r}{k}\right)\gamma_0^2 - 5040\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0 + 5040\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0$$
$$-840\gamma_3\left(\frac{r}{k}\right)\gamma_0 - 2520\zeta(2)^2\gamma_0\left(\frac{r}{k}\right) + 5040\zeta(4)\gamma_0\left(\frac{r}{k}\right) - 5040\zeta(3)\gamma_1\left(\frac{r}{k}\right)$$
$$+2520\zeta(2)\gamma_2\left(\frac{r}{k}\right) - 210\gamma_4\left(\frac{r}{k}\right)\Big]$$
$$+C_2\Big[42\gamma_0\left(\frac{r}{k}\right)\gamma_0^5 + 210\gamma_1\left(\frac{r}{k}\right)\gamma_0^4 - 840\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^3 + 420\gamma_2\left(\frac{r}{k}\right)\gamma_0^3$$
$$+2520\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2 - 2520\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0^2 + 420\gamma_3\left(\frac{r}{k}\right)\gamma_0^2$$
$$+2520\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)\gamma_0 - 5040\zeta(4)\gamma_0\left(\frac{r}{k}\right)\gamma_0 + 5040\zeta(3)\gamma_1\left(\frac{r}{k}\right)\gamma_0$$
$$-2520\zeta(2)\gamma_2\left(\frac{r}{k}\right)\gamma_0 + 210\gamma_4\left(\frac{r}{k}\right)\gamma_0 - 5040\zeta(2)\zeta(3)\gamma_0\left(\frac{r}{k}\right)$$
$$+5040\zeta(5)\gamma_0\left(\frac{r}{k}\right) + 2520\zeta(2)^2\gamma_1\left(\frac{r}{k}\right) - 5040\zeta(4)\gamma_1\left(\frac{r}{k}\right)$$
$$+2520\zeta(3)\gamma_2\left(\frac{r}{k}\right) - 840\zeta(2)\gamma_3\left(\frac{r}{k}\right) + 42\gamma_5\left(\frac{r}{k}\right)\Big]$$
$$+C_1\Big[-7\gamma_0\left(\frac{r}{k}\right)\gamma_0^6 - 42\gamma_1\left(\frac{r}{k}\right)\gamma_0^5 + 210\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^4 - 105\gamma_2\left(\frac{r}{k}\right)\gamma_0^4$$
$$-840\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0^3 + 840\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0^3 - 140\gamma_3\left(\frac{r}{k}\right)\gamma_0^3$$

$$-1260\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)\gamma_0^2+2520\zeta(4)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2-2520\zeta(3)\gamma_1\left(\frac{r}{k}\right)\gamma_0^2$$

$$+1260\zeta(2)\gamma_2\left(\frac{r}{k}\right)\gamma_0^2-105\gamma_4\left(\frac{r}{k}\right)\gamma_0^2+5040\zeta(2)\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0$$

$$-5040\zeta(5)\gamma_0\left(\frac{r}{k}\right)\gamma_0-2520\zeta(2)^2\gamma_1\left(\frac{r}{k}\right)\gamma_0+5040\zeta(4)\gamma_1\left(\frac{r}{k}\right)\gamma_0$$

$$-2520\zeta(3)\gamma_2\left(\frac{r}{k}\right)\gamma_0+840\zeta(2)\gamma_3\left(\frac{r}{k}\right)\gamma_0-42\gamma_5\left(\frac{r}{k}\right)\gamma_0$$

$$+840\zeta(2)^3\gamma_0\left(\frac{r}{k}\right)-2520\zeta(3)^2\gamma_0\left(\frac{r}{k}\right)-5040\zeta(2)\zeta(4)\gamma_0\left(\frac{r}{k}\right)$$

$$+5040\zeta(6)\gamma_0\left(\frac{r}{k}\right)+5040\zeta(2)\zeta(3)\gamma_1\left(\frac{r}{k}\right)-5040\zeta(5)\gamma_1\left(\frac{r}{k}\right)$$

$$-1260\zeta(2)^2\gamma_2\left(\frac{r}{k}\right)+2520\zeta(4)\gamma_2\left(\frac{r}{k}\right)-840\zeta(3)\gamma_3\left(\frac{r}{k}\right)$$

$$+210\zeta(2)\gamma_4\left(\frac{r}{k}\right)-7\gamma_6\left(\frac{r}{k}\right)\Big]$$

$$+C_0\Big[\gamma_0\left(\frac{r}{k}\right)\gamma_0^7+7\gamma_1\left(\frac{r}{k}\right)\gamma_0^6-42\zeta(2)\gamma_0\left(\frac{r}{k}\right)\gamma_0^5+21\gamma_2\left(\frac{r}{k}\right)\gamma_0^5$$

$$+210\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0^4-210\zeta(2)\gamma_1\left(\frac{r}{k}\right)\gamma_0^4+35\gamma_3\left(\frac{r}{k}\right)\gamma_0^4$$

$$+420\zeta(2)^2\gamma_0\left(\frac{r}{k}\right)\gamma_0^3-840\zeta(4)\gamma_0\left(\frac{r}{k}\right)\gamma_0^3+840\zeta(3)\gamma_1\left(\frac{r}{k}\right)\gamma_0^3$$

$$-420\zeta(2)\gamma_2\left(\frac{r}{k}\right)\gamma_0^3+35\gamma_4\left(\frac{r}{k}\right)\gamma_0^3-2520\zeta(2)\zeta(3)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2$$

$$+2520\zeta(5)\gamma_0\left(\frac{r}{k}\right)\gamma_0^2+1260\zeta(2)^2\gamma_1\left(\frac{r}{k}\right)\gamma_0^2-2520\zeta(4)\gamma_1\left(\frac{r}{k}\right)\gamma_0^2$$

$$+1260\zeta(3)\gamma_2\left(\frac{r}{k}\right)\gamma_0^2-420\zeta(2)\gamma_3\left(\frac{r}{k}\right)\gamma_0^2+21\gamma_5\left(\frac{r}{k}\right)\gamma_0^2$$

$$-840\zeta(2)^3\gamma_0\left(\frac{r}{k}\right)\gamma_0+2520\zeta(3)^2\gamma_0\left(\frac{r}{k}\right)\gamma_0+5040\zeta(2)\zeta(4)\gamma_0\left(\frac{r}{k}\right)\gamma_0$$

$$-5040\zeta(6)\gamma_0\left(\frac{r}{k}\right)\gamma_0-5040\zeta(2)\zeta(3)\gamma_1\left(\frac{r}{k}\right)\gamma_0+5040\zeta(5)\gamma_1\left(\frac{r}{k}\right)\gamma_0$$

$$+1260\zeta(2)^2\gamma_2\left(\frac{r}{k}\right)\gamma_0-2520\zeta(4)\gamma_2\left(\frac{r}{k}\right)\gamma_0+840\zeta(3)\gamma_3\left(\frac{r}{k}\right)\gamma_0$$

$$-210\zeta(2)\gamma_4\left(\frac{r}{k}\right)\gamma_0+7\gamma_6\left(\frac{r}{k}\right)\gamma_0+2520\zeta(2)^2\zeta(3)\gamma_0\left(\frac{r}{k}\right)$$

$$-5040\zeta(3)\zeta(4)\gamma_0\left(\frac{r}{k}\right)-5040\zeta(2)\zeta(5)\gamma_0\left(\frac{r}{k}\right)+5040\zeta(7)\gamma_0\left(\frac{r}{k}\right)$$

$$-840\zeta(2)^3\gamma_1\left(\frac{r}{k}\right)+2520\zeta(3)^2\gamma_1\left(\frac{r}{k}\right)+5040\zeta(2)\zeta(4)\gamma_1\left(\frac{r}{k}\right)$$

$$-5040\zeta(6)\gamma_1\left(\frac{r}{k}\right)-2520\zeta(2)\zeta(3)\gamma_2\left(\frac{r}{k}\right)+2520\zeta(5)\gamma_2\left(\frac{r}{k}\right)$$

$$+420\zeta(2)^2\gamma_3\left(\frac{r}{k}\right)-840\zeta(4)\gamma_3\left(\frac{r}{k}\right)+210\zeta(3)\gamma_4\left(\frac{r}{k}\right)-42\zeta(2)\gamma_5\left(\frac{r}{k}\right)$$

$$+\gamma_7\left(\frac{r}{k}\right)\Big]\Big)$$

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